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Luzin and anti-Luzin almost disjoint families

by

Judith Roitman (Lawrence, Kan.) and **Lajos Soukup** (Budapest)

Abstract. Under MA_{ω_1} every uncountable almost disjoint family is either anti-Luzin or has an uncountable Luzin subfamily. This fails under CH. Related properties are also investigated.

0. Introduction. This paper concerns two combinatorial properties of almost disjoint families, Luzin and anti-Luzin, along with some variants.

Let $\mathcal{A}$ be an uncountable almost disjoint family on a countable set W.

DEFINITION 0.1. $\mathcal{A}$ is *Luzin* iff there is an enumeration $\mathcal{A} = \{a_\alpha : \alpha < \omega_1\}$ such that for all $w \in [W]^{<\omega}$ and $\alpha < \omega_1$, $\{\beta < \alpha : a_\alpha \cap a_\beta \subset w\}$ is finite.

DEFINITION 0.2. $\mathcal{A}$ is *anti-Luzin* iff for all $\mathcal{B} \in [\mathcal{A}]^{\omega_1}$ there exist $\mathcal{C}, \mathcal{D} \in [\mathcal{B}]^{\omega_1}$ such that $\bigcup \mathcal{C} \cap \bigcup \mathcal{D}$ is finite.

The notion of "Luzin" is somewhat standard. The notion of "anti-Luzin" is new.

Luzin almost disjoint families are an analogue of Hausdorff gaps: they exist (via a diagonal construction) in ZFC, and because of the finitary nature of the definition the "Luzin" property is upwards absolute. Perhaps because of this similarity, they have been called Luzin *gaps*.

Anti-Luzin families also exist in ZFC, as canonical objects: any uncountable set of branches of a countable tree T is an anti-Luzin family on T.

Luzin and anti-Luzin families are hereditary in the following sense: uncountable subfamilies of Luzin (respectively anti-Luzin) almost disjoint families are Luzin (respectively anti-Luzin).

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Definition 0.3. $\mathcal{A}$ is *near-Luzin* iff for all $\mathcal{C},\mathcal{D} \in [\mathcal{A}]^{\omega_1}$, $\bigcup\mathcal{C} \cap \bigcup\mathcal{D}$ is infinite.

We will rely heavily on the following obvious statement: $\mathcal{A}$ is anti-Luzin iff it has no uncountable near-Luzin subfamily.

Near-Luzin first appeared in [HJ], where it was called ω_1-full: near-Luzin families give rise to compact Hausdorff spaces in which the intersection of any two uncountable open sets is non-empty.

Claim 0.4. *If $\mathcal{A}$ is Luzin then it is near-Luzin.*

Proof. Fix an arbitrary enumeration $\{a_\alpha : \alpha < \omega_1\}$ of $\mathcal{A}$. Suppose $\mathcal{A}$ is not near-Luzin. Then there are uncountable $\mathcal{C},\mathcal{D} \subset \mathcal{A}$ with $\bigcup\mathcal{C} \cap \bigcup\mathcal{D} = w$ finite. So for all $c \in \mathcal{C}$ and $d \in \mathcal{D}$, $c \cap d \subset w$. There is α with $a_\alpha \in \mathcal{C}$ and $\{a_\beta \in \mathcal{D} : \beta < \alpha\}$ infinite. Hence $\mathcal{A}$ is not Luzin. ∎

Corollary 0.5. *An anti-Luzin family is not Luzin.*

While superficially Corollary 0.5 does not reverse (a disjoint union of a Luzin and an anti-Luzin family is neither Luzin nor anti-Luzin), does it reverse in any deep sense? In particular, must any uncountable almost disjoint family which does not embed one embed the other? The answer is yes and no.

Theorem 0.6. *Assume* MA_{ω_1}. *Every uncountable almost disjoint family is either anti-Luzin or contains an uncountable Luzin subfamily.*

Theorem 0.7. *Assume* $\P$. *There is an uncountable almost disjoint family which contains no uncountable anti-Luzin and no uncountable Luzin subfamilies.*

Here $\P$ is the following weakening of CH: There is a family $\mathcal{S} \subset [\omega_1]^\omega$ of size ω_1 so that every uncountable subset of ω_1 contains a set in $\mathcal{S}$.

Theorem 0.6 says that under MA_{ω_1}, almost disjoint families have a lot of structure. Theorem 0.7 says that under $\P$ they do not. This used to be what one would expect, but recent work on iterating totally proper forcing while preserving CH has changed our expectations. In particular, Abraham and Todorčević showed the consistency of "CH + all (ω_1,ω_1)-gaps contain an uncountable Hausdorff subgap". If Luzin almost disjoint families were combinatorially similar to Hausdorff gaps the conclusion of Theorem 0.6 would also be consistent with CH. Thus Theorem 0.7 destroys the parallel between Luzin almost disjoint families and Hausdorff gaps.

In Section 1 we prove Theorem 0.6, in Section 2 we prove Theorem 0.7, in the rest of the paper we explore some of the fine combinatorial structure of these notions.

Conventions. In this paper almost disjoint families are collections of infinite sets whose pairwise intersections are finite; the superscript "*" means "mod finite"; all trees grow upward; and properties are listed consecutively no matter what theorem, lemma, or definition they occur in, so if there is a reference to property 17 the reader can easily find it.

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1. Proof of Theorem 0.6. Assume MA_{ω_1}. Let $\mathcal{A} = \{a_\alpha : \alpha < \omega_1\}$ be an almost disjoint family which is not anti-Luzin. By moving to a possibly smaller subfamily, we may assume $\mathcal{A}$ is near-Luzin.

Let $\mathbb{P}$ be the set $[\omega_1]^{<\omega}$ under the following partial order:

$$p \le q \quad \text{iff} \quad p \supset q \quad \text{and} \quad \forall \alpha \in p \setminus q \ \forall \beta \in q \text{ if } \alpha < \beta \text{ then } a_\alpha \cap a_\beta \not\subset k_q$$

where $k_q = \max \bigcup \{a_\alpha \cap a_\beta : \{\alpha, \beta\} \in [q]^2\}$.

REMARK 1.1. If $\beta > \sup p$ then $p \cup \{\beta\} \le p$.

A quick ad hoc definition: for $s, t \subset \mathrm{ON}$, $s \ll t$ iff $\sup s < \inf t$.

REMARK 1.2. If r, s, t are disjoint, $r \ll s \cup t$, $p = r \cup s$, $q = r \cup t$ and $k_p = k_q = k$, then p and q are compatible iff for all $\alpha \in s$ and $\beta \in t$, $a_\alpha \cap a_\beta \not\subset k$.

By Remark 1.1, enough sets are dense so that if $\mathbb{P}$ is ccc then, by MA_{ω_1}, there is a generic filter G so that $\{a_\alpha : \alpha \in \bigcup G\}$ is Luzin. Remark 1.2 will be used to show ccc.

