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Solution of the $1 : -2$ resonant center problem in the quadratic case

by

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Dedicated to the memory of Wiesław Szlenk

Abstract. The $1 : -2$ resonant center problem in the quadratic case is to find necessary and sufficient conditions (on the coefficients) for the existence of a local analytic first integral for the vector field $(x + A_1x^2 + B_1xy + Cy^2)\partial_x + (-2y + Dx^2 + A_2xy + B_2y^2)\partial_y$. There are twenty cases of center. Their necessity was proved in [4] using factorization of polynomials with integer coefficients modulo prime numbers. Here we show that, in each of the twenty cases found in [4], there is an analytic first integral. We develop a new method of investigation of analytic properties of polynomial vector fields.

1. Introduction. Recall the classical *problem of center* for polynomial real planar vector fields

$$\dot{x} = -y + P(x,y), \quad \dot{y} = x + Q(x,y), \tag{1}$$

where P and Q are polynomials belonging to some natural class (e.g. of degree $\le n$, homogeneous of degree n). One has to find conditions, on the coefficients of P and Q, under which a neighborhood of the origin is covered by periodic solutions of the system (1).

This problem was completely solved only in the two general situations:

(i) when P and Q are homogeneous quadratic polynomials (by Dulac and Kapteyn) and

(ii) when P and Q are homogeneous cubic polynomials (by Sibirskiĭ).

(We do not cite many special results in this field.)

In [5] the following generalization of the center problem was proposed. When we treat (1) as a system in the complex plane (with complex time) then after a simple change of variable it is equivalent to $\dot{x} = x + \ldots$, $\dot{y} = -y +$

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$\dots$, i.e. to a $1:-1$ resonant saddle. The existence of a center is equivalent to the existence of a local analytic first integral of the form $H = xy + \dots$ (Equivalent condition: absence of the resonant terms $(xy)^k(x\partial_x + y\partial_y)$ in the normal form.)

In [5] conditions for the existence of a local analytic first integral $H = x^q y^p + \dots$ (i.e. for the existence of a $p:-q$ resonant center) for the polynomial system

$$\dot{x} = px + \dots, \quad \dot{y} = -qy + \dots \tag{2}$$

were investigated. (Equivalent condition: absence of the resonant terms $(x^q y^p)^k(px\partial_x + qy\partial_y)$ in the normal form. One can also associate with such a center a family of homologically different cycles in the leaves of the holomorphic foliation defined by (2); see [5].) One can also consider the case $p = 0$ (i.e. saddle-node) and the case $pq < 0$ (i.e. resonant node).

A series of relevant results were proven, including a number of sufficient conditions for a $p:-q$ resonant quadratic center (eleven cases for the $1:-2$ resonance). The corresponding first integrals were written down explicitly (see Theorem 2 below).

The only way to get necessary conditions for a center is to compute the $p:-q$ resonant focus numbers, the analogues of the Poincaré–Lyapunov focus quantities. One calculates the successive terms in the Taylor expansion of the supposed first integral and the focus numbers g_k are the coefficients of the obstacles to its existence:

$$\dot{H} = \sum g_k (x^q y^p)^{k+1}, \quad H = x^q y^p + \dots$$

The g_k's are polynomials in the coefficients of (2) and are calculated algorithmically (see [4]).

Unfortunately, they are very complicated; it is the main reason of slow progress in the center problem. One should not only calculate g_k but also factorize them modulo the ideal J_{k-1} generated by g_j, $j < k$. The computation of the Gröbner bases of the ideals J_k requires a lot of computer memory. In the case of $1:-2$ resonance it fills 9000 written pages (over the ring $\mathbb{Z}$).

The success of the paper [4] relied on replacing the ring $\mathbb{Z}[A]$ by the ring $\mathbb{Z}/r\mathbb{Z}[A]$, where A denotes the coefficients and r is some prime number. This idea was elaborated on by Faugère [2], in the so-called "GB" package. Of course, using one prime p would not be sufficient and hence in [4] the calculations were repeated for the primes 65111, 65437, 65213, 65513, 64693, 64919, 65033, 65167, 65521, 31991.

Although this method does not provide a rigorous proof of necessary conditions it is practically sure that they are complete. The probability of the opposite event is lower than 10^{-25}.

The list of the 20 necessary conditions from [4] (see Theorem 1 below) contains 11 sufficient conditions found in [5]. The aim of this paper is to prove that the remaining 9 cases are also sufficient.

In 8 of these 9 cases the first integral is written explicitly (of Darboux type or of Darboux–Schwartz–Christoffel type or of Darboux-hyperelliptic type). In one case, i.e. (2.2), we do not even know if it is integrable in quadratures. We have developed a new method of proving analyticity of the local first integral. This method can be described as follows.

Firstly, assume that there is an invariant rational algebraic curve passing through the center point. There is a rational invertible change of variables such that the invariant curve becomes one of the invariant axes (it is the Abhyankar–Moch theorem [1]). Thus we can write the equation for the phase curves in the form

$$\frac{dz}{du} = \frac{a_1(u)z + \ldots + a_k(u)z^k}{b_0(u) + \ldots + b_l(u)z^l} = c_1(u)z + c_2(u)z^2 + \ldots$$

where a_i and b_j are polynomials. We seek a first integral in the form of the series

$$H = H_1(u)z + H_2(u)z^2 + \ldots$$

Calculations show that H_1 is of Darboux type, a product of powers of linear factors: $H_1 = u^{\alpha_0}(u-u_1)^{\alpha_1}\ldots(u-u_r)^{\alpha_r}$. Amazingly, it has turned out that the other H_k's are also of Darboux type in the case (2.2): $H_k = H_1(u)\cdot W_k(u)$ with rational factors W_k. This is proved by induction.

