FUNDAMENTA
MATHEMATICAE
158 (1998)

Continuous decompositions of Peano plane continua into pseudo-arcs

by

Janusz R. Prajs (Opole)

Abstract. Locally planar Peano continua admitting continuous decomposition into pseudo-arcs (into acyclic curves) are characterized as those with no local separating point. This extends the well-known result of Lewis and Walsh on a continuous decomposition of the plane into pseudo-arcs.

The extraordinary position of the pseudo-arc among compact, connected metric spaces can be compared to the position of the Cantor set among all compacta. First, both these spaces are known to be homogeneous. Second, for any nonempty compactum X with no isolated point the family of all topological Cantor sets is a dense G_δ-set in the hyperspace 2^X of all nonempty compacta in X metrized with the Hausdorff metric. Similarly, if X is: an n-manifold ($n > 1$), the Hilbert cube, the Sierpiński curve, the Menger curve M, or a higher dimensional analogue of M, then the family of all pseudo-arcs in X is a dense G_δ-subset of the hyperspace $C(X)$ of all connected members of 2^X. Evidently, by the Baire theorem, the Cantor set and the pseudo-arc are the only spaces (up to homeomorphism) having the respective properties.

Third, if $X, X_1, X_2, \ldots$ are topological Cantor sets in a metric space, then

(∗) if $\{X_n\}$ converges to X in the sense of the Hausdorff distance, then $\{X_n\}$ converges to X homeomorphically (i.e. there are homeomorphisms $h_n : X \to X_n$ with $\sup_x d(h_n(x), x) \xrightarrow{n} 0$).

1991 *Mathematics Subject Classification*: 54F15, 54F50, 54F65, 54F99.

Key words and phrases: continuous decomposition, locally connected continuum, local separating point, open homogeneity, pseudo-arc, 2-manifold.

The author expresses special thanks to I. Krzemińska and K. Omiljanowski for their patience in reading the manuscript and eliminating mistakes.

Surprisingly, the pseudo-arc also has this property [13], and this property characterizes the pseudo-arc among all nondegenerate metric continua [16].

Moreover, for any compactum X whose copies satisfy $(*)$ we have: (1) if X has infinitely many components, then X is homeomorphic to the Cantor set; and (2) if X has only finitely many components, then X is the finite union of mutually disjoint pseudo-arcs and singletons (see again [16]).

Therefore, no wonder that the pseudo-arc, originally constructed by Knaster [8] as a rather peculiar space, became the subject of highest interest in topology during the last half century (see a survey paper [15]).

Among the most amazing results dealing with the pseudo-arc is the construction of a continuous decomposition of the plane into pseudo-arcs by Lewis and Walsh [17] (originally announced by Anderson [1]). The pseudo-arc is the only known continuum such that the plane admits a continuous decomposition into its topological copies. Though some 2-manifolds admit continuous decompositions into arcs or circles (e.g. the annulus, the torus), these decompositions have the (local or global) product structures connected with the local product structure of the plane. Since the plane contains no product of the pseudo-arc and a 1-dimensional set, the decompositions of the plane (and of surfaces) into pseudo-arcs have no product structure, and they provide essentially new information on the local structure of the plane.

Additionally, note that any continuous decomposition $\mathcal{D}$ into pseudo-arcs (though it need not have the local product structure) is continuous with respect to homeomorphical convergence of pairs (X_n, x_n), where $X_n \in \mathcal{D}$ and $x_n \in X_n$ (i.e. if $x = \lim x_n$ and $x \in X \in \mathcal{D}$, then there are homeomorphisms $h_n : X \to X_n$ with $h_n(x) = x_n$ and $\sup_x d(h_n(x), x) \xrightarrow{n} 0$). In fact, we apply condition $(*)$, the homogeneity of the pseudo-arc and the well-known Effros' theorem to get this stronger kind of continuity of any continuous decomposition into pseudo-arcs. For other information on spaces with continuous decompositions into pseudo-arcs the reader is referred to [14].

Twenty years before the publication of [17] Brown constructed [5] a continuous decomposition of the plane into hereditarily indecomposable continua with a half-line as quotient space. The present paper starts with the proof that each 2-manifold without boundary admits a continuous decomposition into pseudo-arcs. However, neither the results of [17] nor those of [5] provide tools to construct such a decomposition for 2-manifolds with boundary. In the previous paper [20] the author has shown that the annulus can be filled up with a continuous circle of mutually disjoint pseudo-arcs. (This special decomposition cannot be derived from those obtained below because of its quotient.) Applying this result, in the present paper we construct a continuous decomposition of any Peano plane continuum with no local separating point into pseudo-arcs such that the quotient space is again such a continuum. Moreover, all locally planar Peano continua admitting a continuous decomposition into nondegenerate, acyclic continua are characterized as those with no local separating point.

The paper is closely related to some recent results dealing with: continuous decomposition of the Sierpiński curve into acyclic curves (Seaquist [21]), approximating mapping from a compactum to $[0,1]$ by mappings with hereditarily indecomposable fibers (Levin [10]), and a result more general than Levin's, where $[0,1]$ is replaced by any n-manifold (Krasinkiewicz [9]).

We end the paper with some applications to open homogeneity of the constructions obtained. Problems concerning that homogeneity were, actually, the main inspiration for the results of this paper.

Preliminaries. In this paper spaces are assumed to be metric, 2-manifolds are assumed to be compact and connected. A continuum T is said to be *terminal* in a space X provided for any continuum K in X intersecting T we have either $T \subset K$, or $K \subset T$. If a continuum K has a closed basis of neighborhoods composed of homeomorphic copies of the Sierpiński universal plane curve, then it is called an *S-manifold*.

For any compactum X the symbols $C(X)$ and 2^X stand for the hyperspaces of all nonempty subcontinua of X and of all nonempty closed sets in X, respectively, both metrized with the Hausdorff metric, denoted here by dist. For any compactum X we let $C(C(X)) = C^2(X)$ and we denote by dist_2 the Hausdorff distance in $C^2(X)$. Further, for all nonempty subsets A, B of a metric space X, for any $x \in X$ and for any $\varepsilon > 0$ define $\widehat{d}(A,B) = \inf\{d(a,b) : a \in A,\ b \in B\}$, $\widehat{d}(x,A) = \inf\{d(x,a) : a \in A\}$ and $N_\varepsilon(A) = \{y \in X : \widehat{d}(y,A) < \varepsilon\}$.

The sets of all real and complex numbers are denoted by $\mathbb{R}$ and $\mathbb{C}$, respectively. In $\mathbb{R}^2$ the usual Euclidean metric is considered. A sequence $\{A_n\}$ of subsets of a metric space is said to be a *null sequence* provided $\lim \mathrm{diam}(A_n) = 0$.

A surjective mapping $f : X \to Y$ between compacta X, Y is called an *ε-mapping* if $\mathrm{diam}(f^{-1}(y)) < \varepsilon$ for each $y \in Y$. A continuum is said to be *ε-chainable* provided it admits an ε-mapping onto an arc, and it is called *chainable* if it is ε-chainable for each $\varepsilon > 0$.

