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On character and chain conditions in images of products

by

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Abstract. A scadic space is a Hausdorff continuous image of a product of compact scattered spaces. We complete a theorem begun by G. Chertanov that will establish that for each scadic space X, $\chi(X) = w(X)$. A ξ-adic space is a Hausdorff continuous image of a product of compact ordinal spaces. We introduce an either-or chain condition called Property R'_λ which we show is satisfied by all ξ-adic spaces. Whereas Property R'_λ is productive, we show that a weaker (but more natural) Property R_λ is not productive. Polyadic spaces are shown to satisfy a stronger chain condition called Property R''_λ. We use Property R'_λ to show that not all compact, monolithic, scattered spaces are ξ-adic, thus answering a question of Chertanov's.

1. Introduction. For cardinals κ and τ, $(\kappa+1)^\tau$ is the Tikhonov product of τ copies of the compact ordinal space $\kappa+1$. A space X (all of our defined properties will assume Hausdorff henceforth) is *ξ-adic* (Mrówka [Mr70]) if there exist κ and τ such that X is a continuous image of $(\kappa+1)^\tau$. Gerlits [Ge73] has shown that $\chi = w$ for ξ-adic spaces (thus generalizing the classical Essenin–Vol'pin result for dyadic spaces). Every ordinal space is scattered, i.e., every subspace has an isolated point (in the subspace topology). Chertanov [Ch88] introduced *scadic* spaces, i.e., continuous images of products of compact scattered spaces, and was able to extend the Gerlits result to continuous images of products of compact, monolithic, scattered spaces but left open the question of whether all scadic spaces satisfy $\chi = w$. In Section 2 we complete this extension to all scadic spaces. It is a proper extension as Example 1.14 in Chertanov [Ch88] is a scadic space which is not an image of a product of compact, monolithic, scattered spaces. In Section 4 we show that the ξ-adic spaces satisfy a strong chain condition

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called Property R'_λ. Using this result, we show that the Chertanov extension was strict; we do this by producing a compact, monolithic, scattered space which is not ξ-adic. In Section 3 we study Property R'_λ and related properties in their own right; for example, we show that Property R'_λ is preserved by products whereas a weaker (but more natural) Property R_λ is not. An important subclass of the ξ-adic spaces is formed by the *polyadic* spaces, continuous images of $(\alpha\kappa)^\tau$ where $\alpha\kappa$ is the Alexandrov one point compactification of the discrete space κ. In Section 4 we show that polyadic spaces satisfy a stronger chain condition than Property R'_λ; this improves a result in [Be96].

We denote the family of all clopen subsets of X by $\mathrm{CO}(X)$. A *Boolean* space is a compact space such that $\mathrm{CO}(X)$ is a basis. For a point $x \in X$, $\chi(x)$ is the least cardinality of a neighbourhood base at x in X. Then $\chi(X) = \sup\{\chi(x) : x \in X\}$. The least cardinality of a base for X is denoted by $w(X)$. Further, $\{x_\alpha : \alpha < \kappa\} \subset X$ is called a *right-separated κ-sequence* if for each $\alpha < \kappa$, $x_\alpha \notin \overline{\{x_\beta : \beta > \alpha\}}$. The *hereditary Lindelöf number* of X is $\mathrm{hL}(X) = \sup\{\kappa :$ there exists a right-separated κ-sequence in $X\}$. For a cardinal κ, $[X]^\kappa$ denotes the set $\{A \subset X : |A| = \kappa\}$.

A *Δ-system* is a collection $\mathcal{B}$ of sets for which there exists a set R (called the *root* of the Δ-system) such that if A and B are two distinct elements of $\mathcal{B}$, then $A \cap B = R$. A standard fact (Lemma 2.4 of Hodel [Ho84]) is the following: if λ is an uncountable regular cardinal and $\langle F_\alpha : \alpha < \lambda\rangle$ is a λ-sequence of finite sets, then there exists $A \subset \lambda$ with $|A| = \lambda$ such that $\{F_\alpha : \alpha \in A\}$ is a Δ-system.

2. Character and weight coincide for scadic spaces. Here is Chertanov's main theorem on character versus weight for scadic spaces.

