FUNDAMENTA
MATHEMATICAE
157 (1998)

An entropy for $\mathbb{Z}^2$-actions with finite entropy generators

by

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Abstract. We study a definition of entropy for $\mathbb{Z}^+ \times \mathbb{Z}^+$-actions (or $\mathbb{Z}^2$-actions) due to S. Friedland. Unlike the more traditional definition, this is better suited for actions whose generators have finite entropy as single transformations. We compute its value in several examples. In particular, we settle a conjecture of Friedland [2].

0. Introduction. In this note we shall study an entropy which is appropriate for $\mathbb{Z}^2$-actions (or, more generally, $\mathbb{Z}^+ \times \mathbb{Z}^+$-actions) whose generators have finite entropy as single transformations. In the traditional definition of entropy of a $\mathbb{Z}^2$-action a necessary condition for entropy to be positive is that the generators should have infinite entropy as single transformations. The definition of this new entropy was originally proposed by Friedland [2] and was motivated by methods and results in the case of algebraic examples.

We begin by recalling the definition of Friedland. Assume that $S, T : X \to X$ are a pair of (commuting) continuous maps on a compact metric space X. We can define the sequence space

$$\mathcal{X} = \mathcal{X}_{S,T} = \Big\{(x_n)_{n\in\mathbb{Z}^+} \in \prod_{n\in\mathbb{Z}^+} X : T(x_n) = x_{n+1} \text{ or } S(x_n) = x_{n+1}\Big\}$$

of all possible orbits of points $x_0 \in X$ under iterates of S and T. This is a closed subset of the compact space $\prod_{n\in\mathbb{Z}^+} X$ (with the Tikhonov product topology) and so is again compact. A natural metric on $\mathcal{X}$ is

$$d((x_n)_{n\in\mathbb{Z}^+}, (y_n)_{n\in\mathbb{Z}^+}) = \sum_{n=0}^{\infty} \frac{d_X(x_n, y_n)}{2^n}.$$

We can define a natural *shift map* $\sigma : \mathcal{X} \to \mathcal{X}$ on the space $\mathcal{X}$ by $\sigma((x_n)_{n\in\mathbb{Z}^+})$

1991 *Mathematics Subject Classification*: 58, 28.

We would like to thank Shmuel Friedland for introducing us to his definition of entropy. We would also like to thank Klaus Schmidt for some useful comments. We are very grateful to the referee for his careful reading and helpful comments.

$= (x_{n+1})_{n\in\mathbb{Z}^+}$. Thus we have associated in a natural way a $\mathbb{Z}^+$-action (generated by σ) with the $\mathbb{Z}^+ \times \mathbb{Z}^+$-action (generated by S and T).

DEFINITION. We define the *entropy* $e(S,T)$ of the $\mathbb{Z}^+ \times \mathbb{Z}^+$-action generated by S and T to be the topological entropy of the map $\sigma : \mathcal{X} \to \mathcal{X}$, i.e. $h(\sigma) = e(S,T)$.

This definition is motivated by the following simple observation. Let

$$\mathcal{X}_T = \Big\{ (x_n)_{n\in\mathbb{Z}^+} \in \prod_{n\in\mathbb{Z}^+} X : T(x_n) = x_{n+1} \Big\}$$

and $\sigma_T : \mathcal{X}_T \to \mathcal{X}_T$ be defined by $\sigma_T((x_n)_{n\in\mathbb{Z}^+}) = (x_{n+1})_{n\in\mathbb{Z}^+}$. Then it is easily seen that $h(\sigma_T) = h(T)$.

It is also easy to see from the definition that $e(S,T)$ is unchanged by taking an equivalent metric on $\mathcal{X}$. However, changing the generators may result in a change in this entropy.

This paper is arranged as follows. In Section 1 we describe some simple properties of $e(S,T)$. In Section 2 we calculate the entropy of simple examples of linear hyperbolic maps. In particular, we consider multiplication by $p > q \geq 2$ modulo one on the unit interval. In Section 3 we consider some standard models of $\mathbb{Z}^2$-subshifts of finite type using a construction of Ledrappier. We give a simple expression for the entropy in terms of the characterization of the subshift, and show that it has rather surprising features. In Section 4 we show that when $S,T : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ are two rotations then $e(S,T) = 0$. In Section 5 we make some final remarks.

1. Some simple properties. In this section we want to present a few elementary results on the entropy $e(S,T)$ which will prove useful later.

