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On backward stability of holomorphic dynamical systems

by

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Abstract. For a polynomial with one critical point (maybe multiple), which does not have attracting or neutral periodic orbits, we prove that the backward dynamics is stable provided the Julia set is locally connected. The latter is proved to be equivalent to the non-existence of a wandering continuum in the Julia set or to the shrinking of Yoccoz puzzle-pieces to points.

1. Introduction. Let f be a non-affine polynomial considered as a dynamical system on the complex plane:

$$f : \mathbb{C} \to \mathbb{C}.$$

Recall that the *Julia set* J of f is the closure of the repelling periodic points of f. The Julia set of the polynomial f is a non-empty nowhere dense compact set on the plane, and $f^{-1}(J) = J = f(J)$. It is well known [F] that the forward dynamics of f on J is never stable: any arbitrary small disc which intersects the Julia set J of f becomes large (even covers J) under some iterate.

Is the backward dynamics of $f : J \to J$ stable? More precisely, are the components of the preimages of any small disc under the iterates of f small as well? In general, it is not true. Firstly, if there is a neutral fixed point of f which is not linearizable (i.e., belongs to the Julia set), then f is not backward stable at this point (this follows from the classical description of local dynamics). Moreover, there exist polynomials without neutral periodic orbits which are not backward stable on J: see e.g. Remark 2. On the other hand, the asymptotic backward stability of $f : J \to J$ is known to hold for the following classes of polynomials: hyperbolic [F], sub-hyperbolic [DH], and, more generally, for semi-hyperbolic [CJY] polynomials, and for

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Collet–Eckmann maps [Pr]. One of the results of the present paper is that the backward dynamics is stable for any polynomial of the form $z^l + c_1$ having locally connected Julia set (and all periodic points repelling). This is a consequence of Theorem 1 below.

In what follows we assume that all periodic points of the map $f : \mathbb{C} \to \mathbb{C}$ are repelling (for maps with neutral points, see Remark 3). This implies in particular that the compact J is *full*, i.e., the complement $\mathbb{C} \setminus J$ is connected.

By a *continuum* we always mean a connected compact non-one-point set.

Definition. A continuum $K \subset J$ is called *wandering* if

$$f^n(K) \cap f^m(K) = \emptyset \quad \text{for any non-negative } n \neq m.$$

Theorem 1. *Let $f(z) = z^l + c_1$, and assume that the Julia set J of f is connected. Then the following conditions are equivalent*:

(a) *J is locally connected*,

(b) *no continuum $K \subset J$ is wandering.*

Remark 1. The results and the ideas of the paper are expected to be true for all polynomials without attracting or neutral periodic orbits. (In fact, the proofs hold in the general case, except for Theorem 2 of Sect. 3, where modifications are needed: work in progress.) Nevertheless, there is a natural bound for generalizations: in the thesis of Pascale Roesch [R], a rational function is constructed (having attracting fixed points) which has a wandering continuum inside the locally connected Julia set.

In Section 2 we give a characterization of an arbitrary wandering continuum as an intersection of Yoccoz pieces. (This concerns the infinitely renormalizable case as well.) Thus, an equivalent statement of the theorem is:

The Julia set is locally connected if and only if the Yoccoz pieces shrink to points.

From this point of view, the implication (b)⇒(a) (the pieces shrink ⇒ the Julia set is locally connected) is well known and is used (after Yoccoz) to prove local connectivity. The statement (a)⇒(b) is proved in Section 4 and the proof is based on Theorem 2 of Section 3.

Let us write down two consequences of Theorem 1.

The first corollary was, in fact, a motivation for the present paper. One should compare it with Mañé's Lemma [Ma].

Corollary 1 (Backward stability). *If the Julia set J of the polynomial $f(z) = z^l + c_1$ is locally connected*, *then for every $\varepsilon > 0$ there exists $\delta > 0$ such that*, *for any point x of the plane*, *and for any component V of the preimage $f^{-n}(B(x, \delta))$, $n = 1, 2, \ldots$, of the disc $B(x, \delta) = \{z : |z - x| < \delta\}$, the diameter of V is less than ε.*

The following polynomials (without attracting and neutral periodic orbits) are shown to have locally connected Julia set: all finitely renormalizable quadratic polynomials $z^2 + c_1$ ([Y], [H], [Mi2]); special infinitely renormalizable quadratic polynomials [Ly]; all $z^l + c_1$ with c_1 real [LvS].