THEOREM 2.1 (Chertanov [Ch88]). *Let $S = \prod_{\alpha\in A} S_\alpha$ be a product of compact scattered spaces and let X be an image of S with $\chi(X) = \kappa$. Then there exists $B \subset A$ with $|B| \le \kappa$ and for each $\alpha \in B$, an $F_\alpha \subset S_\alpha$ with $|F_\alpha| \le \kappa$ such that X is an image of $\prod_{\alpha\in B} \overline{F}_\alpha$. Consequently, a scadic space X with $\chi(X) = \kappa$ is an image of a product of at most κ compact scattered spaces, each of density at most κ.*

If κ is an infinite cardinal, $A \subset X$, $x \in X$, and for every neighbourhood V of x, $|A \setminus V| < \kappa$, then we say that x is a *κ-limit point* of A. Further, X is *κ-sequentially compact* if for every $A \subset X$ with $|A| = \kappa$ there exists $B \subset A$ with $|B| = \kappa$ and there exists $x \in X$ such that x is a κ-limit point of B.

LEMMA 2.2. *A compact scattered space X is κ-sequentially compact for all infinite cardinals κ.*

P r o o f. We assume the reader is familiar with the scattering height of a compact scattered space ([HBA89], page 275). Supppose that our lemma is true for all compact scattered spaces Y with $\mathrm{ht}(Y) < \alpha$. Let $\mathrm{ht}(X) = \alpha = \beta + 1$ where the βth level X_β is a finite non-empty set F and let $A \in [X]^\kappa$. Assuming that each $x \in F$ is not a κ-limit point of any subset of A of cardinality κ, by a finite recursion, we can get an open set $U \supset F$ such that $|A \setminus U| = \kappa$. As $\mathrm{ht}(X \setminus U) < \alpha$, our inductive hypothesis implies that there exists $B \in [A \setminus U]^\kappa$ and there exists $x \in X \setminus U$ such that x is a κ-limit point of B in $X \setminus U$ but then x is a κ-limit point of B in X. ∎

If $X \subset S = \prod_{\alpha \in A} S_\alpha$ and $F \subset A$, then F is called a *support* of X if $X = \pi_F^{-1}(\pi_F(X))$ where π_F is the projection map of S onto $\prod_{\alpha \in F} S_\alpha$. The open subsets of S with a finite support form a basis for S, which is closed under finite unions. This implies that whenever $K \subset U$, with K compact and U open in S, then there exists an open O with a finite support such that $K \subset O \subset U$.

LEMMA 2.3. *Let $S = \prod_{\alpha<\kappa} S_\alpha$ be a product of κ^+-sequentially compact compact spaces and let X be an image of S with $\chi(X) = \kappa$. Then $w(X) = \kappa$.*

P r o o f. Let f map S continuously onto X. We first show that $\mathrm{hL}(X) \le \kappa$. Assume that $\mathrm{hL}(X) \ge \kappa^+$. It then follows, upon using regularity twice, that we can choose open sets U_α, V_α, W_α in X and points s_α in X, for $\alpha < \kappa^+$, with $\overline{U}_\alpha \subset V_\alpha$, $\overline{V}_\alpha \subset W_\alpha$ and $s_\alpha \in U_\alpha \setminus \bigcup_{\beta<\alpha} W_\beta$. Get open sets O_α and O'_α in S with finite supports F_α such that $f^{-1}(\overline{U}_\alpha) \subset O_\alpha \subset f^{-1}(V_\alpha)$ and $f^{-1}(\overline{V}_\alpha) \subset O'_\alpha \subset f^{-1}(W_\alpha)$. Since there are only κ finite subsets of κ, get a finite $F \subset \kappa$ and an $A \in [\kappa^+]^{\kappa^+}$ such that each O_α and O'_α, for $\alpha \in A$, has F as a support. For each $\alpha \in A$, choose $p_\alpha \in f^{-1}(s_\alpha)$. As $\prod_{\alpha \in F} S_\alpha$ is κ^+-sequentially compact (this property is finitely productive) and $|\{p_\alpha\restriction F : \alpha \in A\}| = \kappa^+$, get $B \in [A]^{\kappa^+}$ and $y \in \prod_{\alpha\in F} S_\alpha$ such that y is a κ^+-limit point of $\{p_\alpha\restriction F : \alpha \in B\}$. Extend y to y' in $\prod_{\alpha<\kappa} S_\alpha$. For each $\beta \in B$, put $q_\beta = p_\beta\restriction F \cup y'\restriction(\kappa \setminus F)$. Then y' is a κ^+-limit point of $\{q_\alpha : \alpha \in B\}$. As $\chi(X) = \kappa$, choose open sets G_α in S, for $\alpha \in \kappa$, such that $f^{-1}(f(y')) = \bigcap_{\alpha<\kappa} G_\alpha$. Choose $\gamma \in B$ such that $q_\gamma \in f^{-1}(f(y'))$. Since $q_\gamma\restriction F = p_\gamma\restriction F$ and F is a support of O_γ and $p_\gamma \in O_\gamma$, we have $q_\gamma \in O_\gamma$. Therefore, $q_\gamma \in f^{-1}(V_\gamma)$. Since $f(q_\gamma) = f(y')$, we have $y' \in f^{-1}(V_\gamma)$. Now, choose $\delta > \gamma$ with $\delta \in B$ such that $q_\delta \in f^{-1}(V_\gamma)$. Since $p_\delta\restriction F = q_\delta\restriction F$ and F is a support of O'_γ and $q_\delta \in O'_\gamma$, we have $p_\delta \in O'_\gamma$. Therefore, $p_\delta \in f^{-1}(W_\gamma)$, whence $s_\delta \in W_\gamma$ yet $\gamma < \delta$, a contradiction. Hence, $\mathrm{hL}(X) \le \kappa$.