LEMMA 1.3. *Suppose $n, k \in \omega$ and let $E \in [[\omega_1]^n]^{\omega_1}$ be pairwise disjoint. Then there are $s \ne t \in E$ with $a_\alpha \cap a_\beta \not\subset k$ for all $\alpha \in s$ and $\beta \in t$.*

To prove Lemma 1.3 we need

SUBLEMMA 1.3.1. *If S, T are uncountable disjoint subsets of ω_1 and $k < \omega$ then there are $S' \in [S]^{\omega_1}$, $T' \in [T]^{\omega_1}$ and $k' > k$ with $k' \in a_\alpha \cap a_\beta$ for all $\alpha \in S'$ and $\beta \in T'$.*

Proof. Otherwise for all $k' > k$ either $S(k') = \{\alpha \in S : k' \in a_\alpha\}$ is countable or $T(k') = \{\alpha \in T : k' \in a_\alpha\}$ is countable. Let $S'' = S \setminus \bigcup \{S(k') : k' > k \text{ and } S(k') \text{ is countable}\}$. Let $T'' = T \setminus \bigcup \{T(k') : k' > k \text{ and } T(k') \text{ is countable}\}$. Then S'' and T'' are uncountable and $\bigcup_{\alpha \in S''}(a_\alpha \setminus k) \cap \bigcup_{\beta \in T''}(a_\beta \setminus k) = \emptyset$, contradicting our hypothesis on $\mathcal{A}$. ■

Proof of Lemma 1.3. For $s \in E$ we write $s = \{\alpha(s, i) : i < n\}$. Let $\{(i_m, j_m) : m < n^2\}$ enumerate $n \times n$. Using Sublemma 1.3.1 iteratively, at each stage m we have uncountable disjoint subsets of E, $E(m)$ and $F(m)$, with $E(m) \supset E(m+1)$ and $F(m) \supset F(m+1)$, and $k_m > k$ with $k_m \in a_{\alpha(s,i_m)} \cap a_{\alpha(t,j_m)}$ for all $s \in E(m)$ and $t \in F(m)$. But then for $\alpha \in s \in E(n^2 - 1)$ and $\beta \in t \in F(n^2 - 1)$, $a_\alpha \cap a_\beta \not\subset k$. ■

LEMMA 1.4. $\mathbb{P}$ *is ccc.*

Proof. Given an uncountable subset F of $\mathbb{P}$ we may without loss of generality assume that for some $n, F \subset [\omega_1]^n$; F is a Δ-system with root r so that each $p \in F$ has the form $r \cup s_p$; there is some k with $k = k_p$ for all $p \in F$; and $E = \{s_p : p \in F\}$ is well-ordered by $\ll$. By Lemma 1.3 there are $p \neq q \in F$ so that for all $\alpha \in s_p$ and $\beta \in s_q$, $a_\alpha \cap a_\beta \not\subset k$. By Remark 1.2, p and q are compatible. ∎

2. Proof of Theorem 0.7. In this section we prove Theorem 0.7. In the next section we will give a stronger version, but the combinatorics is sufficiently complicated that it makes sense to give the weaker proof first and then show how to improve it.

Let $\{S_\alpha : \alpha < \omega_1\}$ be a $\clubsuit$-sequence, i.e. each $S_\alpha \subset \alpha$ and every uncountable $X \subset \omega_1$ contains some S_α.

The family $\mathcal{A} = \{f_\alpha : \alpha < \omega_1\}$ will be a subset of $\wp(\omega \times \omega)$, where each f_α is a function from ω to ω.

We require:

1. If $\alpha \neq \gamma$ then $\{i : f_\alpha(i) = f_\gamma(i)\}$ is finite.
2. If $\beta \leq \alpha$ then $f_\alpha \cap \bigcup_{\gamma \in S_\beta} f_\gamma$ is infinite.
3. If $\beta \leq \alpha$ then there is n such that $\{\gamma \in S_\beta : f_\alpha \cap f_\gamma \subset n \times n\}$ is infinite.

Property 1 makes $\mathcal{A}$ almost disjoint and property 2 makes it near-Luzin (hence prevents an uncountable anti-Luzin subfamily). Finally, property 3 prevents an uncountable Luzin subfamily. Indeed, let $\{f_{\alpha_\nu} : \nu < \omega_1\}$ be an enumeration of an uncountable subfamily of $\mathcal{A}$. Pick $\beta < \omega_1$ with $S_\beta \subset \{\alpha_\nu : \nu < \omega_1\}$. Let $I = \{\nu : \alpha_\nu \in S_\beta\}$ and fix $\mu < \omega_1$ such that $\alpha_\mu \geq \beta$ and $\mu > \sup I$. Then, by 3, there is $m \in \omega$ such that $\{\nu < \mu : f_{\alpha_\nu} \cap f_{\alpha_\mu} \subset m \times m\}$ is infinite, which contradicts the requirement formulated in Definition 0.1. So $\{f_{\alpha_\nu} : \nu < \omega_1\}$ is not Luzin.

Some preliminaries:

DEFINITION 2.1. A set $F \subset \omega \times \omega$ is *fat* iff $\limsup_n |\pi_n F| = \omega$, where $\pi_n F = \{j : (n, j) \in F\}$.

Notice that if there is a finite family of functions U with $F \subset^* \bigcup U$ then F is not fat, and fat sets are infinite.

LEMMA 2.2. *If $\mathcal{C}$ is an infinite almost disjoint family of functions from ω to ω, then $\bigcup \mathcal{C}$ is fat.*

Proof. Fix $m < \omega$. Let $G \in [\mathcal{C}]^{m+1}$. There is r so that if $f \neq g \in G$ then $f \restriction [r, \omega) \cap g \restriction [r, \omega) = \emptyset$. So $m < \limsup_n |\pi_n \bigcup G|$. ∎

DEFINITION 2.3. Let $\mathcal{C}$ be a countable almost disjoint collection of functions from ω to ω. A finite partial function σ from ω to ω is *$\mathcal{C}$-free* iff $\exists^\infty g \in \mathcal{C}$ $g \cap \sigma = \emptyset$.

LEMMA 2.4. *Let $\mathcal{C}$ be a countable collection of functions from ω to ω. There is a function $s : \omega \to \omega + 1$ so that if σ is a finite partial function from ω to ω with $\sigma(i) \neq s(i)$ for all $i \in \operatorname{dom} \sigma$, then σ is $\mathcal{C}$-free.*

Such an s is called *$\mathcal{C}$-tight*.

