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Conformal measures for rational functions revisited

by

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Abstract. We show that the set of conical points of a rational function of the Riemann sphere supports at most one conformal measure. We then study the problem of existence of such measures and their ergodic properties by constructing Markov partitions on increasing subsets of sets of conical points and by applying ideas of the thermodynamic formalism.

Introduction. In this paper we recall from [U2] the notion of conical points and analyze some of its aspects. The idea of conical points has been used implicitly in [DU2], [DU3], [U1], [U3] and other papers of Denker and Urbański. Recently this idea has been used for example in [BMO] to study conformal measures and in [MM] to characterize the Hausdorff dimension and the Poincaré exponent of the Julia sets for certain rational functions. Note that McMullen used the term "radial Julia set" instead of "conical limit set" in analogy with Kleinian groups.

We would also like to remark that our approach here is one possible means for examining these notions in the case of parabolic or "geometrically finite" rational maps, that is, those whose Julia sets contain no critical points but some rationally indifferent periodic points. In fact, in these cases (and others also) our construction shows that the h-dimensional Hausdorff measure, where h is the Hausdorff dimension of the Julia set, is supported on the conical set. From this it is not so hard to show that the dimension of the conical set equals the dimension of the measure, hence also equals the Poincaré exponent defined by McMullen and the dimension of the Julia set.

1. Conical points. Let $f:\overline{\mathbb{C}}\to\overline{\mathbb{C}}$ be a rational function of degree $d\geq 2$. Following [U2], by analogy with the theory of Kleinian groups, we call

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a point z in $J(f)$, the Julia set of f, a *conical point* of f if there exist $\delta > 0$ and an infinite increasing sequence $n_k \geq 1$ of positive integers such that for each k, there exists $f_z^{-n_k}$, a holomorphic inverse branch of f^{n_k}, defined on the disk $B(f^{n_k}(z), \delta)$ and sending $f^{n_k}(z)$ to z. If we want to be more specific we call z a *δ-conical point* to keep track of the radii of the balls around the iterates $f^{n_k}(z)$. We denote the set of conical points by $\mathrm{Con}(f)$ and the set of δ-conical points by $\mathrm{Con}(f, \delta)$. Other alternative definitions of conical points have been later on provided by P. Jones (oral communication), F. Przytycki (see [Pr4]) and Lyubich and Minsky (see [LM]).

Let us begin with some comments concerning conical points. If z is a periodic repelling point, then there is some δ such that for every n, there is a holomorphic inverse branch, f_z^{-n}, defined on the ball $B(f^n(z), \delta)$ and sending $f^n(z)$ to z. Thus, in this case z is a conical point and we may take $n_k = kp$, where p is the period of the point. If z is a δ-conical point and there is a critical point $c \in \omega(z)$, then the corresponding sequence n_k must have gaps of arbitrarily large length.

To see this suppose to the contrary that the gaps are bounded by some constant b. Now, there exists a positive integer n (in fact infinitely many of them) such that $|f^n(z) - c| < \delta \|f'\|^{-b}$, where the supremum norm $\|\cdot\|$ is taken with respect to the spherical metric. Consider the only subscript k such that $n_{k-1} < n \leq n_k$. Then

$$f_{f^n(z)}^{-(n_k-n)}(B(f^{n_k}(z), \delta)) \supset B(f^n(z), \delta \|f'\|^{-(n_k-n)}) \supset B(f^n(z), \delta \|f'\|^{-b}).$$

Since this last set contains the critical point c, we have a contradiction which finishes the argument.

Let PC be the closure of the post-critical set. If $z \in J(f)$ and $\omega(z)$ is not a subset of PC, then z is a conical point. To see this note that there is some $\varepsilon > 0$ and a sequence n_k such that $\mathrm{dist}(f^{n_k}(z), \mathrm{PC}) \geq \varepsilon$. So, by the monodromy theorem there is a holomorphic branch $f_z^{-n_k}$ defined on the ball $B(f^{n_k}(z), \varepsilon)$ such that $f_z^{-n_k}(f^{n_k}(z)) = z$. In particular, note that if the post-critical set is not dense in $J(f)$, then each transitive point is a conical point. This occurs for example for the maps $z^2 + c$, where c is real and $J(f)$ is not a subset of $\mathbb{R}$. As we mention in the course of the paper, for every invariant ergodic measure with positive entropy almost every point of $J(f)$ is a conical point. Notice that the measure of maximal entropy is such a measure and therefore, there are always plenty of conical points. On the other hand, any preimage of a critical point of any order is not conical. So, if $\mathrm{PC} \neq \emptyset$, then there is a dense set of non-conical points.