PROPOSITION 1. *Assume that* $S,T : X \to X$ *are continuous. Then*

(1) $e(S,T) = e(T,S)$ *and* $e(T,T) = h(T)$.

(2) *If* $I : X \to X$ *is the identity map then* $e(T,I) = h(T)$.

(3) *For* $n \geq 1$ *we have* $e(S^n,T^n) \leq ne(S,T)$.

(4) $\max(h(S),h(T)) \leq e(S,T)$.

(5) *If we assume that* $S,T : X \to X$ *are Lipschitz* (*with Lipschitz constants* $L(S)$ *and* $L(T)$) *then*

$$e(S,T) \leq \log(L(S)+L(T)) \quad \textit{and} \quad e(S,T) \leq \log 2 + \max(h(S),h(T)).$$

Proof. The proofs of parts (1), (2) are an easy consequence of the definition.

To prove part (3) we note that if $\sigma : \mathcal{X} \to \mathcal{X}$ is the shift associated with S and T then it is easy to see from the definitions that $\sigma^n : \mathcal{X} \to \mathcal{X}$ is conjugate to the shift associated with S^n and T^n. In particular, by [3, Theorem 7.10(i)] applied to σ we see that

$$e(S^n, T^n) \le h(\sigma^n) = nh(\sigma) = ne(S, T).$$

To prove part (4) consider the subspaces $\mathcal{X}_S, \mathcal{X}_T \subset \mathcal{X}_{S,T}$ given by

$$\mathcal{X}_S = \Big\{(x_n)_{n\in\mathbb{Z}^+} \in \prod_{n\in\mathbb{Z}^+} X : S(x_n) = x_{n+1}\Big\},$$
$$\mathcal{X}_T = \Big\{(x_n)_{n\in\mathbb{Z}^+} \in \prod_{n\in\mathbb{Z}^+} X : T(x_n) = x_{n+1}\Big\}.$$

If $\sigma_S : \mathcal{X}_S \to \mathcal{X}_S$ and $\sigma_T : \mathcal{X}_T \to \mathcal{X}_T$ are the associated shift maps then as observed in the introduction $h(\sigma_S) = h(S)$ and $h(\sigma_T) = h(T)$. However, notice that $\sigma_S = \sigma|\mathcal{X}_S$ and $\sigma_T = \sigma|\mathcal{X}_T$. Thus $h(\sigma_S) \le h(\sigma)$ and $h(\sigma_T) \le h(\sigma)$ (which can be seen immediately from the variational principle [3, Theorem 8.6]), completing the proof of this part.

The first inequality in part (5) is due to Friedland [2, Theorem 3.4]. The second inequality follows by considering the space

$$Y = \Big\{((i_n)_{n\in\mathbb{Z}^+}, x) : (i_n)_{n\in\mathbb{Z}^+} \in \prod_{n\in\mathbb{Z}^+} \{0,1\},\ x \in X\Big\}$$

and the map $\widetilde{\sigma} : Y \to Y$ defined by

$$\widetilde{\sigma}((i_n)_{n\in\mathbb{Z}^+}, x) = \begin{cases} ((i_{n+1})_{n\in\mathbb{Z}^+}, Sx) & \text{if } i_0 = 0, \\ ((i_{n+1})_{n\in\mathbb{Z}^+}, Tx) & \text{if } i_0 = 1. \end{cases}$$

This is a skew product over the shift transformation

$$\sigma_2 : \prod_{n\in\mathbb{Z}^+} \{0,1\} \to \prod_{n\in\mathbb{Z}^+} \{0,1\}$$

defined by $\sigma_2((i_n)_{n\in\mathbb{Z}^+}) = (i_{n+1})_{n\in\mathbb{Z}^+}$.

We can define a map $\pi : Y \to X$ such that $\pi(((i_n)_{n\in\mathbb{Z}^+}, x)) = (x_n)_{n\in\mathbb{Z}^+}$, where $x_0 = x$ and x_n is defined inductively by

$$x_n = \begin{cases} S(x_{n-1}) & \text{if } i_{n-1} = 0, \\ T(x_{n-1}) & \text{if } i_{n-1} = 1. \end{cases}$$

Fix $\varepsilon > 0$ and define $L = \max(L(S), L(T))$. Since S, T are Lipschitz we see that if $((i_n)_{n\in\mathbb{Z}^+}, x), ((j_n)_{n\in\mathbb{Z}^+}, y) \in Y$ with $|x - y| \le \varepsilon$ and $i_n = j_n$ for $0 \le n \le N$ then $|x_n - y_n| \le L^n\varepsilon$. Thus

$$d((x_n)_{n\in\mathbb{Z}^+}, (y_n)_{n\in\mathbb{Z}^+}) \le \Big(\sum_{n=0}^{N} \frac{L^n}{2^n}\Big)\varepsilon + \frac{1/2^N}{1 - 1/\beta}$$

and we conclude that π is continuous.