Let again f be a polynomial $z^l + c_1$ and, moreover, let f be infinitely renormalizable. Then there exists a maximal sequence of nested Julia sets J_i of the renormalizations of f which contain the critical point $c = 0$ (see [McM]). If

$$J_\infty = \bigcap_{i=1}^{\infty} J_i \neq \{0\},$$

such a polynomial is called *non-degenerate* in [P-M]. As J_∞ is wandering, an immediate consequence of Theorem 1 is the following statement (which can also be proved by the methods of [P-M]):

Corollary 2. *Let $f(z) = z^l + c_1$ be infinitely renormalizable and non-degenerate. Then the Julia set of f is not locally connected.*

Remark 2. First striking examples of infinitely renormalizable quadratic polynomials with Julia set not locally connected are due to A. Douady (see [Mi2]). A simpler and more general approach is developed in [P-M]. Note that any known such polynomial is non-degenerate [P-M]. Finally, observe that Douady's polynomials are *not* backward stable.

Remark 3. In fact, the proofs in the paper hold (with minor changes) in the presence of Cremer points (since the Julia set is full in this case as well). As f having a Cremer point is backward stable it gives yet another proof of local disconnectedness of the Julia set of polynomials with such points.

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2. Yoccoz's structure. In this section we describe all wandering continua (if any) via the Yoccoz structure, and prove the easier part of Theorem 1: "no wandering continuum ⇒ Julia set is locally connected". Given a polynomial f of degree $l \geq 2$ with all periodic points repelling, let us build its Yoccoz structure (cf. [H], [Mi2], [Y]).

Denote by $\beta_1, \ldots, \beta_{l-1}$ those fixed points of f which are the landing points of $l-1$ external rays of f of external arguments $0, 1/(l-1), 2/(l-1), \ldots, (l-2)/(l-1)$ respectively (i.e., fixed by f). Let α be the remaining fixed point of f. Assume for simplicity the Julia set is connected. Then α is the landing point of finitely many external rays with non-zero rational combinatorial rotation number p/q, where $q \geq 2$ (see e.g. [Mi1]).

Let $R_1, \ldots, R_{q'}$ be all external rays landing at α. Fix an equipotential curve Γ, and let W_0 be a bounded component of $\mathbb{C} \setminus \Gamma$. The components of $W_0 \setminus \bigcup_{i=1}^{q'} R_i$ are called the *Yoccoz* (*open*) *pieces of depth zero* $Y_0^{(i)}, 1 \leq i \leq q'$. All components of the preimages $f^{-k}(Y_0^{(i)})$, $1 \leq i \leq q'$, are the *open pieces of depth* $k \geq 0$. Let $Y_0 \supset Y_1 \supset Y_2 \supset \ldots$ be a sequence of nested pieces. Denote by

$$K = \bigcap_{n=1}^{\infty} \overline{Y}_n$$

the non-empty intersection of their closures. K is either a point, or a continuum. Now we distinguish two cases.

(1) Every continuum K obtained as above is wandering. Then *the final Yoccoz structure is the union of the pieces of all depths constructed above*.

(2) For some continuum K as above, and for some positive integers n, m $(n \neq m)$, $f^n(K) \cap f^m(K) \neq \emptyset$. By the construction, either

$$f^n(K) = f^m(K)$$

or $f^n(K)$ intersects $f^m(K)$ at the point α (see [McM]). Because the combinatorial rotation number of α is rational, in the latter case again $f^{n'}(K) = f^{m'}(K)$ (with other $n' \neq m'$). In either case an image J_1 of K under an iterate of f must contain a critical point of f (otherwise K would be a point: see e.g. [Mi2]), so that $f^{N_1}(J_1) = J_1$ for some *minimal* $N_1 \geq 2$. Let Y_{n_1} be a piece of depth n_1 so that $J_1 \subset \overline{Y}_{n_1}$ and, moreover,

$$J_1 = \{x \in \overline{Y}_{n_1} : f^{iN_1}(x) \in \overline{Y}_{n_1},\ i = 0, 1, \ldots\}$$