Note that if f is parabolic, then all points of $J(f)$ other than the inverse images of parabolic periodic points are conical. In this case there exists a unique conformal measure with exponent equal to $\mathrm{HD}(J(f))$, the Hausdorff dimension of the Julia set. This measure is supported on the set of conical points (see [ADU]). On the other hand, for all exponents strictly greater than the Hausdorff dimension there also exist conformal measures and all these measures are supported on the complement of the conical points (see [DU2]). This discussion indicates that the property of being a conical point is rather delicate. One of our main goals is to examine conditions under which there is precisely one conformal measure supported on the set of conical points. We prove here that there is always at most one such conformal measure.

Given $t \geq 0$ we say that a Borel probability measure m supported on $J(f)$ is *t-conformal* provided

$$m(f(A)) = \int_A |f'|^t \, dm$$

for all Borel sets $A \subset J(f)$ such that $f : A \to f(A)$ is 1-to-1.

Let us now collect some properties of conical points.

LEMMA 1.1. *The set of conical points is a Borel set, in fact it is a $G_{\delta\sigma}$-set.*

Proof. Given $\delta > 0$ and an integer $n \geq 1$ let $F_n(\delta)$ be the union of all connected components C of $f^{-n}(B(z, \delta/2))$, $z \in J(f)$, such that $\widetilde{C}$, the only connected component of $f^{-n}(B(z, \delta))$ containing C, is disjoint from the set of critical points of f^n. Since for every δ,

$$\mathrm{Con}(f, \delta) \subset F(\delta) = \bigcap_{n \geq 1} \bigcup_{k \geq n} F_k(\delta) \subset \mathrm{Con}(f, \delta/2),$$

it follows that $\mathrm{Con}(f) = \bigcup_{n \geq 1} F(1/n)$. Since all the sets $F(\delta)$ are G_δ, the proof is complete. ■

It follows from [DU1] and [Pr1] that $\mathrm{HD}(\mathrm{Con}(f)) = \mathrm{DD}(J(f)) = e(f)$, where $\mathrm{DD}(J(f))$ is the *dynamical dimension* of $J(f)$ defined as the supremum of the dimensions of f-invariant ergodic probability measures of positive entropy and $e(f)$ is the minimal exponent allowing a conformal measure. It follows from [PU] that $\mathrm{DD}(f)$ coincides with the hyperbolic dimension introduced in [Sh]. In the case of rational functions with no recurrent critical points in $J(f)$ (they include hyperbolic, subhyperbolic, and parabolic maps) $\mathrm{Con}(f)$ is the whole Julia set with a countable set formed by all the inverse images of critical points and rationally neutral periodic points (see [U1], comp. [ADU]) deleted. Moreover, in this case there exists a unique conformal measure supported on the set of conical points. Conformal measures concentrated on the set of conical points also exist for some subclasses of Collet–Eckmann maps (see [Pr2] and [Pr3]).

Recall that a Borel σ-finite measure μ supported on $J(f)$ is said to be *ergodic* if all f-invariant sets on $J(f)$ (a set $A \subset J(f)$ is *f-invariant* if $f^{-1}(A) = A$) are of measure 0 or their complements are of measure 0, and μ is said to be *conservative* if $\sum_{n \geq 0} 1_A \circ f^n = \infty$ μ-a.e. for all Borel sets A

of positive measure. Of course, by the Poincaré recurrence theorem every finite f-invariant measure is conservative, but if finiteness is relaxed, this implication may fail; we will return to this point in Theorem 2.9. Let us also mention that if f-invariance is relaxed, the implication may also fail. In fact, there are non-conservative t-conformal measures, e.g., in the parabolic case for any t larger than the Hausdorff dimension.

Our main result in this section is the following.

THEOREM 1.2. *There exists at most one value of t for which a t-conformal measure exists and is supported on the set of conical points of f. Additionally, for such a t there is exactly one t-conformal measure supported on the set of conical points of f.*

Proof. Let m be a t-conformal measure and let z be a δ-conical point. First, using a normal family argument, we observe that there is a subsequence n_k of the sequence associated with z as a conical point such that $\lim_{k\to\infty} \operatorname{diam}(f_z^{-n_k}(B(f^{n_k}(z),\delta))) = 0$. In view of the Koebe distortion theorem, there are constants $C > 0$ and $0 < \eta \le 1/2$ depending on δ such that

$$f_z^{-n_k}(B(f^{n_k}(z),\eta\delta)) \subset B(z, C|(f^{n_k})'(z)|^{-1}\delta) \subset f_z^{-n_k}(B(f^{n_k}(z),\delta/2)).$$