It follows from the definitions that $\pi : Y \to X$ is a semi-conjugacy (i.e. $\pi \circ \widetilde{\sigma} = \sigma \circ \pi$). It is also easy to see that π is surjective since for any point $(x_n)_{n\in\mathbb{Z}^+} \in X$ we can construct $((i_n)_{n\in\mathbb{Z}^+}, x) \in Y$ by setting $x = x_0$ and choosing i_n such that

$$i_n = \begin{cases} 0 & \text{if } S(x_n) = x_{n+1}, \\ 1 & \text{if } T(x_n) = x_{n+1}. \end{cases}$$

There always exists at least one such choice, and in the event that there is an ambiguous choice we can choose either.

Since $\pi : Y \to X$ is a surjective semi-conjugacy we see that $e(S,T) = h(\sigma) \le h(\widetilde{\sigma})$ [3, Theorem 7.2].

Finally, it only remains to recall that by a result of Bowen [1] we have

$$h(\widetilde{\sigma}) \le h(\sigma_2) + \max\{h(S), h(T)\} \le \log 2 + \max\{h(S), h(T)\}$$

(i.e. the topological entropy $h(\sigma_2) = \log 2$ of the base map plus the bound $\max\{h(S), h(T)\}$ on the topological entropies of the fibre maps).

Remarks. In [2], Friedland gives examples due to M. Boyle of (non-Lipschitz) homeomorphisms $S, T : X \to X$ such that $h(S) = h(T) = 0$ but $h(S,T) = +\infty$. This shows that the bounds in (5) of Proposition 1 cannot be extended to general continuous $\mathbb{Z}^+ \times \mathbb{Z}^+$-actions.

As one would imagine, this entropy is also an invariant in classifying certain $\mathbb{Z}^2$-actions.

Definition. Consider two $\mathbb{Z}^+ \times \mathbb{Z}^+$-actions $A : \mathbb{Z}^+ \times \mathbb{Z}^+ \times X \to X$ and $B : \mathbb{Z}^+ \times \mathbb{Z}^+ \times Y \to Y$ with generators

$$S = A((1,0),\cdot) \quad \text{and} \quad T = A((0,1),\cdot),$$
$$S' = B((1,0),\cdot) \quad \text{and} \quad T' = B((0,1),\cdot).$$

We say these are *conjugate* if there exists a homeomorphism $\psi : X \to Y$ such that $\psi(A((n,m),x)) = B((n,m),\psi(x))$ (i.e. ψ is simultaneously a conjugacy between S and S' and between T and T'). We say that they are *semi-conjugate* if there exists a continuous surjective map $\psi : X \to Y$ such that

$$\psi(A((n,m),x)) = B((n,m),\psi(x))$$

(i.e. ψ is simultaneously a semi-conjugacy between S and S' and between T and T').

Proposition 2. (1) *If the actions A and B are conjugate then $e(S,T) = e(S',T')$.*

(2) *If the actions A and B are semi-conjugate then $e(S,T) \ge e(S',T')$.*

P r o o f. This is again an easy consequence of the definitions, since ψ gives rise to a (semi-)conjugacy $\psi : \mathcal{X}_{S,T} \to \mathcal{X}_{S',T'}$, $\widehat{\psi}((x_n)_{n\in\mathbb{Z}^+}) = (\psi(x_n))_{n\in\mathbb{Z}^+})$, between the two corresponding shift maps $\sigma : \mathcal{X}_{S,T} \to \mathcal{X}_{S,T}$ and $\sigma : \mathcal{X}_{S',T'} \to \mathcal{X}_{S',T'}$.