Since $\chi(X \times X) = \kappa$ and $X \times X$ is an image of $S \times S$, we can apply the preceding to deduce that $\mathrm{hL}(X \times X) \le \kappa$. It is a basic result for compact spaces (Corollary 7.6 of Hodel [Ho84]) that this implies $w(X) \le \kappa$. The author would like to thank Paul Gartside for showing this quick way to end the proof of the lemma. ■

Thus, Theorem 2.1 followed by Lemmas 2.2 and 2.3 yields

THEOREM 2.4. *If X is a scadic space, then $\chi(X) = w(X)$.*

3. Either-or chain conditions. This section expands upon the results about Property R_λ in [Be96]. For clarity of thought and ease of presentation, we are going to present the clopen version of our either-or chain conditions; see the remarks after Theorem 4.3 for the non-Boolean version. Let λ be an infinite cardinal. We say that a Boolean space has *Property* R'_λ if whenever $\langle U_\alpha, V_\alpha\rangle_{\alpha<\lambda}$ is a sequence of pairs of clopen sets, then there exists a $K \subset \lambda$ with $|K| = \lambda$ such that *either* for every $\alpha < \beta$ in K, $U_\alpha \cap V_\beta = \emptyset$, *or* for every $\alpha < \beta$ in K, $U_\alpha \cap V_\beta \neq \emptyset$. If, in the if-clause of the definition of Property R'_λ, we require that $U_\alpha = V_\alpha$ for all $\alpha < \lambda$, then this is the weaker *Property* R_λ of [Be96]. If, in the then-clause of the definition of Property R'_λ, we replace both occurrences of $\alpha < \beta$ by $\alpha \neq \beta$, then this is the stronger *Property* R''_λ. We say that a Boolean space has *Property* T_λ if whenever $\langle U_\alpha\rangle_{\alpha<\lambda}$ is a sequence of clopen sets, then there exists a $K \subset \lambda$ with $|K| = \lambda$ such that *either* for every $\alpha < \beta$ in K, $U_\alpha \cap U_\beta = \emptyset$, *or* for every finite $F \subset K$, $\bigcap_{\alpha\in F} U_\alpha \neq \emptyset$. *Properties* R'', R', R and T denote Properties R''_{ω_1}, R'_{ω_1}, R_{ω_1} and T_{ω_1} respectively. For brevity, in all the above properties, we will refer to a K with the attributes in the corresponding either-or clauses as a *correct* K. By taking inverse images, it is easily seen that Properties R''_λ, R'_λ, R_λ and T_λ are preserved by continuous images.

The classical Ramsey theorem for ω shows us that every Boolean space has Property R'_ω and an easy example shows that no infinite Boolean space can have Property R''_ω. No crowded (i.e. without isolated points) Boolean space can have Property T_ω whereas for every κ, $\alpha\kappa$ has Property T_ω. Our interest is in these properties when λ is uncountable and regular.