This integral usually cannot be analytic near the origin because it contains negative powers of u; namely, we always have $W_k = u^{-k}\cdot Z_k(u)$, $Z_k(0) \neq 0$. However, after blowing-up we obtain a new singular point (with another resonance) which is orbitally linearizable. It is easy to show, using the formal orbital normal forms theory, that also the point $u = v = 0$ has trivial formal orbital normal form.

Therefore the study of the quadratic $1 : -2$ resonant case is finished. It is the third general case which is solved completely, after the quadratic $1 : -1$ resonant and homogeneuos cubic $1 : -1$ resonant cases.

Before passing to the theorems and proofs we want to underline some general characteristics, which distinguish the $p : -q$ resonances from the $1 : -1$ resonance.

(a) All known $1 : -1$ cases of polynomial center contain at least one modulus, i.e. the quotient of the center variety by the action of affine changes of coordinates and time has all components of positive dimension. In the quadratic $p : -q$ resonant case there are components without moduli, i.e. of maximal possible codimension. In particular, there is no analogue of Bautin's theorem about the bound 3 for the number of small limit cycles bifurcating from a quadratic $1 : -1$ resonant center. In [5] it was shown that the number of such cycles can be ≥ 4.

(b) In all known $p : -q$ resonant polynomial centers at least one of its separatrices lies in an invariant algebraic curve. This is not the case for the $1 : -1$ resonance; for example, in the Lotka–Volterra quadratic center or in the reversible quadratic center.

(c) In all known real elementary polynomial centers ($1 : -1$ case with additional antisymmetry), the vector field is either reversible (in a generalized rational sense) or has a first integral of liouvillian type. In this paper we present some cases of $p : -q$ resonant center with a first integral expressed by means of hypergeometric functions (see Theorem 2). The latter have non-solvable monodromy group and cannot be of liouvillian type (with solvable monodromy). They are not reversible either.

A rigorous estimate of the cyclicity of a quadratic $1 : -2$ resonant center from above would be difficult because we need some radicality property of the ideal generated by the focus numbers. However, it seems that using the calculations from [4] it should be possible to get an almost sure upper bound. Namely, one has to compute the first five $1 : -2$ resonant focus numbers in order to get the center conditions from Theorem 1. It seems that the further focus numbers can be expressed by means of the first five; i.e. they lie in the ideal generated by $g_1, \ldots, g_5$, which should be radical. (Fronville checked that the next two focus numbers lie in this ideal.) In such a case one could apply the method of Françoise and Yomdin [3] which allows showing that there are no more than 5 limit cycles bifurcating from the center. One could also use the arguments from the Appendix of [5] where the analogue of Bautin's theorem in the complex case is proven.

If the above conjecture is true, then we would have the inequalities

$$4 \leq \text{number of small limit cycles} \leq 5$$

where the first inequality is rigorous and the second one is to be proven.

2. Results. Consider the system

$$\dot{x} = x + A_1x^2 + B_1xy + Cy^2, \quad \dot{y} = -2y + Dx^2 + A_2xy + B_2y^2. \tag{3}$$

(Here the notation A_i, B_j, C, D is chosen in such a special way because of the additional symmetry structure. Namely, the change of variable $x \to \mu x,\ y \to \mu^{-2}y,\ \mu \in \mathbb{C}^* = \mathbb{C} \setminus 0$, results in the action of the torus $\mathbb{C}^*$ on the coefficients: $A_i \to \mu A_i,\ B_j \to \mu^{-2}B_j,\ C \to \mu^{-5}C,\ D \to \mu^4 D$. The focus numbers are invariant with respect to this action.)

Definition. We say that the system (3) has a $1 : -2$ *resonant center* iff it has a local analytic first integral $H = xy^2 + \ldots$ near $x = y = 0$.

THEOREM 1. *The system* (3) *has a* $1:-2$ *resonant center iff one of the following twenty conditions holds*:

(1.1) $C = D = 2A_1B_1 - A_1B_2 - A_2B_2 = 0$,

(2.1) $A_2 = D = 0$,

(2.2) $A_1 + 2A_2 = B_1 = B_2 = D = 0$,

(2.3) $A_1 - 2A_2 = B_1 = B_2 = D = 0$,

(2.4) $4A_1 + A_2 = 2B_1 + 3B_2 = D = 0$,

(2.5) $A_1 + A_2 = B_1 = D = 0$,

(3.1) $B_1 = B_2 = C = 0$,

(3.2) $2A_1A_2 + 2A_2^2 + B_2D = B_1 = C = 0$,

(3.3) $A_2 = B_1 + B_2 + C + 0$,

(3.4) $3A_1 + A_2 = B_1 + B_2 = C = 0$,

(3.5) $A_1 - 3A_2 = B_1D + 3A_2^2 = B_2D + 2A_2^2 = C = 0$,

(3.6) $2A_1 - 11A_2 = 121B_1D + 30A_1^2 = 121B_2D + 12A_1^2 = C = 0$,

(3.7) $A_2 = 2B_1 - B_2 = C = 0$,

(4.1) $A_1 - A_2 = 11B_1 - 8B_2 = 25B_1D - 16A_2^2 = 4A_2^3 - 25D^2C = 0$,

(4.2) $4A_1 - 19A_2 = B_1 - B_2 = 125A_2^3 + 16D^2C = 4B_1D + 35A_2^2 = 0$,

(4.3) $A_1 + 2A_2 = 2B_1D - B_2D + 2A_2^2 = A_2B_2D - 2A_2^3 + CD^2 = 0$,

(4.4) $28A_1 + 29A_2 = 17B_1 + B_2 = 196B_1D + 5A_2^2 = 125A_2^3 + 5488CD^2 = 0$,

(4.5) $4A_1 - 7A_2 = 4B_1 - B_2 = 2B_1D + A_2^2 = 5A_2^3 + 4CD^2 = 0$,

(4.6) $8A_1 + A_2 = B_1 + 5B_2 = 32B_1D - 5A_2^2 = 5A_2^3 - 128CD^2 = 0$,

(4.7) $65A_2^4 - 40A_1CD^2 + 104A_2CD^2 = 55A_2^5 + 148A_2^2CD^2 + 40B_1CD^3 = 45A_2^5 + 32A_2^2CD^2 + 40B_2CD^3 = 5A_2^6 + 8A_2^3CD^2 - 16C^2D^4 = 0$.

(The labelling is taken from [4]. All cases are divided into four groups, labelled by the first number: 1. $C = D = 0$, 2. $C \neq 0 = D$, 3. $C = 0 \neq D$, 4. $C \neq 0 \neq D$. The above conditions are written in the general (equivariant) form; in [4] they were simplified with $C = 1$ in 2., $D = 1$ in 3., and $C = 5$, $D = 2$ in 4.)

The proof of Theorem 1 consists of two parts: necessity of the conditions (1.1)–(4.7) and their sufficiency. The first part was done in [4] using computer programs; recall that this proof is almost surely complete but not absolutely rigorous.

In [5] it was shown that eleven of the above conditions are sufficient by finding explicit formulas for first integrals. Below we recall these formulas in some special affine coordinates and for generic coefficients. In these formulas,

$$F_1(z) = F(a, b; c; z) = 1 + \frac{ab}{c1!}z + \frac{a(a+1)b(b+1)}{c(c+1)2!}z^2 + \dots$$

is the hypergeometric function, and $F_2(z) = F(a-c+1, b-c+1; 2-c; z)$.

THEOREM 2 [5]. *The following functions define first integrals for the system* (3) *in the eleven cases below.*

(1.1) $x^2y(1+x+y)^a$ (case (7) in [5]),

(2.1) $x^{1/2}\{[x+cy/(ab)]F_1(y) - y(1-y)F_1'(y)\}/\{[1/2+x+(c/(ab)-1/2)y]F_2(y) - y(1-y)F_2'(y)\},\ c = 3/2$ (case (1) in [5]),

(2.3) $(y^2+5x)^4y^2(y^2+4x+1)^{-5}$ (case (12) in [5]),

(2.4) $(x+x^2+axy+y^2)^2y$ (case (6) in [5]),

(3.1) $(1+x)^{-a}[1+\binom{a}{1}x+\binom{a}{2}x^2+\binom{a}{3}x^3+x^2y]$ (case (4) in [5]),

(3.2) $x^2\{[y+cx/(ab)]F_1(x)-x(1-x)F_1'(x)\}/\{[2+y+(c/(ab)-2)y]F_2(x)-x(1-x)F_2'(x)\},\ c = 3$ (case $(3)_1$ in [5]),

(3.3) $[(1+x+xy-(a-2)x^2/6)^a]/[1+ax+axy+(2a-1)x^2/6]$ (case (8) in [5]),

(3.4) $x^2(y+x^2+axy+y^2)$ (case (5) in [5]),

(3.7) $[1+y+(a+1)x]^a[1+y+(a-2)x]^{1-a}[1+y+(4a-2)x+9a(a-1)x^2/2]^{-1}$ (case (9) in [5]),

(4.1) $[(x+y)^2+6x]^2[(x+y)^2+12y][x+y+2]^{-6}$ (case (10) in [5]),

(4.6) $[1+(x+y-1)(-2x+y+1)]^2[1+(x+y-1)(x-3y+1)]$ (case (13) in [5]).

REMARK 1. In [5] also the case $A_1 = -1$, $A_2 = 1$, $B_1 = D = 0$, $B_2 = 7$, $C = -5$, i.e. (14), was investigated. It is included in (2.5) as a subcase.

The cases (11) and (15) from [5] (for $p = 1$, $q = 2$) are particular subcases of (3.7).

The cases (2) and $(3)_k$, $k > 1$, from [5] concern other resonances (different p, q).

In Sections 3–13 we will prove the analyticity of the first integrals in the remaining cases, i.e. (2.2), (2.5), (3.5), (3.6), (4.2), (4.3), (4.4), (4.5), (4.7).

We also have one result concerning the case of saddle with general $1 : -q$ resonance. Namely, we consider the Lotka–Volterra case

$$\dot{x} = x(1+A_1x+B_1y), \quad \dot{y} = y(-q+A_2x+B_2y). \tag{4}$$

PROPOSITION 1. *The system* (4) *has a* $1 : -q$ *resonant center at* $(0,0)$ *iff*

$$A_2[A_1+A_2][2A_1+A_2]\dots[(q-2)A_1+A_2] \times [qA_1B_1-(q-1)A_1B_2-A_2B_2] = 0. \tag{5}$$

We will prove this in the last section.