Set $r_k(z) = C|(f^{n_k})'(z)|^{-1}\delta$. Since by topological exactness of f on the Julia set, the measure m is positive on non-empty open sets, using the above two inclusions and employing conformality of the measure m along with the Koebe distortion theorem, we see there is a constant $C_\delta \ge 1$ such that

$$C_\delta^{-1} \le \frac{m(B(z, r_k(z)))}{r_k(z)^t} \le C_\delta. \tag{1.1}$$

Since z is a conical point, $\lim_{k\to\infty} |(f^{n_k})'(z)| = \infty$ and consequently

$$\lim_{k\to\infty} r_k(z) = 0. \tag{1.2}$$

Now, formulas (1.1) and (1.2) show that if we have two conformal measures m_t and m_s with two distinct exponents t and s respectively (say $s > t$), then $m_s(\operatorname{Con}(f,\delta)) = 0$ for all $\delta > 0$ and consequently $m_s(\operatorname{Con}(f)) = 0$. This proves the first part of our theorem.

Notice that formulas (1.1) and (1.2) also show that any two t-conformal measures restricted to the set of δ-conical points are equivalent. Since $\operatorname{Con}(f) = \bigcup_{n\ge1} \operatorname{Con}(f, 1/n)$, any two such measures are equivalent. Now, suppose a t-conformal measure μ supported on $\operatorname{Con}(f)$ is not ergodic. Then $\operatorname{Con}(f) = A \cup B$ where $A \cap B = \emptyset$ and $\mu(A) \ne 0$, $\mu(B) \ne 0$ and A, B are invariant. Then after normalization we obtain two t-conformal measures: $\mu_1 = (\mu|_A)/\mu(A)$ and $\mu_1 = (\mu|_B)/\mu(B)$ which are mutually singular. This contradicts the statement above: any two t-conformal measures on $\operatorname{Con}(f)$

are equivalent. Thus, every t-conformal measure on $\mathrm{Con}(f)$ must be ergodic. This implies there can only be one t-conformal measure supported on $\mathrm{Con}(f)$. ∎

2. Markov partitions and associated maps. There already exists a fairly rich flow of papers aiming toward exhibiting and understanding various quasi-Markovian properties of rational functions. In what follows we provide a partial contribution toward this end by further developing some ideas contained in [DNU] and [MU]. In particular, we focus on the subset X_0 of conical points with some natural dynamical properties. We begin by recalling [DU, Lemma 7] (comp. also [Ma]):

Fix an ergodic invariant probability measure μ of positive entropy. Let $1 > \lambda > 0$. Then there exist an integer $m \geq 1$, $C > 0$, an open topological disk U containing no critical values of f up to order m and analytic inverse branches $f_i^{-mn} : U \to \overline{\mathbb{C}}$ of f^{mn} ($i = 1, \ldots, k_n \leq d^{nm},\ n \geq 0$), satisfying:

$$\forall_{n\geq 0}\ \forall_{1\leq i\leq k_{n+1}}\ \exists_{1\leq j\leq k_n}\quad f^m \circ f_i^{-m(n+1)} = f_j^{-mn}, \tag{2.1}$$

$$\mathrm{diam}(f_i^{-mn}(U)) \leq c\lambda^n \quad \text{for } n = 0, 1, \ldots \text{ and } i = 1, \ldots,\ k_n, \tag{2.2}$$

(2.3) for each fixed $n \geq 1$, for all $i = 1, \ldots, k_n$ the sets $\overline{f_i^{-mn}(U)}$ are pairwise disjoint and $\overline{f_i^{-mn}(U)} \subset U$,

$$\mu\Big(\bigcup_{n=1}^{\infty}\bigcup_{i=1}^{k_n} f_i^{-mn}(U)\Big) = 1. \tag{2.4}$$

In the sequel, in order to simplify exposition, we will take $m = 1$. In what follows we suppress the dependence of this construction upon μ and U unless otherwise noted.

Remark 2.1. It follows from the proof of [DU, Lemma 7] that there exists $K \geq 1$ such that for every $i = 1, \ldots, k_1$, every $n \geq 2$, every $j = 1, \ldots, k_n$ such that $f^{n-1}(f_j^{-n}(U)) = f_i^{-1}(U)$, and every pair of points $x, y \in f_i^{-1}(U)$ we have

$$\frac{|(f_j^{-n} \circ f)'(y)|}{|(f_j^{-n} \circ f)'(x)|} \leq K.$$

Let us also state as a lemma the following consequence of (2.1) and (2.3).