We say that a continuum X is *ε-hereditarily indecomposable* if for any two subcontinua K, L of X with nonempty intersection we have either $K \subset N_\varepsilon(L)$, or $L \subset N_\varepsilon(K)$. One can see that if X is ε-hereditarily indecomposable for each $\varepsilon > 0$, then X is hereditarily indecomposable in the usual sense.

A point p is called a *local separating point* of a continuum X provided p has a connected, closed neighborhood P such that $P - \{p\}$ is not connected. A point p is said to be an *accessible point* of a continuum K in a space X if there is a nondegenerate continuum L in X satisfying $p \in L \subset (X-K) \cup \{p\}$.

Continuous decompositions into pseudo-arcs. We start with a number of results concerning continuous decompositions of 2-manifolds.

PROPOSITION 1. *Let $\mathcal{D}$ be any continuous decomposition of the plane $\mathbb{R}^2$ into acyclic continua with quotient mapping $q:\mathbb{R}^2\to\mathbb{R}^2/\mathcal{D}$. Then for each continuum K in $\mathbb{R}^2$ such that $q(K)$ is nondegenerate and $K\neq q^{-1}(q(K))$ we have* $\operatorname{int} q(K)\neq\emptyset$.

Proof. By the assumptions there is D_0 in $\mathcal{D}$ such that some maximal subcontinuum K_0 of K contained in D_0 is nonempty and $K_0\neq D_0$.

Enlarge K_0 a little to obtain a continuum $K_1\neq K_0$ such that $K_0\subset K_1\subset K$ and K_1 contains no element of $\mathcal{D}$. Then the set $G=q^{-1}(q(K_1))$ is a nondegenerate continuum (for q is monotone). Take any arc $ab\subset\mathbb{R}^2$ such that $ab\cap G=\{a\}$. The set $A=q^{-1}(q(a))$ does not separate the plane, thus there is an arc bc such that $ac=ab\cup bc$ is an arc and $bc\cap G=\{c\}$ with $C=q^{-1}(q(c))\neq A$. Further, take a continuum $K_2\subset K_1$ irreducible with respect to the property that it intersects A and C, and take some $D\in\mathcal{D}$ intersecting K_2 with $A\neq D\neq C$. Then the continuum $A\cup K_2\cup C\cup ac$ separates the plane and it contains an irreducible separator L with $ac\cup K_2\subset L$.

Let x be any point in $K_2\cap D$, and y be any point in $D-K_2$. For a sufficiently small neighborhood U of x and any $x'\in U$ the set $q^{-1}(q(x'))$ intersects the component of $\mathbb{R}^2-L$ containing y, and does not intersect $ac\cup A\cup C$. By the irreducibility of the separator L, for some open set $U'\subset U$ the points $x'\in U'$ belong to some component of $\mathbb{R}^2-L$ different from that containing y. Therefore $q^{-1}(q(x'))\cap K_2\neq\emptyset$ for all $x'\in U'$. Hence $\emptyset\neq\operatorname{int} q(K_2)\subset\operatorname{int} q(K)$ by the openness of q. ■

Now we formulate a version of the well-known Moore theorem for 2-manifolds (compare [18] and [4]).

THEOREM 2 (Moore). *If $\mathcal{D}$ is an upper semicontinuous decomposition of a 2-manifold M without boundary into acyclic planar continua, then the quotient space $M/\mathcal{D}$ is homeomorphic to M.*

PROPOSITION 3. *Let $\mathcal{D}$ be any continuous decomposition of a 2-manifold M without boundary (resp. of the plane $\mathbb{R}^2$) into acyclic planar continua and let $q:M\to M$ (resp. $q:\mathbb{R}^2\to\mathbb{R}^2$) be the quotient mapping. Then for any curve L in M (in $\mathbb{R}^2$) and any $p\in L$ the continuum $q^{-1}(p)$ is terminal in $q^{-1}(L)$.*

Proof. First, assume $\mathcal{D}$ is a decomposition of $\mathbb{R}^2$. Suppose $q^{-1}(p)$ is not terminal in $q^{-1}(L)$. Then there is a continuum K in $q^{-1}(L)$ intersecting $q^{-1}(p)$ and such that $q^{-1}(p)-K\neq\emptyset\neq K-q^{-1}(p)$. So $K\neq q^{-1}(q(K))$. Applying Proposition 1 we have $\operatorname{int} q(K)\neq\emptyset$, and thus $\operatorname{int} L\neq\emptyset$, for $q(K)\subset L$. Since $\mathbb{R}^2/\mathcal{D}$ is homeomorphic to $\mathbb{R}^2$ by the Moore theorem, L cannot be a curve, a contradiction.

If $\mathcal{D}$ is a decomposition of M, we take a neighborhood U of p in $M/\mathcal{D}$ homeomorphic to $\mathbb{R}^2$ ($M/\mathcal{D}$ is homeomorphic to M by the Moore theorem—Theorem 2). Then we prove that $q^{-1}(p)$ is terminal in $q^{-1}(L_0)$, where L_0 is the component of $L \cap U$ containing p. This implies that $q^{-1}(p)$ is also terminal in $q^{-1}(L)$. ∎

The following theorem is a generalized version of Lewis' result from [11].

THEOREM 4 (Lewis). *Let X be a curve and $\mathcal{D}$ be a continuous decomposition of X such that each element of $\mathcal{D}$ is either a point or a pseudo-arc terminal in X, and the quotient space is a pseudo-arc. Then X is a pseudo-arc.*

Proof. Let $q_0 : X \to \mathcal{D}$ be the quotient mapping, and let $\widehat{X}$ be a continuous curve of pseudo-arcs with quotient space X and quotient map q_1 defined in [12]. Consider the composition $q = q_0 \circ q_1$, which is open and monotone, and observe that the point inverses of q are terminal in $\widehat{X}$. Let $D \in \mathcal{D}$. If D is a point, then $q^{-1}(D)$ is a pseudo-arc. If D is a pseudo-arc, then $q^{-1}(D)$ is a pseudo-arc by [11]. Thus $\widehat{X}$ is a pseudo-arc by [11]. Since X is a monotone image of the pseudo-arc $\widehat{X}$, it is itself a pseudo-arc. ∎

In [17] Lewis and Walsh proved Anderson's announcement stating that the plane admits a continuous decomposition into pseudo-arcs. Now we prove a version of this theorem for any 2-manifold without boundary.

THEOREM 5. *Each 2-manifold without boundary admits a continuous decomposition into pseudo-arcs.*

Proof. Let $C_1, \ldots, C_n$ be 2-cells in M such that $M = \operatorname{int} C_1 \cup \ldots \cup \operatorname{int} C_n$. For any $i \in \{1, \ldots, n\}$ take a continuous decomposition of $\operatorname{int} C_i$ into pseudo-arcs (see [17]) such that the diameters of the elements approximating $\operatorname{bd} C_i$ converge to 0. Extend this decomposition to M taking singletons in $M - \operatorname{int} C_i$. Then the quotient space is homeomorphic to M (Theorem 2). Let $q_i : M \to M$ be the quotient mapping additionally satisfying $q_i(x) = x$ for all $x \in M - C_i$. Then the q_i are monotone and open, and the sets $q_i^{-1}(p)$ are either points or pseudo-arcs. Let $q = q_n \circ \ldots \circ q_1$. Then q is a monotone, open mapping from M onto M with all point inverses nondegenerate.