Let $\mathcal{B} \subset \mathrm{CO}(X)$. We say that X has *Property* $R'_\lambda(\mathcal{B})$ if whenever $\langle U_\alpha, V_\alpha\rangle_{\alpha<\lambda}$ is a sequence of pairs of clopen sets of $\mathcal{B}$, then there exists a correct $K \in [\lambda]^\lambda$.

LEMMA 3.1. *Let $\mathcal{B} \subset \mathrm{CO}(X)$ be a fixed base for the open sets of a Boolean space X and let λ be a cardinal of uncountable cofinality. If X has Property $R'_\lambda(\mathcal{B})$, then X has Property R'_λ.*

Proof. Let $\langle U_\alpha, V_\alpha\rangle_{\alpha<\lambda}$ be a sequence of pairs of clopen sets of X. As λ has uncountable cofinality and $\mathcal{B}$ is a base, by thinning to a subsequence of cardinality λ, we will assume that there exist $m, n < \omega$ such that for each $\alpha < \lambda$, $U_\alpha = \bigcup_{i<m} A^i_\alpha$ and $V_\alpha = \bigcup_{i<n} B^i_\alpha$ where $A^i_\alpha, B^j_\alpha \in \mathcal{B}$ for all $i < m$, $j < n$ and $\alpha < \lambda$. We now assume that there does not exist a $K \in [\lambda]^\lambda$ such that $\alpha < \beta$ in K implies $U_\alpha \cap V_\beta \neq \emptyset$. For each $i < m$, $j < n$ and $H \in [\lambda]^\lambda$ we can apply Property $R'_\lambda(\mathcal{B})$ to the sequence $\langle A^i_\alpha, B^j_\alpha\rangle_{\alpha\in H}$ to get $K \in [H]^\lambda$ such that $\alpha < \beta$ in K implies $A^i_\alpha \cap B^j_\beta = \emptyset$. Thus, after $m \times n$ successive applications, we get $K \in [\lambda]^\lambda$ such that $\alpha < \beta$ in K implies $U_\alpha \cap V_\beta = \emptyset$. ∎

LEMMA 3.2. *For each uncountable*, *regular cardinal* λ, *Property* R'_λ *is productive.*

Proof. Let κ be a cardinal and let $\{X_\alpha : \alpha < \kappa\}$ be a family of Boolean spaces with Property R'_λ. Put $X = \prod_{\alpha<\kappa} X_\alpha$. Let $\langle U_\alpha, V_\alpha\rangle_{\alpha<\lambda}$ be a sequence of pairs of clopen sets of X. For each $\alpha < \lambda$, let $F_\alpha \subset \kappa$ be a finite support of U_α and of V_α. Choose $A \in [\lambda]^\lambda$ such that $\{F_\alpha : \alpha \in A\}$ is a Δ-system with root R. Then, for each $\alpha < \beta$ in A, $U_\alpha \cap V_\beta = \emptyset \Leftrightarrow \pi_R(U_\alpha) \cap \pi_R(V_\beta) = \emptyset$. So, it suffices to show that Property R'_λ is finitely productive, i.e., that Property R'_λ is 2-productive. To this end, let X and Y be Boolean spaces with Property R'_λ. Invoking Lemma 3.1, we start with a sequence of pairs of standard basic clopen sets $\langle A_\alpha \times B_\alpha, C_\alpha \times D_\alpha\rangle_{\alpha<\lambda}$ in $X \times Y$. Now we apply Property R'_λ to $\langle A_\alpha, C_\alpha\rangle_{\alpha<\lambda}$ to get a correct $H \in [\lambda]^\lambda$ and then we apply Property R'_λ to $\langle B_\alpha, D_\alpha\rangle_{\alpha\in H}$ to get a correct $K \in [H]^\lambda$. Then $\langle A_\alpha \times B_\alpha, C_\alpha \times D_\alpha\rangle_{\alpha\in K}$ is our required correct subsequence. ∎