Proof. Consider $\mathcal{C}$ as a subset of the compact space $(\omega+1)^\omega$. Let s be an accumulation point of $\mathcal{C}$ in $(\omega+1)^\omega$. If σ is a finite partial function from ω to ω with $s \cap \sigma = \emptyset$ then $s \notin \{f : f \cap \sigma \neq \emptyset\}$ which is closed, so $\mathcal{C} \setminus \{f : f \cap \sigma \neq \emptyset\}$ is infinite. ∎

The construction is a straightforward induction, given the following

LEMMA 2.5. *Let $\mathcal{C}$ be a countable almost disjoint collection of functions from ω to ω, let $\mathcal{C}_n \subset \mathcal{C}$ for each $n < \omega$, and let F_n be fat for each $n < \omega$. Then there is a function $f : \omega \to \omega$ so that*:

4. *$\{f\} \cup \mathcal{C}$ is almost disjoint.*
5. *For each n, $f \cap F_n$ is infinite.*
6. *$\forall n \ \exists m_n \ \{g \in \mathcal{C}_n : f \cap g \subset m_n \times \{f(i) : i < m_n\}\}$ is infinite.*

The family $\{f_\alpha : \alpha < \omega_1\}$ will be constructed recursively in ω_1 steps. Assume that $\{f_\alpha : \alpha < \beta\}$ is already constructed. Fix an enumeration $\{\beta_n : n < \omega\}$ of β. Let $\mathcal{C} = \{f_\alpha : \alpha < \beta\}$, $\mathcal{C}_n = \{f_\alpha : \alpha \in S_{\beta_n}\}$ and $F_n = \bigcup \mathcal{C}_n$. Now we can apply Lemma 2.5 to get f_β as f.

In the next section, we will need to deal with many more fat sets, which is why 2.5 is stated in its current generality.

Proof of Lemma 2.5. Let $\mathcal{C} = \{g_i : i < \omega\}$. At stage j we construct a finite set U_j ("U" is short for "used up") of functions in $\mathcal{C}$ where $U_{j-1} \subset U_j$; we define $m_j \geq j$, and define f on $(m_{j-1}, m_j]$.

So suppose we are at stage j. We know m_k for each $k < j$, $f \restriction m_{j-1} + 1$, U_{j-1}, and, for each $k < j$, we have a $\mathcal{C}_k$-tight s_k. Our induction hypothesis is that

$$\forall k < j \ f \restriction (m_{k-1}, m_{j-1}] \cap s_k \restriction (m_{k-1}, m_{j-1}] = \emptyset.$$

Let s_j be $\mathcal{C}_j$-tight. Since F_k is fat, for $k \leq j$ there is some $r_{k,j} > m_{j-1}$ with $r_{k,j} < r_{k+1,j}$, and some $(r_{k,j}, t_{k,j}) \in F_k \setminus \bigcup U_{j-1}$. Let $m_j = r_{j,j}$. Let $r_{-1,j} = m_{j-1}$.

For $k \leq j$, let $f(r_{k,j}) = t_{k,j}$. (This is towards property 5.)

For $i \in (r_{-1,j}, r_{j,j})$ with $i \neq r_{k,j}$ for each k, let $f(i)$ be any $m \notin \{g(i) : i \in U_{j-1}\} \cup \{s_k(i) : k \leq j\}$. (This is towards properties 4 and 6.)

For each $k \le j$,

$$f \restriction (m_{k-1}, m_j] \cap s_k \restriction (m_{k-1}, m_j] = \emptyset$$

so $f \restriction (m_{k-1}, m_j]$ is $\mathcal{C}_k$-free, for each $k \le j$. Hence, for each $k \le j$ there is $g_{k,j} \in \mathcal{C}_k \setminus U_{j-1}$ with

$$g_{k,j} \cap f \restriction (m_{k-1}, m_j] = \emptyset.$$

Let

$$U_j = U_{j-1} \cup \{g_{k,j} : k \le j\} \cup \{g_j\}.$$

Property 4 is satisfied: If $n \le j$ and $i \in (m_{j-1}, m_j]$ then $g_n(i) \ne f(i)$, so

$$f \cap g_n \subset m_n \times \{f(i) : i < m_n\}.$$

Property 5 is satisfied: $f(r_{n,j}) = t_{n,j}$ for all $j \ge n$ and $(r_{n,j}, t_{n,j}) \in F_n$, so $f \cap F_n$ is infinite.

Property 6 is satisfied: If $j \ge n$ then $f \restriction (m_{n-1}, m_j] \cap g_{n,j} = \emptyset$, so $g_{n,j} \cap f \subset m_n \times \{f(i) : i < m_n\}$. ∎

3. A strengthening of Theorem 0.7. In this section we strengthen Theorem 0.7.

Definition 3.1. An uncountable almost disjoint family $\mathcal{A}$ is *strongly near-Luzin* iff, for every $\mathcal{C}_0, \ldots, \mathcal{C}_n \in [\mathcal{A}]^{\omega_1}$, $\bigcap_{i \le n} \bigcup \mathcal{C}_i$ is infinite.

Strongly near-Luzin families appear in [JN], where they are called strong Luzin families. They cannot exist under $\mathrm{MA} + \neg\,\mathrm{CH}$. The following theorem shows that they need not be Luzin.

Theorem 3.2. *Assume* $\clubsuit$. *There is an uncountable almost disjoint family which is strongly near-Luzin*, *but has no uncountable Luzin subfamilies.*

The proof is somewhat like that of Theorem 0.7, but the combinatorics is more complicated, so complicated that we will invoke elementary submodels to avoid stating it explicitly.

So let $\{S_\alpha : \alpha < \omega_1\}$ be a $\clubsuit$-sequence. We begin by strengthening property 2 to

7. If $\beta_0, \ldots, \beta_n < \alpha$ and $\sup S_{\beta_i} \cup (\beta_i + 1) < \inf S_{\beta_{i+1}}$ for each i, then $f_\alpha \cap \bigcap_{i \le n} \bigcup_{\gamma \in S_{\beta_i}} f_\gamma$ is infinite.

We will be done if the sequence of f_α's has properties 1, 7, and 3. Indeed, as we have seen in the proof of Theorem 0.7, property 3 implies that $\mathcal{A}$ does not contain an uncountable Luzin subfamily. Property 1 yields that $\mathcal{A}$ is almost disjoint. So we need to show that if property 7 holds then $\mathcal{A}$ is strongly near-Luzin. Let $n \in \omega$ and $I_0, \ldots, I_{n-1} \in [\omega_1]^{\omega_1}$. Since $\{S_\alpha : \alpha < \omega_1\}$ is a $\clubsuit$-sequence we can find $\beta_0 < \beta_1 < \ldots < \beta_{n-1} \in \omega_1$ such that $S_{\beta_i} \subset I_i$ and $\sup S_{\beta_{i-1}} \cup (\beta_{i-1} + 1) < \min S_{\beta_i}$. Then, by property 7,

$\bigcap_{i<n}\bigcup\{f_\gamma : \gamma \in I_i\} \supset \bigcap_{i<n}\bigcup\{f_\gamma : \gamma \in S_{\beta_i}\}$ is infinite, which was to be proved.

In applying 2.5 in the previous section we had the luxury of knowing that each $\bigcup\mathcal{C}_n$ was fat. But an intersection of fat sets need not be fat. So we must ensure that the following property holds:

8. If $\beta_0,\ldots,\beta_n < \alpha$ and $\sup S_{\beta_i} \cup (\beta_i+1) < \inf S_{\beta_{i+1}}$ for each i, then $\bigcap_{i\le n}\bigcup_{\gamma\in S_{\beta_i}} f_\gamma$ is fat.

Property 8 allows us to construct a family in which property 7 holds. How will we build a family in which property 8 holds?

DEFINITION 3.4. For $F \subset \omega\times\omega$ and $E \in [\omega]^\omega$ let $\pi_E F = F \cap (E\times\omega)$.