Lemma 2.2. *For each n, let $\mathcal{N}_n = \bigcup\{f_j^{-n}(U) : j = 1, \ldots, k_n\}$ and let $\mathcal{N} = \bigcup \mathcal{N}_n$. Then $\mathcal{N}$ is a net, i.e. any two sets in $\mathcal{N}$ are either disjoint or one is a subset of the other.*

Set now

$$U_\infty = \bigcup_{n=1}^{\infty}\bigcup_{i=1}^{k_n} f_i^{-n}(U).$$

We define inductively a partition γ_0 of U_∞ into elements of the form $f_i^{-n}(U)$, $n \geq 1$, $i = 1, \ldots, k_n$. First, all the sets $f_i^{-1}(U)$ are in γ_0 and secondly $f_i^{-n}(U) \in \gamma_0$, $n \geq 2$, if and only if $f_i^{-n}(U) \cap \bigcup_{j=1}^{n-1} \bigcup_{i=1}^{k_j} f_i^{-j}(U) = \emptyset$. Notice that

$$\bigcup_{A \in \gamma_0} A = U_\infty,$$

since by the net property (see Lemma 2.2) either $f_i^{-n}(U) \cap \bigcup_{j=1}^{n-1} \bigcup_{l=1}^{k_j} f_l^{-j}(U) = \emptyset$ or $f_i^{-n}(U) \subset \bigcup_{j=1}^{n-1} \bigcup_{l=1}^{k_j} f_l^{-j}(U)$.

The partition γ_0 gives rise to a map $F_0 : U_\infty \to U$ as follows: take $x \in U_\infty$ and consider the unique element $\gamma_0(x) \in \gamma_0$ such that $x \in \gamma_0(x)$. By the definition of U_∞, there exists a minimal $j \geq 1$ such that $f^j(\gamma_0(x)) \in \gamma_0 \cup \{U\}$. We now define $F_0(x)$ to be $f^j(x)$ and we set

$$X_0 = J(f) \cap \bigcap_{n=0}^{\infty} F_0^{-n}(U_\infty) = \bigcap_{n=0}^{\infty} F_0^{-n}(U_\infty).$$

Then $F_0(X_0) \subset X_0$ and we may consider the dynamical system $F_0 : X_0 \to X_0$. To see that $\bigcap_{n=0}^{\infty} F_0^{-n}(U_\infty) \subset J(f)$, notice that by (2.2) for each $\varepsilon > 0$, if n is sufficiently large, then $(f_i^{-n}(U))$ lies in the ε-neighborhood of $J(f)$.

Notice also that $X_0 = X_0(\mu)$ is a subset of the set of conical points of f and by (2.4), $\mu(X_0) = 1$. It is a G_δ set by construction. Also, note that if μ has full support (for example if μ is the measure of maximal entropy) then $X_0(\mu)$ is dense in $J(f)$. In particular, if the conformal measure admits an equivalent invariant measure μ, then the conformal measure is supported on the set $X_0(\mu)$. Examples of such maps can be found for instance in [ADU], [Pr3], and [U3].

Finally, X_0 may be a proper subset of the set of conical points. This is the case for example for the map $z \mapsto z^2$, where we take U to be the bounded component of the complement of $[0, 1+\varepsilon] \cup H \cup G$, where H is the circle centered at the origin with radius $3/2$ and G is the closed disk centered at the origin with radius $1/2$. In fact, in this case, the dyadic points on the unit circle are not included in X_0.

At this moment we want to raise two problems.

Problem A. Does there always exist a conformal measure supported on the set of conical points?

Problem B. Suppose that for a conformal measure m the set of conical points is of measure 1. Is it true that $m(X_0(\mu)) = 1$ for some ergodic invariant measure μ of positive entropy?

If we keep the same symbol γ_0 for the partition $\gamma_0|_{X_0}$, property (2.3) along with our construction gives the following.

LEMMA 2.3. *The partition γ_0 is a Markov partition for the dynamical system $F_0 : X_0 \to X_0$, i.e., the image of any element of γ_0 under F_0 can be represented as a union of some elements of γ_0. Additionally, if $x \in \bigcup_{n=2}^{\infty} \bigcup_{i=1}^{k_n} f_i^{-n}(U)$, then $F_0(\gamma_0(x)) \in \gamma_0$.*

Proof. By the construction of γ_0 and F_0 the second part of Lemma 2.3 is obvious. Now, we only need to remark that for every $i = 1, \ldots, k_1$, we have $F_0(X_0 \cap f_i^{-1}(U)) = X_0$ and X_0 is the union of all elements of γ_0. ∎

Observe that although we have gained a Markov property, the map F_0 may fail to have Rényi's property (distortion) because the elements of γ_0 may accumulate arbitrarily close to the boundary of U and consequently arbitrarily close to the critical values of order 1 of f. In order to remedy this possible failure we introduce below a family of induced maps F_k as follows. Given $k \ge 1$ and $x \in X_0$, let

$$N_k(x) = \min\{j \ge 1 : F_0^j(\gamma_0(x)) \in \{f_i^{-s}(U) : 1 \le s \le k,\ i = 1, \ldots, k_s\}\}.$$

Set $E_k = \bigcup\{f_i^{-s}(U) : 1 \le s \le k,\ i = 1, \ldots, k_s\}$ and let

$$\begin{aligned} X_k &= X_0 \cap E_k \\ &= \{x \in X_0 \cap E_k : N_k(F_0^n(x)) < \infty \text{ for infinitely many } n\text{'s}\}. \end{aligned}$$

This last equality holds since if $x \in X_k$, then its forward trajectory under F_0 must pass through E_1 and consequently E_k infinitely often.