(a renormalization of f). Call N_1 the *period* of this renormalization. Define the *Yoccoz structure of the first renormalization* as the union of the previous pieces up to depth $n_1 - 1$. Then repeat the procedure. Namely, let $\alpha_1 \in J_1$ be a dividing fixed point of $f^{N_1} : J_1 \to J_1$. It is the landing point of finitely many (but at least two) external rays of f. The forward images of these rays under iterates of f divide the piece Y_{n_1} and all other pieces of depth n_1 of the previous (first) renormalization, which are met by the forward trajectory of J_1, into finitely many components (since the rays are periodic). Call them *the zero depth pieces of the structure of the second renormalization*. Taking all the components of their preimages under f^k, we either finish the construction, or come to the next renormalization. Then we proceed as before, constructing *the Yoccoz structure of the third renormalization*, and so on.

Define the *height* of a piece Y as the unique $h \geq 0$ such that the boundary of Y contains arcs of equipotential $f^{-h}(\Gamma)$.

By construction, the intersection of a nested sequence Y^h of (closed) pieces, where h is the height of the piece Y^h, is either a point, or a wandering continuum as $h \to \infty$ (we have proved this if the number of renormalizations is finite; if it is infinite, the intersection is wandering because the periods of the renormalizations tend to infinity). If there are no such continua, the intersection is a point. Then this gives a sequence of shrinking connected neighbourhoods in J for any point of J different from the boundary points of the pieces. (As for a boundary point x of a piece, the Julia set is always locally connected at x; see [Mi2] for details.) Thus, "no wandering continua $\Rightarrow$ the Yoccoz pieces shrink to points", and, therefore, the Julia set is locally connected.

3. Trees in the Julia set. Here we construct a tree in the Julia set which will be used to prove: "Julia set J is locally connected $\Rightarrow$ any continuum in J is non-wandering". (By a "tree" we mean a finite or countable union of arcs Γ_j so that $\bigcup_{j=1}^{i} \Gamma_j$ intersects Γ_{i+1} at its end point.) Our standing assumption is that the Julia set J is locally connected.

Given a non-dividing fixed point of f, we construct the tree in J which generalizes the Hubbard tree for critically finite polynomials (see [DH]). Fix such a fixed point β (one of the points $\beta_1, \ldots, \beta_{l-1}$ introduced at the beginning of Section 2). Let, say, the external argument of β be zero.

Since J is connected, locally connected, full (i.e., the complement $\mathbb{C} \setminus J$ is connected), and J is nowhere dense, for any two points $x, y \in J$ there exists a unique arc $[x, y]$ (homeomorphic image of $[0, 1]$) joining x and y inside J. Observe that any iterate $f^n([x, y])$ is either the arc $[f^n(x), f^n(y)]$, or a finite tree which contains $[f^n(x), f^n(y)]$. (There are no closed loops in J.) Moreover, if the latter is the case, then $[x, y]$ contains a critical point of f^n, and any extreme point of this tree different from $f^n(x)$, $f^n(y)$ is a critical value of f^n.

For the fixed point β, let us construct the tree T_β as follows. Let $\gamma_1, \ldots, \gamma_{l-1}$ be all the f-preimages of β different from β. Consider first the union (tree)

$$t = [\gamma_1, \beta] \cup \ldots \cup [\gamma_{l-1}, \beta].$$

Then define

$$T_\beta = \bigcup_{n=0}^{\infty} f^n(t).$$

Then T_β is a tree with (generally speaking) infinitely many branch points having remarkable properties (some of them) listed below.

T1. The tree t (and, hence, T_β) contains the unique dividing fixed point α. Indeed, at least two different external rays land at α. Since α is fixed by f, all iterates of these rays again land at α, and also have different first digits in the l-base expansions of their arguments. So they are in different sectors (components) of $\mathbb{C} \setminus (t \cup r)$ where r is the union of the external rays to β and γ_i, $i = 1, \ldots, l-1$. So the meeting point α belongs to t.