To get property 8 to hold, we need to start with enough fat C's, then have enough E's so that the resulting $\pi_E C$'s are fat, iterate the process... Rather than try to define the precise combinatorics of "enough", we take advantage of elementary submodels which provide all the fat sets we need.

Along with constructing our sequence of functions f_α, then, we will construct a sequence of large enough countable elementary submodels $\{N_\alpha : \alpha < \omega_1\}$ where

9. $f_\alpha, S_\alpha, \{N_\beta : \beta \le \alpha\} \in N_{\alpha+1}$ for each α, $\{N_\beta : \beta < \alpha\} \subset N_\alpha$, and $\{S_\alpha : \alpha < \omega_1\} \in N_0$.

Further requirements are:

1. If $\alpha \ne \gamma$ then $\{i : f_\alpha(i) = f_\gamma(i)\}$ is finite.
10. If $C \in N_\alpha$ is fat and $E = \{n : (n, f_\alpha(n)) \in C\}$ then $\pi_E C$ is fat.
11. If $C \in N_\alpha$ is fat, then $f_\alpha \cap C$ is infinite.
12. If $S \in N_\alpha \cap [\alpha]^\omega$ then there is $m \in \omega$ such that $\{\gamma \in S : f_\gamma \cap f_\alpha \subset m\times m\}$ is infinite.

Note that property 11 follows from property 10.

As before, property 1 yields that $\{f_\alpha : \alpha < \omega_1\}$ is almost disjoint. It remains to show that properties 10 and 11 imply property 8 (which implies property 7, which implies strongly near-Luzin), and property 12 implies there are no uncountable Luzin subfamilies.

LEMMA 3.5. *Suppose* $\mathcal{A} = \{f_\alpha : \alpha < \omega_1\}$ *and* $\{N_\alpha : \alpha < \omega_1\}$ *satisfy* 1, 9, 10, 11 *and* 12. *Then*

(a) $\mathcal{A}$ *has property* 8.

(b) $\mathcal{A}$ *has no uncountable Luzin subfamily.*

Proof. (a) We show by induction on n that if $\beta_0 < \ldots < \beta_n$ and $\sup S_{\beta_i} < \inf S_{\beta_{i+1}}$ for each i then $\bigcap_{i\le n}\bigcup_{\gamma\in S_{\beta_i}} f_\gamma$ is fat. So suppose $\beta_0 < \ldots < \beta_n$, $\sup S_{\beta_i} < \inf S_{\beta_{i+1}}$ for all i, and $C = \bigcap_{i<n}\bigcup_{\gamma\in S_{\beta_i}} f_\gamma$ is fat. Let $\alpha < \inf S_{\beta_n}$ with $\beta_{n-1} \in N_\alpha$. Then $C \in N_\alpha$. Let $\{\gamma_j : j < \omega\} \subset$

S_{β_n} with $\gamma_j < \gamma_{j+1}$. Define $\{E_j : j < \omega\}$ and $\{C_j : j < \omega\}$ as follows: $E_0 = \{k : (k, f_{\gamma_0}(k)) \in C\}$, $C_0 = \pi_{E_0} C$, $E_{j+1} = \{k : (k, f_{\gamma_{j+1}}(k)) \in C_j\}$, $C_{j+1} = \pi_{E_{j+1}} C_j$. By property 9, $C_j, E_j \in N_{\beta_{j+1}}$ for all i. By property 10, each C_j is fat.

But then, by property 11, each $C_j \cap f_{\gamma_j}$ is infinite, so since $\mathcal{A}$ is almost disjoint and $C_j \supset C_{j+1}$, $C \cap \bigcup\{f_{\gamma_j} : j < \omega\}$ is fat.

(b) Given an uncountable subfamily $\mathcal{B}$ of $\mathcal{A}$ and an enumeration $\mathcal{B} = \{g_\alpha : \alpha < \omega_1\}$, where $g_\alpha = f_{\phi(\alpha)}$, there are $\alpha \le \beta < \omega_1$ so that $S_\alpha \subset \operatorname{ran}\phi \cap \beta$ and $\phi''\beta = \operatorname{ran}\phi \cap \beta$. Let $\delta = \phi(\beta) \ge \beta$. Note that $\alpha \in N_\alpha \subset N_\beta \subset N_\delta$, so, by property 12, there is m such that $\{\gamma \in S_\alpha : f_\gamma \cap f_\delta \subset m \times m\}$ is infinite. Since $g_\beta = f_\delta$ and $\phi^{-1} S_\alpha \subset \beta$ it follows that $\mathcal{B}$ is not Luzin. ∎

Now notice that the construction used in the proof of 2.5 easily adapts to a construction of a family with properties 1, 9, 11, and 12. To get property 10, $r_{k,j}$ is required to satisfy

$$|\pi_{r_{k,j}} C_k| > j,$$

which can be done because C_k is fat.

4. Trees and anti-Luzin families. The canonical example of an anti-Luzin family is a set of branches of a countable perfect tree, i.e. a countable tree such that there are two incomparable nodes above every node. What about the reverse? Must every anti-Luzin family look like the branches of a tree?

Definition 4.1. An uncountable almost disjoint family $\mathcal{A}$ is a *tree family* iff there is a tree ordering $\mathcal{T} = (\bigcup \mathcal{A}, \prec)$ so that for every $a \in \mathcal{A}$ there is a branch b of $\mathcal{T}$ with $a =^* b$.

We will show that under CH + "there exists a Suslin line" there is an anti-Luzin family which contains no uncountable tree families.

Question 4.2. *Is there* (*under ZFC alone*) *an anti-Luzin family which contains no uncountable tree families?*

We do not know the answer to this question, but we have a related MA + ¬CH result.

Definition 4.3. An almost disjoint family $\mathcal{A}$ is a *hidden tree family* iff for some infinite $T \subset \bigcup \mathcal{A}$ the set $\{a \cap T : a \in \mathcal{A}\}$ is a tree family.

Hidden tree families need not be anti-Luzin. For example, let $\mathcal{A} = \{a_\alpha : \alpha < \omega_1\}$ be a tree family on the set of even integers, and let $\mathcal{B} = \{b_\alpha : \alpha < \omega_1\}$ be Luzin on the set of odd integers. Then $\{a_\alpha \cup b_\alpha : \alpha < \omega_1\}$ is both Luzin and a hidden tree family.

In fact, under MA + ¬CH all uncountable almost disjoint families of size $< 2^\omega$ are hidden tree families.

THEOREM 4.4. *Assume* MA(*precaliber* ω_1). *Then every uncountable almost disjoint family of size* $< 2^\omega$ *on* ω *is a hidden tree family.*

Proof. We define $p \in \mathbb{P}$: $p = (T_p, \prec_p, \mathcal{A}_p, h_p)$ where

- $T_p \in [\omega]^{<\omega}$, $(T_p, \prec_p)$ is a finite tree,
- $\mathcal{A}_p \subset \mathcal{A}$ is finite,
- $h_p : \mathcal{A}_p \to \omega$ is a function,
- each $(a \setminus h_p(a)) \cap T_p$ is linearly ordered by $\prec_p$,
- $\forall a \in \mathcal{A}_p\ \forall n \in a \cap (T_p \setminus h_p(a))\ \forall k \in T_p \setminus h_p(a)$ if $k \prec_p n$ then $k \in a$.