2. Commuting hyperbolic maps. We begin by considering the simple example of $S, T : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ where $S(x) = 2x \pmod 1$ and $T(x) = 3x \pmod 1$. In this section we shall compute the entropy $e(S,T)$. In [2, p. 342], Friedland observed that $e(S,T) \le \log 5$ (this is a consequence of the first inequality in Proposition 1(5)) and conjectured that this should be an equality. We answer this conjecture affirmatively.

THEOREM 1. *If* $S(x) = 2x \pmod 1$ *and* $T(x) = 3x \pmod 1$ *then* $e(S,T)$
$= \log 5$.

In fact, we prove the more general result:

THEOREM 2. *If* $p, q \ge 2$ *and* $p \ne q$ *then* $e(\times p, \times q) = \log(p+q)$.

We present a proof which is relatively straightforward and depends on studying the skew product $\widetilde{\sigma} : \mathbb{R}/\mathbb{Z} \times \prod_{n\in\mathbb{Z}^+}\{0,1\}$ defined by

$$\widetilde{\sigma}(x, (i_n)_{n\in\mathbb{Z}^+}) = \begin{cases} (S(x), (i_{n+1})_{n\in\mathbb{Z}^+}) & \text{if } i_0 = 0, \\ (T(x), (i_{n+1})_{n\in\mathbb{Z}^+}) & \text{if } i_0 = 1. \end{cases}$$

We need the following lemma.

LEMMA 1. *If* $S(x) = px \pmod 1$ *and* $T(x) = qx \pmod 1$ *then* $h(\sigma) = \log(p+q)$.

P r o o f. Consider the partition of $\widetilde{\sigma} : \mathbb{R}/\mathbb{Z} \times \prod_{n\in\mathbb{Z}^+}\{0,1\}$ consisting of the sets

$$\begin{array}{llll} \left[0, \dfrac{1}{pq}\right] \times [0]_0, & \left[\dfrac{1}{pq}, \dfrac{2}{pq}\right] \times [0]_0, & \ldots, & \left[\dfrac{pq-1}{pq}, 1\right] \times [0]_0, \\ \left[0, \dfrac{1}{pq}\right] \times [1]_0, & \left[\dfrac{1}{pq}, \dfrac{2}{pq}\right] \times [1]_0, & \ldots, & \left[\dfrac{pq-1}{pq}, 1\right] \times [1]_0. \end{array}$$

We can associate with this partition a $2pq \times 2pq$ transition matrix A. This will take the form

$$A = \begin{pmatrix} P & P \\ \vdots & \vdots \\ P & P \\ Q & Q \\ \vdots & \vdots \\ Q & Q \end{pmatrix}$$

where P and Q are the $q \times pq$ and $p \times pq$ matrices, respectively, given by

$$P = \begin{pmatrix} 1\ldots1 & 0\ldots0 & \ldots & 0\ldots0 & \vdots & 1\ldots1 & 0\ldots0 & \ldots & 0\ldots0 \\ 0\ldots0 & 1\ldots1 & \ldots & 0\ldots0 & \vdots & 0\ldots0 & 1\ldots1 & \ldots & 0\ldots0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \underbrace{0\ldots0}_{\times p} & \underbrace{0\ldots0}_{\times p} & \ldots & \underbrace{1\ldots1}_{\times p} & \vdots & \underbrace{0\ldots0}_{\times p} & \underbrace{0\ldots0}_{\times p} & \ldots & \underbrace{1\ldots1}_{\times p} \end{pmatrix}$$

and

$$Q = \begin{pmatrix} 1\ldots1 & 0\ldots0 & \ldots & 0\ldots0 & \vdots & 1\ldots1 & 0\ldots0 & \ldots & 0\ldots0 \\ 0\ldots0 & 1\ldots1 & \ldots & 0\ldots0 & \vdots & 0\ldots0 & 1\ldots1 & \ldots & 0\ldots0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \underbrace{0\ldots0}_{\times q} & \underbrace{0\ldots0}_{\times q} & \ldots & \underbrace{1\ldots1}_{\times q} & \vdots & \underbrace{0\ldots0}_{\times q} & \underbrace{0\ldots0}_{\times q} & \ldots & \underbrace{1\ldots1}_{\times q} \end{pmatrix}$$

(i.e. A has $2pq$ entries) We can let $\widehat{\sigma} : \Sigma \to \Sigma$ denote the associated subshift of finite type.

We observe that the column sums in this transition matrix are all equal to $p+q$. In particular, we conclude that A has maximal eigenvalue equal to $p+q$. Therefore the associated subshift of finite type $\widehat{\sigma}$ has topological entropy $\log(p+q)$. This completes the proof of Lemma 1.

