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Computing Reidemeister classes

by

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Abstract. In order to compute the Nielsen number $N(f)$ of a self-map $f : X \to X$, some Reidemeister classes in the fundamental group $\pi_1(X)$ need to be distinguished. In this paper some algebraic results are given which allow distinguishing Reidemeister classes and hence computing the Reidemeister number of some maps. Examples of computations are presented.

1. Introduction. Let X be a finite CW-complex and $F : X \to X$ be a given map. The generalized Lefschetz number $\mathcal{L}(F)$ is defined to be the alternating sum of the Reidemeister traces of F (see [Hu]). It lies in $\mathbb{Z}\mathcal{R}(F)$, the free $\mathbb{Z}$-module generated by the F-Reidemeister classes in $\pi_1(X)$. The Nielsen number $N(F)$ is the minimum number of nonzero summands in the sum representing $\mathcal{L}(F)$ and gives a lower bound for the number of fixed points of maps homotopic to F. For background on Nielsen fixed point theory, the best general references are [B, J].

As McCord pointed out in [McC], a great deal of work has been devoted to the question of computation of $N(F)$. Several results of this work can be easily written in terms of $\mathcal{L}(F)$. For example, if X is a Jiang space then $\mathcal{L}(F)$ is an integral multiple of a certain known element (see [J, McC]); if F is a fibre map then formula (1) of Section 5.2 below holds true (see [Y, HKW]); if X is a nilmanifold or a solvmanifold (see [McC]) then F can be factored through a sequence of fibrations and hence formula (1) can be used. Moreover, X is defined to be of *Jiang type* if $L(F) \neq 0 \Rightarrow N(F) = \#\mathcal{R}(F)$, where $\#\mathcal{R}(F)$ is the Reidemeister number, and $L(F) = 0 \Rightarrow N(F) = 0$. It has been recently proved by Wong [W] that a wide class of homogeneous spaces are of Jiang type. For such spaces therefore the question is to compute the Reidemeister number of the map. Again, the useful trace-formula of [Hu] allowed Fadell and Husseini [FaHu] to compute generalized Lefschetz numbers and

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hence some Nielsen numbers on surfaces. Davey, Hart and Trapp [DHT] improved upon this algebraic method, and again one of the essential steps was to distinguish Reidemeister classes. In brief, once either $\mathcal{L}(F)$ is known or X is of Jiang type, it is only left to distinguish Reidemeister classes.

The aim of this article is to give some algebraic results which allow one to distinguish Reidemeister classes and hence to compute $\mathcal{R}(F)$. Let G be a group and $f : G \to G$ an endomorphism. Let $\mathcal{R}(f)$ denote the set of Reidemeister classes in G. The most used and known method is the abelianization of G: if $[G, G]$ is the commutator subgroup of G, then the induced projection $q_* : \mathcal{R}(f) \to \mathcal{R}(f; [G, G])$ can distinguish classes with distinct images in $\mathcal{R}(f; [G, G])$. Moreover, in the abelianized group $G/[G, G]$ classes can be distinguished as left cosets of $\mathrm{Im}(1 - \bar{f})$, where $\bar{f}$ is the endomorphism induced on G/H. As shown in [B, J], if f is eventually commutative, then q_* is a bijection. Another kind of result is given in [FeHi]: if G is a finite group, then $\#\mathcal{R}(f)$ is the number of ordinary conjugacy classes $\langle\omega\rangle$ in G such that $\langle f(\omega)\rangle = \langle\omega\rangle$.

The main idea we use to compute $\mathcal{R}(f)$ is the following: consider a normal subgroup $H \trianglelefteq G$ such that $f(H) \subset H$; let $q_* : \mathcal{R}(f) \to \mathcal{R}(f; H)$ be the projection onto the quotient. Let $[x_1]$ denote the Reidemeister class of $x_1 \in G$. Then $\mathcal{R}(f)$ is the disjoint union of all the counter-images $q_*^{-1}(q_*([x_1]))$ for $[x_1] \in \mathcal{R}(f; H)$. For any $x_1 \in G$ there is a natural surjection $i_* : \mathcal{R}(f_H^{x_1^{-1}}) \to q_*^{-1}(q_*([x_1]))$ induced by the inclusion $i : H \to G$, where $f_H^{x_1^{-1}} : H \to H$ is defined by $f_H^{x_1^{-1}}(x) := x_1 f(x) x_1^{-1}$ for all $x \in H$ (see Section 2). Using the results of Sections 3 and 4, for any x_1 we find a subgroup $T_{x_1} \trianglelefteq H$ and a surjection in the opposite direction $A : q_*^{-1}(q_*([x_1])) \to \mathcal{R}(f_H^{x_1^{-1}}; T_{x_1})$ which is injective under the hypotheses of Lemma 3.2. The main result is Theorem 4.1, which allows us to split the question of computing Reidemeister classes in G into two problems: computation of classes in G/H and in H/T_{x_1} for some $x_1 \in G$.

The paper is organized as follows. In Section 2 some preliminaries on Reidemeister classes are given; in Section 3 the main idea is developed: the subgroup T and the surjection A are defined for the simpler case of base point $[x_1] = [1]$. In Section 4 Theorem 4.1 is proved, and as corollaries some additive formulae are given which estimate $\mathcal{R}(f)$ and allow computing it exactly in some interesting cases. In Section 5 we directly apply these results to fibre maps; after some preliminaries Theorem 5.1 and Corollaries 5.2 and 5.3 are directly deduced from Theorem 4.1 and Corollaries 4.2, 4.3 of Section 4.

Examples are given to illustrate the method and the results. Example 1 is an easy example of an endomorphism f and a subgroup $H \trianglelefteq G$ such that the induced map i_* is not injective. In Example 2 the method is used to distinguish two Reidemeister classes, defined in [DHT], in a purely algebraic way. In Examples 3, 4, 5, 6 and 8, Theorem 4.1 and its corollaries are used to compute exactly $\#\mathcal{R}(f)$. In Example 7 a non-eventually commutative endomorphism f is defined such that $\mathcal{R}(f) = \mathcal{R}(f; [G, G])$.

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2. The Reidemeister action. Let G be a group and $f : G \to G$ an endomorphism. The *Reidemeister* (*left*) *action* induced by f on G is defined by setting $g \cdot x := gxf(g^{-1})$ for all $g, x \in G$. The orbit set $\mathcal{R}(f)$ is called the *Reidemeister set* of f. If $F : X \to X$ is a self-map of a space X and $F_\pi : \pi_1(X) \to \pi_1(X)$ is the endomorphism induced on the fundamental group, a function $\mathrm{cd} : \mathrm{Fix}(F) \to \mathcal{R}(F_\pi)$ can be given such that $\mathrm{cd}(y_1) = \mathrm{cd}(y_2)$ if and only if y_1 and y_2 belong to the same Nielsen fixed point class, for all fixed points $y_1, y_2 \in \mathrm{Fix}(F) := \{y \in X \mid F(y) = y\}$. For full details see Section 5 and [B, Hu, J, McC]. An orbit $[x] \in \mathcal{R}(f)$ is also called a *Reidemeister class*. Note that the orbits of the Reidemeister action induced by the identity are exactly the ordinary conjugacy classes in G.

Just as the fixed point class functor of [J], it can be shown that $\mathcal{R}$ is actually a functor: let $\vec{Gr}$ be the category of group endomorphisms (an object f is a group endomorphism $f : G \to G$; a morphism $h : f_1 \to f_2$ from $f_1 : G_1 \to G_1$ to $f_2 : G_2 \to G_2$ is a group homomorphism $h : G_1 \to G_2$ such that $f_2 h = h f_1$) and Set_* the category of pointed sets. Then $\mathcal{R} : \vec{Gr} \to Set_*$ is well defined if the base-point of $\mathcal{R}(f)$ is $[1]$, and it is a functor if we define the base-point preserving function $\mathcal{R}(h) : \mathcal{R}(f_1) \to \mathcal{R}(f_2)$ by $\mathcal{R}(h)([x]) := [h(x)]$ for all $x \in G_1$ and any morphism $h : f_1 \to f_2$. For brevity we set $h_* := \mathcal{R}(h)$.