We define $p \le q$ iff

(a) $(T_q, \prec_q)$ is an initial subtree of $(T_p, \prec_p)$,
(b) $\mathcal{A}_q \subset \mathcal{A}_p$,
(c) $h_q \subset h_p$.

$\mathbb{P}$ is easily seen to have precaliber ω_1.

SUBCLAIM 4.4.1. *For each* $a \in \mathcal{A}$, *the set* $D_a = \{p \in \mathbb{P} : a \in \mathcal{A}_p\}$ *is dense in* $\mathbb{P}$.

Proof. Assume that $a \notin \mathcal{A}_p$. Let $q = (T_p, \prec_p, \mathcal{A}_p \cup \{a\}, h_p \cup (a, n))$, where $n > \max T_p$. Then $q \in \mathbb{P}$ because $a \cap (T_p \setminus h_p(a)) = \emptyset$. The relation $q \le p$ is clear. ∎

SUBCLAIM 4.4.2. *For each* $n \in \omega$ *and* $a \in \mathcal{A}$,

$$D_{a,n} = \{p \in \mathbb{P} : a \in \mathcal{A}_p \text{ and } a \cap (T_p \setminus n) \ne \emptyset\}$$

is dense in $\mathbb{P}$.

Proof. Let $p \in \mathbb{P}$. By Subclaim 4.4.1 we can assume that $a \in \mathcal{A}_p$. Let

$$k \in a \setminus \bigcup (\mathcal{A}_p \setminus \{a\}) \setminus \max\{T_p \cup n + 1\}.$$

Let $q = (T_p \cup \{k\}, \prec_q, \mathcal{A}_p, h_p)$, where $\prec_p \subset \prec_q$ and $l \prec_q k$ for each $l \in a \cap (T_p \setminus h_p(a))$. Since $k \notin a'$ for $a' \in \mathcal{A}_p \setminus \{a\}$ and $a \cap (T_p \setminus h_p(A)$ is linearly ordered by $\prec_p$ we have $q \in \mathbb{P}$ and clearly $q \le p$. ∎

Let

$$\mathcal{D} = \{D_{a,n} : a \in \mathcal{A},\ n \in \omega\}.$$

By MA(precaliber ω_1) we have a $\mathcal{D}$-generic filter $\mathcal{G}$. Then

$$\mathcal{T} = (T, \prec) = \Big(\bigcup_{p \in \mathcal{G}} T_p, \bigcup_{p \in \mathcal{G}} \prec_p \Big)$$

witnesses that $\mathcal{A}$ is a hidden tree family: taking $h = \bigcup_{p \in \mathcal{G}} h_p$ we see that $(T \cap a) \setminus h(a)$ is a tail of a branch of $\mathcal{T}$. ∎

In contrast we get under CH + "there exists a Suslin line" an anti-Luzin family which has no uncountable hidden tree families. In fact, we get something stronger.

DEFINITION 4.5. An uncountable almost disjoint family $\mathcal{A}$ is a *weak tree family* iff there is a tree ordering $\mathcal{T} = (\bigcup \mathcal{A}, \prec)$ and a 1-1 function $\phi : \mathcal{A} \to \mathrm{Br}(T)$ (here $\mathrm{Br}(T)$ is the set of branches of T) where range ϕ is pairwise disjoint, and each $a \subset^* \phi(a)$.

$\mathcal{A}$ is a *hidden weak tree family* iff, for some T, $\{a \cap T : a \in \mathcal{A}\}$ is a weak tree family.

Weak tree families appeared in [V] where they are called neat families. Velickovic proved the following result (Lemma 2.3 of [V]): *Assume* $\mathrm{MA}_{\aleph_1}$. *If* $\mathcal{A} \subset [\omega]^\omega$ *is an almost disjoint family then there is an uncountable family* $\mathcal{B} \subset \mathcal{A}$ *and a partition* $b = b_0 \cup b_1$ *for each* $b \in \mathcal{B}$ *such that* $\mathcal{B}_i = \{b_i : b \in \mathcal{B}\}$ *is a weak tree family for* $i \in 2$.

REMARK. One can consider the following weakening of the notion of weak tree families. An uncountable almost disjoint family $\mathcal{A}$ is a *very weak tree family* iff there are a tree ordering $\mathcal{T} = (\bigcup \mathcal{A}, \prec)$ and a function $\phi : \mathcal{A} \to [\mathrm{Br}(T)]^{<\omega}$ such that the range of ϕ is pairwise disjoint and each $a \subset^* \bigcup \phi(a)$. $\mathcal{A}$ is a *hidden very weak tree family* iff, for some T, $\{a \cap T : a \in \mathcal{A}\}$ is a very weak tree family.

However, as observed by the referee, a hidden very weak tree family can be split into countably many hidden tree families: for every element x of $\mathcal{A}$ fix a node of the tree such that above this node x is covered by a single branch, and split $\mathcal{A}$ accordingly. ∎

THEOREM 4.6. *Assume* CH + *"there exists a Suslin line". Then there is an uncountable anti-Luzin almost disjoint family which contains no uncountable hidden weak tree families.*

First, a quick lemma.

LEMMA 4.7. *Let* $\mathcal{T}^*$ *be Aronszajn, and* B *an uncountable set of branches of* $\mathcal{T}^*$ *so that no two elements of* B *have the same order type. Then there are incompatible elements* $s, t \in \mathcal{T}^*$ *so that* $\{b \in B : s \in b\}$ *and* $\{b \in B : t \in b\}$ *are uncountable.*

Proof. By contraposition, suppose B is a set of branches of T^* with different order types so that if s and t are incompatible then either $B(s) = \{b \in B : s \in b\}$ is countable or $B(t) = \{b \in B : t \in b\}$ is countable. Then $S = \{s : B(s) \text{ is uncountable}\}$ forms a chain, hence is countable. So there is α with $T^*(\alpha) \cap S = \emptyset$, where $T^*(\alpha)$ is the set of elements of $\mathcal{T}^*$ of height α. But all but countably many elements of B are in $\bigcup_{s \in T^*(\alpha)} B(s)$, so B is countable. ∎

Proof of Theorem 4.6. Let $\mathcal{T}^*$ be a Suslin tree so that every element has successors at arbitrarily high levels, and for each $t \in \mathcal{T}^*$ construct $c_t \in [\omega]^\omega$ so that if $s < t$ then $c_s \supset^* c_t$ and if s, t are not comparable then $c_s \cap c_t =^* \emptyset$. Let B be an uncountable set of branches of $\mathcal{T}^*$ so that no two elements of B have the same order type and every element of $\bigcup B$ is in uncountably many branches of B.

Let $B = \{b_\alpha : \alpha < \omega_1\}$. We will define $\mathcal{A} = \{a_\alpha : \alpha \in \omega_1\}$ where:

13. For all $s \in b_\alpha$, $c_s \supset^* a_\alpha$.

$\mathcal{A}$ will clearly be almost disjoint.