Proof of Theorem 2. Consider the map $\varrho : \Sigma \to \mathcal{X}$ defined by:

(1) $\varrho(z_n) = (x_n)$ with $x_0 = \bigcap_{n=0}^{\infty} I_n(x)$ where

$$I_n(x) = \bigcap_{k=0}^{n-1} (T_{n-1} \ldots T_0)^{-1} \left[\frac{i_k}{pq}, \frac{i_k+1}{pq}\right],$$

and

$$T_i = \begin{cases} S & \text{if } z_n = \left(0, \left[\frac{i_k}{pq}, \frac{i_k+1}{pq}\right]\right), \\ T & \text{if } z_n = \left(1, \left[\frac{i_k}{pq}, \frac{i_k+1}{pq}\right]\right); \end{cases}$$

(2) the points x_n are defined inductively by

$$x_{n+1} = \begin{cases} S(x_n) & \text{if } z_n = \left(0, \left[\frac{i_k}{pq}, \frac{i_k+1}{pq}\right]\right), \\ T(x_n) & \text{if } z_n = \left(1, \left[\frac{i_k}{pq}, \frac{i_k+1}{pq}\right]\right). \end{cases}$$

It is easy to see that ϱ is continuous, surjective and a semi-conjugacy. In particular, since $\varrho : \Sigma \to \mathcal{X}$ is a semi-conjugacy, we see that $e(S,T) = h(\sigma) \le h(\widehat{\sigma})$ [3, Theorem 7.2].

It only remains to show that $h(\sigma) \ge \log(p+q)$. Observe that although the map ϱ is surjective, it can fail to be injective. We claim that the set on which injectivity fails is "small". Assume that $\varrho((z_n)_{n\in\mathbb{Z}^+}) = \varrho((z'_n)_{n\in\mathbb{Z}^+})$, but $(z_n)_{n\in\mathbb{Z}^+} \ne (z'_n)_{n\in\mathbb{Z}^+}$. In particular, assume that $z_i = z'_i$ for $0 \le i \le n-1$, but $z_n = z'_n$. This can only happen if $x_n \in \{0, 1/(pq), \ldots, (pq-1)/(pq), 1\}$. In particular, we see that

$$\Omega = \{(z_n) \in \Sigma : \operatorname{Card} \varrho^{-1}(\varrho((z_n)_{n\in\mathbb{Z}^+})) \ge 2\}$$

is a countable set.

Since $\widehat{\sigma} : \Sigma \to \Sigma$ is a transitive subshift of finite type there is a unique measure of maximal entropy, i.e. μ is the unique σ-invariant probability measure with entropy $h(\widehat{\sigma}) = \log(p+q)$. Moreover, since μ is Markov it is clear that $\mu(\Omega) = 0$ and so $\varrho : (\Sigma, \mu) \to (\mathcal{X}, \varrho^*\mu)$ is an isomorphism. By the variational principle [3, Theorem 8.6] we see that

$$\begin{aligned} h(\sigma) &= \sup\{h_m(\sigma) : m = \sigma\text{-invariant probability measure}\} \\ &\ge h_{\varrho^*\mu}(\sigma) = h_\mu(\widehat{\sigma}) = \log(p+q). \end{aligned}$$

This completes the proof of Theorem 2.

Remark. One could define a *zeta-function* by

$$\zeta(z) = \exp\bigg(-\sum_{n=1}^{\infty} \frac{z^n}{n} N(n)\bigg)$$

where $N(n)$ denotes the number of strings $(x_0, x_1, \ldots, x_{n-1})$ such that either $T(x_n) = x_{n+1}$ or $S(x_n) = x_{n+1}$. From the above proof we see that the $N(n)$ corresponds to the number of periodic points for the symbolic shift $\widehat{\sigma} : \Sigma \to \Sigma$ (except an additional contribution from the fixed point 0). In particular, we can write

$$\zeta(z) = \frac{1-z}{\det(1-zA)}.$$

Corollary 1. *For $S = \times p$ and $T = \times q$ the zeta function $\zeta(z)$ is a rational function. $\zeta(z)$ is analytic for $|z| < 1/(p+q)$ and $z = 1/(p+q)$ is a simple pole.*

3. Symbolic examples of Ledrappier type. Ledrappier introduced an important class of $\mathbb{Z} \times \mathbb{Z}$-subshifts of finite type. Amongst their many properties, they generally have zero $\mathbb{Z}^2$-entropy in the usual sense. In this section we shall compute explicitly their entropies.