It was shown in [Be96] that $\alpha\omega_1$ has Property T but $(\alpha\omega_1)^2$ does not (although $(\alpha\omega_1)^2$ does have Property R) and the question was raised whether Property R is productive. We will show that there is a Boolean space with Property T whose square does not have Property R. The following graph spaces will be used again in the next section. G is a *graph* on a set X if $G \subset [X]^2$. For $\{x, y\} \in G$, we will write $x -_G y$ or just $x - y$ if the graph is understood. A subset C of X such that $x - y$ for all $x \neq y$ in C is said to be *complete*. A subset I of X such that $x \not- y$ for all $x \neq y$ in I is said to be *independent*. For $x \in X$, define $x^+ = \{C \subset X : C \text{ is complete and } x \in C\}$ and define $x^- = \{C \subset X : C \text{ is complete and } x \notin C\}$. Use $\{x^+, x^- : x \in X\}$ as a closed (also open) subbase for a topology on $G^* = \{C \subset X : C \text{ is complete}\}$. G^* with this topology is a Boolean space. Every clopen subset b of G^* has a finite support, i.e., there is a finite $F \subset X$ such that for each $C \in G^*$ one has $C \in b$ iff $C \cap F \in b$. A graph G on a set X is said to be *twofold* if $X = C \cup I$ where C is complete and I is independent.

THEOREM 3.3. *If* G *is a twofold graph and* λ *is an uncountable*, *regular cardinal*, *then* G^* *has Property* T_λ.

Proof. Let G be a graph on $X = C \cup I$ where C is complete and I is independent. Let $\langle b_\alpha : \alpha < \lambda\rangle$ be a sequence of clopen sets of G^*. For each $\alpha < \lambda$, let $F_\alpha \subset X$ be a finite support of b_α. We may assume that $\{F_\alpha : \alpha < \lambda\}$ forms a Δ-system with root R.

Case 1: *There exist λ α's for which there exists $X_\alpha \subset F_\alpha \cap C$ with $X_\alpha \in b_\alpha$.* Choose $A \in [\lambda]^\lambda$, $S \subset R$ and for each $\alpha \in A$, an $X_\alpha \subset F_\alpha \cap C$ with $X_\alpha \in b_\alpha$ and $X_\alpha \cap R = S$. Then $\{b_\alpha : \alpha \in A\}$ is centered. To see this, take a finite $H \subset A$. Since $Z = \bigcup_{\alpha \in H} X_\alpha \subset C$, we have $Z \in G^*$. If $\beta, \alpha \in H$ and $\beta \neq \alpha$, then $X_\beta \cap F_\alpha = X_\beta \cap F_\beta \cap F_\alpha = X_\beta \cap R = S = X_\alpha \cap R \subset X_\alpha$; hence, for $\alpha \in H$, $Z \cap F_\alpha = X_\alpha \cap F_\alpha$; so $Z \in b_\alpha$.

Case 2: *There exists $x \in R \cap I$ and there exist λ α's for which there exists $X_\alpha \in x^+ \cap b_\alpha$.* Choose $A \in [\lambda]^\lambda$, $x \in R \cap I$ and $S \subset R$ such that for each $\alpha \in A$, $X_\alpha \in x^+ \cap b_\alpha$ and $X_\alpha \cap R = S$. Then $\{b_\alpha : \alpha \in A\}$ is centered. To see this, take a finite $H \subset A$. Put $Z = \bigcup_{\alpha \in H}(X_\alpha \cap F_\alpha)$. Since I is independent, $Z \cap I = \{x\}$ and so $Z \in G^*$. If $\beta, \alpha \in H$ and $\beta \neq \alpha$, then $X_\beta \cap F_\beta \cap F_\alpha = X_\beta \cap R = S = X_\alpha \cap R \subset X_\alpha$; hence, for $\alpha \in H$, $Z \cap F_\alpha = X_\alpha \cap F_\alpha$; so $Z \in b_\alpha$.

Case 3: *There exist λ α's such that for each $Z \subset F_\alpha$ with $Z \in b_\alpha$ we have $Z \cap I \neq \emptyset$ and for each $x \in R \cap I$ we have $x^+ \cap b_\alpha = \emptyset$.* Let A be the set of all such α's. Then $A \in [\lambda]^\lambda$ and we claim that for $\alpha \neq \beta$ in A, $b_\alpha \cap b_\beta = \emptyset$. Striving for a contradiction, take $Z \in b_\alpha \cap b_\beta$ where $\alpha \neq \beta$ in A. Then $Z \cap F_\alpha \in b_\alpha$, so $Z \cap F_\alpha \cap I \neq \emptyset$. Similarly, $Z \cap F_\beta \cap I \neq \emptyset$. As I is independent, this means we can choose $x \in I$ such that $x \in Z \cap F_\alpha \cap F_\beta = Z \cap R$; but then $x \in R \cap I$ and $Z \in x^+ \cap b_\alpha$, a contradiction. ∎

We will use the Sierpiński graph S on ω_1 (cf. [EHMR84], page 123). The key property of S that we use is that there does not exist an uncountable complete subset nor an uncountable independent subset of ω_1.