Let $\{\mathcal{T}_\alpha = (T_\alpha, \prec_\alpha) : \alpha < \omega_1\}$ enumerate all perfect trees whose underlying set is some infinite subset of ω.

We further require:

14. For all $\beta < \alpha$ either for some $s \in b_\alpha$, $c_s \cap T_\beta$ is contained, mod finite, in a branch of $\mathcal{T}_\beta$, or $a_\alpha \cap T_\beta$ is not a subset, mod finite, of a branch of $\mathcal{T}_\beta$.

The family $\mathcal{A}$ is constructed recursively in ω_1 steps. In the αth step we apply Lemma 4.8 below to get a_α.

LEMMA 4.8. *Suppose* $\{a_\beta : \beta < \alpha\}$ *has property* 13. *Then there is a set* a *with properties* 13 *and* 14.

Proof. Rather than describe the proof as an induction, we will (equivalently) use the Rasiowa–Sikorski lemma (see [K, Theorem 2.21]), defining a countable set of forcing conditions and countably many dense sets so that any generic filter meeting the dense sets gives rise to the desired object.

The partial order is as follows: $\mathbb{P}$ consists of all pairs $p = (a_p, b_p)$ where a_p is a finite subset of ω and b_p is a finite subset of b_α. The order is as follows: $p \le q$ iff $a_p \supset a_q$, $b_p \supset b_q$, and $a_p \setminus a_q \subset \bigcap_{t \in b_q} c_t$.

Clearly $\{p : |a_p| > n\}$ and $\{p : s \in b_p\}$ are dense for each $n < \omega$ and $s \in b_\alpha$, so if $\mathcal{G}$ is a filter meeting each of these dense sets then $\bigcup_{p \in \mathcal{G}} a_p$ has property 13.

Towards property 14, fix $\mathcal{T} = \mathcal{T}_\beta$ and $T = T_\beta$. We may assume that for every $t \in b_\alpha$, $c_t \cap T$ is not a subset, mod finite, of a branch of $\mathcal{T}$. For each $n < \omega$ define

$$D(\mathcal{T}, n) = \{p : a_p \setminus n \text{ contains } \mathcal{T}\text{-incomparable elements}\}.$$

We show that $D(\mathcal{T}, n)$ is dense for each n.

Fix $q \notin D(\mathcal{T}, n)$. Let $c = \bigcap_{t \in b_q} c_t$. Since $c \cap T \setminus n$ is not a subset, mod finite, of a branch of $\mathcal{T}$, there are two $\prec_\beta$-incompatible elements, ξ and η, of $T \cap (c \setminus n)$. Set $a_p = a_q \cup \{\xi, \eta\}$ and $b_p = b_q$. Then $p \in D(\mathcal{T}, n)$ and $p \le q$.

If, for all n, $\mathcal{G}$ meets $D(\mathcal{T}, n)$, $\bigcup_{p \in \mathcal{G}} a_p \cap T$ will not be a subset, mod finite, of any branch of $\mathcal{T}$. ∎

The following two lemmas, once proved, will complete the proof of Theorem 4.6.

LEMMA 4.9. *If property* 14 *holds*, *then $\mathcal{A}$ has no uncountable hidden weak tree families.*

Proof. This is where we use the fact that $\mathcal{T}^*$ is Suslin.

So suppose $\mathcal{B}$ is an uncountable subset of $\mathcal{A}$, $T \subset \omega$ is infinite, and $\mathcal{C} = \{a \cap T : a \in \mathcal{B}\}$ is a collection of infinite sets. We show that $\mathcal{C}$ is not a weak tree family.

Let $\mathcal{T} = (T, \prec)$ and $\phi : \mathcal{C} \to \mathrm{Br}(\mathcal{T})$. We show that ϕ does not have the properties of Definition 4.5.

For some β, $\mathcal{T} = \mathcal{T}_\beta$.

Let $S = \{s \in \mathcal{T}^* : c_s \cap T$ is a subset, mod finite, of a branch of $\mathcal{T}\}$. If $S = \emptyset$, then by property 14 for all but countably many $a \in \mathcal{B}$, $a \cap T \not\subset^* \phi(a \cap T)$.

Suppose $s \in S$ and $c_s \cap T \subset^* d$, where d is a branch of $\mathcal{T}$. Then if $s \in b_\alpha$ we have $a_\alpha \cap T \subset^* d$. Let S' be the set of $\mathcal{T}^*$-minimal elements of S. Since $\mathcal{T}^*$ is Suslin, S' is countable. So there is $s \in S'$ with $\{b_\alpha : a_\alpha \in \mathcal{B}$ and $s \in b_\alpha\}$ uncountable. But then either there are uncountably many α with $a_\alpha \not\subset \phi(a_\alpha \cap T)$, or ϕ is not 1-1. ∎

LEMMA 4.10. *If property* 13 *holds*, *then $\mathcal{A}$ is anti-Luzin.*

Proof. Suppose $\mathcal{B}$ is an uncountable subset of $\mathcal{A}$. By Lemma 4.7 there are incompatible $s, t \in \mathcal{T}^*$ with $\mathcal{C} = \{b_\alpha \in \mathcal{B} : s \in b_\alpha\}$ and $\mathcal{D} = \{b_\alpha \in \mathcal{B} : t \in b_\alpha\}$ both uncountable. Without loss of generality, we may assume that for some $n \ge \sup c_s \cap c_t$ if $b_\alpha \in \mathcal{C}$ then $a_\alpha \setminus c_s \subset n$ and if $b_\alpha \in \mathcal{D}$ then $a_\alpha \setminus c_t \subset n$. But then $\bigcup \mathcal{C} \cap \bigcup \mathcal{D} \subset n$, as desired. ∎

5. Between near-Luzin and strongly near-Luzin

DEFINITION 5.1. An uncountable almost disjoint family $\mathcal{A}$ is *k-near-Luzin* iff for every $\mathcal{C}_0, \ldots, \mathcal{C}_{k-1} \in [\mathcal{A}]^{\omega_1}$, $\bigcap_{i<k} \bigcup \mathcal{C}_i$ is infinite.

The purpose of this section is to show that these notions are (consistently) distinct.

Clearly near-Luzin is 2-near-Luzin and so every Luzin family is 2-near-Luzin, but does not necessarily contain a 3-near-Luzin subfamily, as we will see in Theorem 5.9.

THEOREM 5.2. *The following is consistent*: *for each $k \in [2, \omega)$ there is an uncountable almost disjoint family $\mathcal{A}_k$ which is k-near-Luzin*, *contains no uncountable Luzin subfamilies*, *and contains no uncountable* $(k+1)$*-near-Luzin subfamilies.*

The proof proceeds by showing that for every k there is a partial order $\mathbb{P}_k$ with precaliber ω_1 forcing $\mathcal{A}_k$ to exist, and iterating with precaliber ω_1. It is easy to see that both k-near-Luzin and "no Luzin subfamilies" are preserved by precaliber ω_1 forcing. The way we ensure no $(k+1)$-near-Luzin subfamilies will also be preserved by precaliber ω_1 forcing.