Now let $f : G \to G$ be an endomorphism of a group G. A normal subgroup $H \trianglelefteq G$ is said to be *f-invariant* if $f(H) \subset H$ and *fully invariant* if this happens for every endomorphism of G. Let $i : H \to G$ be the inclusion homomorphism and $q : G \to G/H$ be the quotient homomorphism. Let $f_H : H \to H$ denote the restriction of f to H and $\bar{f} : G/H \to G/H$ the endomorphism induced on the left coset group G/H. The Reidemeister set $\mathcal{R}(\bar{f})$ is also called the *Reidemeister set of f relative to H* and it is denoted by $\mathcal{R}(f; H) := \mathcal{R}(\bar{f})$.

EXAMPLE 1. Let $f : G \to G$ be an endomorphism of an abelian (additive) group and H an f-invariant subgroup. Then $\mathcal{R}(f) \cong \mathrm{Coker}(I_G - f)$, $\mathcal{R}(f_H) \cong \mathrm{Coker}(I_H - f_H)$ and $\mathcal{R}(f; H) \cong \mathrm{Coker}(I_{G/H} - \bar{f})$ where I_G, I_H and $I_{G/H}$ are the identity endomorphisms of G, H and G/H. Moreover, if $i : H \to G$

and $q : G \to G/H$ are the inclusion and the quotient homomorphisms, then i_* and q_* are actually group homomorphisms, and the short sequence

$$\mathcal{R}(f_H) \xrightarrow{i_*} \mathcal{R}(f) \xrightarrow{q_*} \mathcal{R}(f;H) \to \{0\}$$

is exact.

Let $G := \mathbb{Z}$ be the additive group of integers; let $n, k \in \mathbb{Z}$ be given; let $H := n\mathbb{Z} \subset G$ be the fully invariant subgroup of multiples of n; let $f : G \to G$ be the endomorphism given by $f(x) := kx$ for all $x \in G$. Then $\mathcal{R}(f) = \mathcal{R}(f_H) = \mathbb{Z}/(1-k)\mathbb{Z}$ and $\mathcal{R}(f;H) = \mathbb{Z}/d\mathbb{Z}$ where $d := \gcd(1-k, n)$ is the greatest common divisor of $1-k$ and n; the short sequence

$$\mathbb{Z}/(1-k)\mathbb{Z} \xrightarrow{n} \mathbb{Z}/(1-k)\mathbb{Z} \to \mathbb{Z}/d\mathbb{Z} \to 0$$

is exact, where $n := i_*$ is the homomorphism induced by the inclusion $i : H \to G$. It is worth while seeing that $n = i_* : \mathcal{R}(f_H) \to \mathcal{R}(f)$ is not injective unless $d = 1$. ∎

The main problem is to know whether two elements x_1 and x_2 of the group G are in the same Reidemeister class or not. This is a difficult problem, even in the simpler case of conjugacy classes. If G is abelian then $[x_1] = [x_2]$ if and only if $x_1 - x_2 \in \operatorname{Im}(1-f)$; it is the problem of taking the quotient of G modulo a subgroup, as seen in the previous example. If G is not abelian we try to take as $H \subset G$ the commutator subgroup $[G,G]$ of G and to look at the abelianized images $q_*([x_1])$ and $q_*([x_2])$ in $\mathcal{R}(f;[G,G])$; in this case $\mathcal{R}(f;[G,G])$ is computable as a quotient group, and if the images are distinct as abelianized classes, they are distinct also in $\mathcal{R}(f)$. But what can we say if they have the same abelianized image in $\mathcal{R}(f;[G,G])$? If f is *eventually commutative*, i.e. there exists a positive integer n such that $f^n(x_1x_2x_1^{-1}x_2^{-1}) = 1$ for each $x_1, x_2 \in G$, then (see [J]) $\mathcal{R}(f) \cong \mathcal{R}(f;[G,G])$ and therefore the answer is that they are in the same class even in $\mathcal{R}(f)$.

In the following we give examples of endomorphisms such that q_* is not injective. More generally, a method to distinguish classes could be the following: first choose an f-invariant normal subgroup H of G, e.g. the derived subgroup, then look at $\mathcal{R}(f;H)$ and see if $q_*([x_1]) \neq q_*([x_2])$. In this case we are done, else, we try to see what happens to the counter-images of $q_*([x_1]) = q_*([x_2])$.

For any $x \in G$, let f^x denote the endomorphism of G defined by $f^x(g) := x^{-1}f(g)x$ for all $g \in G$. It is the composition of f with the inner automorphism induced by x. Then there is a canonical bijection of the Reidemeister sets of f and f^x denoted by $x_* : \mathcal{R}(f) \to \mathcal{R}(f^x)$ and given by $x_*([g]) := [gx]$. Hence, by replacing f with $f^{x_1^{-1}}$, it can be supposed that $x_1 = 1$ and hence that $q_*([x_1]) = q_*([x_2]) = [1]$. We want to study the surjection $i_* : \mathcal{R}(f_H) \to q_*^{-1}([1])$.

2. Computing Reidemeister classes. As seen in Example 1 for a simpler case, given $f : G \to G$ and an f-invariant normal subgroup $H \trianglelefteq G$, the exact sequence in $G\vec{r}$

$$\{1\} \to f_H \xrightarrow{i} f \xrightarrow{q} \bar{f} \to \{1\}$$

induces an exact sequence of pointed sets

$$\mathcal{R}(f_H) \xrightarrow{i_*} \mathcal{R}(f) \xrightarrow{q_*} \mathcal{R}(f;H) \to \{1\}$$

which is an exact sequence of groups if G is abelian. We have seen that i_* need not be injective. In any case, $i_*(\mathcal{R}(f_H)) = q_*^{-1}([1])$, where $1 \in G$; moreover, we try to find a normal f-invariant subgroup $T \trianglelefteq H$ such that the map $i'_* : \mathcal{R}(f_H;T) \rightleftharpoons q_*^{-1}([1])$ induced by i_* is a bijection. If this happens, then we could work in a group different from G, namely in H/T, because in this case, for all $h \in H$, $[h] = [1]$ in $\mathcal{R}(f)$ if and only if the same equation holds in $\mathcal{R}(f_H;T)$. Therefore the sequence of pointed sets

$$\{1\} \to \mathcal{R}(f_H;T) \xrightarrow{i'_*} \mathcal{R}(f) \xrightarrow{q_*} \mathcal{R}(f;H) \to \{1\}$$

would be exact.

For any endomorphism φ we define the subgroup $\mathrm{Fix}(\varphi)$ to be $\{x \in G \mid \varphi(x) = x\}$; if $\bar{f} : G/H \to G/H$ is defined as above, the subgroup $q^{-1}\,\mathrm{Fix}(\bar{f}) \subset G$ is f-invariant and $H \subset q^{-1}\,\mathrm{Fix}(\bar{f})$. Therefore there exists at least one f-invariant subgroup $K \subset q^{-1}\,\mathrm{Fix}(\bar{f})$ such that $KH = q^{-1}\,\mathrm{Fix}(\bar{f})$. Let $[K,H]$ denote the subgroup of G generated by all $khk^{-1}h^{-1}$ such that $k \in K$ and $h \in H$. If K^G is defined as the smallest normal subgroup of G containing K, we see that the subgroup $[K^G,H] = [K,H]^G$ of G is normal and f-invariant. Let the set O_fK be defined by $O_fK := \{kf(k^{-1}) \mid k \in K\}$. For any such subgroup K let the subgroup $T_f(K)$ be defined as

$$T_f(K) = [K^G,H] \cup O_fK,$$

the smallest subgroup of G containing both $[K^G,H]$ and O_fK.