Example 3.4. *Property R is not productive.*

Proof. Let $A = \{a_\alpha : \alpha < \omega_1\}$ and $B = \{b_\alpha : \alpha < \omega_1\}$ be two disjoint sets of cardinality ω_1. We define a twofold graph G on $X = A \cup B$ as follows: A is complete and B is independent. Let S be the Sierpiński graph on the set ω_1. We put $a_\alpha - b_\beta \Leftrightarrow (\alpha -_S \beta$ or $\alpha = \beta)$. Theorem 3.3 implies that G^* has Property T and therefore Property R. In $G^* \times G^*$, for $\alpha < \omega_1$, put $U_\alpha = (a_\alpha^+ \times b_\alpha^+) \cup (b_\alpha^+ \times a_\alpha^+)$. We claim that for $\alpha \neq \beta$, $\alpha -_S \beta \Leftrightarrow U_\alpha \cap U_\beta \neq \emptyset$. Hence, $G^* \times G^*$ does not have Property R. To see this, take $\alpha \neq \beta$. If $\alpha -_S \beta$, then $(\{a_\alpha, b_\beta\}, \{b_\alpha, a_\beta\}) \in U_\alpha \cap U_\beta$. If $\alpha \not-_S \beta$, then

- $(a_\alpha^+ \times b_\alpha^+) \cap (a_\beta^+ \times b_\beta^+) = \emptyset$ because $b_\alpha \not- b_\beta$.
- $(a_\alpha^+ \times b_\alpha^+) \cap (b_\beta^+ \times a_\beta^+) = \emptyset$ because $a_\alpha \not- b_\beta$.
- $(b_\alpha^+ \times a_\alpha^+) \cap (a_\beta^+ \times b_\beta^+) = \emptyset$ because $a_\alpha \not- b_\beta$.
- $(b_\alpha^+ \times a_\alpha^+) \cap (b_\beta^+ \times a_\beta^+) = \emptyset$ because $b_\alpha \not- b_\beta$. ∎

4. ξ-adic spaces. A compact space X is *monolithic* (Arkhangel'skiĭ [Ar76]) if for every $A \subset X$ we have $w(\overline{A}) \leq |A|$. It is mentioned in [Ch88] that if all factor spaces S_α are monolithic, then Theorem 2.1 implies $\chi(X) = w(X)$. Thus, as compact ordinal spaces are monolithic, this generalizes the result of J. Gerlits [Ge73] that $\chi = w$ for ξ-adic spaces. In [Ch88] the question is asked whether there exists a compact, monolithic, scattered space which is not ξ-adic. We will use Property R'_λ to show that there is such an example.

If $A, B \subset \lambda$ and $\alpha < \beta$ whenever $\alpha \in A$ and $\beta \in B$, then we write $A < B$. In our theorem we will need two basic facts about clopen intervals in a compact ordinal space.

FACT 4.1. *Let κ and λ be infinite cardinals with λ regular. For every sequence of clopen intervals $\langle I_\alpha = [l_\alpha, r_\alpha]\rangle_{\alpha<\lambda}$ in $\kappa+1$, there exists $A \in [\lambda]^\lambda$ such that one of the following is true*:

O1. *$\alpha < \beta$ in A implies $I_\alpha \subset I_\beta$.*

O2. *$\alpha < \beta$ in A implies $I_\beta \subset I_\alpha$.*

O3. *$\alpha < \beta$ in A implies $I_\alpha < I_\beta$.*

O4. *$\{l_\alpha : \alpha \in A\} < \{r_\alpha : \alpha \in A\}$ and $\alpha < \beta$ in A imply $l_\alpha < l_\beta$ and $r_\alpha < r_\beta$.*