In contrast to our earlier constructions, $|\mathcal{A}_k| = 2^\omega$ for each k. Each $\mathcal{A}_k$ is again a family of functions, but instead of functions on ω the domains come from a k-linked not $(k+1)$-linked family $\mathcal{E}_k$ with special properties. This family was first constructed by Hajnal; the construction appeared in [JS].

LEMMA 5.3. *For all $k < \omega$ there is a family $\mathcal{E}_k \subset [\omega]^\omega$ with $|\mathcal{E}_k| = 2^\omega$ so that*:

15. *If $e_0, \ldots, e_{k-1} \in \mathcal{E}_k$ then $\bigcap_{i<k} e_i$ is infinite.*
16. *If $e_0, \ldots, e_k$ are distinct elements of $\mathcal{E}_k$ then $|\bigcap_{i\le k} e_i| < \omega$.*
17. *If $X \in [\mathcal{E}_k]^{\omega_1}$ then there are $X_0, \ldots, X_k \in [X]^{\omega_1}$ with*

$$\Big|\bigcap_{i\le k}\bigcup X_i\Big| < \omega.$$

Proof. Let $S_k = \{[^n2]^k : n < \omega\}$. We will construct $\mathcal{E}_k$ as a subset of $[S_k]^\omega$; since $|S_k| = \omega$, this construction proves the lemma. For $f : \omega \to 2$ let $e_f = \bigcup_{n<\omega}\{s \in [2^n]^k : f\restriction n \in s\}$. Let $\mathcal{E}_k = \{e_f : f \in 2^\omega\}$.

We show that property 15 holds: Given $f_0, \ldots, f_{k-1}$ distinct, pick m so $\{f_i\restriction m : i < k\}$ are distinct. But then for each $n \ge m$,

$$\{f_i\restriction n : i < k\} \in \bigcap_{i<k} e_{f_i}.$$

We show that property 16 holds: Given $f_0, \ldots, f_k$ distinct, there is m so that $\{f_i\restriction m : i \le k\}$ are distinct. But then $\bigcap_{i\le k} e_{f_i} \subset \bigcup_{j<m}[2^j]^k$.

We show that property 17 holds: Fix $X \in [\mathcal{E}]^{\omega_1}$. Let $I = \{f : e_f \in X\}$ and let $g_0, \ldots, g_k$ be distinct complete accumulation points of I in the usual topology on 2^ω. Fix n so that $g_0\restriction n, \ldots, g_k\restriction n$ are distinct. Define $X_i = \{e_f \in \mathcal{E} : f\restriction n = g_i\restriction n\}$. Then

$$\bigcap_{i\le k}\bigcup X_i \subset \bigcap_{j<n}[2^j]^k. \ \blacksquare$$

Note that by construction properties 15 and 16 are absolute in the following sense: Let $\mathcal{E} = (\mathcal{E}_k)^M$, and let $M \subset N$, where M, N are models of enough set theory. Then 15 and 16 hold for $\mathcal{E}$ in N.

The next lemma says that property 17 is preserved in some models. In this and succeeding proofs we will refer to the following easy fact about ccc forcing:

Fact 5.4. *If $\mathbb{P}$ is a ccc partial order, P is an uncountable subset of $\mathbb{P}$ and $\dot{G}$ names the generic filter, then there is $p \in \mathbb{P}$ such that $p \Vdash$ "$P \cap \dot{G}$ is uncountable".*

Lemma 5.5. *Suppose $\mathcal{E}$ has the property that if $X \in [\mathcal{E}]^{\omega_1}$ then there are $X_0, \ldots, X_k \in [X]^{\omega_1}$ with*

$$\Big|\bigcap_{i\le k}\bigcup X_i\Big| < \omega.$$

Then $\mathcal{E}$ will still have this property in a forcing extension by a precaliber ω_1 partial order.

Proof. Let $\mathbb{P}$ have precaliber ω_1, and suppose $\Vdash_{\mathbb{P}} \dot{X} = \{\dot{x}_\alpha : \alpha < \omega_1\} \in [\mathcal{E}]^{\omega_1}$. Fix $p \in \mathbb{P}$. For each α pick $p_\alpha \le p$ so that for some $e_\alpha \in \mathcal{E}$, $p_\alpha \Vdash \dot{x}_\alpha = e_\alpha$. Then there is an uncountable centered family $\{p_\alpha : \alpha \in I\}$. Let $Y = \{e_\alpha : \alpha \in I\}$. By hypothesis there are $I_0, \ldots, I_k \in [I]^{\omega_1}$ and $m < \omega$ with

$$\bigcap_{i\le k}\bigcup\{e_\alpha : \alpha \in I_i\} \subset m.$$

Let $\dot{G}$ be the generic filter, and define

$$\dot{J}_i = \{\alpha \in I_i : p_\alpha \in \dot{G}\}.$$

List each I_i as $\{\alpha^i_\gamma : \gamma < \omega_1\}$. Let $q_\gamma \le p_{\alpha^i_\gamma}$ for all $i \le k$. Let $Q = \{q_\gamma : \gamma < \omega_1\}$. By Fact 5.4 there is $p \in \mathbb{P}$ with $p \Vdash |Q \cap \dot{G}| = \omega_1$. So $p \Vdash$ "$\dot{J}_i$ is uncountable" for each $i \le k$, which by a density argument completes the proof. ∎

Lemma 5.6. *Suppose $\mathcal{E} \subset \wp(\omega)$ satisfies the following: if $X \in [\mathcal{E}]^{\omega_1}$ there are $X_0, \ldots, X_k \in [X]^{\omega_1}$ with*

$$\Big|\bigcap_{i\le k}\bigcup X_i\Big| < \omega.$$

If $\mathcal{A} = \{f_e : e \in \mathcal{E}\}$ where each $f_e : e \to \omega$ then $\mathcal{A}$ has no uncountable $(k+1)$-near Luzin subfamilies.

Proof. If for some finite m,

$$\bigcap_{i\le k}\bigcup X_i \subset m$$

then

$$\bigcap_{i\le k}\bigcup_{e\in X_i} f_e \subset m \times \omega.$$

Let $Y_i \in [X_i]^{\omega_1}$ so $\exists \sigma_i \ \forall e \in Y_i \ f_e \restriction m = \sigma_i$. Then

$$\bigcap_{i \le k} \bigcup_{e \in Y_i} f_e \subset \bigcap_i \sigma_i.$$

This completes the proof. ■

Proof of Theorem 5.2. Let $\mathcal{E}_k \subset [\omega]^\omega$ be a family satisfying 15–17 of Lemma 5.3. We have $\mathcal{A}_k = \{f_e : e \in \mathcal{E}_k\}$, where $f_e : e \to \omega$ is a function for $e \in \mathcal{E}_k$. By Lemma 5.6, this assumption guarantees that $\mathcal{A}_k$ has no $(k+1)$-near-Luzin subfamily.