Similar considerations show that any point in J with at least 2 external arguments eventually (under iterates) hits t.

T2. The arc $[\alpha, \beta]$ contains a critical point of f. Indeed, otherwise f maps $[\alpha, \beta]$ onto itself and one-to-one. Then $f : [\alpha, \beta] \to [\alpha, \beta]$ has an attracting fixed point. A contradiction. Similarly, each arc $[\gamma_i, \gamma_j]$ and $[\gamma_i, \beta]$, $i \neq j$, $i, j = 1, \ldots, l-1$, contains a critical point of f (because $f(\gamma_i) = f(\gamma_j) = f(\beta)$). So all the arcs meet at the critical point $c = 0$, which is therefore a branch point in t of order l. (Namely the number of edges of t landing at c is l. Formally this is a branch point if $l > 2$.) Note that this argument holds if one does not exclude the presence of Cremer fixed points. Then one gets in $[\alpha, \beta]$ a point attracting at least from one side (this is either α or β), which is not possible by the "snail argument" [Mi1].

T3. The set T_β is forward invariant, i.e. $f : T_\beta \to T_\beta$.

T4. T_β is a tree. The branch points (we call them *vertices*) of the tree T_β have the properties described in the following

THEOREM 2. *Let $f(z) = z^l + c_1$. Then any vertex of T_β is either a preimage under iterates of a critical point of f, or is a* (*pre*)*periodic point for f having at least three external arguments.*

This result (for $l = 2$) follows from the main theorem of the theory of Thurston's laminations [Th] (see also [Ki]), but we give an independent, completely elementary, and purely topological proof of this fundamental fact (for all l).

Proof of Theorem 2. Set $t_1 = f(t)$. If $t_1 \subset t$, then there is nothing to prove (since in this case $f^n(t) \subset t$ for any $n > 0$). Otherwise $t_1 \setminus t$ is an arc. Now we apply

LEMMA. *Let $T' \subset T_\beta$ be a finite subtree such that $t \subset T'$. Assume that $f(T') \setminus T'$ consists of a unique arc t', and denote by x' the base point of t', i.e.*,

$$x' = T' \cap \overline{t'}.$$

Then there exists $p \geq 1$ *such that*

$$f^k(\overline{t'}) \cap T' = \{f^k(x')\} \quad \text{for } k = 0, \ldots, p-1,$$

and one of the following possibilities holds:

(1) $f^{p-1}(x')$ *is the critical point of* f;
(2) $f^{p-1}(x')$ *is an extreme point of* T';
(3) $f^p(x')$ *is a vertex of* $T' \cup f(T')$.

REMARK 4. Case (2) can happen only if: either x' itself is an extreme point in T', or case (1) happens for some p. This case is distinguished for the completeness of the scenario, it is not discussed in Theorem 2.

CLAIM. *Assume that all iterates* $f^k(x')$ *lie in* T', $k = 0, 1, 2, \ldots,$ *and* x' *is neither a preimage of a critical point of* f *under iterates nor a preimage of an extreme point of* T'. *Then either* x' *is a vertex of* T', *or there is* $q \geq 1$ *such that* $f^q(x')$ *is a vertex of* $T' \cup f(T')$.

Proof. If x' is already a vertex of T', the claim holds. Otherwise a neighborhood of x' in T' is an arc. If, for every k, the point $f^k(x')$ is not a vertex of $T' \cup f(T')$, then $f^k(t')$ is an arc, which sticks out of T' at $f^k(x')$. Then two external rays to x' which are not separated by T', but are separated by t', will never be separated by T' by the forward iterates by f. A contradiction. This proves the claim.

By the Claim, if all $f^k(x')$ lie in T', then either (1), (2), or (3) happens. If some $f^k(x')$ leaves T' for the first time, and neither (1), (2), nor (3) happens, then the arcs $t', \ldots, f^{k-1}(t')$ stick out of T' while the arc $f^k(t')$ sticks out of t'. Then we iterate the tree $t' \cup f^k(t')$ (which grows up from the tree T' at the point x') k times until it leaves T' and sticks out of the tree $t' \cup f^k(t')$, then we iterate the tree $t' \cup f^k(t') \cup f^k(t' \cup f^k(t'))$, and so on, so that x' will never return to T', in particular to t, under the iterates. This contradicts the last assertion of **T1**. The Lemma is proved.