3. Blowing-up of resonant saddles. Consider the resonant analytic vector field (2), i.e. $\dot{x} = px + \ldots$, $\dot{y} = -qy + \ldots$

Its formal orbital normal form (i.e. with respect to formal changes of the coordinates x, y and multiplication by formal nonzero functions) is either

$$\dot{\widetilde{x}} = p\widetilde{x}, \quad \dot{\widetilde{y}} = -q\widetilde{y}(1 + az/(1 + bz)), \quad z = \widetilde{x}^{kq}\widetilde{y}^{kp} \tag{6}$$

where k is a positive integer, $a \neq 0$ (it is the $p : -q$ resonant focus number and can be normalized to 1), and b is a formal modulus, or

$$\dot{\widetilde{x}} = p\widetilde{x}, \quad \dot{\widetilde{y}} = -q\widetilde{y}.$$

In the latter case the change is analytic and the initial system has the analytic first integral $\widetilde{x}^q\widetilde{y}^p$. It is just the $p : -q$ resonant center.

Consider now the blowing-up of this singularity. It means that we apply the change $(x, y) \to (\xi, \eta) = (\widehat{x}, \widehat{y}/\widehat{x})$, where $\widehat{x}, \widehat{y}$ is some linear system of coordinates. The point $x = y = 0$ is replaced by the line $\xi = 0$, which contains two singular points of the resolved field of directions. These singular points correspond to the separatrices of (2) and are saddles:

- P_1 which is $(p + q) : -p$ resonant, and
- P_2 which is $(p + q) : -q$ resonant.

LEMMA 1. *If one of the points P_1 or P_2 is orbitally analytically linearizable then the point $x = y = 0$ is a $p : -q$ resonant center.*

Proof. The formal system (6) is equivalent to

$$\dot{\widetilde{x}} = p\widetilde{x}, \quad \dot{z} = -aqz^2/(1 + bz)$$

with first integral $\frac{1}{p}\ln\widetilde{x} + \frac{b}{aq}\ln z - \frac{1}{aqz}$ containing logarithmic singularities along the separatrices $\widetilde{x} = 0$ and $\widetilde{y} = 0$. (The separatrices of a saddle always form analytic curves.)

After the blowing-up $\xi = \widetilde{x}$, $\eta = \widetilde{y}/\widetilde{x}$ (for example) the above first integral transforms to $\frac{1}{p}\ln\xi + \frac{b}{aq}\ln z - \frac{1}{aqz}$, $z = (\xi^{p+q}\eta^p)^k$. It is a first integral of the system $\dot{\xi} = p\xi$, $\dot{\eta} = -(p + q)\eta(1 + \widetilde{a}z/(1 + bz))$, $\widetilde{a} = aq/(p + q)$, in its normal form.

Thus if $x = y = 0$ is not a center then neither is $\xi = \eta = 0$. ∎

As a corollary we get some useful property which will be used in the next sections. It allows us to replace the expansion of a first integral in the Taylor series by an expansion of another first integral in a series containing terms with negative powers.

PROPOSITION 2. *Assume that there is a Taylor series F of two variables and a polynomial resolution map $\Phi : \mathbb{C}^2 \to \mathbb{C}^2$ (which defines a diffeomorphism between $\mathbb{C}^2 \setminus \Phi^{-1}(0)$ and $\mathbb{C}^2 \setminus 0$) such that the series $F \circ \Phi^{-1}(x, y)$ is* *a formal first integral of the resonant saddle* (2). *Then the point* $x = y = 0$ *is a* $p : -q$ *resonant center for* (2).

For example, if the series $\sum_{n=0}^{\infty} W_n(y) \cdot x^{q/p-n} y^{n+1}$, $W_n(0) \neq 0, \infty$, is a formal first integral for a $p : -q$ resonant point then we have a center.

4. Darboux form of the Chebyshev function. Liouville showed that the Chebyshev integral

$$\Phi(x) = \int^{x} \tau^{\alpha-1}(1-\tau)^{\beta-1}\, d\tau \tag{7}$$

is elementary iff one of the numbers $\alpha, \beta, \alpha+\beta$ is an integer. We ask whether it is of Darboux type

$$x^{\alpha}(1-x)^{\beta} P(x) \tag{8}$$

(up to a constant) with a polynomial P for both α and β noninteger.

PROPOSITION 3. *If neither* α *nor* β *is an integer then the function* (7) *has the form* (8) *iff* $\alpha + \beta$ *is a nonpositive integer. Moreover*, *in that case* $\deg P \leq -(\alpha + \beta)$.

Proof. We can define the *principal value* of the Chebyshev function at the singular points $0, 1$.

Let $x_0 \neq 0, 1$ be the initial limit in the integral (7). Then near $x = 0$ we have the representation $\Phi(x) = c_0 + x^{\alpha} \times$ analytic function. Similarly near $x = 1$ we have $\Phi(x) = c_1 + (x-1)^{\beta} \times$ analytic function.

The difference $c_1 - c_0$ is equal to the Euler Beta-function

$$\int_0^1 \tau^{\alpha-1}(1-\tau)^{\beta-1}\, d\tau = \Gamma(\alpha)\Gamma(\beta)/\Gamma(\alpha+\beta),$$

where the right-hand side defines its analytic prolongation to the domain with $\operatorname{Re}\alpha \leq 0$ or $\operatorname{Re}\beta \leq 0$.

The first part of Proposition 3 follows from the fact that $\Gamma(\alpha)\Gamma(\beta) \neq 0$ and the Gamma-function has poles at nonpositive integers.

The estimate for the degree follows from the fact that $\Phi(x)$ tends to a constant as x tends to infinity. ∎

COROLLARY. *If* α, β *are not integers*, n *is a nonnegative integer and* $\alpha + \beta + n$ *is a nonpositive integer then for any polynomial* Q_n *of degree* n *the integral*

$$\int^{x} Q_n(\tau)\tau^{\alpha-1}(1-\tau)^{\beta-1}\, d\tau$$

is of Darboux type (8) *with* P *of degree* $\leq -(\alpha+\beta)$.