Let $\mathcal{S} \subset \mathbb{Z}^2$ be a finite set. We can define a space

$$X = \Big\{ (x_{n,m})_{(n,m)\in\mathbb{Z}^2} \in \prod_{(n,m)\in\mathbb{Z}^2} \{0,1\} : \sum_{(r,s)\in\mathcal{S}} x_{(n+r,m+s)} = 0 \pmod 2 \text{ for all } n,m\in\mathbb{Z} \Big\}$$

and a $\mathbb{Z}^2$-action generated by

$$S = \sigma_{(1,0)} : (x_{(m,n)}) \mapsto (x_{(m+1,n)}),$$
$$T = \sigma_{(0,1)} : (x_{(m,n)}) \mapsto (x_{(m,n+1)}).$$

The entropy $e(S,T)$ is given by the following theorem.

THEOREM 3. *The entropy e of the Ledrappier type actions can take at most a countable number of values with* $\log 2 < e(S,T) \le \log 4$. *Moreover*:

(1) *In the special case when we can choose*

$$\mathcal{S} = \bigcup_{r=1}^{M} (\{(i,l_r) : k_r \le i \le k_{r+1}\} \cup \{(k_r,j) : l_r \le j \le l_{r+1}\})$$

where $0 \le k_1 \le \ldots \le k_M$ *and* $0 \le l_1 \le \ldots \le l_M$ *are not all zero* (*i.e. S consists of a "staircase" consisting of a horizontal string followed by a vertical string*), *we have* $e(S,T) < \log 4$.

(2) *For all other cases we have* $e = \log 4$.

Proof. The key to the proof is to consider the "triangles"

$$\mathcal{T}_k = \{(n,m)\in\mathbb{Z}^2 : 0 \le n,m \le k \text{ and } n+m \le k\}, \quad k \ge 1.$$

Given $S \subset \mathbb{Z}^2$ we define $N(k)$ to be the number of ways of allocating 0 or 1 to each of the sites in $\mathcal{T}_k$. To begin, we fix $\varepsilon = 1/2$. This means that we need to distinguish between orbits $\sigma^i(x), \sigma^i(y) \in Y \subset \prod_{n=0}^{\infty} X$ $(i = 0,\ldots,k-1)$ seen with a "coarseness" which only distinguishes up to the partition $[0]_{(0,0)} \cup [1]_{(0,0)}$ for each of the points $\sigma^i(x), \sigma^i(y) \in Y$.

More specifically, we want to estimate the maximal number of $(n,1/2)$ spanning sets (or the minimal number of $(n,1/2)$ separating sets).

We recall that a typical point $x = (x_n)_{n=0}^{\infty}$ satisfies $x_{n+1} = T(x_n)$ or $S(x_n)$. From this perspective, we need to estimate:

(i) The number of distinct combinations of the two generators S and T for the $\mathbb{Z}^2$-action. It is convenient to visualise this as paths joining lattice points in $\mathbb{Z}^2$ with vertical segments (corresponding to strings of S's) and horizontal segments (corresponding to strings of T's) of total length k from $(0,0) \in \mathbb{Z}^2$ to the line $\{(n,m)\in\mathbb{Z}^2 : n+m = k\}$. Clearly, the number of such paths totals 2^k.

(ii) The two elements in the partition lead us to consider the number of allowable "two-shade colourings" of the lattice points visited by these paths (where a colouring corresponds to associating an element of the partition $[0]_{(n,m)} \cup [1]_{(n,m)}$ to each site (n, m) on the above paths).

If we assume that each of the 2^k paths can be coloured freely then there are clearly 2^k such colourings. We claim that this is the case except where $\mathcal{S}$ is as described in case (1). To see this, we observe that for a given path only a translate of sets $\mathcal{S}$ of the form described in (1) can fit wholly within the path (and thus impose restrictions on allowed shadings). Thus we conclude that in case (2) we have $N(k) = 2^k \cdot 2^k$.