Define $\mathbb{P}_k$, a precaliber ω_1 forcing which adds generic almost disjoint functions $f_e : e \to \omega$ for $e \in \mathcal{E}_k$, as follows:

- elements of $\mathbb{P}_k$ have the form $p = \{\sigma_{p,e} : e \in E_p\}$,
- E_p is a finite subset of $\mathcal{E}_k$,
- each $\sigma_{p,e}$ is a finite function from e to ω.

The order is: $p \le q$ iff

- $E_p \supset E_q$,
- for $e \in E_q$, $\sigma_{p,e} \supset \sigma_{q,e}$,
- for $e \ne e' \in E_q$, $\sigma_{p,e} \cap \sigma_{p,e'} = \sigma_{q,e} \cap \sigma_{q,e'}$.

$\mathbb{P}_k$ is easily seen to have precaliber ω_1. We define $p \Vdash \dot{f}_e(i) = j$ iff $e \in E_p$ and $\sigma_{p,e}(i) = j$. By a standard genericity argument, $\Vdash_{\mathbb{P}_k}$ [dom $\dot{f}_e = e$ and if $e \ne e'$ then $|\dot{f}_e \cap \dot{f}_{e'}| < \omega$].

Lemma 5.7. $\Vdash_{\mathbb{P}_k}$ "*$\dot{A}_k$ is k-near-Luzin*".

Proof. Working in $V^{\mathbb{P}_k}$, suppose that for each $i < k$ we have an uncountable subset $\dot{X}_i$ of $\mathcal{E}_k$. We want to show that $\bigcap_{i<k} \bigcup \{\dot{f}_e : e \in \dot{X}_i\}$ is infinite.

We may assume that the $\dot{X}_i$'s are disjoint, and each $\dot{X}_i$ has a 1-1 enumeration $\{\dot{e}_{\alpha,i} : \alpha < \omega_1\}$. Fix $p \in \mathbb{P}_k$. For each α there is $p_\alpha \le p$, for all α, i there is $d_{\alpha,i}$ so that

$$p_\alpha \Vdash \forall i < k \ \dot{e}_{\alpha,i} = d_{\alpha,i},$$

and for each α the $d_{\alpha,i}$'s are distinct.

We may assume the p_α's are centered. Then, since the enumeration is 1-1, for each i we have $d_{\alpha,i} \ne d_{\beta,i}$ if $\alpha \ne \beta$.

Pick distinct $\alpha_0, \ldots, \alpha_{k-1}$ so that $d_{\alpha_i,i} \ne d_{\alpha_j,j}$ for $i \ne j$ and (by a Δ-system argument) $d_{\alpha_i,i} \notin E_{p_{\alpha_j}}$. By property 15,

$$\Big| \bigcap_{i<k} d_{\alpha_i,i} \Big| = \omega.$$

Let $m \in \bigcap_{i<k} d_{\alpha_i,i}$, $m > \sup \operatorname{dom} \sigma_{p_{\alpha_i},e}$ for all i and all $e \in E_{p_{\alpha_i}}$. There is $q < p_{\alpha_i}$ for all i with $\sigma_{q,d_{\alpha_i,i}}(m) = 0$. So $q \Vdash \exists m > n\ (m,0) \in \bigcap_{i<k} \bigcup\{\dot{f}_e : e \in \dot{X}_i\}$. A density argument completes the proof. ■

Lemma 5.8. $\Vdash_{\mathbb{P}_k}$ *"$\dot{A}_k$ has no uncountable Luzin subfamilies".*

Proof. Suppose $\{\dot{a}_{\dot{e}_\alpha} : \alpha < \omega_1\} \subset \mathcal{A}_k$ and $p \Vdash$ "the enumeration $\{\dot{a}_{\dot{e}_\alpha} : \alpha < \omega_1\}$ witnesses that the family is Luzin". Choose $p_\alpha \le p$ and $d_\alpha \in \mathcal{E}_k$ with $p_\alpha \Vdash \dot{e}_\alpha = d_\alpha \in E_{p_\alpha}$. We may assume that:

18. The p_α's are centered.
19. $\{E_{p_\alpha} : \alpha < \omega\}$ is a Δ-system with root E.
20. $\exists n\ \forall \alpha\ E_{p_\alpha} = \{e_{\alpha,i} : i < n\}$.
21. $\forall i\ \exists \sigma_i\ \forall \alpha\ \sigma_{p_\alpha,e_{\alpha,i}} = \sigma_i$.

By necessity,

22. $\exists m\ \forall i\ \sigma_i \subset m \times m$.

There is $q \le p_\omega$ and k such that $q \Vdash \forall i > k\ \dot{a}_{\dot{e}_i} \cap \dot{a}_{\dot{e}_\omega} \not\subset m \times m$. By property 19 there is $j \ge k$ with $E_q \cap E_{p_j} = E$.

We define $r \le q$:

- $E_r = E_q \cup E_{r_j}$,
- for $e \in E_q$, $\sigma_{r,e} = \sigma_{q,e}$,
- for $e \in E_{r_j} \setminus E$, $\sigma_{r,e} = \sigma_{r_j,e}$.

Then $r \Vdash \dot{a}_{\dot{e}_j} \cap \dot{a}_{\dot{e}_\omega} = \dot{a}_{d_j} \cap \dot{a}_{d_\omega} \subset m \times m$, a contradiction. Theorem 5.2 is proved. ■

Finally, we note that Luzin does not imply 3-near-Luzin:

Theorem 5.9. *There is a Luzin almost disjoint family with no uncountable 3-near-Luzin subfamily.*

Proof. Let $\mathcal{E} = \mathcal{E}_2$ be as in Lemma 5.3. As in Theorem 5.2, we construct $\mathcal{A} = \{f_e : e \in \mathcal{E}\}$ where $\operatorname{dom} f_e = e$ for all e, so that $\mathcal{A}$ has no uncountable 3-near-Luzin subfamily. Here is how we get Luzin.

Let $\mathcal{E} = \{e_\alpha : \alpha < \omega_1\}$ and $f_\alpha = f_{e_\alpha}$. Our induction hypothesis at stage α is that for all $\beta < \alpha$ and all $n < \omega$, $\{\gamma < \beta : f_{e_\gamma} \cap f_{e_\beta} \subset n \times \omega\}$ is finite. This will certainly give us Luzin.

At stage α fix a 1-1 enumeration $\{\beta_n : n < \omega\}$ of α. In the nth step of the construction of f_α we ensure that $\operatorname{dom} f_\alpha \cap n = e_\alpha \cap n$ and $f_\alpha \cap f_{\beta_n} \not\subset n \times \omega$, without increasing $f_\alpha \cap f_{\beta_m}$ for $m < n$. Since $e_\alpha \cap e_{\beta_n} \setminus \bigcup\{e_{\beta_m} : m < n\}$ is finite, this can be done, and the construction is complete.

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Department of Mathematics
University of Kansas
Lawrence, Kansas 66045
U.S.A.
E-mail: roitman@math.ukans.edu

Mathematical Institute
Hungarian Academy of Sciences
POB 127
H-1364 Budapest, Hungary
E-mail: soukup@math-inst.hu

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