To finish the proof of Theorem 2, it is enough to find a sequence $k_n \to \infty$ such that the trees

$$T'_n := \bigcup_{i=0}^{k_n} f^i(t), \qquad n = 0, 1, 2, \ldots,$$

have the following properties:

($**$) any vertex of T'_n is either preperiodic or precritical,
($***$) $f(T'_n) \setminus T'_n$ consists of a unique arc.

(If, for some n, this arc degenerates to a point, the final tree has finitely many vertices.)

To this end, we set $T'_0 = t$. Obviously, ($**$)–($***$) are true for the tree T'_0.

We now describe the procedure of passage from T'_n to T'_{n+1}, proving by induction the properties $(**)$–$(***)$.

The set $f(T'_n)\setminus T'_n$ consists of one arc t' with base point x', by the induction hypothesis. Apply the Lemma to the tree T'_n, and take the corresponding $p \geq 1$. By the Lemma, either (1), (2), or (3) holds (in particular, x' is either precritical, or an extreme point of T'_n (see Remark 4), or preperiodic, by the induction hypothesis). If (1) or (3) holds, we set

$$T'_{n+1} = T'_n \cup f(T'_n) \cup \ldots \cup f^p(T'_n).$$

Observe that

$$T'_{n+1} = T'_n \cup t' \cup f(t') \cup \ldots \cup f^{p-1}(t').$$

If (2) holds, we keep iterating until a q-iterate ($q > 0$) of the point x' becomes a non-extreme point of the tree T'_n, and then set

$$T'_{n+1} = T'_n \cup f(T'_n) \cup \ldots \cup f^q(T'_n).$$

(If $q = \infty$, the proof of the theorem is finished.) Observe that

$$T'_{n+1} = T'_n \cup t' \cup f(t') \cup \ldots \cup f^{q-1}(t').$$

and that $x', \ldots, f^{q-1}(x')$ are not vertices of T'_{n+1}.

Then $(**)$–$(***)$ hold for the tree T'_{n+1}, and the induction step is completed.

Remark 5. In the Lemma, one can choose $p' \geq p$ in such a way that the statements of the Lemma hold with p' instead p, and with an extra property in case (3): $f^{p'}(\overline{t'}) \cap (T' \cup f(T'))$ is a subarc of $T' \cup f(T')$ (this shows how new vertices can appear under the iterates of the initial tree t).

Let us prove this. First, let $f^p(x')$ be fixed by f (more generally, by an iterate of f). Then iterating the arc $f^p(t')$ further, we find p' with the desired property. Indeed, take two external rays R_1 and R_2 which land at $x := f^p(x')$ and are such that no other rays land at x in the component of $\mathbb{C} \setminus (R_1 \cup R_2)$ which contains the arc $\gamma := f^q(t')$. Since x is not a precritical point, there exists $q \geq 0$ such that the rays $f^q(R_1)$ and $f^q(R_2)$ start with different digits (in their d-expansions). Hence, $f^q(x) \in t$ and, moreover, the rays $f^q(R_1)$ and $f^q(R_2)$ land at $f^q(x)$ from different sides of the tree t (i.e., separated by t). If $f^q(\gamma) \cap t$ is not an arc (i.e., just the one-point set $f^p(x)$), then there exists a ray R different from $f^q(R_1)$ and $f^q(R_2)$ that lands at $f^q(x)$ and lies in the same component of $\mathbb{C}\setminus(f^q(R_1) \cup f^q(R_2))$ which contains $f^q(\gamma)$. Pulling back R by f^q from $f^q(x)$ to x, we get another ray landing at x between R_1 and R_2, which contradicts the choice of these rays. Thus $f^q(\gamma) \cap t$ is an arc.