We show that all these point inverses are pseudo-arcs. Indeed, for any $p \in M$ the set $P = q_1^{-1}(q_2^{-1}(p))$ is either a curve or a singleton, for the mappings q_1, q_2 are open and monotone. Thus the set P has a continuous decomposition with elements either points or terminal pseudo-arcs (Proposition 3), and with quotient space either a pseudo-arc or a point. Hence P is either a pseudo-arc (Theorem 4) or a point. Continuing this argument inductively, we prove all the sets $q^{-1}(p)$ to be pseudo-arcs. These sets form the required decomposition. ∎

The following lemma is crucial. To prove it, we employ the most essential tool in this paper, a continuous decomposition of the annulus into pseudo-arcs, presented by the author in [20].

LEMMA 6. *Let D be an element of a continuous decomposition $\mathcal{D}$ of the plane $\mathbb{R}^2$ into pseudo-arcs, and let $p \in D$. Then for each $\varepsilon > 0$ there is a $\delta > 0$ such that for each 2-cell neighborhood C of p with* $\operatorname{diam} C < \delta$ *there exists a continuous decomposition $\mathcal{D}'$ of $M = \mathbb{R}^2 - \operatorname{int} C$ into pseudo-arcs such that* $\operatorname{dist}(D, D') < \varepsilon$ *for each $D' \in (\mathcal{D}' - \mathcal{D}) \cup (\mathcal{D} - \mathcal{D}')$.*

Proof. Let $q : \mathbb{R}^2 \to \mathbb{R}^2$ be a quotient mapping of $\mathcal{D}$ with $p' = q(p)$. Let $\sigma > 0$ be so small that $\operatorname{dist}(q^{-1}(r), D) < \varepsilon$ for each $r \in N_\sigma(p')$. Consider the following decomposition $\mathcal{D}_1$ of $\mathbb{R}^2$:

$$\mathcal{D}_1 = \{q^{-1}(y) : y \in \operatorname{cl} N_{\sigma/4}(p')\} \cup \{\{x\} : q(x) \notin \operatorname{cl} N_{\sigma/4}(p')\}.$$

It is upper semicontinuous, and, by the Moore theorem, the quotient space is the plane. Define $X = \mathbb{R}^2 - q^{-1}(N_{\sigma/2}(p'))$ and let $\delta > 0$ be such that $N_\delta(p) \subset \mathbb{R}^2 - X$. Take a quotient mapping $q_1 : \mathbb{R}^2 \to \mathbb{R}^2$ of $\mathcal{D}_1$, additionally assuming that $q_1(p) = p$, $q_1(x) = x$ for $x \in X$, and $C = q_1(q^{-1}(\operatorname{cl} N_{\sigma/4}(p')))$ is an arbitrarily chosen 2-cell neighborhood of p in $N_\delta(p) \subset \mathbb{R}^2 - X$. Thus $M = \mathbb{R}^2 - \operatorname{int} C = \mathbb{R}^2 - q_1(q^{-1}(N_{\sigma/4}(p')))$.

In [20], actually, two essentially different decompositions of the annulus into pseudo-arcs were constructed: with a circle and with an annulus as quotient space. Now we use the second one, modifying it as follows. Choose one of the boundary circles of the annulus, and shrink to points all those pseudo-arcs in the decomposition which intersect this circle. We obtain again a topological annulus (by the Moore theorem) with a continuous decomposition into pseudo-arcs and singletons in one of the boundary simple closed curves.

Consider such a decomposition $\mathcal{D}_2'$ of the annulus $A = \{x \in \mathbb{R}^2 : \sigma/4 \le d(p', x) \le \sigma\}$, additionally assuming that $\mathcal{D}_2'$ has singletons in $\{x \in \mathbb{R}^2 : d(p, x) = \sigma\}$, and each element of $\mathcal{D}_2'$ intersects the annulus $A_1 = \{x \in \mathbb{R}^2 : \sigma/2 \le d(p', x) \le \sigma\}$. Extend $\mathcal{D}_2'$ to the decomposition $\mathcal{D}_2$ of $Y = \mathbb{R}^2 - N_{\sigma/4}(p')$ taking singletons in $Y - A$. Then

$$\mathcal{D}' = \{q_1(q^{-1}(F)) : F \in \mathcal{D}_2\}$$

is the required decomposition of M. Indeed, obviously, it is well defined and continuous. Further, since q is open, the sets $q^{-1}(F)$ are curves for $F \in \mathcal{D}_2$. Their continuous decompositions into elements of $\mathcal{D}$ consist of terminal pseudo-arcs only (Proposition 3), and the quotient space is either a pseudo-arc or a point. Thus the sets $q^{-1}(F)$ are pseudo-arcs (Theorem 4). Hence so are their nondegenerate, monotone images $q_1(q^{-1}(F))$.

Let $D' \in \mathcal{D}' - \mathcal{D}$. Observe that then $D' = q_1(q^{-1}(F))$ for some nondegenerate $F \in \mathcal{D}_2'$. Thus F contains a point r of A_1, and D' contains the pseudo-arc $q^{-1}(r) = q_1(q^{-1}(r)) \in \mathcal{D}$ which satisfies $\mathrm{dist}(q^{-1}(r), D) < \varepsilon$ by the assumption. Therefore $D \subset N_\varepsilon(D')$. On the other hand, $D' \subset q_1(q^{-1}(N_\sigma(p'))) \subset q_1(N_\varepsilon(D)) \subset N_\varepsilon(D)$. (The last inclusion follows from the inclusion $q^{-1}(N_{\sigma/2}(p')) \subset N_\varepsilon(D)$ and the definition of q_1.) Hence $\mathrm{dist}(D, D') < \varepsilon$.

If $D' \in \mathcal{D} - \mathcal{D}'$, then $D' = q^{-1}(r)$ for some $r \in N_\sigma(p')$, and thus $\mathrm{dist}(D, D') < \varepsilon$ by the assumption. ∎

Remark 6a. Observe that the property described in the above lemma is topological, and does not depend on the particular metric on $\mathbb{R}^2$.

Now we formulate a variant of Lemma 6 for any 2-manifold M (with or without boundary). Let $\mathcal{D}$ be a continuous decomposition of M into pseudo-arcs with quotient map q, and let D be an element of $\mathcal{D}$ containing no boundary point of M. By the Moore theorem there exists an open neighborhood U of D homeomorphic to the plane such that $q^{-1}(q(U)) = U$. Applying Lemma 6 to U (see Remark 6a) we obtain the next lemma.