PROPOSITION 3.1. *The subgroup $T_f(K)$ is normal in H, f-invariant and the equality $T_f(K) = \{xkf(k^{-1}) \mid x \in [K^G,H],\ k \in K\}$ holds true.*

Proof. By definition $q(k) = q(f(k))$ for all $k \in K$, hence $O_fK \subset H$. Since H is normal, $[K^G,H] \subset H$ and therefore $T_f(K) \subset H$. Because

$$\begin{aligned} hkf(k^{-1})h^{-1} &= (kf(k^{-1}))f(k)(k^{-1}hkh^{-1})f(k^{-1})(f(k)hf(k^{-1})h^{-1}) \\ &= (kf(k^{-1}))f(k)[k^{-1},h]f(k^{-1})[f(k),h] \end{aligned}$$

for all $h \in H$ and $k \in K$, we deduce that $T_f(K)$ is normal in H.

Again, observe that for $k, k_1, k_2 \in K$,

$$\begin{gathered} k_1f(k_1^{-1})k_2f(k_2^{-1}) = (k_1f(k_1^{-1})k_2(k_1f(k_1^{-1}))^{-1}k_2^{-1})k_2k_1f(k_2k_1)^{-1}, \\ f(k)k^{-1} = k^{-1}f(k)(f(k^{-1})kf(k)k^{-1}), \end{gathered}$$

and thus, each element of $T_f(K)$ can be written as stated. In fact, by definition each $g \in T_f(K)$ is a finite product $g = \prod_{i=0}^{j} g_i$ where for all $i = 0, \dots, j$ either $g_i \in [K^G, H]$ or $g_i \in O_f K \cup (O_f K)^{-1}$ (where $(O_f K)^{-1}$ is the set of inverses of elements in $O_f K$). Because $[K^G, H]$ is normal in G, without loss of generality we can assume that $g_i \in [K^G, H] \Leftrightarrow i = 0$. By the second displayed identity above we can assume that $g_i \in O_f K$ for all $j \geq 1$, and by the first one that $j = 1$. Therefore

$$T_f(K) = \{g_0 g_1 \mid g_0 \in [K^G, H],\ g_1 \in O_f K\}$$

and the last claim is proved.

As we have seen in Example 1 the natural surjection $i_* : \mathcal{R}(f_H) \to q_*^{-1}([1])$ may not be injective. Nevertheless the following lemma shows that for a suitable subgroup $T := T_f(K) \subset H$ there exists a natural surjection in the opposite direction $q_*^{-1}([1]) \to \mathcal{R}(f_H; T)$.

Lemma 3.2. *For any f-invariant subgroup K of G such that*

$$KH = q^{-1} \operatorname{Fix}(\bar{f})$$

there exists a surjection

$$A : q_*^{-1}([1]) = i_* \mathcal{R}(f_H) \to \mathcal{R}(f_H; T_f(K))$$

defined by $A([h]) := [p(h)]$ where p is the projection $p : H \to H/T_f(K)$; A is injective whenever $\mathcal{R}(f) = \mathcal{R}(f; [K^G, H])$.

Proof. Consider the natural projections

$$i_*(\mathcal{R}(f_H)) \xleftarrow{i_*} \mathcal{R}(f_H) \xrightarrow{p_*} \mathcal{R}(f_H; T_f(K)).$$

We will show that $p_* i_*^{-1}[h]$ is a single element in $\mathcal{R}(f_H; T_f(K))$ for all $h \in H$. If $h' = g h f(g^{-1})$ with $h, h' \in H$ and $g \in G$, then $q(g) = q(f(g))$, and hence $g \in q^{-1} \operatorname{Fix}(\bar{f}) = KH$; then $g = k_1 h_1$ with $k_1 \in K$ and $h_1 \in H$. Therefore $h' = k_1 h_1 h f(h_1^{-1}) f(k_1^{-1})$ and the equality

$$p(h') = p(h_1 h f(h_1^{-1})) p(k_1 f(k_1^{-1})) = p(h_1) p(h) p(f(h_1^{-1}))$$

shows that A is well defined and surjective. Note that $p(hk) = p(kh)$ for all $h \in H$ and $k \in K$.

Now assume that $\mathcal{R}(f) = \mathcal{R}(f; [K^G, H])$. If $A([h_1]) = A([h_2])$ then there exists $h \in H$ such that

$$p(h_2) = p(h) p(h_1) p(f(h^{-1})).$$

Thus, from Proposition 3.1, we can find $h \in H$, $k \in K$ and $x \in [K, H]^G$ such that $h_2 = h h_1 f(h^{-1}) x k f(k^{-1})$; as a consequence, there exist $h \in H$, $k \in K$ and $x' \in [K^G, H]$ such that $h_2 = k h h_1 f(h^{-1}) f(k^{-1}) x'$, i.e. $[h_1] = [h_2] \in \mathcal{R}(f; [K^G, H])$; now, from the assumption $\mathcal{R}(f) = \mathcal{R}(f; [K^G, H])$ we conclude that $[h_1] = [h_2] \in \mathcal{R}(f)$, that is to say, A is injective. ∎

COROLLARY 3.3. *If* $\mathrm{Fix}(\bar{f}) = \{1\}$ *then*

$$i_* : \mathcal{R}(f_H) \to q_*^{-1}([1]) \subset \mathcal{R}(f)$$

is a bijection.

Proof. In this case we can define K to be the trivial subgroup $\{1\}$. Then $T_f(K) = \{1\}$, $[K^G, H] = \{1\}$ and hence the assertion follows from Lemma 3.2. ∎

EXAMPLE 2. Let

$$G := \langle a, b, c, d \mid aba^{-1}b^{-1}cdc^{-1}d^{-1} = 1\rangle$$

be the fundamental group of the double torus. For any integer $n \geq 2$ let f be the automorphism of G defined by

$$\begin{aligned} a &\mapsto c^{-n+1}d^{-1}, & c &\mapsto a,\\ b &\mapsto dc^n, & d &\mapsto b. \end{aligned}$$

It is the homomorphism induced on G by the self-map of the double torus defined in Example 4 of [DHT]. In [DHT] the question arises if the elements 1 and $bab^{-1}a^{-1}$ belong to the same Reidemeister class in $\mathcal{R}(f)$. With purely topological arguments the authors prove that they do not. Here we give an algebraic proof.

Let $H := [G, G]$. Then $G/H = \mathbb{Z}^4$ and, if $\overline{F}$ is the matrix representing $\bar{f} : G/H \to G/H$, we see that $\det(I - \overline{F}) = n \neq 0$ and hence $\mathrm{Fix}(\bar{f}) = 0$. Therefore we can use Corollary 3.3 to show that $[bab^{-1}a^{-1}] = [1]$ in $\mathcal{R}(f)$ if and only if the same equality holds in $\mathcal{R}(f_H)$.