Finally, we define the induced map $F_k : X_k \to X_k$ by setting

$$F_k(x) = F_0^{N_k(x)}(x).$$

We also introduce a partition γ_k of X_k corresponding to F_k as follows:

$$\gamma_k = \bigcup_{l \ge 0} (\gamma)_0^l \cap N_k^{-1}(l)|_{X_k},$$

where $(\gamma)_0^l = \bigvee_{j=0}^{l} F_0^{-j}(\gamma_0)$. We then have

LEMMA 2.4. *Fix $k \ge 1$ and suppose that $X_k \ne \emptyset$. Then the system (X_k, F_k, γ_k) is a Markov system with the bounded distortion property in the sense that there exists a constant $K_k \ge 1$ such that*

$$|(F_k^n)'(y)| \le K_k |(F_k^n)'(x)|$$

for all $n \ge 1$, $G \in (\gamma)_0^n$ and all $x, y \in G$.

Proof. This lemma follows immediately from Lemma 2.3 which is responsible for the Markov property along with Remark 2.1 and the fact that the number of sets of the form $f_i^{-s}(U)$, $1 \le s \le k$, $i = 1, \ldots, k_s$, is finite, which are responsible for bounded distortion. ∎

For each $k \geq 0$ and $t \geq 0$, define the *topological pressure* $\mathrm{P}_k(t)$ of the system F_k with respect to the potential $-\log|F_k'|$ as follows:

$$\mathrm{P}_k(t) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{\nu \in \nu_k(n)} \|(F_{k,\nu}^{-n})'\|^t,$$

where $\nu_k(n)$ enumerates all the inverse branches of F_k^n. The limit exists since for each k, the sequence $a_k(n) = \log \sum_{\nu\in\nu_k(n)} \|(F_{k,\nu}^{-n})'\|^t$ is subadditive. Notice that $\mathrm{P}_k(t)$ is convex, continuous in its domain of finiteness, and strictly decreasing on that domain by (2.1). Using the Koebe distortion theorem we obtain $a_k(m+n) \geq a_k(m) + a_k(n) - t\log K_k$. So, we conclude that $\mathrm{P}_k(t) \geq a_k(1) - t\log K_k > -\infty$. Since $\sum_{\nu\in\nu_k(n)} \|(F_{k,\nu}^{-n})'\|^2 \leq K_k^2 \times$ (area of U) $< \infty$, we get $\inf\{t : \mathrm{P}_k(t) < \infty\} \leq 2$. Following [MU], we denote this infimum by $\theta = \theta(F_k)$. In fact, we have the following little lemma.

LEMMA 2.5. *For each k, $\theta(F_k) = \inf\{t : \sum_{\nu\in\nu_k(1)} \|(F_{k,\nu}^{-1})'\|^t < \infty\}$ and $\theta = \theta(F_k)$ is independent of k.*

Proof. The first statement immediately follows from subadditivity estimates from above and below of the sequence $a_k(n) = \log \sum_{\nu\in\nu_k(n)} \|(F_{k,\nu}^{-n})'\|^t$.

In order to see why the second statement is true observe that the series $\sum_{\nu\in\nu_k(1)} \|(F_{k,\nu}^{-1})'\|^t$ and $\sum_{\nu\in\nu_{k+1}(1)} \|(F_{k+1,\nu}^{-1})'\|^t$ actually differ by only finitely many summands. To be more precise, if F_ν^{-1} is an inverse branch of F_k (resp. F_{k+1}) defined on an element of the form $f_i^{-s}(U)$, $1 \leq s \leq k-1$, $i = 1, \ldots, k_s$, then it is simultaneously an inverse branch of F_{k+1} (resp. F_k). If now $F_{k,\nu}^{-1}$ is an inverse branch defined on an element $f_i^{-k}(U)$, $1 \leq i \leq k_k$, then $F_{k,\nu}^{-1}(f_i^{-k}(U)) \subset f_i^{-1}(U)$ for some $i = 1, \ldots, k_1$ and $F_{k,\nu}^{-1} = F_{k+1,\mu}^{-1} \circ f_j^{-1}$ for some μ and $j = 1, \ldots, k_1$. If in turn $F_{k+1,\nu}^{-1}$ is an inverse branch defined on $f_i^{-k+1}(U)$, $i = 1, \ldots, k_{k+1}$, then $F_0|_{f_i^{-(k+1)}(U)}$ is a composition of at most k mappings f and $F_{k+1,\nu}^{-1} \circ (F_0|_{f_i^{-(k+1)}(U)})^{-1}$ is an inverse branch of F_k.