Lemma 7. *Let D be an element of a continuous decomposition $\mathcal{D}$ of a 2-manifold M (with or without boundary) into pseudo-arcs such that D contains no boundary point of M, and let $p \in D$. Then for each $\varepsilon > 0$, there is a $\delta > 0$ such that for each 2-cell neighborhood C of p with $\mathrm{diam}\, C < \delta$ there exists a continuous decomposition $\mathcal{D}'$ of $M' = M - \mathrm{int}\, C$ into pseudo-arcs satisfying*:

(i) $\mathrm{dist}(D, D') < \varepsilon$ *for each* $D' \in (\mathcal{D}' - \mathcal{D}) \cup (\mathcal{D} - \mathcal{D}')$,

(ii) *each element of $\mathcal{D}$ containing a boundary point of M belongs to $\mathcal{D}'$, and*

(iii) *if no element of $\mathcal{D}$ contains two different boundary points of M, then $\mathcal{D}'$ has the same property in M'.*

Remark 7a. Observe that $\mathrm{dist}_2(\mathcal{D}, \mathcal{D}') < 2\varepsilon$ for $\mathcal{D}$, $\mathcal{D}'$ as in the lemma.

Any 2-manifold with boundary may be obtained from a 2-manifold without boundary by deleting finitely many interiors of 2-cells. Thus, by Theorem 5 and Lemma 7, each 2-manifold M with boundary admits a continuous decomposition into pseudo-arcs. Moreover, this decomposition can be chosen to have only elements containing at most one boundary point. Applying the Moore theorem one can easily verify that the quotient space is then homeomorphic to M.

Theorem 8. *Each 2-manifold M (with or without boundary) admits a continuous decomposition into pseudo-arcs with quotient space homeomorphic to M.*

There is no internal characterization of all continua embeddable in surfaces. Evidently, such continua must be locally planar. The example of a solenoid shows that this condition is not sufficient. However, locally planar Peano continua are known to be embeddable into surfaces [2]. Since this result of Bajguz has not been published yet, we present the following less general (but suitable for our purposes) proposition.

PROPOSITION 9. *Each locally connected, locally planar continuum X with no local separating point is embeddable in a 2-manifold. Moreover, the embedding $e : X \to M$ can be chosen so that M is a 2-manifold without boundary, and*

$$e(X) = M - \bigcup_n \operatorname{int} C_n$$

for some (possibly finite or empty) null sequence $\{C_n\}$ of mutually disjoint 2-cells in M.

Proof. By the assumptions X admits a finite cover $\{U_1, \ldots, U_n\}$ of open, connected subsets such that each U_i is contained in the interior of a 2-cell D_i. Additionally, we can assume that $D_1 \cup \ldots \cup D_n$ is a compact metric space satisfying $D_i \cap D_j \subset X$ for $i \neq j$ and $D_i \cap X = \operatorname{cl} U_i$. Let $\varepsilon > 0$ be the Lebesgue number of the covering $\{U_1, \ldots, U_n\}$ of X. Consider the family $\mathcal{F}$ of all simple closed curves in X having diameters less than $\varepsilon/2$ which do not separate X. Observe that if $Y \in \mathcal{F}$, then $Y \subset U_i$ for some i, and if $Y \subset U_i$ for some i, then Y is the boundary of some complementary domain $C(Y, i)$ of U_i in D_i. Further, note that the elements of $\mathcal{F}$ are mutually disjoint. Indeed, if two different elements Y_1, Y_2 of $\mathcal{F}$ had a common point p, then they would be contained in some U_i, and thus p would be a local separating point of U_i. Consequently, it would be a local separating point of X, a contradiction.

We prove that all the sets $C(Y, i)$ for $Y \in \mathcal{F}$ (and thus also the elements of $\mathcal{F}$) form a null sequence. In fact, otherwise some D_i would contain an infinite sequence of mutually disjoint complementary domains $C(Y_k, i)$ with diameters greater than some fixed $\delta > 0$, converging to some $Y_0 \subset D_i \cap X$, and thus X would not be locally connected at some points of Y_0, contrary to the assumption.

For any $Y \in \mathcal{F}$ fix one set $C_Y = C(Y, i)$. Observe that different C_Y are mutually disjoint, form a null sequence and $X \cap C_Y = \emptyset$ for $Y \in \mathcal{F}$. Let $M' = X \cup \bigcup\{C_Y : Y \in \mathcal{F}\}$. Given $Y \in \mathcal{F}$, by the Schönflies theorem, each 2-cell $Y \cup C(Y, j)$ can be homeomorphically mapped onto $Y \cup C_Y$ with all points of Y invariant. Thus M' has a finite covering composed of the sets homeomorphic to the sets $U_i \cup \bigcup\{C(Y, i) : Y \in \mathcal{F},\ Y \subset U_i\} \subset D_i$. So, M' is a locally connected, 2-dimensional continuum with no local separating point such that all sufficiently small simple closed curves separate M'. Hence it is a surface by Young's result ([24], Th. 4.1). Finally, M' can be considered as a subspace of a 2-manifold M without boundary such that $M - M'$ is a finite union of interiors of mutually disjoint 2-cells. We can take these 2-cells together with the 2-cells $Y \cup C_Y$ for the required sets C_n. ■

The next theorem was essentially proved by Whyburn [23] for the 2-sphere, and generalized for any 2-manifold by Borsuk [4].

THEOREM 10 (Whyburn, Borsuk). *Let M be a 2-manifold without boundary, and let $\{C_n\}$ and $\{C'_n\}$ be null sequences of mutually disjoint 2-cells in M such that*

$$\mathrm{cl}\Big(\bigcup_n C_n\Big) = \mathrm{cl}\Big(\bigcup_n C'_n\Big) = M.$$

Then there is a homeomorphism $h : M \to M$ such that

$$h\Big(\bigcup_n C_n\Big) = \bigcup_n C'_n.$$

The following lemma is a variation of a known property. For example it was used (for $\mathcal{F}$ consisting of a single continuum) to prove that all pseudo-arcs in $\mathbb{R}^n$ $(n > 1)$ form a G_δ-set [3]. The proof is natural and easy, so we omit it.

LEMMA 11. *Let $\mathcal{F}$ be a compact family of ε-chainable* (*resp. ε-hereditarily indecomposable*) *continua in a compact space X. Then there exists a $\delta > 0$ such that for each continuum $K \in N_\delta(\mathcal{F}) \subset C(X)$, K is ε-chainable* (*resp. ε-hereditarily indecomposable*)*. In other words*, *the family of all ε-chainable* (*resp. ε-hereditarily indecomposable*) *continua is open in $C(X)$.*

Now we introduce the following concept of an *ε-continuous decomposition*, which will be used in the main construction of the paper.

DEFINITION 1. Let X be a metric space, and let $\mathcal{F}$ be a compact family (as a subspace of 2^X) of compact subsets of X. Then $\mathcal{F}$ is called an *ε-continuous decomposition* provided there exists a $\delta > 0$ such that for all $A, B \in \mathcal{F}$ if $\widehat{d}(A, B) < \delta$, then $\mathrm{dist}(A, B) < \varepsilon$.

Note that in the above definition the elements of $\mathcal{F}$ may not be mutually disjoint, and they may not cover X.