For each $x = (x_1, x_2, x_3, x_4) \in \mathbb{Z}^4$ define the elements

$$\begin{aligned} g_{x,a} &:= a^{x_1}b^{x_2}c^{x_3}d^{x_4}ad^{-x_4}c^{-x_3}b^{-x_2}a^{-1-x_1},\\ g_{x,b} &:= a^{x_1}b^{x_2}c^{x_3}d^{x_4}bd^{-x_4}c^{-x_3}b^{-1-x_2}a^{-x_1},\\ g_{x,c} &:= a^{x_1}b^{x_2}c^{x_3}d^{x_4}cd^{-x_4}c^{-1-x_3}b^{-x_2}a^{-x_1}, \end{aligned}$$

of H. According to [MKS], pp. 86–98, H is generated by the sets of generators $I_1 := \{g_{x,a} \mid x \in \mathbb{Z}^4,\ |x_2| + |x_3| + |x_4| \neq 0\}$, $I_2 := \{g_{x,b} \mid x \in \mathbb{Z}^4,\ |x_3| + |x_4| \neq 0\}$ and $I_3 := \{g_{x,c} \mid x \in \mathbb{Z}^4,\ |x_4| \neq 0\}$. The two generators

$$g_{(0,1,0,0),a} = bab^{-1}a^{-1} = g_{(0,0,0,1),c} = dcd^{-1}c^{-1}$$

coincide and therefore $I_1 \cup I_2 \cup I_3 - \{g_{(0,0,0,1),c}\}$ is a set of free generators for the free group H. Let the homomorphism $\delta : H \to \mathbb{Z}_2$ be defined by

$$\delta(g_{x,\alpha}) = \begin{cases} 1 & \text{if } x = (0,1,0,0) \text{ and } \alpha = a,\\ 0 & \text{otherwise,} \end{cases}$$

and let H_1 be the kernel of δ. Using the Reidemeister–Schreier rewriting process applied to the images under f_H of the generators (we refer to [MKS] for full details) it can be shown that $f(H_1) \subseteq H_1$. Therefore the quotient map $q'_* : \mathcal{R}(f_H) \to \mathcal{R}(f_H; H_1)$ is well defined. But $f(bab^{-1}a^{-1}) = (bab^{-1}a^{-1})^{-1}$, hence the endomorphism induced on $H/H_1 = \mathbb{Z}_2$ is the identity and $\mathcal{R}(f_H; H_1) \cong \mathbb{Z}_2$. Since

$$q'_*([1]) \neq q'_*([g_{(0,1,0,0),a}])$$

it follows that $[1] \neq [bab^{-1}a^{-1}] \in \mathcal{R}(f_H)$ and hence 1 and $bab^{-1}a^{-1}$ do not belong to the same class in $\mathcal{R}(f)$. ∎

4. Additive formulae for Reidemeister sets. In this section we prove some additive formulae for Reidemeister sets. The main idea is the following: for a given endomorphism $f : G \to G$ and an f-invariant subgroup $H \trianglelefteq G$ with quotient homomorphism $q : G \to G/H$, the Reidemeister set $\mathcal{R}(f)$ is split into the disjoint counter-images $q_*^{-1}(j)$ of all $j \in \mathcal{R}(f; H)$. Lemma 3.2 can be applied to any such counter-image and so we can prove the following theorem. Let S_1 and S_2 be sets. Then we write $S_1 \geq S_2$ if there exists a surjection $S_1 \to S_2$. If there is a bijection between S and the disjoint union $\bigsqcup_{j\in Z} S_j$ then we write $S = \sum_{j\in Z} S_j$. Let $\#S$ denote the cardinality of S.

THEOREM 4.1. *For all $j \in \mathcal{R}(f; H)$ let $x_j \in G$ be such that $[q(x_j^{-1})] = j$; for any $j \in \mathcal{R}(f; H)$ let K_j be an f^{x_j}-invariant subgroup of G such that*

$$q^{-1}\operatorname{Fix}(\overline{f}^{q(x_j)}) = K_j H.$$

Then

$$\mathcal{R}(f) \geq \sum_{j\in\mathcal{R}(f;H)} \mathcal{R}(f_H^{x_j}; T_{f^{x_j}}(K_j))$$

and equality holds if $\mathcal{R}(f) = \mathcal{R}(f; [K_j^G, H])$ for all j.

Proof. We have seen that for all $x \in G$ there is a bijection $x_* : \mathcal{R}(f) \to \mathcal{R}(f^x)$ where f^x is defined by $f^x(g) := f(g)^x = x^{-1}f(g)x$ and $x_*([g]) := ([gx])$. Moreover, for each $y \in G$ we can apply Lemma 3.2 to the endomorphism f^y. Let $q_{*y} : \mathcal{R}(f^y) \to \mathcal{R}(f^y; H)$ denote the function induced by q on $\mathcal{R}(f^y)$. On the other hand, for each $y \in G$, if we choose an f^y-invariant subgroup K_y of G such that $K_y H = q^{-1}\operatorname{Fix}(\overline{f}^{q(y)})$ then there exists a surjection

$$A_y : q_{*y}^{-1}([1]) \to \mathcal{R}(f_H^y; T_{f^y}(K_y))$$

defined by $A_y([h]) := [p_y(h)]$ where $p_y : H \to H/T_{f^y}(K_y)$; according to Lemma 3.2, A_y is injective whenever $\mathcal{R}(f^y) = \mathcal{R}(f^y; [K_y^G, H])$. Now, $\mathcal{R}(f)$ is the disjoint union of $q_*^{-1}(j)$ for all $j \in \mathcal{R}(f; H)$; if $y \in G$ is such that $[q(y^{-1})] = j$ then the bijection $y_* : \mathcal{R}(f) \to \mathcal{R}(f^y)$ induces a bijection $y'_* : q_*^{-1}(j) \to q_{*y}^{-1}([1])$. Thus, because of the choice of x_j there is a surjection $A_{x_j} x'_{j*} : q_*^{-1}(j) \to \mathcal{R}(f_H^{x_j}; T_{f^{x_j}}(K_j))$ for all j, which gives the desired inequality.

Moreover, $\mathcal{R}(f^{x_j}) = \mathcal{R}(f^{x_j}; [K_j^G, H])$ for all j if and only if $\mathcal{R}(f) = \mathcal{R}(f; [K_j^G, H])$ for all j, and hence the proof is complete using again Lemma 3.2. ∎

COROLLARY 4.2. *If* $\mathrm{Fix}(\bar{f}^{q(x_j)}) = \{1\}$ *for all* $j \in \mathcal{R}(f; H)$ *then*

$$\mathcal{R}(f) = \sum_{j \in \mathcal{R}(f;H)} \mathcal{R}(f_H^{x_j}).$$

P r o o f. As in Corollary 3.3 it suffices to define $K_j = 1$ for all j. ∎

EXAMPLE 3 (Semidirect product of finitely generated free abelian groups). Let $H = \mathbb{Z}^n$ and $A = \mathbb{Z}^k$ be two (additive) finitely generated free abelian groups and let $M : A \to \mathrm{Aut}(\mathbb{Z}^n)$ be a homomorphism from A to the automorphism group of H, i.e. to the group of all nonsingular integer matrices with determinant ± 1. Denote by $M_a := M(a)$ the image of each $a \in A$. Let G be the external semidirect product of H and A via M; it is the set of all pairs $(a, h) \in A \times H$, with the group operation $(a_1, h_1) + (a_2, h_2) = (a_1 + a_2, M_{a_2}(h_1) + h_2)$. The subgroup $H \cong 0 \times H \trianglelefteq G$ is normal in G and $G/H \cong A \cong A \times \{0\} \subseteq G$. Let $f : G \to G$ be an endomorphism such that $f(H) \subseteq H$. Then $\bar{f} : A \to A$ and $f_H : H \to H$ are defined by two matrices $\overline{F} \in \mathrm{Mat}_{k,k}(\mathbb{Z})$ and $F_H \in \mathrm{Mat}_{n,n}(\mathbb{Z})$. Note that $\mathrm{Fix}(\bar{f}) \neq 0$ if and only if $\det(I - \overline{F}) = 0$, and this happens if and only if $\#\mathcal{R}(\bar{f}) = \infty$. Therefore either $\mathcal{R}(f)$ and $\mathcal{R}(f; H)$ are infinite or $\mathrm{Fix}(\bar{f}) = 0$. In this last case, because A is abelian and so $\bar{f}^{q(x)} = \bar{f}$ for all $x \in A$, $\mathrm{Fix}(\bar{f}^{q(x)}) = \{0\}$ for all $x \in A$ and therefore Corollary 4.2 can be used. For each $j \in \mathcal{R}(f; H)$ let $a_j \in A$ be such that $[a_j^{-1}] = j$; thus

$$\mathcal{R}(f) = \sum_{j \in \mathcal{R}(f;H)} \mathcal{R}(f_H^{a_j})$$

by the Corollary; moreover, because

$$\mathcal{R}(f; H) = A/\mathrm{Im}(I - \bar{f})$$

it follows that $\#\mathcal{R}(f; H) = |\det(I - \overline{F})|$. For all $a \in A$ and $h \in H$ the conjugation is $-(a, 0) + (0, h) + (a, 0) = (0, M_a h)$, hence $f_H^{a_j}$ is defined by the matrix $M_{a_j} F_H$ and so $\#\mathcal{R}(f_H^{a_j}) = |\det(I - M_{a_j} F_H)|$ if this determinant is $\neq 0$, otherwise it is infinite. Thus if all the determinants involved are different from 0 then

$$\#\mathcal{R}(f) = \sum_{j=1}^{|\det(I - \overline{F})|} |\det(M_{a_j}^{-1} - F_H)|.$$