Therefore, the only inverse branches of F_{k+1} which do not correspond to any inverse branches of F_k are of the form $(f|_{f_i^{-(k+1)}(U)})^{-1}$, where $1 \leq i \leq k_{k+1}$ and $f_i^{-(k+1)}(U) \in \gamma_0$, and there are only finitely many of them. The proof is finished. ∎

LEMMA 2.6. *If $\mathrm{P}_k(t) < \infty$, then there exists a $|F_k'|^t e^{\mathrm{P}_k(t)}$-conformal measure for $F_k : X_k \to X_k$.*

Proof. The proof employs the Perron–Frobenius argument and most directly the reasoning given in [MU]. Indeed, for every bounded function $\phi : \overline{E}_k \to \mathbb{R}$ define $\mathcal{L}(\phi) : \overline{E}_k \to \mathbb{R}$ by setting

$$\mathcal{L}(\phi)(x) = \sum_{\nu\in\nu_k(1,x)} |F_{k,\nu}^{-1}(x)|^t e^{-\mathrm{P}_k(t)} \phi(F_k^{-1}(x)),$$

where the summation is taken over all inverse branches of F_k that are defined on that element of the family $\{f_i^{-s}(U) : 1 \le s \le k,\ 1 \le i \le k_s\}$ which contains x.

Denote by $\mathcal{L}^* : C(\overline{E}_k) \to C(\overline{E}_k)$ the operator dual to $\mathcal{L}$. Consider the continuous map $\mu \mapsto \mathcal{L}^*(\mu)/\mathcal{L}^*(\mu)(1)$ defined on the space of Borel probability measures on $\overline{E}_k$ treated as a subspace of $C(\overline{E}_k)^*$. In view of the Schauder–Tikhonov theorem, this map has a fixed point, say m. Writing $\lambda = \mathcal{L}^*(m)(1)$ we thus have $\mathcal{L}^*(m) = \lambda m$ and consequently $\mathcal{L}^{*n}(m) = \lambda^n m$ for every $n \ge 1$. Fix $\varepsilon > 0$. Then for every n large enough, $\sum_{\nu\in\nu_k(n)} \|(F_{k,\nu}^{-n})'\|^t \le e^{(\mathrm{P}_k(t)+\varepsilon)n}$. Therefore, for those n,

$$\lambda^n = \lambda^n m(1) = \mathcal{L}^{*n}(m)(1) = \int \mathcal{L}^n(1)\,dm$$
$$\le \int \sum_{\nu\in\nu_k(n)} \|(F_{k,\nu}^{-n})'\|^t e^{-\mathrm{P}_k(t)n}\,dm = \sum_{\nu\in\nu_k(n)} \|(F_{k,\nu}^{-n})'\|^t e^{-\mathrm{P}_k(t)n} \le e^{\varepsilon n}.$$

Hence, letting $n \to \infty$, we get $\lambda \le e^{\varepsilon}$, and letting $\varepsilon \searrow 0$ gives $\lambda \le 1$.

In order to get the opposite inequality, first notice that if $x \in X_k$, then $f_i^{-s}(x) \in X_k$ for every $s = 1,\dots,k$, and every $i = 1,\dots,k_s$. This enables us to show that $m(A) > 0$ for every set A of the form $f_i^{-s}(U)$, $1 \le s \le k$, $i = 1,\dots,k_s$. Indeed, since $X_k \subset \bigcup_{s=1}^k \bigcup_{i=1}^{k_s} \overline{f_i^{-s}(U)}$ and $m(\overline{E}_k) = 1$, there exist $1 \le j \le k$ and $1 \le l \le k_j$ such that $m(\overline{f_i^{-j}(U)}) > 0$. Set $A = f_i^{-j}(U)$. Then

$$m(A) = m(1_A) = \lambda^{-1} m(\mathcal{L}(1_A))$$
$$\ge \lambda^{-1} \int_{\overline{f_i^{-j}(U)}} \sum_{\nu\in\nu_k(1,x)} |(F_{k,\nu}^{-1})'(x)|^t e^{-\mathrm{P}_k(t)} 1_A(F_{k,\nu}^{-1}(x))\,dm(x).$$