OBSERVATION 12. *Let X and $\mathcal{F}$ be as in the above definition. Then $\mathcal{F}$ is a continuous decomposition of $\bigcup \mathcal{F}$ if and only if $\mathcal{F}$ is an ε-continuous decomposition for each $\varepsilon > 0$.*

LEMMA 13. *Let X and $\mathcal{F}$ be as above. If $\mathcal{F}$ is an ε-continuous decomposition*, *then there exists a $\sigma > 0$ such that for each compact family $\mathcal{F}'$ of compact subsets of X satisfying* $\mathrm{dist}_2(\mathcal{F}, \mathcal{F}') < \sigma$, *the family $\mathcal{F}'$ is also an ε-continuous decomposition. In other words*, *the set of all ε-continuous decompositions is open in 2^{2^X}.*

P r o o f. Let $\mathcal{F} \subset 2^X$ be an ε-continuous decomposition. Suppose there are compact families $\mathcal{F}_n \subset 2^X$ converging to $\mathcal{F}$ such that no $\mathcal{F}_n$ is an ε-continuous decomposition. Let $\delta > 0$ be the number guaranteed by the definition of the ε-continuous decomposition for $\mathcal{F}$. Then for each n there are $K_n, L_n \in \mathcal{F}_n$ such that $\widehat{d}(K_n, L_n) < \delta/2$ and $\operatorname{dist}(K_n, L_n) \geq \varepsilon$. Since $\bigcup \mathcal{F} \cup \bigcup\{\bigcup \mathcal{F}_n : n = 1, 2, \ldots\}$ is compact, we may assume that the sets K_n, L_n converge to some $K, L \in \mathcal{F}$, respectively. Thus $\widehat{d}(K, L) \leq \delta/2 < \delta$, while $\operatorname{dist}(K, L) \geq \varepsilon$, contrary to the definition of δ. ∎

Now we start the construction of continuous decompositions into pseudo-arcs of Peano continua in 2-manifolds with no local separating point.

The Main Construction. Let $\mathcal{A}$ be the set of all sequences $(a_0, a_1, \ldots) \in \{0, 1\}^\omega$ with $a_0 = 0$. For sequences in $\mathcal{A}$ with almost all entries 0 we let $(a_0, \ldots, a_n, 0, 0, \ldots) = (a_0, \ldots, a_n)$.

Fix any 2-manifold M without boundary. We construct a null sequence $\{C_n\}$ of mutually disjoint 2-cells in M, a family of decompositions $\mathcal{D}(\boldsymbol{a})$ of $M(\boldsymbol{a}) = M - \bigcup\{\operatorname{int} C_i : a_i = 1\}$ (where $\boldsymbol{a} = (a_0, a_1, \ldots) \in \mathcal{A}$) into pseudo-arcs and positive numbers ε_n such that:

(1) $3\varepsilon_0 < \inf\{\operatorname{diam} D : D \in \mathcal{D}(a_0)\}$,

(2) for each $\boldsymbol{a} = (a_0, \ldots, a_n) \in \mathcal{A}$ and each $D \in \mathcal{D}(\boldsymbol{a})$, D contains at most one boundary point of a complementary domain of $M(\boldsymbol{a})$ in M,

(3) for each $\mathcal{F} \in C^2(M)$ and for each $(a_0, \ldots, a_n) \in \mathcal{A}$ if $\operatorname{dist}_2(\mathcal{F}, \mathcal{D}(a_0, \ldots \ldots, a_n)) < 2\varepsilon_n$, then $\mathcal{F}$ is a $(1/(n+1))$-continuous decomposition composed of $(1/(n+1))$-chainable and $(1/(n+1))$-hereditarily indecomposable continua,

(4) $\operatorname{dist}_2(\mathcal{D}(a_0, \ldots, a_n), \mathcal{D}(a_0, \ldots, a_n, 1)) < \varepsilon_n$ for each $(a_0, \ldots, a_n) \in \mathcal{A}$,

(5) $\operatorname{diam} C_n < \varepsilon_{n-1}$,

(6) $2\varepsilon_n < \varepsilon_{n-1}$, and

(7) $\operatorname{cl}(\bigcup\{C_i : i = 1, 2, \ldots\}) = M$.

Let $M(a_0) = M$ and let $\mathcal{D}(a_0)$ be a fixed continuous decomposition of M into pseudo-arcs (see Theorem 5). Fix an $\varepsilon_0 > 0$ so small that (1) and (3) hold for $n = 0$ (see Observation 12 and Lemmas 11 and 13).

Assume that for some fixed k there are already constructed: mutually disjoint 2-cells $C_1, \ldots, C_k$ in M, positive numbers $\varepsilon_0, \ldots, \varepsilon_k$ and continuous decompositions $\mathcal{D}(\boldsymbol{a})$ of $M(\boldsymbol{a})$ for all $\boldsymbol{a} = (a_0, \ldots, a_k) \in \mathcal{A}$ satisfying conditions (2)–(6) for $0 \leq n \leq k$. For any $\boldsymbol{a} = (a_0, \ldots, a_k) \in \mathcal{A}$ let $B(\boldsymbol{a})$ be the union of all elements of $\mathcal{D}(\boldsymbol{a})$ containing a boundary element of $M(\boldsymbol{a})$, i.e. containing a point from $\operatorname{bd} C_i$ for some $i \in \{1, \ldots, k\}$ such that $a_i = 1$. Observe that $B(\boldsymbol{a})$ is a boundary set in $M(\boldsymbol{a})$. Let $B_k = \bigcup\{B(\boldsymbol{a}) : \boldsymbol{a} = (a_0, \ldots, a_k) \in \mathcal{A}\}$, and $M_k = M(0, 1, \ldots, 1) = M - (\operatorname{int} C_1 \cup \ldots \cup \operatorname{int} C_k)$. Then $B_k \cap M_k$ is a closed boundary set in M_k.

Fix a point $p_{k+1} \in M_k - B_k$ such that some ball in M contained in $M_k - B_k$ with center p_{k+1} has the maximal radius among all balls in M contained in $M_k - B_k$. For any $\boldsymbol{a} = (a_0, \ldots, a_k) \in \mathcal{A}$ we take a $\delta > 0$ guaranteed by Lemma 7 for $M(\boldsymbol{a})$, the decomposition $\mathcal{D}(\boldsymbol{a})$, $p = p_{k+1}$ and for $\varepsilon = \varepsilon_k/2$. Then we take a 2-cell neighborhood $C_{k+1} \subset M_k - B_k$ of p_{k+1} with diameter less than ε_k and less than each δ chosen above for any $(a_0, \ldots, a_k) \in \mathcal{A}$. Putting $\boldsymbol{a}' = (a_0, \ldots, a_k, 1)$, for any $\boldsymbol{a} = (a_0, \ldots, a_k) \in \mathcal{A}$ we take a decomposition $\mathcal{D}(\boldsymbol{a}')$ of $M(\boldsymbol{a}') = M(\boldsymbol{a}) - \operatorname{int} C_{k+1}$ guaranteed by Lemma 7 for $M(\boldsymbol{a})$, $\mathcal{D}(\boldsymbol{a})$, $p = p_{k+1}$ and $\varepsilon = \varepsilon_k/2$.

Observe that $\operatorname{dist}_2(\mathcal{D}(\boldsymbol{a}'), \mathcal{D}(\boldsymbol{a})) < \varepsilon_k$ (see Remark 7a). Next, for any $(a_0, \ldots, a_k) \in \mathcal{A}$ let $\mathcal{D}(a_0, \ldots a_k, 0) = \mathcal{D}(a_0, \ldots, a_k)$. Finally, we choose ε_{k+1} to satisfy (6), and, by Lemmas 11 and 13, to satisfy (3) for $n = k+1$. Thus conditions (3)–(6) are satisfied for $n = k+1$ and (2) is satisfied for all $(a_0, \ldots, a_{k+1}) \in \mathcal{A}$.