5. The case (2.2). The equation for the phase curves is

$$3\frac{dy}{dx} = -\frac{(2-A_2x)y}{x(1-2A_2x)+Cy^2} \tag{9}$$

where, after some simple changes, we can assume that $A_2 = -1$, $C = -1$. (If $A_2 = 0$ or $C = 0$ then one can easily integrate (9).)

We expand the right-hand side of (9) in a power series

$$\frac{dy}{dx} = a(x)y\Big[1 + \sum y^{2n}/b(x)^n\Big]$$

where $a(x) = -2/x + 3/(1+2x)$, $b(x) = x(1+2x)$. We seek a first integral also in the form of a power series

$$H = H_1(x)y + H_3(x)y^3 + H_5(x)y^5 + \dots$$

The functions H_n satisfy the system of equations

$$\begin{aligned}
&H_1' + aH_1 = 0,\\
&H_3' + 3aH_1 + (a/b)H_1 = 0,\\
&\dots\dots\dots\dots\dots\dots\dots\dots\\
&H_{2n+1}' + (2n+1)H_{2n+1} + \frac{(2n-1)a}{b}H_{2n-1}\\
&\qquad + \frac{(2n-3)a}{b^2}H_{2n-3} + \dots + \frac{a}{b^n}H_1 = 0.\\
&\dots\dots\dots\dots\dots\dots\dots\dots
\end{aligned}$$

The first of them has the solution

$$H_1 = x^2(1+2x)^{-3/2}$$

and the others are solved in the form

$$H_{2n+1} = H_1(x)^{2n+1}\int^x H_1(s)^{-2n-1}\Psi_n(s)\,ds$$

where Ψ_n is calculated algebraically using the previous steps.

The next lemma in combination with Proposition 2 proves the existence of a center in the case (2.2).

LEMMA 2. *We have*

$$H_{2n+1} = x^{2-n}(1+2x)^{-3n-3/2}P_{2n}(x)$$

where $P_{2n}(x)$ are polynomials of degree $2n$.

Proof. We use induction with respect to n. Of course H_1 has the form from Lemma 2.

Next

$$H_{2n+3} = \frac{-x^{4n+6}}{(1+2x)^{3n+9/2}} \times \int^{x} \frac{(1+2s)^{3n+9/2}}{s^{4n+6}} \cdot \frac{a}{b}(s) \cdot \left[\sum_{j=0}^{n} \frac{(2j+1)H_{2j+1}}{b^{n-j}}\right] ds. \tag{10}$$

Here each summand in the integrand equals

$$\frac{(1+2s)^{3n+9/2}}{s^{4n+6}} \cdot \frac{s+2}{s^2(1+2s)^2} \cdot \frac{1}{s^{n-j}(1+2s)^{n-j}} \cdot \frac{(2j+1)P_{2j}(s)}{s^{j-2}(1+2s)^{3j+3/2}} = (2j+1)\frac{(s+2)(1+2s)^{2n-2j+1}P_{2j}(s)}{s^{5n+6}}.$$

We see that the degree of the numerator is $2n+2$ and is smaller than the degree of the denominator (power of s) minus 1. Therefore, we can integrate this expression term-by-term to obtain $Q^{(j)}(x) \cdot x^{-5n-5}$, with polynomial $Q^{(j)}$ of degree $2n+2$.

This together with (10) gives the result. ∎

Remark 2. After applying the change $y^2 = xz$ the system (9) is transformed to

$$\dot{x} = x(1+2x-z), \quad \dot{z} = z(-5-4x+z), \tag{11}$$

which has a center at $x = z = 0$ (by Proposition 1 with $q = 5$). The above change is the composition $(x,y) \to (x,y^2) \to (x,y^2/x)$ of a fold map and a blowing-up map. Therefore, applying the arguments from Section 3 we obtain another proof of the existence of a center for the system (9).

The analyticity of a first integral of (9) can also be obtained by refining the proof of Proposition 1 from Section 14. Namely, one seeks a first integral of the Lotka–Volterra system (11) in the form $H = x^5 y[H_0(x)+H_1(x)z+\ldots]$. In the proof of Proposition 1 it is shown that $H_j = (1+2x)^{-3(j+1)}P_{2j}(x)$, where P_{2j} are polynomials of degree $2j$.

In the special case of the system (11) it can be shown that $P_{2j} = x^{j-4}Q_{j+4}(x)$, $j > 4$, where Q_m are of degree m. This allows us to show the analyticity of the first integral $H(x, y^2/x)$ near $x = y = 0$.

6. The case (2.5). By using the changes $x \to \lambda x$, $y \to \mu y$ the variety described by the four general conditions (2.5) can be reduced to one line in the parameter space. More precisely, we can assume that $A_1 = -A_2 = 1$, $C = -1$, $B_2 = -b \in \mathbb{C}$. We obtain the system

$$\dot{x} = x(x+1) - y^2, \quad \dot{y} = -(x+2)y - by^2. \tag{12}$$

One can check that the point $x = -1$, $y = 0$ is a $1 : 1$ resonant node with diagonal linear part. This implies that the system (12) has three invariant lines passing through $(-1, 0)$ and through the singular points at infinity.