In case (1), any path containing a translation of the configuration $\mathcal{S}$ has the colouring in the final position determined by the others. To estimate the total number of such colourings, we can recode a full shift on two symbols (corresponding to S and T) by blocks of length $N = (k_1 + \ldots + k_M) + (l_1 + \ldots + l_M)$. The associated $2^{N+1} \times 2^{N+1}$ transition matrix takes the form $A = \left(\begin{smallmatrix} I & I \\ I & I \end{smallmatrix}\right)$. We define a new matrix B by replacing the entry in each row *except that of the cylinder corresponding to* $\mathcal{S}$ by 2:

$$B = \begin{pmatrix}
2 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 2 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \\
0 & 2 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 2 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \\
0 & 0 & 2 & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 & 2 & \ldots & 0 & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & 2 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 2 & 0 & 0 & \ldots & 0 \\
0 & 0 & 0 & \ldots & 0 & 1 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 1 & 0 & \ldots & 0 \\
0 & 0 & 0 & \ldots & 0 & 0 & 2 & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 & 2 & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 2 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 2
\end{pmatrix}.$$

We can then write $N(k) = \sum_{i,j} B^k(i,j)$.

By recoding, we can make similar estimates for each $\varepsilon = 1/2^p$ with $p \geq 2$, to conclude that: in case (1) we have $e(S,T) = \log 2 + \log \lambda$, where $1 < \lambda < 2$ is the maximal positive eigenvalue for B; and in case (2) we have $e(S,T) = \log 4$.

Example. Consider the original example of Ledrappier with

$$\mathcal{S} = \{(0,0), (0,1), (1,1)\}.$$

Following the proof of the above proposition we see that the associated matrix is $B = \left(\begin{smallmatrix} 2 & 1 \\ 2 & 2 \end{smallmatrix}\right)$. This has maximal eigenvalue $\lambda = 2 + 2^{1/2}$ and so we conclude that $e(S,T) = \log(2 + 2^{1/2})$.

Particularly surprising is that if we consider the related set

$$\mathcal{S}' = \{(0,1),(1,1),(1,0)\}$$

then we have the different value $e(S,T) = \log 4$.

Remark. Klaus Schmidt has pointed out to us that most other definitions of entropy have the property that the entropy of such "algebraic examples" for S is equivalent to that of the convex hull of S. We see from this proposition that F-entropy does not have this property.

These examples are particularly useful for comparing other definitions of entropy. Let us consider two other "natural" definitions.

Let $M(k)$ denote the number of permissible ways of labelling the $k \times k$ square $\{(n,m) \in \mathbb{Z}^2 : 0 \le n,m \le k-1\}$. In the Ledrappier example above, we have $M(k) = 2^{2k-1}$ (any labelling for the entire square being determined by the 2^{2k-1} choices on the bottom and left hand sides). Let $L(k)$ denote the number of permissible ways of labelling the triangle $\{(n,m) \in \mathbb{Z}^2 : 0 \le n,m \le k-1,\ n+m \le k\}$. In the Ledrappier example, we have $L(k) = 2^{2k-1}$ (any labelling for the entire triangle again being determined by the 2^{2k-1} choices on the bottom and left hand sides). Thus

$$h_M = \lim_{k\to\infty} \frac{1}{k} \log M(k) = 2\log 2 \quad \text{and} \quad h_L = \lim_{k\to\infty} \frac{1}{k} \log L(k) = 2\log 2,$$

which is different from the entropy $e(S,T)$.

4. Rotations on a circle. In this section we consider another simple example. Let $X = \mathbb{R}/\mathbb{Z}$ and let $S(x) = x + \alpha \pmod 1$ and $T(x) = x + \beta \pmod 1$.

Theorem 4. *If $\alpha \ne \beta$ then $e(S,T) = \log 2$. If $\alpha = \beta$ then $e(S,T) = 0$.*

Proof. First assume that $\alpha \ne \beta$. Fix $\varepsilon > 0$ and consider the finite sets $Y = \{k\varepsilon : 0 \le k \le [1/\varepsilon] - 1\}$ and

$$\Lambda = \{(x_k)_{k=0}^{n-1} : S(x_{k+1}) = x_k \text{ or } T(x_{k+1}) = x_k \text{ with } x_0 \in Y\}.$$