By induction we have constructed C_n, ε_n for all positive integers n, and $\mathcal{D}(\boldsymbol{a})$ for all $\boldsymbol{a} \in \mathcal{A}$ with almost all entries 0, such that (1)–(6) are fulfilled. Observe that (7) is satisfied by the choice of the points p_n.

Let $\boldsymbol{a} = (a_0, a_1, \ldots) \in \mathcal{A}$ be a fixed sequence with infinitely many entries 1. Observe that the sequence $\mathcal{D}_n = \mathcal{D}(a_0, \ldots, a_n)$ converges in $C^2(M)$ by (4) and (6). Let $\mathcal{D}(\boldsymbol{a}) = \operatorname{Lim} \mathcal{D}_n$, and note that $\bigcup \mathcal{D}(\boldsymbol{a}) = M(\boldsymbol{a})$. Further, observe that for each n we have $\operatorname{dist}_2(\mathcal{D}(\boldsymbol{a}), \mathcal{D}_n) < 2\varepsilon_n$ again by (4) and (6). Therefore, for each n, the family $\mathcal{D}(\boldsymbol{a})$ is a $(1/n)$-continuous decomposition composed of $(1/n)$-chainable and $(1/n)$-hereditarily indecomposable continua by (3). The elements of $\mathcal{D}(\boldsymbol{a})$ are nondegenerate by (1), (6) and (4). Hence $\mathcal{D}(\boldsymbol{a})$ is a continuous decomposition (Observation 12) of $M(\boldsymbol{a})$ into nondegenerate, chainable, hereditarily indecomposable continua, i.e. into pseudo-arcs [3].

Take two different points x, y in $\operatorname{bd} C_i$, $\operatorname{bd} C_j$, respectively, such that $a_i = a_j = 1$. Then, for all $n \geq \max\{i, j\}$, they belong to different and invariant elements of $\mathcal{D}_n = \mathcal{D}(a_0, \ldots, a_n)$. Therefore (2) is satisfied for $\mathcal{D}(\boldsymbol{a}) = \operatorname{Lim} \mathcal{D}_n$. The construction is complete. ∎

In the next proposition we use the following observation.

OBSERVATION 14. *Let x and y be any two different points of a continuum X. Then the continuum $Y = X/\{x, y\}$, obtained from X by identifying x and y to a point, admits an essential mapping onto the circle.*

Indeed, let $q : X \to Y$ be the quotient mapping. There exists a continuous surjection $f : X \to [0,1]$ such that $f^{-1}(0) = \{x\}$ and $f^{-1}(1) = \{y\}$ by the Urysohn lemma. Put $g(t) = \cos 2\pi t + i \sin 2\pi t \in \mathbb{C}$ for any $t \in [0,1]$, and for any $p \in Y$ define $h(p) = g(f(q^{-1}(p)))$. Then h is a well-defined and continuous mapping from Y onto the unit circle in the complex plane.

Moreover, it is nonhomotopic to the constant map. The details are left to the reader. ∎

Now we establish some relationship between monotone, continuous decompositions of continua and their local separating points.

PROPOSITION 15. *Let $\mathcal{D}$ be a nontrivial continuous decomposition of a continuum X into continua, and let p be a local separating point of X. Then the element D of $\mathcal{D}$ containing p is either degenerate, or it admits an essential mapping onto the circle.*

Proof. Let P be a closed, connected neighborhood of p in X such that $P = A \cup B$, where A and B are closed and satisfy $A - \{p\} \neq \emptyset \neq B - \{p\}$ and $A \cap B = \{p\}$. Define X' as the quotient space of $\mathrm{cl}(X-P) \cup A \times \{1\} \cup B \times \{2\}$ under the obvious identification of $(A \cap \mathrm{bd}\, P) \times \{1\}$ with $A \cap \mathrm{bd}\, P \subset \mathrm{cl}(X-P)$ and of $(B \cap \mathrm{bd}\, P) \times \{1\}$ with $B \cap \mathrm{bd}\, P$. Then X' is a compact metric space and the natural projection $q : X' \to X$ is continuous and has only one nontrivial point inverse, which is $q^{-1}(p) = \{(p,1)\} \cup \{(p,2)\}$.

Denote by $D(x)$ the element of $\mathcal{D}$ containing x (for $x \in X$). Assume $D(p)$ is nondegenerate. Then the component C_1 containing p either of $D(p) - A$ or of $D(p) - B$ is nondegenerate. Assume C_1 is a component of $D(p) - B$. Let $\{b_n\}$ be a sequence of points in $B - D(p)$ converging to p (we have $\mathrm{int}\, D(p) = \emptyset$ by the continuity of $\mathcal{D}$). Then for each $\varepsilon > 0$ we have $C_1 \subset N_\varepsilon(D(b_n))$ for almost all n. Thus some points of $D(b_n) \cap A$ approximate p. Therefore the continua $q^{-1}(D(b_n))$ converge to $q^{-1}(D(p)) = q^{-1}(D(p) - \{p\}) \cup \{(p,1),(p,2)\}$. So, $q^{-1}(D(p))$ is a continuum.

From Observation 14 applied to $D(p)$ and $q^{-1}(D(p))$, it follows that $D(p)$ admits an essential mapping onto the circle. ∎

The following result summarizes the above investigation. It comprises a complete characterization of all locally planar Peano continua admitting a continuous decomposition into pseudo-arcs (into nondegenerate acyclic continua).

MAIN THEOREM 16. *For each locally planar, locally connected continuum X the following conditions are equivalent*:

(a) *X admits a continuous decomposition into pseudo-arcs*,

(b) *X admits a continuous decomposition into nondegenerate, acyclic continua*,

(c) *X has no local separating point*,

(d) *there exist a* 2*-manifold M without boundary and a null sequence C_n (possibly finite or empty) of mutually disjoint* 2*-cells in M such that X is homeomorphic to $M - \bigcup_n \mathrm{int}\, C_n$.*

Moreover, the decomposition into pseudo-arcs in (a) *can be so chosen that the quotient space again satisfies each of the equivalent conditions* (a)–(d).

P r o o f. The implication (a)⇒(b) is obvious. Proposition 15 implies (b)⇒(c), and (c)⇒(d) follows by Proposition 9.