If $|\det(M_{a_j}^{-1} - F_H)|$ are constant then we get the product formula

$$\begin{aligned}\#\mathcal{R}(f) &= |\det(I - \overline{F})| \cdot |\det(M_{a_1}^{-1} - F_H)| \\ &= |\det(I - \overline{F})| \cdot |\det(I - F_H)|\end{aligned}$$

because without loss of generality $M_{a_1} = I$. ∎

Example 4 (The Klein bottle). Let G be the fundamental group of the Klein bottle, i.e. $G := \langle \alpha, \beta \mid \beta\alpha = \alpha^{-1}\beta \rangle$. The subgroup $H := \langle \alpha \rangle \trianglelefteq G$ is a fully invariant normal subgroup of G and if $M : \mathbb{Z} \to \mathrm{Aut}(\mathbb{Z}) = \{1, -1\}$ is the homomorphism defined by $M(x) = (-1)^x$ for all $x \in \mathbb{Z}$ then G is the semidirect product of H and $A := \mathbb{Z}$ via M. So let $f : G \to G$ be an endomorphism. As in the previous example, $f_H : H \to H$ and $\bar{f} : G/H \cong A \to A$ are defined by elements of $\mathrm{Mat}_{1,1}(\mathbb{Z})$, thus they are integers u and w. In other words, $f_H(x) = ux$ for all $x \in H$ and $\bar{f}(y) = wy$ for all $y \in G/H \cong A$. If $w = 1$ then $\#\mathcal{R}(f; H) = \infty$, otherwise, if $(-1)^j - u \neq 0$ for all $j = 1, \ldots, |1 - w|$, then as in the previous example

$$\#\mathcal{R}(f) = \sum_{j=1}^{|1-w|} |(-1)^j - u|,$$

and if $u = \pm 1$ then $\#\mathcal{R}(f) = \infty$. It can be seen that if w is even then u must be zero, hence if w is even then $\#\mathcal{R}(f) = |1 - w|$; on the other hand, w odd, $w \neq 1$, $u \neq \pm 1$ implies $\#\mathcal{R}(f) = |u(1 - w)|$. Thus

$$\#\mathcal{R}(f) = \begin{cases} |(1 - w)| & \text{if } w \neq 1 \text{ odd and } u \neq 0, \pm 1, \\ |1 - w| & \text{if } w \text{ even}, \\ \infty & \text{otherwise}. \end{cases}$$

The commutator subgroup of G is $[G, G] = \langle \alpha^2 \rangle$ and the quotient $G/[G, G]$ is the direct sum $\mathbb{Z}_2 \oplus \mathbb{Z}$. Let $a := (1, 0)$ and $b := (0, 1)$ be the generators of $\mathbb{Z}_2$ and $\mathbb{Z}$ respectively. Let φ denote the endomorphism induced on the abelianized group $\mathbb{Z}_2 \oplus \mathbb{Z}$. Then $\varphi(a) = (u, 0)$ and $\varphi(b) = (v, w)$ for an integer v mod 2 and w even implies $u = 0$. We want to compute $\mathrm{Coker}(1 - \varphi)$. It is not difficult to see that

$$\#\mathcal{R}(f; [G, G]) = \begin{cases} |2(1 - w)| & \text{if } w \neq 1 \text{ and } u(v - 1) \equiv 1 \bmod 2, \\ |1 - w| & \text{if } w \neq 1 \text{ and } u(v - 1) \equiv 0 \bmod 2, \\ \infty & \text{if } w = 1, \end{cases}$$

and so if w is even then $\mathcal{R}(f) = \mathcal{R}(f; [G, G])$ but if w is odd then either both $\mathcal{R}(f)$ and $\mathcal{R}(f; [G, G])$ are infinite or $\#\mathcal{R}(f)$ is strictly greater than $\#\mathcal{R}(f; [G, G])$. ∎

Example 5. Let N be a nilpotent, finitely generated and torsion free group and $f : N \to N$ an endomorphism. Then, as shown e.g. in [McC], there exists a fully invariant central series $\{N_i\} \subset N$ with torsion free and finitely generated factors. In other words, for $i = 0, \dots, n+1$ there exist subgroups $N_i \subset N$ such that $1 = N_{n+1} \subset N_n \subset \dots \subset N_1 \subset N_0 = N$, $[N_{i-i}, N] \subset N_i$, $f(N_i) \subset N_i$ and the factors N_{i-1}/N_i are torsion free abelian finitely generated groups. By Corollary 4.2 it is easy to see that if $\mathrm{Fix}(f) \neq 0$ then $\mathcal{R}(f)$ is infinite. Hence, as shown in [McC], either $\mathcal{R}(f)$ is infinite or

$$\mathcal{R}(f) = \prod_{i=0}^{n} |\det(I - f|_{N_i/N_{i+1}})|$$

where $f|_{N_i/N_{i+1}} : N_i/N_{i+1} \to N_i/N_{i+1}$ are the restrictions of f to N_i/N_{i+1} and the determinants are all nonzero.

Now let G be a group and $f : G \to G$ be an endomorphism such that there exists an f-invariant series

$$1 = G_{n+1} \subset G_n \subset \dots \subset G_1 \subset G_0 = G$$

with nilpotent torsion free finitely generated factors. Again, either $\#\mathcal{R}(f) = \infty$ or we can compute $\mathcal{R}(f)$ step by step: for any $i = 0, \dots, n-1$ and $y \in G$ the sequence

$$\{1\} \to G_{i+1} \to G_i \to G_i/G_{i+1} \to \{1\}$$

is exact; $\mathrm{Fix}(f_{G_i/G_{i+1}}) = 0$ because otherwise $\#\mathcal{R}(f) = \infty$, and hence Corollary 4.2 can be used to obtain

$$\mathcal{R}(f^y_{G_i}) = \sum_{j \in \mathcal{R}(f^y_{G_i/G_{i+1}})} \mathcal{R}((f^y)^{x_j}_{G_{i+1}})$$

where $[x_j^{-1}] = j$; $\mathcal{R}(f^y_{G_i/G_{i+1}})$ can be computed as in the previous example because G_i/G_{i+1} is nilpotent torsion free and finitely generated; we omit the quotient homomorphisms in writing for simplicity. In the following lines we write j instead of x_j for the same reason. Using the previous formula we can thus prove that either $\mathcal{R}(f) = \infty$ or

$$\begin{aligned}
\mathcal{R}(f) &= \sum_{j_1 \in \mathcal{R}(f_{G_0/G_1})} \sum_{j_2 \in \mathcal{R}(f^{j_1}_{G_1/G_2})} \mathcal{R}(f^{j_1 j_2}_{G_2}) \\
&= \sum_{j_1 \in \mathcal{R}(f_{G_0/G_1})} \sum_{j_2 \in \mathcal{R}(f^{j_1}_{G_1/G_2})} \sum_{j_3 \in \mathcal{R}(f^{j_1 j_2}_{G_2/G_3})} \mathcal{R}(f^{j_1 j_2 j_3}_{G_3}) \\
&= \dots = \\
&= \sum_{j_1 \in \mathcal{R}(f_{G_0/G_1})} \sum_{j_2 \in \mathcal{R}(f^{j_1}_{G_1/G_2})} \sum_{j_3 \in \mathcal{R}(f^{j_1 j_2}_{G_2/G_3})} \cdots \sum_{j_n \in \mathcal{R}(f^{j_1 j_2 \cdots j_{n-1}}_{G_{n-1}/G_n})} \mathcal{R}(f^{j_1 j_2 \cdots j_n}_{G_n}). \quad \blacksquare
\end{aligned}$$

As seen in the previous examples, the easiest way of computing $\mathcal{R}(f)$ is to reduce it to the abelian case in which only the Coker of some endomorphisms has to be computed. In the following two corollaries we apply this idea to more general settings. We want to compute $\mathcal{R}(f)$ as a sum of quotients of subgroups.