Proof. Use the partition relation $\lambda \to (\lambda, \omega)^2$ (cf. [EHMR84], page 70) to get $B \in [\lambda]^\lambda$ such that $\alpha < \beta$ in B implies that $l_\alpha \le l_\beta$ and $r_\alpha \le r_\beta$. If there exists $\alpha \in B$ such that for λ β's in B, $l_\alpha = l_\beta$, then we achieve O1. If there exists $\alpha \in B$ such that for λ β's in B, $r_\alpha = r_\beta$, then we achieve O2. Otherwise, by regularity of λ, we can extract $C \in [B]^\lambda$ such that for $\alpha < \beta$ in C, we have $l_\alpha < l_\beta$ and $r_\alpha < r_\beta$. Put $l = \sup\{l_\alpha : \alpha \in C\}$ and $r = \sup\{r_\alpha : \alpha \in C\}$ (l or r may equal κ). If $l = r$, then we achieve O3, by recursion, for some $D \in [C]^\lambda$. If $l < r$, then we achieve O4 for some tail of C. ∎

FACT 4.2. *Let κ and λ be infinite cardinals. Let U be a clopen interval in $\kappa+1$. Let $\langle V_\alpha = [l_\alpha, r_\alpha]\rangle_{\alpha<\lambda}$ be a sequence of clopen intervals in $\kappa+1$ that satisfies* O1, O2, O3 *or* O4 *with $A = \lambda$. Then either $|\{\alpha < \lambda : U \cap V_\alpha = \emptyset\}| < \lambda$ or $|\{\alpha < \lambda : U \cap V_\alpha \neq \emptyset\}| < \lambda$.*

Proof. Assume not, i.e., there are λ α's such that $U \cap V_\alpha = \emptyset$ and there are λ α's such that $U \cap V_\alpha \neq \emptyset$. Clearly, the V_α's cannot satisfy O1, O2, or O3. So, the V_α's must satisfy O4. Let $l = \sup\{l_\alpha : \alpha < \lambda\}$ and let $r = \sup\{r_\alpha : \alpha < \lambda\}$. Then $l < r$. If $U \cap [l, r) \neq \emptyset$, then $|\{\alpha < \lambda : U \cap V_\alpha = \emptyset\}| < \lambda$. If $U \cap [l, r) = \emptyset$, then $|\{\alpha < \lambda : U \cap V_\alpha \neq \emptyset\}| < \lambda$. In either case, we get a contradiction. ∎

THEOREM 4.3. *Every ξ-adic space has Property R'_λ, for all uncountable regular cardinals λ.*

Proof. By Lemma 3.2 and the fact that Property R'_λ is preserved by continuous images it will suffice to show that $\kappa+1$ has Property R'_λ. Using

Lemma 3.1 we start with $\langle U_\alpha, V_\alpha\rangle_{\alpha<\lambda}$, a sequence of pairs of clopen intervals of $\kappa+1$. Apply Fact 4.1 to the sequence $\langle V_\alpha = [l_\alpha, r_\alpha]\rangle_{\alpha<\lambda}$ to get an $A \in [\lambda]^\lambda$ such that one of O1, O2, O3 or O4 holds.

Now we use Fact 4.2 for each U_α, $\alpha \in A$, to deduce that either there exist λ α's in A such that $|\{\beta \in A : U_\alpha \cap V_\beta = \emptyset\}| < \lambda$ or there exist λ α's in A such that $|\{\beta \in A : U_\alpha \cap V_\beta \neq \emptyset\}| < \lambda$. Now, a straightforward recursion, using the regularity of λ, will extract a correct $K \in [A]^\lambda$. ■

Theorem 4.3 cannot be improved to Property R''_λ as the sequence $U_\alpha = [0, \alpha]$ and $V_\alpha = \{\alpha + 1\}$ in the ordinal space $\omega_1 + 1$ will testify. To make Property R'_λ meaningful for arbitrary compact spaces, we employ the standard device of replacing a clopen set by a pair of open sets (U, V) with $\overline{U} \subset V$. Property R'_λ becomes Property Q'_λ: whenever $\langle A_\alpha, B_\alpha, C_\alpha, D_\alpha\rangle_{\alpha<\lambda}$ is a sequence of quadruples of open sets such that $\overline{A}_\alpha \subset B_\alpha$ and $\overline{C}_\alpha \subset D_\alpha$, then there exists $K \subset \lambda$ with $|K| = \lambda$ such that *either* for every $\alpha < \beta$ in K, $A_\alpha \cap C_\beta = \emptyset$, *or* for every $\alpha < \beta$ in K, $B_\alpha \cap D_\beta \neq \emptyset$. All of our results on Property R'_λ for Boolean spaces are true for Property Q'_λ for compact spaces.