Since $m(\overline{f_i^{-j}(U)}) > 0$ and $\sum_{\nu\in\nu_k(1,x)} |(F_{k,\nu}^{-1})'(x)|^t e^{-\mathrm{P}_k(t)} 1_A(F_{k,\nu}^{-1}(x)) > 0$ for every $x \in \overline{f_i^{-j}(U)}$, we conclude that $m(A) > 0$. Since the family $\{\overline{f_i^{-s}(U)} : 1 \le s \le k,\ i = 1,\dots,k_s\}$ is finite and X_k is contained in its union, it follows from the definition of $\mathrm{P}_k(t)$ that there exists $1 \le s \le k$ such that

$$\mathrm{P}_k(t) = \limsup_{n\to\infty} \frac{1}{n} \log \sum_{\nu\in\nu_n(s,i)} \|(F_{k,\nu}^{-n})'\|^t,$$

where $\nu_n(s,i)$ enumerates all the inverse branches of F_k^n which begin with f_i^{-s}. Therefore, fixing $\varepsilon > 0$, taking $n \ge 1$ sufficiently large and using Lemma 2.3, we get

$$\lambda^n = \int \sum_{\nu\in\nu_n(x)} |(F_{k,\nu}^{-n})'(x)|^t e^{-\mathrm{P}_k(t)n}\,dm(x)$$

$$
\begin{aligned}
&\geq \int_{f_i^{-s}(U)} \sum_{\nu\in\nu_n(x)} |(F_{k,\nu}^{-n})'(x)|^t e^{-\mathrm{P}_k(t)n}\, dm(x)\\
&\geq K^{-t} \sum_{\nu\in\nu_n(s,i)} \|(F_{k,\nu}^{-n})'\|^t e^{-\mathrm{P}_k(t)n} m(\overline{f_i^{-s}(U)})\\
&\geq K^{-t} e^{(\mathrm{P}_k(t)-\varepsilon)n} e^{-\mathrm{P}_k(t)n} m(\overline{f_i^{-s}(U)})\\
&= K^{-t} m(\overline{f_i^{-s}(U)}) e^{-\varepsilon n},
\end{aligned}
$$

where as above $\nu_n(x)$ enumerates all the inverse branches of F_k^n that are defined on that element of the family $\{f_i^{-s}(U) : 1 \leq s \leq k,\ 1 \leq i \leq k_s\}$ which contains x. Thus, $\log \lambda \geq -\varepsilon$ and letting $\varepsilon \to 0$, we get $\lambda = 1$.

Our aim now is to show that m is $|F_k'|^t e^{\mathrm{P}(t)}$-conformal. Indeed, for every set $A \subset \overline{f_i^{-s}(U)}$, $s = 1,\ldots,k$, $i = 1,\ldots,k_s$, and every inverse branch $F_{k,\nu}^{-n}$ of F_n defined on $f_i^{-s}(U)$, we have

$$
\begin{aligned}
m(F_{k,\nu}^{-n}(A)) &= \int_{\overline{E}_k} \sum_{q\in\nu_n(x)} |(F_{k,q}^{-n})'(x)|^t e^{-\mathrm{P}_k(t)n} 1_{F_{k,\nu}^{-n}(A)} \circ F_{k,q}^{-n}(x)\, dm(x)\\
&= \int_A |(F_{k,\nu}^{-n})'(x)|^t e^{-\mathrm{P}_k(t)n}\, dm(x),
\end{aligned}
$$

where the last equality holds since the sets $\overline{f_i^{-s}(U)}$ are mutually disjoint. Thus the conformality requirement is satisfied and we only need to show that $m(X_k) = 1$. In order to do this put $U_n = \bigcup_{\nu\in\nu_n} \overline{f_i^{-s}(U)}$. Then $1_{U_n} \circ F_{k,\nu}^{-n}(x) = 1$ for all $\nu \in \nu_k(n)$ and all $x \in U$. Therefore

$$
\begin{aligned}
m(U_n) &= \int_{\overline{E}_k} \sum_{\nu\in\nu_n(x)} |(F_{k,\nu}^{-n})'(x)|^t e^{-\mathrm{P}_k(t)n} 1_{U_n} \circ F_{k,\nu}^{-n}(x)\, dm(x)\\
&= \int_{\overline{E}_k} \sum_{\nu\in\nu_n(x)} |(F_{k,\nu}^{-n})'(x)|^t e^{-\mathrm{P}_k(t)n}\, dm(x) = \int_{\overline{E}_k} 1\, dm = 1.
\end{aligned}
$$

Since U_n is a descending family and $\bigcap_{n\geq 1} U_n = X_k$, we conclude that $m(X_k) = 1$. The proof is finished. ■

Lemma 2.6 makes up the central component of the following main result of this section.