Moreover, this system can be integrated by means of a Darboux–Schwartz–Christoffel integral. Indeed, in the variables $u = y/(x + 1)$, $z = x + 1$ we have the linear equation

$$\frac{dz}{du} = \frac{1 + (u^2 - 1)z}{u(2 + bu - u^2)}$$

with first integral

$$H = zu^{1/2}(u - u_2)^{a_1}(u - u_2)^{a_2} - \int^u s^{-1/2}(s - u_1)^{a_1-1}(s - u_2)^{a_2-1}\, ds.$$

Here $u_{1,2} = (b \pm \sqrt{b^2 + 8})/2$, $a_{1,2} = (\pm 3b + \sqrt{b^2 + 8})/(2\sqrt{b^2 + 8})$. ∎

Remark 3. One can prove the existence of a center in the case (2.5) in the same way as in the case (2.2), i.e. using the arguments from Section 3. Namely, one expands the first integral in a power series $H = \sum H_j(x)y^j$ and shows (by induction) that $H_{2n+1} = P_n(x)x^{2-n}(x+1)^{-2n-1}$ and $H_{2n} = Q_{n-1}(x)x^{3-n}(x + 1)^{-2n}$, where P_j and Q_j are polynomials of degree j.

Remark 4. The case (2.5) can be generalized in the following way. The system

$$\dot{x} = px(1 + x) - y^2, \quad \dot{y} = y(-q + (p - q)x - by)$$

has the singular point $(-1, 0)$, which is a $1 : 1$ resonant and linearizable node with three invariant lines. It also has a Darboux–Schwartz–Christoffel integral.

The corresponding conditions for center (in the equivariant form) are

$$pA_2 + (q - p)A_1 = B_1 = D = 0.$$

7. The case (3.5). This case is of codimension four and after a suitable reduction we can assume that $A_1 = 3$, $A_2 = 1$, $B_1 = -3$, $B_2 = -2$, $C = 0$, $D = 1$ or that

$$\dot{x} = x(1 + 3x - 3y), \quad \dot{y} = -2y(1 + y) + xy + x^2.$$

One checks that the curves

$$f_1 = 4(x - y)^3 + x^2 - 20xy - 8y^2 - 4y = 0,$$
$$f_2 = (x - y)^2 + 4x + 2y + 1 = 0$$

are invariant and $\dot{f}_1 = 2(-1+3x-3y)f_1$, $\dot{f}_2 = 4(x-y)f_2$. This means that the function $H = x^2 f_1/f_2^3$ is a first integral. ∎

Remark 5. If X_H denotes the Hamiltonian vector field with Hamilton function H as above then it is of the form $X_H = xf_3^{-4}V_H$, where V_H is a quintic vector field. It turns out that V_H has nonisolated critical points. Namely, $1-(27/4)H = f_3^2/f_2^3$, $f_3 = (x-y)^3 - 15x^2/2 - 3xy - 3y^2 - 6x - 3y - 1$, which means that X_H vanishes at $f_3 = 0$. Thus the vector field V_H/f_3 is proportional to our quadratic vector field.

REMARK 6. Using the change $z = y - z$ and expansion of the first integral $H = H_2(z)x^2 + H_3(z)x^3 + \ldots$ one can show the existence of a center using the method described in Section 3 (as in the case (2.2)). By induction, one proves that $H_n = P_{n-2}(z)z^{3-n}(1+z)^{-2n}$, where P_j denotes a polynomial of degree j.

8. The case (3.6). After some change the problem is reduced to the study of the system

$$\dot{x} = x + 11x^2 + 30xy, \quad \dot{y} = -2y - x^2 + 2xy + 12y^2.$$

It is useful to make the change $z = x + 6y - 1$ (the point $(1 : -6 : 0)$ at infinity is singular). Then we get

$$\dot{x} = x(6 + 6x + 5z), \quad \dot{z} = 6x + 2z + 3xz + 2z^2.$$

One checks that this system has the following invariant curves:

$$f = z^3 - 18xz - 36x = 0,$$
$$g = z^4 - 24xz^2 - 48xz + 72x^2 = 0.$$

More precisely, $\dot{f} = 3(2 + 3x + 2z)f$, $\dot{g} = 4(2 + 3x + 2z)g$. This means that $H = g^3/f^4$ is a first integral. ■

REMARK 7. Notice that

$$g = z^4\left(1 - 18\frac{x}{z^2} - 36\frac{x}{z^3}\right)^{4/3} + O[1] = z^4(f/z^3)^{4/3} + O[1]$$

where $O[n] = O(|(x,z)|^n)$ as $|(x,z)| \to \infty$. This implies that the vector field $V_H = g^{-2}f^5X_H$ is cubic. But $H - 1 = x^2(y + \ldots)$, which implies that the line $x = 0$ is critical for V_H. Thus $V_H = xV$ with quadratic vector field V.

REMARK 8. Initially, the authors used the method of Section 3 to show the existence of a center in the case (3.6). However, that proof needs a lot of preliminary transformations and the calculations were quite complicated.

9. The case (4.2). After reduction we obtain the system

$$\dot{x} = x - 19x^2 - 14xy + 5y^2, \quad \dot{y} = -2y + 10x^2 - 4xy - 14y^2.$$

Here the curves

$$f_1 = 2y - 5x^2 - 64xy + 22y^2 + 18(x+y)^2(7x - 2y) + 81(x+y)^4 = 0,$$
$$f_2 = x + (x+y)^2 = 0,$$
$$f_3 = 1 - 24x + 12y = 0$$

are invariant. More precisely, $\dot f_1 = -2[1+18(x+y)]f_1$, $\dot f_2 = [1-18(x+y)]f_2$, $\dot f_3 = -24(x+y)f_3$, and $H = f_1 f_2^2/f_3^3$ is a first integral. ∎

REMARK 9. Here $V_H = f_2^{-1} f_3^4 X_H$ is a sixtic vector field with a quartic curve $f_4 = 1-18(2x-y)+54(2x-y)^2+108(8x-y)(x+y)^2+486(x+y)^4 = 0$ of nonisolated critical points. We have $V = V_H/f_4$.