Second, let $f^p(x')$ be not fixed by f. Assume that case (3) never happens. Consider $f^{p+1}(t')$. It is an arc which sticks out of T' at $f^{p+1}(t')$ (otherwise $p' = p + 1$). Repeating again the considerations at the beginning of the proof of the claim, we find that some $f^{q_1}(x')$, $q_1 > q$, is again a vertex of $T' \cup f(T')$, and so on till we get back to one of the points already passed through. Hence, $f^p(x')$ is (pre)periodic, and p' exists.

Concluding the remark, let us note that the periodicity is needed here in order that branching of T' at an image is not larger than at $f^p(x')$.

4. Proof of the main statements

Proof of the implication (a)⇒(b) *of the Theorem*. Let $K \subset J$ be a continuum in the locally connected J. Then K contains a non-trivial arc γ (for example, given two points $x \neq y$ in K, the arc $[x, y]$ must belong to K, because J does not separate the plane). Since any point of γ different from the end points has at least two external arguments, it eventually lands at the tree t. Moreover, there exists n so that $f^n(\gamma)$ contains a subarc (denote it again by γ) which lies in the tree T_β. Indeed, x and y eventually hit t, and one can assume that $f^i(x) \neq f^j(y)$ for all i, j.

By a *semineighborhood* of a point z in γ we mean one of the two components of $U \setminus \gamma$, where U is a small enough neighborhood of z. It defines two sides of the point z. If x is any point of γ close enough to z, there is an external ray of f which lands at x on a given side of z. Take two such points x_1 and x_2 in γ and two corresponding external rays R_1 and R_2 on the same side of z. Then some iterate of f "separates" R_1 and R_2: they will land on different sides of the tree T_β (different "sectors" in the complement of the union of t and the rays to $f^{-1}(\beta)$). Therefore, this (or earlier) image of $[x_1, x_2]$ under an iterate of f covers either the critical point or a vertex of T_β. In the former case, we can repeat the considerations. In the latter case, because of property **T4** (Theorem 2) of the tree, we always end up with capturing either a critical point for two different iterates, or a periodic vertex. Thus in either case K is not wandering.

Proof of Corollary 1. Let J be locally connected. Consider the Yoccoz structure starting, for example, from the fixed point α, or from any other periodic point with rational non-zero combinatorial rotation number. Then by Theorem 1, the intersection of any nested sequence Y^h of closed pieces is a point as h tends to ∞. Moreover, since for any two open pieces with non-empty intersection one contains the other, "no wandering continuum" again implies that the maximum of the diameters of all pieces of a given height h tends to zero as h tends to ∞.

Let us prove the backward stability. Fix $\varepsilon > 0$. Choose also a closed neighborhood W of J bounded by some equipotential. By normality considerations, there exists $\delta_1 > 0$ such that for any point x outside W, and for any component V of $f^{-n}(B(x, \delta_1))$, $n = 1, 2, \ldots$, the diameter of V is less than ε. It remains to find $\delta_2 > 0$ with this property uniformly for all $x \in W$.

By a standard compactness argument, it is enough to find δ_2 for any $x \in W$ separately (i.e., $\delta_2 > 0$ depending on $x \in W$). If $x \in W$ is outside J, δ_2 exists by the same normality considerations. Fix $x \in J$. If x is a boundary point of a piece, then either it is one of the dividing periodic points, or it becomes an interior point of pieces of higher heights. In the former case, by a standard argument [Mi2], one can consider a finite union of pieces with this point at the boundary. Thus, in any case it is enough to deal with the point $x \in J$ which is inside pieces. Then choose the height h so great that the diameter of any piece of this height is less than ε, and take $\delta_2 > 0$ such that $B(x, \delta_2)$ is inside a piece of this height.

Added in proof. Using the method of the paper, Theorem 2 has been generalized to any polynomial with all periodic points repelling; see A. Blokh and G. Levin, "Trees in Julia sets". Consequently, this yields a generalization of Corollary 1 and Theorem 1 ("no wandering continuum" if and only if "J is locally connected") to such polynomials. J. Kiwi has recently told me how a different proof of the latter statement follows from [Ki]. I thank him for helpful discussions.

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