Corollary 4.3. *For all $j \in \mathcal{R}(f;H)$ put $K_j := q^{-1}\operatorname{Fix}(\bar{f}^{q(x_j)})$ where as before $[q(x_j^{-1})] = j$. Then*

$$\mathcal{R}(f) \geq \sum_{j\in\mathcal{R}(f;H)} H/T_{f^{x_j}}(K_j)$$

and equality holds whenever $\mathcal{R}(f) = \mathcal{R}(f;[K_j^G, H])$ for every j.

Proof. By Theorem 4.1 it suffices to show that, for every j,

$$\mathcal{R}(f_H^{x_j}; T_{f^{x_j}}(K_j)) = H/T_{f^{x_j}}(K_j)$$

where the right hand side is a quotient of groups. In fact, since $H \subseteq K_j$, the homomorphism φ_j induced by $f_H^{x_j}$ on the quotient

$$\varphi_j : H/T_{f^{x_j}}(K_j) \to H/T_{f^{x_j}}(K_j)$$

is the identity homomorphism and hence $\mathcal{R}(\varphi_j) = \mathcal{R}(f_H^{x_j}; T_{f^{x_j}}(K_j))$ is the set of conjugacy classes in $H/T_{f^{x_j}}(K_j)$. But $H/T_{f^{x_j}}(K_j)$ is an abelian group because $[H,H] \subset [K_j, H] \subset T_{f^{x_j}}(K_j)$ for all j and hence

$$\mathcal{R}(\varphi_j) = H/T_{f^{x_j}}(K_j),$$

which completes the proof. ∎

Corollary 4.4. *If $\mathcal{R}(f) = \mathcal{R}(f;[G,H])$, then*

$$\mathcal{R}(f) = \sum_{j\in\mathcal{R}(f;H)} H/T_{f^{x_j}}(q^{-1}\operatorname{Fix}(\bar{f}^{q(x_j)})).$$

In particular, if $\mathcal{R}(f) = \mathcal{R}(f;[G,G])$ (for example, if f is eventually commutative), then

$$\mathcal{R}(f) = G/T_f(G).$$

Proof. The first assumption implies that $\mathcal{R}(f) = \mathcal{R}(f;[K_j,H]^G)$ for each $j \in \mathcal{R}(f;H)$, where $K_j = q^{-1}\operatorname{Fix}(\bar{f}^{q(x_j)})$. Applying Corollary 4.3 we obtain the stated formula. If we take $H = G$ we obtain

$$\mathcal{R}(f) = \sum_{j\in\mathcal{R}(f;G)=\{1\}} G/T_f(G) = G/T_f(G). \ ∎$$

Example 6. Let G be a nilpotent group. Then there exists a fully invariant central series

$$1 = G_{n+1} \subset G_n \subset G_{n-1} \subset \ldots \subset G_1 \subset G_0 = G$$

where $[G_{i-1}, G] \subset G_i$ for all $i = 1, \ldots, n+1$. For any $i, k = 0, \ldots, n+1$ let $f_{G_i/G_k} : G_i/G_k \to G_i/G_k$ denote the endomorphism induced by f on

G_i/G_k. Because $[G_{i-1}/G_i, G/G_i] = 1$, the hypothesis of Corollary 4.4 is true for the short exact sequence

$$\{1\} \to G_{i-1}/G_i \to G/G_i \xrightarrow{q_i} G/G_{i-1} \to \{1\}$$

and therefore

$$\mathcal{R}(f_{G/G_i}) = \sum_{j \in \mathcal{R}(f_{G/G_i}; G_{i-1}/G_i)} \frac{G_{i-1}/G_i}{T_{f^{x_j}_{G/G_i}}(q_i^{-1}\operatorname{Fix}(f^{q_i(x_j)}_{G/G_{i-1}}))}$$

where $x_j \in G/G_i$ is such that $[q_i(x_j^{-1})] = j$ for all j. Now since

$$\mathcal{R}(f_{G/G_i}; G_{i-1}/G_i) = \mathcal{R}(f_{G/G_{i-1}})$$

and $\mathcal{R}(f_{G/G_1}) = \operatorname{Coker}(1 - f_{G/G_1})$ we can compute $\mathcal{R}(f)$ starting from $\mathcal{R}(f_{G/G_1})$ and using the previous formula a finite number of times if every $\mathcal{R}(f_{G/G_i})$ is finite. We recall that

$$T_{f^{x_j}_{G/G_i}}(q_i^{-1}\operatorname{Fix}(f^{q_i(x_j)}_{G/G_{i-1}})) = (1 - f^{x_j}_{G/G_i})(q_i^{-1}\operatorname{Fix}(f^{q_i(x_j)}_{G/G_{i-1}}))$$

and that G_{i-1}/G_i is abelian. Hence the problem becomes that of computing quotients of abelian groups. ∎

The following corollary is an easy consequence of the preliminaries in Sections 2 and 3.

COROLLARY 4.5. *Let f be an endomorphism of a group G and $H \subset G$ a normal f-invariant subgroup. If $\mathcal{R}(f^y_H) = 1$ for all $y \in G$ then $\mathcal{R}(f) = \mathcal{R}(f; H)$.*

P r o o f. For all $j \in \mathcal{R}(f; H)$ let $x_j \in G$ be such that $[q(x_j^{-1})] = j$, where as usual q denotes the projection $q : G \to G/H$. For each $y \in G$ we have $i_{*y}\mathcal{R}(f^y_H) = q_{*y}^{-1}([1])$, where $q_{*y} : \mathcal{R}(f^y) \to \mathcal{R}(f^y; H)$ and $i_{*y} : \mathcal{R}(f^y_H) \to \mathcal{R}(f)$ are the maps induced by $i : H \to G$ and q. Hence if $\mathcal{R}(f^y_H) = 1$ for all $y \in G$ then $q_{*y}^{-1}([1]) = \{[1]\}$. Moreover, as in the proof of 4.1, if $y = x_j$ then the bijection $y_* : \mathcal{R}(f) \to \mathcal{R}(f^y)$ induces a bijection $y'_* : q_*^{-1}(j) \to q_{*y}^{-1}([1])$ and hence each counter-image $q_*^{-1}(j)$ consists of a single element. Thus $\mathcal{R}(f) = \mathcal{R}(f; H)$ as claimed. ∎

If we apply this corollary to the case of the commutator subgroup $H := [G, G]$ then a weaker condition occurs than the eventual commutativity of f. In fact, if f is eventually commutative, then $\mathcal{R}(f^y_H) = 1$ for all y; as shown in the following example, the converse is not true.

