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A note on noninterpretability in o-minimal structures

by

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Abstract. We prove that if M is an o-minimal structure whose underlying order is dense then $\mathrm{Th}(M)$ does not interpret the theory of an infinite discretely ordered structure. We also make a conjecture concerning the class of the theory of an infinite discretely ordered o-minimal structure.

Introduction. In [9], Świerczkowski proves that $\mathrm{Th}(\langle\omega,<\rangle)$ is not interpretable (with parameters) in RCF (the theory of real closed fields) by showing that a pre-ordering with successors is not definable in $\mathbb{R}$.

(We recall that a *pre-ordering with successors* is a reflexive and transitive binary relation $\ll$, satisfying $\forall x\forall y\ (x \ll y \vee y \ll x)$ and $\forall x\exists y\ \mathrm{Succ}(x,y)$, where

$$\mathrm{Succ}(x,y) \Leftrightarrow x \ll y \wedge x \not\approx y \wedge \forall z\ (x \ll z \ll y \to z \approx x \vee z \approx y),$$

and $x \approx y$ means $x \ll y \wedge y \ll x$.)

Recall that a structure $(M,<,R_i)_{i\in I}$ is said to be *o-minimal* if $<$ is a total ordering on M and every definable (with parameters) subset of M is a finite union of points in M and open intervals (a,b), where $a \in M\cup\{-\infty\}$ and $b \in M \cup \{\infty\}$. Recall also that if M is o-minimal, then all $N \models \mathrm{Th}(M)$ are o-minimal, where $\mathrm{Th}(M)$ is the theory of M (see [1]).

Certain properties of RCF are used in the proof of the main result of [9], such as o-minimality and definable Skolem functions. We show that this result remains true in the more general setting of o-minimal densely ordered structures.

Noninterpretability results. We show the following:

THEOREM. *Let M be an o-minimal structure whose underlying order is dense. Then $\mathrm{Th}(M)$ does not interpret the theory of a preordered structure with successors.*

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The proof is done in the following three lemmas. We closely follow [9].

We assume familiarity with the notions of interpretability, definability and o-minimality. The reader should consult [1], [2], [4], [7] and [8].

Suppose that M is a densely ordered o-minimal structure.

LEMMA 1. *Let $L_0 \subset L_1 \subset \ldots$ be a strictly increasing sequence of infinite subsets of M such that the boundary ∂L_n of each L_n has at most K elements, for a fixed positive integer K. Then the difference $L_{i+1} \setminus L_i$ is infinite for infinitely many i.*

Proof. It is exactly the same as the proof of Lemma 3.1 of [9], and remains true for any Hausdorff space M without isolated points. ∎

LEMMA 2. *There does not exist a definable pre-ordering of M for which there is an infinite sequence $x_0, x_1, \ldots$ such that every x_{i+1} is an immediate successor of x_i.*

Proof. Compare with the proof of Lemma 3.2 in [9].

Suppose that $\ll$ is a definable pre-ordering of M, and $(x_i)_{i<\omega}$ is a sequence such that $M \models \mathrm{Succ}(x_i, x_{i+1})$ for each i. Let $L_i = \{x \in M : x \ll x_i\}$, $i < \omega$. These sets are definable by a formula $\Lambda(x, x_i)$, whose parameters include the x_i's. Let $\approx$ denote the equivalence relation associated with $\ll$. By the o-minimality of M (using Theorem 0.3(a) of [2]), there is a $K < \omega$ such that $|\partial L_n| \le K$ for all $n < \omega$. Therefore, Lemma 1 applies. This means that there are infinitely many infinite $\approx$-classes in M, a contradiction to o-minimality, by [5], Proposition 2.1. ∎

LEMMA 3. *There is no definable pre-ordering with successors in M^d, $d \ge 1$.*

Proof. We do this by induction on $d \ge 1$, the case of $d = 1$ being treated in Lemma 2.

Now, suppose that the result is true for $d = 1, \ldots, n$ and let M be an o-minimal structure such that on M^{n+1} there is a definable preorder $\ll$ with successors and $\{\overline{a}_i\}_{i\in\omega}$ is a sequence in M^{n+1} satisfying $\mathrm{Succ}(\overline{a}_i, \overline{a}_{i+1})$ for all i. The equivalence relation $\approx$ on M^{n+1} corresponding to $\ll$ is definable, so there are only finitely many $\approx$-classes $[\overline{a}] \subseteq M^{n+1}$ which have nonempty interior (i.e., contain an $(n+1)$-cell). Thus, there is no loss in generality if we assume that none of the equivalence classes $[\overline{a}_i]$ contains an $(n+1)$-cell, whence none of the closed intervals for the preorder $\ll$

$$[\overline{a}_0, \overline{a}_n] = [\overline{a}_0] \cup \ldots \cup [\overline{a}_n], \qquad n \in \omega,$$

contains an $(n+1)$-cell (see the Cell Decomposition Theorem in [1] and [2]). It follows that the set $\Sigma(\overline{x}) = \{\overline{a}_i \ll \overline{x} \wedge [\overline{a}_0, \overline{x}]$ does not contain an $(n+1)$-cell: $i \in \omega\}$ of formulas is consistent with the theory of $(M, \overline{a}_i)_{i\in\omega}$.

Now let M_1 be an ω_1-saturated elementary extension of M. Then M_1 is o-minimal, by [6]. Also, by the saturation property, there is an $\overline{a}_\infty$ in M^{n+1} such that $\overline{a}_i \ll \overline{a}_\infty$ for all $i \in \omega$ and the closed interval $I = [\overline{a}_0, \overline{a}_\infty]$ does not contain an $(n+1)$-cell. By the Cell Decomposition Theorem, I is the union of finitely many cells of dimension at most n. So there is a k-cell $X \subseteq I$ $(k \leq n)$ intersecting infinitely many $\approx$-equivalence classes $[\overline{a}_i]$. Using a definable homeomorphism $h : X \to M^k$ and the induction hypothesis we arrive at the required contradiction. ∎

We finish by stating the following conjecture. But firstly we recall from [8] the following definition.

DEFINITION. A structure M is said to be a *discrete o-minimal structure in the broad sense* if it is an infinite linearly ordered structure such that the set of points which have no immediate predecessor or immediate successor is finite and the definable sets in M are finite unions of intervals with endpoints in $M \cup \{\pm\infty\}$.

CONJECTURE. Let M be an infinite o-minimal discretely ordered structure (in the broad sense as above) in a countable language. Then $\mathrm{Th}(M)$ interprets $\mathrm{Th}(\omega, <)$. We conjecture that, conversely, $\mathrm{Th}(\omega, <)$ interprets $\mathrm{Th}(M)$.

(By [6] and [8], the theory of such an M is not *rich* enough to define too many sets or functions.)

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