Another form of a first integral is f_4^2/f_3^3.

REMARK 10. Passing to the variables $u = x+y$, $z = x+u^2$ we can apply the method of Section 3. We obtain $H = H_2(u)z^2 + H_3(u)z^3 + \ldots$, where $H_n = P_{n-2}(u)u^{3-n}(1+6u)^{-2n}$. This gives another proof of the existence of a center.

10. The case (4.3). Reduction gives the 1-parameter family of systems

$$\dot x = x - 2x^2 + bxy - 2by^2,$$
$$\dot y = -2y + x^2 + xy + (2b+2)y^2.$$

This system has invariant algebraic curves

$$f = (x+2y)^2 - 4y = 0, \qquad g_1^2 - f = 0, \qquad g_2^2 - f = 0$$

where $g_{1,2} = x + 2y + \alpha_{1,2}/(b+2)$ and $\alpha_{1,2} = 3 \pm \sqrt{1-4b}$.

We have presented the invariant curves in this special form because it suggests the existence of a first integral of a special form. We have

$$(g_{1,2} \pm \sqrt{f})^{\cdot} = [4(b+2)y - \alpha_{2,1}(x+2y \pm \sqrt{f})](g_{1,2} \pm \sqrt{f})/4.$$

This means that we have the following first integral of Darboux-hyperelliptic type:

$$H = \left(\frac{g_1 - \sqrt{f}}{g_1 + \sqrt{f}}\right)^{\alpha_1} \left(\frac{g_2 + \sqrt{f}}{g_2 - \sqrt{f}}\right)^{\alpha_2}.$$

REMARK 11. After introducing the new coordinates $u = x + 2y$, $z = 4y - u^2$ one obtains

$$\frac{dz}{du} = \frac{[2(b+2)u^2 - 8]z + 2(b+2)z^2}{u[(b+2)u^2 - 6u + 4] - [6 - (b+2)u]z}$$
$$= \left(\frac{-2}{u} + \frac{2\varphi'(u)}{\varphi(u)}\right)z + \sum \frac{Q_n(u)}{u^n \varphi^n(u)} z^n$$

where $\varphi = (b+2)u^2 - 6u + 4$ and Q_n are polynomials of degree n. Next, expanding the first integral $H = H_1 z + H_2 z^2 + \ldots$ we get $H_1 = u^2/\varphi^2$ and $H_n = P_{3n-4}(u)u^{2-n}\varphi^{-2n}(u)$.

This provides another proof of the existence of a center in this case.

11. The case (4.4). After reduction we arrive at the system

$$\dot{x} = x - 29x^2 - 2xy - 5y^2, \quad \dot{y} = -2y + 10x^2 + 28xy + 34y^2.$$

Here the curves

$$\begin{aligned} f_1 &= 2y - 5(x^2 + 16xy + 10y^2) + 6(x + 5y)(19x^2 + 25xy + 10y^2) \\ &\quad + 9(x + 5y)^2(5x + y)^2 = 0, \\ f_2 &= x - (5x + y)^2 = 0, \\ f_3 &= 1 - 24x - 12y = 0 \end{aligned}$$

are invariant with $\dot{f}_1 = 2[-1 - 6x + 42y]f_1$, $\dot{f}_2 = [1 - 54x + 18y]f_2$ and $\dot{f}_3 = 24(y - x)f_3$. Thus $H = f_1 f_2^2 / f_3^5$ is a first integral. ∎

Remark 12. Here V_H is a sixtic vector field and it has a quartic curve $1 - 30(2x + y) + 270(2x + y)^2 - 108(8x^2 + 5xy + 5y^2)(5x + y) - 162(x + 5y)(5x + y)^3 = 0$ of nonisolated singular points.

Remark 13. The existence of a center in this case can also be proved using Propositions 2 and 3 (and its Corollary).

After introducing the variables $u = y + 5x$, $z = x - u^2$ one obtains the equation

$$\begin{aligned} \frac{dz}{du} &= \frac{(1 + 24u)(6u - 1)z + 144z^2}{2u(6u - 1)^2 + 3(24u - 1)z} \\ &= \left(\frac{-1}{2u} + \frac{15}{6u - 1}\right) + \sum \frac{Q_{n-2}(u)}{u^n(6u - 1)^{2n-1}} z^n. \end{aligned}$$

We look for $H = H_2(u)z^2 + \ldots$ and find

$$\begin{aligned} H_2 &= u(6u - 1)^{-5}, \\ H_3 &= \frac{-u^{3/2}}{(6u - 1)^{7+1/2}} \int^u \frac{(6u - 1)^{7+1/2}}{s^{1+1/2}} \cdot \frac{2Q_0 s\, ds}{s^2(6s - 1)^{3+5}}. \end{aligned}$$

We integrate the latter function using Proposition 3 and its Corollary. We get

$$H_3 = \frac{-u^{3/2}}{(6u - 1)^{7+1/2}} \int^u \frac{2Q_0\, ds}{s^{2+1/2}(6s - 1)^{1/2}} = \frac{P_1}{(6u - 1)^7}.$$