THEOREM 2.7. *For each k and t, there exists exactly one $|F_k'|^t e^{\mathrm{P}_k(t)}$-conformal measure for $F_k : X_k \to X_k$ and $\mathrm{P}_k(t)$ is the only number α_t which admits a $|F_k'|^t e^{\alpha_t}$-conformal measure. There also exists exactly one F_k-invariant probability measure μ_k absolutely continuous with respect to m. This measure is ergodic, positive on non-empty open sets and equivalent to m.*

Proof. First notice that Lemma 2.2 easily implies topological exactness of $F_k : X_k \to X_k$ and this along with the conformality condition implies that any conformal measure is positive on non-empty open sets of X_k. In particular, all the sets $f_i^{-s}(U)$, $1 \le s \le k$, $1 \le i \le k_s$, have positive measure. Therefore using the conformality condition and the bounded distortion property we easily deduce the existence of a constant $C \ge 1$ such that if m_1 and m_2 are respectively $|F_k'|^t e_1^\alpha$- and $|F_k'|^t e_2^\alpha$-conformal measures then

$$e^{(\alpha_1-\alpha_2)n}C^{-1} \le \frac{m_2(F_{k,\nu}^{-n}(U))}{m_1(F_{k,\nu}^{-n}(U))} \le Ce^{(\alpha_1-\alpha_2)n}$$

for all $n \ge 1$ and all the inverse branches of F_k^n. Using now Besicovitch covering theorem type arguments, we conclude that $\alpha_1 = \alpha_2$ and the measures m_1 and m_2 are equivalent with bounded Radon–Nikodym derivatives. Now, Rényi's condition along with topological exactness imply that each conformal measure has an equivalent F_k-invariant measure. Since all such measures must also be mutually equivalent, there can exist at most one such measure. Invoking now Lemma 2.6 finishes the proof. ■

THEOREM 2.8. *If* $\mathrm{P}_k(t) = 0$ *for some* $k \ge 1$, *then* $\mathrm{P}_n(t) = 0$ *for all* $n \ge k$ *and* $m_n|_{X_k}$ *coincides with* m_k *up to a multiplicative constant. Moreover, there exists a* σ-*finite* $|F_0'|^t$-*conformal measure for* $F_0 : X_0 \to X_0$ *which is finite on all the sets* X_n *and whose restriction to* X_n *coincides with* m_n *up to a multiplicative constant.*

Proof. Since $X_n \cap E_k = X_k \cap E_k = X_k$ and $m_n(X_n \cap E_k) > 0$ and since F_k restricted to any atom of its Markov partition can be expressed as a composition of at most two mappings F_n, using the chain rule we conclude that after normalization $m_n|_{X_k}$ is $|F_k'|^t$-conformal. Thus applying Theorem 2.7 finishes the proof of the first part.

In order to prove the second part consider the sequence of measures m_n', $n \ge k$, defined inductively as follows: $m_k' = m_k$ and $m_{n+1} = c_{n+1}m_{n+1}$, where c_n is chosen such that $m_{n+1}'|_{X_n} = m_n'$. Thus the formula $\nu(A) = m_n'(A)$ for $A \subset X_n$ defines a measure on $\bigcup_{n\ge k} X_n = X_0$ which has all the required properties. ■

As a converse to Theorem 2.8 we prove the following.

THEOREM 2.9. *Up to a multiplicative constant, there exists at most one* σ-*finite* $|F_0'|^t$-*conformal measure for* $F_0 : X_0 \to X_0$ *which is finite on all the sets* X_k, $k \ge 1$. *If such a measure exists, then* $\mathrm{P}_k(t) = 0$ (*so such a* t *is also uniquely determined*) *for all* $k \ge 1$ *and* m *is conservative ergodic with respect to* F_0.

Proof. All the claims of this theorem except ergodicity and conservativity follow immediately from Theorem 2.7 combined with the remark that for every $k \geq 1$, $m|_{X_k}$ is $|F_k'|^t$-conformal for $F_k : X_k \to X_k$. Conservativity and ergodicity follow now from the fact that $X_0 = \bigcup_{k\geq 1} X_k$ and from the fact that all the maps $F_k : X_k \to X_k$ are conservative (conservativity of at least one of these maps would be sufficient for us as every point in X_0 visits X_1 under F_0 infinitely often), which in turn is a consequence of the second part of Theorem 2.7 producing F_k-invariant ergodic probability measures equivalent to $m|_{X_k}$. ∎

We finish the paper with the following problem.

Problem C. Suppose that m is a σ-finite $|F_0'|^t$-conformal measure for $F_0 : X_0 \to X_0$. Is m finite on the sets X_k, $k \geq 1$?

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