The set Λ corresponds to an (n,ε)-spanning set for $\sigma : \mathcal{X} \to \mathcal{X}$ and has cardinality $2^n \times [1/\varepsilon]$. From this we conclude that $e(S,T) \le \log 2$. (This also follows from Proposition 1, part (5).) To get the reverse inequality, we choose $|\alpha-\beta| > \varepsilon > 0$. We claim that Λ corresponds to an $(n,\varepsilon/2)$-separated set. Given two finite sequences $(x_k)_{k=0}^{n-1}, (x_k')_{k=0}^{n-1} \in \Lambda$ with $|x_k - x_k'| < \varepsilon/2$ for $k = 0,\ldots,n-1$, we immediately see that $x_0 = x_0'$. Assume that we have shown inductively that $x_i = x_i'$ for $0 \le i \le k-1 < n-1$. If $x_k \ne x_k'$ then $|x_k - x_k'| = |\alpha - \beta| > \varepsilon$, giving a contradiction. Thus we conclude that $(x_k)_{k=0}^{n-1} = (x_k')_{k=0}^{n-1}$, and so Λ does correspond to an $(n,\varepsilon/2)$-separated set for $\sigma : \mathcal{X} \to \mathcal{X}$, and so $e(S,T) \le \log 2$.

If $\alpha = \beta$ then $e(S,T) = h(S) = 0$ by Proposition 1, part (1).

REMARKS. (1) A similar result holds for any $\mathbb{Z}^2$-action by isometries.

(2) The definition of entropy generalises to $\mathbb{Z}^n$. Theorem 4 also has a natural generalisation.

5. Comments and problems. In this section, we conclude with a few observations and problems.

(1) Friedland [2] considered the example of $S, T : \widehat{\mathbb{R}} \to \widehat{\mathbb{R}}$ on the extended real line $\widehat{\mathbb{R}}$ given by $S(x) = x + p$ and $T(x) = x - q$ (where $p, q > 0$ are distinct positive integers). In this case

$$e(S,T) = \frac{p}{p+q}\log\left(\frac{p}{p+q}\right) + \frac{q}{p+q}\log\left(\frac{q}{p+q}\right).$$

Despite some superficial similarity with Theorem 4, the entropy in Friedland's example is less than that in the commuting rotations because of the concentration of orbits near ∞.

(2) An interesting example of a group action other than $\mathbb{Z}^2$ is a Schottky group $\Gamma \subset \mathrm{SL}(2,\mathbb{C})$. This corresponds to a choice of n pairs of circles (C_i, C_i') $(i = 1, \ldots, n)$ in the extended complex plane $\widehat{\mathbb{C}}$. We assume that all of the circles (and their interiors) are disjoint. We choose generators for the group Γ to be the linear fractional transformations $g_i \in \mathrm{SL}(2,\mathbb{C})$ which map the interior of C_i to the exterior of C_i'. Following the original definition of Friedland, we can define the entropy $h(\Gamma)$ for this group in terms of the entropy of the shift map $\sigma : \mathcal{X} \to \mathcal{X}$ on the space of sequences

$$\mathcal{X} = \Big\{(x_n) \in \prod_{n=0}^{\infty} \widehat{\mathbb{C}} : \forall n \geq 0\ \exists g_i \text{ with } g_i(x_n) = x_{n+1}\Big\}$$

given by $\sigma(x_n) = (x_{n+1})$. It is now easy to see that $h(\Gamma) = h(\sigma) = \log n$.

(3) Consider two independent commuting hyperbolic toral automorphisms $S, T : \mathbb{B}^n/\mathbb{Z}^n \to \mathbb{B}^n/\mathbb{Z}^n$. We would conjecture that

$$\begin{aligned} e(S,T) = &\sum_{|\lambda_i|^{|\lambda_i|}.|\mu_i|^{|\mu_i|}>1} \log_+(|\lambda_i| + |\mu_i|) \\ &+ \sum_{|\lambda_i|^{|\lambda_i|}.|\mu_i|^{|\mu_i|}<1} \log_+(\max\{|\lambda_i|, |\mu_i|\}). \end{aligned}$$

(4) What are the continuity properties of the entropy $e(S,T)$? (Notice that by Theorem 4, $(S,T) \mapsto e(S,T)$ is not always continuous. However, if $S, T : \mathbb{T}^n \to \mathbb{T}^n$ are commuting hyperbolic toral automorphisms then locally this is the case since any nearby action is conjugate.)

(5) Is the entropy $e(S,T)$ a useful conjugacy or semi-conjugacy invariant? (Probably this reduces to asking if there are cases where the entropy is easier to compute than the entropy of the corresponding generators.)

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Received 11 September 1997;
in revised form 16 February 1998