The general formula is

$$H_{2n} = \frac{P_{3n-4}(u)}{u^{2n-3}(6u - 1)^{5n}}, \quad H_{2n+1} = \frac{P_{3n-2}(u)}{u^{2n-2}(6u - 1)^{5n+2}}.$$

12. The case (4.5). Here the problem is reduced to the analysis of the system

$$\dot{x} = x + 7x^2 - 2xy - 5y^2, \quad \dot{y} = -2y + 4x^2 + 4xy - 8y^2. \tag{13}$$

LEMMA 3. *The system* (13) *has the first integral*

$$H = [y - (x-y)^2][x - (x-y)^2]^2[1 + 6x + 3y - 9(x-y)^2]^{-3}.$$

Proof. One checks that the curves $f = y - (x-y)^2 = 0$ and $g = x - (x-y)^2 = 0$ are invariant. More precisely,

$$\dot{f} = 2f(-1 - 3f + 3g), \quad \dot{g} = g(1 - 6f + 6g).$$

The latter is a Lotka–Volterra system with first integral

$$f^{-1/3}g^{-2/3}(1 + 3f + 6g)$$

equivalent to H. ∎

REMARK 14. In fact, this type of integral also appears in the case of general $p : -q$ resonance. It is of the form

$$[y-(x-y)^2]^p[x-(x-y)^2]^q[pq+2q(p+q)x+2p(p+q)y-2(p+q)^2(x-y)^2]^{-p-q}$$

and the corresponding conditions for center (in invariant form) are

$$\begin{aligned} 2qA_1 - (3p+2q)A_2 &= 2q^2B_1D + p(2p+q)A_2 \\ &= 4q^2B_2D + (2p+3q)(2p+q)A_2^2 \\ &= (p+2q)(2p+q)A_2^3 + 8q^3CD^2 = 0. \end{aligned}$$

13. The case (4.7). This case is also of codimension four but it cannot be reduced to one system with rational coefficients. In fact, we obtain a polynomial vector field with coefficients in the number field $\mathbb{Q}(\sqrt{6})$. An analyst would say that we have two systems, depending on the choice of the solution of the equation $\lambda^2 - 6 = 0$.

The reduced system is

$$\begin{aligned} \dot{x} &= x + 13(1 \pm \sqrt{6})x^2 - 2(26 \pm 11\sqrt{6})xy + 10(1 \pm \sqrt{6})y^2, \\ \dot{y} &= -2y + 5x^2 + 10xy + 2(1 \mp 9\sqrt{6})y^2. \end{aligned}$$

In the further analysis we choose the + sign in $\pm$.

Here the curves

$$\begin{aligned} f_1 &= 6 - \sqrt{6} + 90\sqrt{6}(x+y) + 270[5x^2 - 8(1+\sqrt{6})xy + 2(7+2\sqrt{6})y^2] = 0, \\ f_2 &= 1 + 24(1+\sqrt{6})(x+y) \\ &\quad + 12[(72+17\sqrt{6})x^2 2 - 4(3+8\sqrt{6})xy + 50(3+\sqrt{6})y^2] \\ &\quad - 72(16+\sqrt{6})[x - (2+\sqrt{6})y]^2[x - (\sqrt{6}-1)y] = 0 \end{aligned}$$

are invariant and $\dot{f}_1 = 18(1+\sqrt{6})(x-2y)f_1$, $\dot{f}_2 = 24(1+\sqrt{6})(x-2y)f_2$. Thus $H = f_1^4/f_2^3$ is a first integral.

Here V_H is a quartic vector field divisible by the polynomial $2(7+3\sqrt{6})x + [x - (2+\sqrt{6}y]^2$. ∎

14. Proof of Proposition 1. We seek a first integral in the form

$$H = x^q y[H_0(x) + H_1(x)y + \ldots]$$

where H_j satisfy the following system of equations:

$$x(1 + A_1 x)H_j' + [-jq + (qA_1 + (j+1)A_2)x]H_j + B_1 x H_{j-1}' + (qB_1 + jB_2)H_{j-1} = 0.$$

We find

$$H_0 = (1 + A_1 x)^{-A-q}, \quad A = A_2/A_1.$$

Calculation of the next term gives

$$H_1 = x^q (1 + A_1 x)^{-2(A+q)} \int^x s^{-q-1}(1 + A_1 s)^{A+q-2} R_1(s)\, ds$$

where $R_1(x) = (A_2 B_1 - A_1 B_2)x - (qB_1 + B_2)$.

Vanishing of the residue at $s = 0$ of the above integrand is a necessary condition for center. This residue equals

$$qA_1^{q-1}\binom{A+q+2}{q-1}[(A_2 B_1 - A_1 B_2)q - (qB_1 + B_2)(A_2 - A_1)] = \frac{q}{(q-1)!} \times (\text{the expression } (5)).$$

If $qA_1B_1 + (1-q)A_1B_2 - A_2B_2 = 0$ (see (5)), then we have the case (7) of center from [5]. If $A_2 = 0$, then we have a subcase of the case (1) from [5]. It is enough to show that, if $A = -i$, $i = 1, \ldots, q-2$, then we have a center.

Let $A = -i$. We can assume that $A_1 = 1$. We calculate successively the H_j's. We get

$$H_1 = (1 + x)^{2(i-q)} P_{q-i-1}(x)$$

where P_{q-i-1} is a polynomial of the indicated degree. By induction we show that

$$H_j = (1 + x)^{(j+1)(i-q)} P_{j(q-i-1)}(x)$$

(in the proof we obtain integrals of the form $\int^x s^{-k} R_m(s)\, ds$ with $m < k+1$).

This completes the proof of Proposition 1. ■

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