Assume (d) is satisfied, i.e. $X = M - \bigcup_n \operatorname{int} C_n$ for some M and C_n as in (d). We will apply the Main Construction for the surface M to obtain the desired decomposition. Take a null sequence $\{F_n\}$ of mutually disjoint 2-cells in M such that $\{C_1, C_2, \ldots\} \subset \{F_1, F_2, \ldots\}$ and $\operatorname{cl}(\bigcup_n F_n) = M$. Let C'_n be a null sequence of 2-cells in M obtained by the Main Construction for M. Then, by Theorem 10, there is a homeomorphism $h : M \to M$ such that $h(\bigcup_n F_n) = \bigcup_n C'_n$, or equivalently, $h(M - \bigcup_n \operatorname{int} F_n) = M - \bigcup_n \operatorname{int} C'_n$. Let $a_0 = 0$, $a_n = 1$ if $h^{-1}(C'_n) = C_i$ for some i, and $a_n = 0$ otherwise. Then, by the construction, the decomposition $\mathcal{D}(\boldsymbol{a})$ of $M(\boldsymbol{a})$ into pseudo-arcs is obtained for $\boldsymbol{a} = (a_0, a_1, \ldots)$. Thus $X = h^{-1}(M(\boldsymbol{a}))$, being a homeomorphic copy of $M(\boldsymbol{a})$, admits a continuous decomposition into pseudo-arcs. Hence we have (d)⇒(a).

Moreover, $\mathcal{D}(\boldsymbol{a})$ is so chosen that each of its elements has at most one boundary point of a complementary domain of $M(\boldsymbol{a})$. Extending $\mathcal{D}(\boldsymbol{a})$ to M by taking singletons in $M - M(\boldsymbol{a})$, we obtain a topological copy of M as quotient space by the Moore theorem. Therefore the quotient space of $\mathcal{D}(\boldsymbol{a})$ is topologically M with the interiors of some null sequence of mutually disjoint 2-cells deleted. Hence it is locally planar and locally connected with no local separating point. ∎

Fix a 2-manifold M without boundary. Among locally connected subcontinua of M obtained by deleting interiors of a null sequence (possibly finite or empty) $\{C_n\}$ of mutually disjoint 2-cells in M, there is a topologically unique curve. In fact, it is obtained if $\operatorname{cl}(\bigcup_n C_n) = M$. Its topological uniqueness follows from Theorems 16 and 10. Denote this curve by $S(M)$.

By the Whyburn characterization [23] of the Sierpiński universal plane curve S one can observe that $S(M)$ has a basis of closed neighborhoods composed of topological copies of S. Therefore $S(M)$ is an S-manifold. One can easily verify (compare [4]) that for all 2-manifolds M_1, M_2 without boundary, $S(M_1)$ and $S(M_2)$ are homeomorphic if and only if M_1 and M_2 are homeomorphic. From Theorem 16 we obtain the following characterization of S-manifolds.

COROLLARY 17. *For any continuum X the following conditions are equivalent*:

(a) *X is an S-manifold*,

(b) *X is a locally connected, locally planar curve with no local separating point*,

(c) *X is a locally connected, locally planar curve admitting a continuous decomposition into nondegenerate acyclic continua*,

(d) *X is a locally connected, locally planar curve admitting a continuous decomposition into pseudo-arcs such that the quotient space is homeomorphic to X.*

In the planar case (i.e. for M homeomorphic to the 2-sphere), we obtain the following extension of Whyburn's characterization [23] of the Sierpiński curve.

COROLLARY 18. *For any continuum X the following conditions are equivalent*:

(a) *X is homeomorphic to the Sierpiński universal plane curve*,

(b) *X is a locally connected planar curve with no local separating point*,

(c) *X is a locally connected planar curve admitting a continuous decomposition into nondegenerate acyclic continua*,

(d) *X is a locally connected planar curve admitting a continuous decomposition into pseudo-arcs such that the quotient space is homeomorphic to X.*

Applications to open homogeneity. Investigating continuous decompositions of Peano continua into pseudo-arcs was mainly inspired by questions concerning open homogeneity.

A space X is said to be *openly homogeneous* provided for any two points $x, y \in X$ there is an open surjective mapping $f : X \to X$ with $f(x) = y$. If the mappings f are to be monotone and open, we say that X is *homogeneous with respect to monotone open mappings* (abbr. (*m.o.*)-*homogeneous*). This generalization of the usual homogeneity was introduced by J. J. Charatonik.

Spaces X, Y are called *openly equivalent* if there are open surjections $f_1 : X \to Y$ and $f_2 : Y \to X$. If f_1, f_2 are monotone and open, then X, Y are called *equivalent with respect to monotone open mappings* (abbr. (*m.o.*)-*equivalent*).

It is an elementary observation that if X and Y are openly equivalent (resp. (m.o.)-equivalent), then X is openly homogeneous (resp. (m.o.)-homogeneous) if and only if so is Y.

The classical result of Mazurkiewicz [19] says that the only homogeneous, plane, locally connected, nondegenerate continuum is the simple closed curve. Over a decade ago J. J. Charatonik ([7], Problem 5) asked whether this result remains true if we replace the usual homogeneity by the open homogeneity. Some years ago the author realized that the answer is no:

Example 19. The 2-cell is openly homogeneous.

Proof. It suffices to show that the 2-cell $C = \{c \in \mathbb{C} : |c| \le 1\}$ is openly equivalent to the unit 2-sphere S^2 in $\mathbb{R}^3$, for S^2 is even homogeneous. Having the obvious, open projection of S^2 onto C, we will construct an open surjection $g : C \to S^2$, and thus the proof will be finished.

Let $C_0 = \{c \in \mathbb{C} : |c| = 1\}$ and let $q : C \to X$ be the quotient mapping identifying to a point $x_0 \in X$ all points of C_0. Then X is a topological 2-sphere. Let $\mathcal{D}$ a be continuous decomposition of X into pseudo-arcs (Theorem 5) such that x_0 is an inaccessible point of the element D_0 of $\mathcal{D}$ containing x_0. Then the quotient space is again a topological 2-sphere by the Moore theorem. Let $q' : X \to S^2$ be the quotient map of $\mathcal{D}$. We claim that the composition $g = q' \circ q$ is open.

Indeed, since $\{x_0\} = q(C_0)$ is inaccessible in D_0, observe that C_0 is a boundary set in $g^{-1}(q'(x_0))$. Therefore the decomposition $\mathcal{D}' = \{g^{-1}(x) : x \in S^2\}$ is continuous at $g^{-1}(q'(x_0))$ by the continuity of $\mathcal{D}$. The continuity of $\mathcal{D}'$ at the other elements is obvious again by the continuity of $\mathcal{D}$. ∎

Now we apply the results of the previous chapter to prove that each curve locally homeomorphic to the Sierpiński universal plane curve is (m.o.)-homogeneous. For the Sierpiński curve itself this was recently shown by Seaquist [22].

First, we need some auxiliary notions and facts. Recall that a point of the Sierpiński curve S is said to be *rational* provided it belongs to the boundary of a complementary domain of S. Otherwise it is called *irrational*. The notion of a rational point of S is known to be a topological invariant. Similarly, we consider rational and irrational points of any S-manifold defined as follows. A point p of an S-manifold X is called *rational* (resp. *irrational*) provided p has a neighborhood N in X homeomorphic to the Sierpiński curve such that p is a rational (irrational) point of N.

Now we formulate some results concerning rational and irrational points of an S-manifold. For the Sierpiński curve S they are well known (see [6]). The general arguments for any S-manifold are similar to those for S, so we omit the proofs.