EXAMPLE 7. Let $G := U(3, \mathbb{Z})$ be the group of 3×3 (upper) unitriangular matrices over $\mathbb{Z}$, that is, matrices with 1 on the diagonal and 0 below it. It is a nilpotent torsion free group, generated by the elements $a := 1 + E_{13}$, $b := 1 + E_{12}$ and $c := 1 + E_{23}$ where E_{ij} is the matrix with 1 in the ij entry and 0 elsewhere. It is easy to see that $ab = ba$, $ac = ca$ and

$bc = cba$. The commutator subgroup $H := [G, G] = \langle a \rangle \cong \mathbb{Z}$ is generated by a and the quotient is $G/H \cong \mathbb{Z} + \mathbb{Z}$. Let $f : G \to G$ be the endomorphism defined by $f(a) := a^2$, $f(b) := b^{-1}$ and $f(c) := c^{-2}$. It is well defined because $f(bc) = f(cba)$. Then $f_H^y : H \to H$ is the multiplication by 2 for all $y \in G$, therefore $\mathcal{R}(f_H^y) = 1$ for all y and hence the corollary implies that $\mathcal{R}(f; H) = \mathcal{R}(f)$. On the other hand, f is not eventually commutative, because $f^n(bcb^{-1}c^{-1}) = f^n(a) \neq 1$ for all n.

For $i \in \{1, 2\}$ let the integers u, x_i, y_i and z_i be given. Then it is possible to define an endomorphism $f : G \to G$ by setting $f(a) := a^u$, $f(b) := a^{x_1} b^{y_1} c^{z_1}$ and $f(c) := a^{x_2} b^{y_2} c^{z_2}$ if and only if $u = y_1 z_2 - z_1 y_2$. Every endomorphism of G can be defined in this way. Again, if $u = 2$ it turns out that $\mathcal{R}(f) = \mathcal{R}(f; [G, G])$ even if f is not eventually commutative. ■

5. Nielsen numbers of fibre maps. In this section the previous results are directly applied to the study of Reidemeister sets of fibre maps. In a straightforward way Theorem 4.1 and Corollaries 4.2, 4.3 are translated into Theorem 5.1 and Corollaries 5.2, 5.3. Some preliminaries are needed; results are in Subsection 5.3.

5.1. *The generalized Lefschetz number.* Let X be a connected, finite CW-complex. A self-map f of X with a path w in X such that $f(w(0)) = w(1)$ is called *path-based* and denoted by (f, w). Let $\mathcal{AM}_{\rm pb}$ be the category of all path-based self-maps. If (f, w) and (g, v) are path-based self-maps of X and Y respectively, then a morphism $h : (f, w) \to (g, v)$ is a map $h : X \to Y$ such that $gh = hf$ and $h(w) = v$.

We are going to define a functor $\pi : \mathcal{AM}_{\rm pb} \to G\vec{r}$ which can be viewed as an extension of the concept of the fundamental group functor. Let (f, w) be as before and let $\pi(f, w)$ be the endomorphism of $\pi_1(X, w(0))$ defined by $\pi(f, w)(\alpha) := wf(\alpha)w^{-1}$ for every $\alpha \in \pi_1(X, w(0))$. If $h : (f, w) \to (g, h(w))$, then $\pi(h) := (\pi_1(h) : \pi_1(X, w(0)) \to \pi_1(Y, h(w(0))))$. Now we take the composition of π and $\mathcal{R}$ to obtain a functor $\mathcal{R}\pi : \mathcal{AM}_{\rm pb} \to \mathit{Set}$ which, by an abuse of notation, we still call $\mathcal{R}$.

Let $\mathrm{Fix}(f) = \{x \in X \mid f(x) = x\}$ be the space of fixed points of f. Define the function $\mathrm{cd} : \mathrm{Fix}(f) \to \mathcal{R}(f, w)$ as follows: for every $x \in \mathrm{Fix}(f)$ choose a path λ_x from $w(0)$ to x and set $\mathrm{cd}(x) := [\lambda_x f(\lambda_x^{-1}) w^{-1}]$. The counter-image $\mathrm{cd}^{-1}(\xi)$ of every $\xi \in \mathcal{R}(f, w)$ is a set of fixed points, to which we can associate an integer $\mathrm{Ind}(\xi) := \mathrm{Ind}(\mathrm{cd}^{-1}(\xi))$ (cf. [B, J]). The integer $\mathrm{Ind}(\xi)$ is the *Fixed Point Index* of ξ. We now define the *generalized Lefschetz number* (cf. [Hu])

$$\mathcal{L}(f, w) := \sum_{\xi \in \mathcal{R}(f, w)} \mathrm{Ind}(\xi) \cdot \xi$$

as an element of the free abelian group $\mathbb{Z}\mathcal{R}(f, w)$ generated by the elements of $\mathcal{R}(f,w)$. The number of $\xi \in \mathcal{R}(f,w)$ such that $\mathrm{Ind}(\xi) \neq 0$ is the *Nielsen number* of f (cf. [Hu]); the sum of the indices coincides with the classical Lefschetz number.

Let w and w' be two base paths for f. If λ is a third path such that $\lambda(0) = w(0)$ and $\lambda(1) = w'(0)$, define the bijection $\lambda_* : \mathcal{R}(f,w) \to \mathcal{R}(f,w')$ (called "change of coordinates") by $\lambda_*[\alpha] := [\lambda^{-1}\alpha w f(\lambda) w'^{-1}]$; it is easy to check that at the free group level, $\lambda_*\mathcal{L}(f,w) = \mathcal{L}(f,w')$ because $\mathrm{Ind}(\lambda_*\xi) = \mathrm{Ind}(\xi)$ for every $\xi \in \mathcal{R}(f,w)$.

5.2. *Fibre maps.* Let $p : (f,w) \to (\bar{f}, p(w))$ be a morphism of $\mathcal{AM}_{\mathrm{pb}}$ represented by a commutative diagram

$$\begin{array}{ccc} E & \xrightarrow{p} & B \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \bar{f}} \\ E & \xrightarrow{p} & B \end{array}$$

If $p : E \to B$ is a fibration with path-connected fibres, the map f is called a *fibre map.* Some of the results about fibre maps can be summed up in the formula

$$\mathcal{L}(f,w) = \sum_{j \in \mathcal{R}(\bar{f};p(w))} \mathrm{Ind}(j) \cdot \lambda_{j*}^{-1} i_* \mathcal{L}(f_{b_j}; v_j) \tag{1}$$

in which b_j is chosen arbitrarily in $\mathrm{cd}^{-1}(j)$, f_{b_j} is the restriction of f to $p^{-1}(b_j)$, v_j is a base path for f_{b_j} in the fibre, i_* is induced by $i : p^{-1}(b_j) \to E$, and $\lambda_{j*} : \mathcal{R}(f,w) \to \mathcal{R}(f,v_j)$ is the change of coordinates determined by the path λ_j such that $\lambda_j(0) = w(0)$ and $\lambda_j(1) = v_j(0)$ (cf. [He, Y, HKW, J]). The formula holds true also in the case in which we have empty fixed point classes since in that situation, although there is no f_{b_j}, the index is zero.

In order to establish a connection between the Nielsen numbers of f and of f_{b_j} and $\bar{f}$, we note that from the previous formula we deduce that

$$N(f) = \sum_{j \in \mathcal{ER}(\bar{f},p(w))} c(b_j) \tag{2}$$

where $\mathcal{ER} = \{\xi \in \mathcal{R}(\bar{f},p(w)) \mid \mathrm{Ind}(\xi) \neq 0\}$ is the set of all the *essential classes* in $\mathcal{R}(\bar{f})$, and $c(b_j)$ is the minimum number of distinct, nonzero summands in $i_*\mathcal{L}(f_{b_j}, v_j)$ viewed as an element of $\mathbb{Z}\mathcal{R}(f,v_j)$ (for full details see [HKW]). To see this, observe that $\lambda_{j_1*}^{-1} i_* \mathcal{R}(f_{b_{j_1}}; v_{j_1}) \cap \lambda_{j_2*}^{-1} i_* \mathcal{R}(f_{b_{j_2}}; v_{j_2}) \neq \emptyset$ if and only if $j_1 = j_2$. Thus, we must study $c(b_j)$.