EXAMPLE 4.4. *There exists a compact*, *monolithic*, *scattered space X which is not ξ-adic.*

Proof. Let S be the Sierpiński graph on the set ω_1. As all complete subsets are countable, S^* is a Corson compact space, hence S^* is monolithic. Let $X = \{C \in S^* : |C| \le 2\}$. Then X is closed in S^*, so X is a compact monolithic space. Since X is the union of 3 discrete subspaces, it is scattered. For each $\alpha < \omega_1$, put $U_\alpha = \alpha^+ \cap X$. As all complete and all independent subsets of ω_1 are countable, the sequence $\langle U_\alpha\rangle_{\alpha<\omega_1}$ witnesses the fact that X does not have Property R. Theorem 4.3 implies that X is not ξ-adic. ■

THEOREM 4.5. *Every polyadic space has Property R''_λ*, *for all uncountable regular cardinals λ.*

Proof. Lemmas 3.1 and 3.2 are true if R'_λ is replaced by R''_λ. Just replace all occurrences of $\alpha < \beta$ in the proofs by $\alpha \neq \beta$. So, just as in Theorem 4.3, we need only show that $\alpha\kappa$ has Property R''_λ. We employ Lemma 3.1 with the base $\mathcal{B} = \{\{\alpha\} : \alpha < \kappa\} \cup \{\alpha\kappa \setminus F : F \text{ is a finite subset of } \kappa\}$. Start with $\langle U_\alpha, V_\alpha\rangle_{\alpha<\lambda}$ where $U_\alpha, V_\alpha \in \mathcal{B}$. We may assume that all the U_α's are singletons or all the U_α's are cofinite. Similarly for the V_α's. If all the U_α's and all the V_α's are cofinite, then the sequence is correct. Let us assume that all the U_α's are singletons and all the V_α's are cofinite. If there exists $H \in [\lambda]^\lambda$ and $\gamma < \kappa$ such that for every $\alpha \neq \beta$ in H, $U_\alpha = U_\beta = \{\gamma\}$, then choose $K \in [H]^\lambda$ such that either for every $\alpha \in K$, $\gamma \in V_\alpha$, or for every $\alpha \in K$, $\gamma \notin V_\alpha$. Then K is correct. Otherwise, if no such H exists, then as λ

is regular, we may assume that for $\alpha < \beta$, $U_\alpha \neq U_\beta$. As λ is uncountable and regular, we can choose $H \in [\lambda]^\lambda$ such that $\{\alpha\kappa \setminus V_\alpha : \alpha \in H\}$ is a Δ-system with root R. Discarding a finite set of α's we may assume that for every $\alpha \in H$, $U_\alpha \cap R = \emptyset$. Let us assume that u_α is the unique element of U_α. We recursively extract a correct $K \in [H]^\lambda$ such that $\alpha \neq \beta$ in K implies that $u_\alpha \in V_\beta$. If $A \in [H]^{<\lambda}$ has already been chosen such that for $\alpha \neq \beta$ in A we have $u_\alpha \in V_\beta$, then we proceed as follows. For each $\alpha \in A$, there exists at most one β in H such that $u_\alpha \in \alpha\kappa \setminus V_\beta$; otherwise we would have $u_\alpha \in R$. Thus, $B = \{\beta \in H : \text{there exists } \alpha \in A \text{ with } u_\alpha \notin V_\beta\}$ has cardinality $\leq |A| < \lambda$. Also, $C = \bigcup_{\alpha \in A} \alpha\kappa \setminus V_\alpha$ has cardinality $\leq |A| + \omega < \lambda$. So, if we choose $\beta \in H \setminus (A \cup B \cup C)$, then for every $\alpha \in A$, $u_\alpha \in V_\beta$ and $u_\beta \in V_\alpha$, and we can continue the recursion. In an analogous manner, we can deal with the case where all the U_α's are cofinite and all the V_α's are singletons. The remaining case where all the U_α's and all the V_α's are singletons is easily disposed of. ∎

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