Proposition 20. *The notion of a rational* (*resp. an irrational*) *point of an S-manifold is a topological invariant.*

Proposition 21. *Let p be a point of an S-manifold X contained in a* 2*-manifold M without boundary. Then p is a rational point of X if and only if p belongs to the boundary of some complementary domain of X in M.*

Proposition 22. *For any two points p, q of an S-manifold X there exists a homeomorphism $h : X \to X$ with $f(p) = q$ if and only if p and q are either both rational points of X, or both irrational points of X.*

Now we formulate the announced result.

THEOREM 23. *Each S-manifold is (m.o.)-homogeneous.*

Proof. In view of Proposition 22 it suffices to show that there are a rational (resp. an irrational) point p of X, an irrational (resp. a rational) point q of X and an open monotone mapping $f : X \to X$ with $f(p) = q$.

Let M be a 2-manifold without boundary such that X is embeddable in M as the complement of the dense union of the interiors of a null sequence of mutually disjoint 2-cells in M. Take a 2-cell C in M and a quotient mapping $q : M \to M'$ with C and singletons as all point inverses. Then M and M' are homeomorphic by the Moore theorem (Theorem 2). Assume X is contained in M' so that for some continuous decomposition $\mathcal{D}'$ of X into pseudo-arcs (Corollary 17) and for some element $D' \in \mathcal{D}'$ containing only irrational points of X, the singleton $\{p\} = q(C)$ is an inaccessible point of D' in M'. Let $X_0 = \operatorname{bd} C \cup q^{-1}(X - \{p\})$. Then X_0 is the complement of the dense union of the interiors of a null sequence of mutually disjoint 2-cells in M, and thus X_0 and X are homeomorphic by Theorem 10.

Let $g : M' \to M''$ be the quotient mapping on M' defined by the decomposition $\mathcal{D}'$ on X and by singletons on $M' - X$. Then M'' is homeomorphic to M' (by the Moore theorem), and thus to M. Moreover, $X_1 = g(X)$ is a complement of the dense union of the interiors of a null sequence of mutually disjoint 2-cells in M''. Thus X_1 is homeomorphic to X by Theorem 10.

Let $q_0 = g \circ q$, $q_1 = q_0|X_0$ and $\mathcal{D} = \{q^{-1}(D) : D \in \mathcal{D}'\} = \{q_0^{-1}(x) : x \in X_1\}$. Similarly to the proof of Example 19, applying inaccessibility of p in D', one can verify that $\mathcal{D}$ is a continuous decomposition of X_0. Thus q_1 is open and monotone. Take any point $a \in \operatorname{bd} C$ (a is rational in X_0 by Proposition 21), and any irrational point $b \in X_0$ contained in an element D of $\mathcal{D}$ such that D contains a rational point of X_0. Then $q_1(a)$ is an irrational point of X_1, and $q_1(b)$ is a rational point of X_1. ■

COROLLARY 24 (Seaquist [22]). *The Sierpiński universal plane curve is (m.o.)-homogeneous.*

In the following remark we announce two results which extend Example 19 and Corollary 24, respectively. Their proofs exceed the scope of this paper and are deferred to a future publication.

REMARK 25. (a) The family of all planar continua, openly equivalent to the 2-sphere (and thus openly homogeneous) has infinitely many topologically different elements. In fact, any planar one-point union of finitely many 2-cells is openly equivalent to the 2-sphere.

(b) There are infinitely many topologically different planar continua (m.o.)-equivalent to the Sierpiński curve (thus they are (m.o.)-homogeneous).

We end the paper with the following two questions concerning openly homogeneous continua. In these questions X is an arbitrary, nondegenerate, locally connected, plane continuum different from the simple closed curve.

QUESTION 1. If X is openly homogeneous, must X be openly equivalent either to the 2-cell or to the Sierpiński curve?

QUESTION 2. If X is (m.o.)-homogeneous, must X be (m.o.)-equivalent to the Sierpiński curve?

References

[1] R. D. Anderson, *On collections of pseudo-arcs*, Abstract 337t, Bull. Amer. Math. Soc. 56 (1950), 350.

[2] W. Bajguz, *Remark on embedding curves in surfaces*, preprint.

[3] R. H. Bing, *Concerning hereditarily indecomposable continua*, Pacific J. Math. 1 (1951), 43–51.

[4] K. Borsuk, *On embedding curves into surfaces*, Fund. Math. 59 (1966), 73–89.

[5] M. Brown, *Continuous collections of higher dimensional continua*, Ph.D. thesis, University of Wisconsin, 1958.

[6] J. J. Charatonik, *Mappings of the Sierpiński curve onto itself*, Proc. Amer. Math. Soc. 92 (1984), 125–132.

[7] —, *Generalized homogeneity of the Sierpiński universal plane curve*, in: Topology. Theory and Applications (Eger, 1983), Colloq. Math. Soc. János Bolyai 41, North-Holland, 1985, 153–158.

[8] B. Knaster, *Un continu dont tout sous-continu est indécomposable*, Fund. Math. 3 (1922), 247–286.

[9] J. Krasinkiewicz, *On mappings with hereditarily indecomposable fibers*, Bull. Polish Acad. Sci. Math. 44 (1996), 147–156.

[10] M. Levin, *Bing maps and finite-dimensional maps*, Fund. Math. 151 (1996), 47–52.

[11] W. Lewis, *Pseudo-arc of pseudo-arcs is unique*, Houston J. Math. 10 (1984), 227–234.

[12] —, *Continuous curves of pseudo-arcs*, ibid. 11 (1985), 225–236.

[13] —, *Observations on the pseudo-arc*, Topology Proc. 9 (1984), 329–337.

[14] —, *Continuous collections of hereditarily indecomposable continua*, Topology Appl. 74 (1996), 169–176.

[15] —, *The pseudo-arc*, in: Contemp. Math. 117, Amer. Math. Soc. 1991, 103–123.

[16] —, *Another characterization of the pseudo-arc*, Bull. Polish Acad. Sci., to appear.

[17] W. Lewis and J. J. Walsh, *A continuous decomposition of the plane into pseudo-arcs*, Houston J. Math. 4 (1978), 209–222.

[18] R. L. Moore, *Concerning upper semicontinuous collections of continua*, Trans. Amer. Math. Soc. 27 (1925), 416–428.

[19] S. Mazurkiewicz, *Sur les continus homogènes*, Fund. Math. 5 (1924), 137–146.

[20] J. R. Prajs, *A continuous circle of pseudo-arcs filling up the annulus*, Trans. Amer. Math. Soc., to appear.

[21] C. R. Seaquist, *A continuous decomposition of the Sierpiński curve*, in: Continua (Cincinnati, Ohio, 1994), Lecture Notes in Pure and Appl. Math. 170, Marcel Dekker, 1995, 315–342.

[22] C. R. Seaquist, *Monotone open homogeneity of the Sierpiński curve*, Topology Appl., to appear.
[23] G. T. Whyburn, *Topological characterization of the Sierpiński curve*, Fund. Math. 45 (1958), 320–324.
[24] G. S. Young, *Characterization of 2-manifolds*, Duke Math. J. 14 (1947), 979–990.

Institute of Mathematics
Opole University
Oleska 48
45-052 Opole, Poland
E-mail: jrprajs@math.uni.opole.pl

Received 11 December 1996;
in revised form 18 November 1997 and 14 April 1998