5.3. *Additive formulae for fibre maps.* Associated with every $j \in \mathcal{ER}(\bar{f}, p(w))$ there is the exact sequence

$$\pi_1(F_{b_j}, v_j(0)) \xrightarrow{i_\pi} \pi_1(E, v_j(0)) \xrightarrow{p_\pi} \pi_1(B, b_j) \to 1$$

where F_b is the fibre over $b \in B$. Let

$$k : \pi_1(F_{b_j}, v_j(0)) \to \pi_1(F_{b_j}, v_j(0))/\mathrm{Ker}(i_\pi) \simeq \mathrm{Ker}(p_\pi)$$

be the quotient homomorphism; moreover, let $f_{\pi j} := \pi(f, v_j)$ and $\bar{f}_{\pi j} := \pi(\bar{f}, p(v_j))$ be the images by the functor π.

Theorem 5.1. *Suppose that, for every* $j \in \mathcal{ER}(\bar{f}, p(w))$, *there exists a subgroup* $K_j \subseteq \pi_1(E, v_j(0))$ *such that*

$$K_j \,\mathrm{Ker}(p_\pi) = p_\pi^{-1} \,\mathrm{Fix}(\bar{f}_{\pi j})$$

and $f_{\pi j}(K_j) \subseteq K_j$. *Then*

$$N(f) \geq \sum_{j \in \mathcal{ER}(\bar{f}, p(w))} N(f_{b_j}; H_j)$$

where $H_j := k^{-1}(T_{f_{\pi j}}(K_j))$. *If*

$$\mathcal{R}(\pi(f, v_j)) = \mathcal{R}(\pi(f, v_j); [K_j, \mathrm{Ker}(p_\pi)]^{\pi_1(E)})$$

for every j, *then*

$$N(f) = \sum_{j \in \mathcal{ER}(\bar{f}, p(w))} N(f_{b_j}; H_j).$$

Proof. Let

$$\pi_1(F_{b_j}, v_j(0))/\mathrm{Ker}(i_\pi) \xrightarrow{i_\pi} \pi_1(E, v_j(0)) \xrightarrow{p_\pi} \pi_1(B, b_j)$$

be the exact sequence associated with $j \in \mathcal{ER}(\bar{f}, p(w))$. Now, using Lemma 3.2, we obtain a surjection

$$A_j : i_*(\mathcal{R}(f_{b_j})) \to \mathcal{R}(f_{b_j}; H_j)$$

with $H_j := k^{-1}(T_{f_{\pi j}}(K_j))$. Thus, $c(b_j) \geq N(f_{b_j}; H_j)$. If

$$\mathcal{R}(\pi(f, v_j)) = \mathcal{R}(\pi(f, v_j); [K_j, \mathrm{Ker}(p_\pi)]^{\pi_1(E)})$$

then A_j turns out to be a bijection and hence $c(b_j) = N(f_{b_j}; H_j)$ because essential classes correspond to essential ones. The conclusion stated is reached on adding over all j and using formula (2). ■

Corollary 5.2. *If* $\mathrm{Fix}(\bar{f}_{\pi j}) = 1$ *for every* $j \in \mathcal{ER}(\bar{f}, p(w))$, *then*

$$N(f) = \sum_{j \in \mathcal{ER}(\bar{f}, p(w))} N(f_{b_j}; \mathrm{Ker}(i_\pi)).$$

Proof. Apply Corollary 4.2.

Corollary 5.3. *If there is* $b \in \mathrm{Fix}(\bar{f})$ *such that* $\mathcal{R}(f; [\pi_1(E), i_\pi(\pi_1(F_b))]) = \mathcal{R}(f)$, *then*

$$N(f) = \sum_{j \in \mathcal{ER}(\bar{f}, p(w))} N(f_{b_j}; H_j)$$

where $H_j := q^{-1}(T_{f_{\pi j}}(p_\pi^{-1}\operatorname{Fix}(\bar f_{\pi j})))$; *we can distinguish the fixed point classes by just computing the quotient groups* $\operatorname{Ker}(p_\pi)/T_{f_{\pi j}}(q^{-1}\operatorname{Fix}(\bar f_{\pi j}))$.

Proof. Note that if the hypothesis holds true for a $b \in \operatorname{Fix}(\bar f)$ then it is satisfied for every $b \in \operatorname{Fix}(\bar f)$. Then apply 4.4.

Example 8 (Well known, the Klein bottle). The computations of the Nielsen classes for the Klein bottle were first given by Halpern [Ha]. Here we show a different approach, following the lines of this paper. Let G be the subgroup of the isometries of $\mathbb{R}^2$ generated by $a : (x, y) \mapsto (x+1, y)$ and $b : (x, y) \mapsto (1-x, y+1)$. The orbit space $\mathbb{R}^2/G$ is the Klein bottle K, and G is the fundamental group of K with presentation $\langle a, b \mid ba = a^{-1}b\rangle$. Two self-maps of K are homotopic (with base point fixed) if and only if the induced endomorphisms on G are the same, because the Klein bottle is a $K(\pi_1(K), 1)$. It is easy to see that for all $u, v, w \in \mathbb{Z}$ such that $u(1 + (-1)^w) = 0$ the endomorphism $f_\pi : a, b \mapsto a^u, a^v b^w$ is well defined, and every endomorphism of G belongs to this family. Let $\widetilde{f}_0 : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$\widetilde{f}_0(x, y) = \begin{cases} (vy, wy) & \text{if } w \text{ is even,} \\ (ux + \frac{1}{2}(v-u+1)(1-\cos \pi y), wy) & \text{if } w \text{ is odd.} \end{cases}$$

It is an equivariant map and it induces a self-map $f : K \to K$. Moreover, $f_\pi : a, b \mapsto a^u, a^v b^w$. Therefore every self-map of K is homotopic to such a self-map.

We want to compute the Nielsen number $N(f)$ and the minimum number $\operatorname{MF}(f)$ of fixed points among all the maps homotopic to f. Consider the projection $p : K \to S^1$ of the Klein bottle onto the unit circle $S^1 = \mathbb{R}/\mathbb{Z}$ given by $p(x, y) := [y]$. Then p is a fibration with total space K, base space S^1 and fibre S^1. The map induced on the base space is $\bar f : S^1 \ni [y] \mapsto [wy] \in S^1$. If $w \neq 1$ then all the $|1-w|$ classes of $\mathcal{R}(\bar f) = \{0, \ldots, |1-w|-1\}$ are essential and consist of the points $\{b_j := [j/(w-1)]\}$ of S^1 for $j = 0, \ldots, |1-w|-1$; on the other hand, if $w = 1$ then $N(f) = 0$ and it is possible to find a map homotopic to f without fixed points, just "rotating" along the y-axis. So consider the case $w \neq 1$. For every $j = 0, \ldots, |1-w|-1$, $\operatorname{Fix}(\bar f) = 1$, therefore, conditions of Corollary 5.2 are satisfied and hence as in Example 4,

$$N(f) = \sum_{j=0}^{|1-w|-1} N(f_{b_j}).$$

Let us remark that $\operatorname{Ker}(i_\pi) = 1$. We have to compute the maps f_{b_j} restricted to the fibres F_{b_j}. If w is even they are constant maps, hence $N(f_{b_j}) = 1$ and $N(f) = |1-w|$. If w is odd then f_{b_j} is defined as follows: if we let $d_j(\alpha) := (-1)^j + \frac{1}{2}(1-(-1)^j)$ then

$$f_{b_j} : [x] \mapsto \left[d_j\left(ux + \frac{1}{2}(v-u+1)\left(1 - \cos\frac{\pi j}{1-w}\right)\right)\right],$$

which is a self-map of degree $(-1)^j$ on S^1. Therefore $N(f_{b_j}) = |1-(-1)^j u|$ and hence $N(f) = |u(1-w)|$. Note that for all the maps involved the number of fixed points of f is equal to $N(f)$, up to homotopy in case $w = 1$. This means that the Klein bottle is Wecken, i.e. $N(f) = \mathrm{MF}(f)$ for all self-maps f.

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