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A cut salad of cocycles

by

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Abstract. We study the centraliser of locally compact group extensions of ergodic probability preserving transformations. New methods establishing ergodicity of group extensions are introduced, and new examples of squashable and non-coalescent group extensions are constructed.

1. Introduction. Let T be an ergodic probability preserving transformation of the probability space $(X,\mathcal{B},m)$. Let $(G,\mathcal{T})$ be a locally compact, second countable, topological group ($\mathcal{T}=\mathcal{T}(G)$ denotes the family of open sets in the topological space G), and let $\varphi : X\to G$ be a measurable function.

The (left) *skew product* or G-*extension* $T_\varphi : X\times G\to X\times G$ is defined by

$$T_\varphi(x,y)=(Tx,\varphi(x)y).$$

The skew product preserves the measure $\mu=m\times m_G$ where m_G is left Haar measure on G. There is an ergodic skew product $T_\varphi : X\times G\to X\times G$ iff the group G is amenable (see [G-S], references therein, and [Z]). In this paper, we are mainly concerned with Abelian G. Recall that on any locally compact, Abelian, second countable topological group G, there is defined a *norm* $\|\cdot\|_G$ (satisfying $\|x\|=\|-x\|\ge 0$ with equality iff $x=0$, and $\|x+y\|\le\|x\|+\|y\|$) which generates the topology of G.

Recall that a measurable function $f:X\to G$ is called a T-*coboundary* if $f=(h\circ T)^{-1}h$ for some measurable function $h:X\to G$ and that measurable functions $f,g:X\to G$ are said to be T-*cohomologous*, written $f\overset{T}{\sim}g$, if there is $h:X\to G$ measurable such that $f=(h\circ T)^{-1}gh$. In case G is Abelian, $f\overset{T}{\sim}g$ iff $f-g$ is a T-coboundary.

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The centraliser. Recall that the *centraliser* of a non-singular transformation $R : X \to X$ is the collection of *commutors* of R, that is, non-singular transformations of X which commute with R. The collection of invertible commutors (the *invertible centraliser*) is denoted by $C(R)$.

We study those commutors Q of T_φ of the form

$$Q(x,y) = (Sx, f(x)w(y)) \tag{$*$}$$

where $w : G \to G$ is a surjective, continuous group endomorphism, S is a commutor of T, and $f : X \to G$ is measurable.

It is evident that Q of the form $(*)$ satisfies $T_\varphi \circ Q = Q \circ T_\varphi$ iff for a.e. $x \in X$,

$$S \circ T(x) = T \circ S(x), \qquad \varphi(Sx)f(x) = f(Tx)w(\varphi(x)).$$

It is shown in Proposition 1.1 of [A-L-M-N] that if T is a Kronecker transformation, and T_φ is ergodic, then every commutor of T_φ is of the form $(*)$.

Let $\mathrm{End}(G)$ denote the collection of surjective, continuous group endomorphisms of G (a semigroup under composition) and let

$$\mathcal{E}_\varphi = \{w \in \mathrm{End}(G) : \text{there is a commutor } Q \text{ of } T_\varphi \text{ of the form } (*) \text{ with } w = w_Q\},$$

a sub-semigroup of $\mathrm{End}(G)$. Evidently

$$\mathcal{E}_\varphi = \{w \in \mathrm{End}(G) : \text{there is a commutor } S \text{ of } T \text{ with } \varphi \circ S \overset{T}{\sim} w \circ \varphi\}.$$

The study of $\mathcal{E}_\varphi$ yields counterexamples:

• if $\mathcal{E}_\varphi$ contains non-invertible endomorphisms, then T_φ is not *coalescent*, i.e. its centraliser contains some non-invertible transformation (see [H-P]); and

• if $\mathcal{E}_\varphi$ contains endomorphisms which do not preserve m_G (a possibility only for non-compact G), then T_φ is *squashable*, i.e. its centraliser contains some non-singular transformation which is not measure preserving (see [A1] and below). Counterexamples like these (and others) will be discussed below.

Semigroup homomorphisms. Let $\mathcal{L}_\varphi$ denote the collection of those commutors S of T for which there is a commutor Q of T_φ of the form $(*)$ with $S = S_Q$. As can be easily seen,

$$\mathcal{L}_\varphi = \{S \text{ a commutor of } T : \text{there is } w \in \mathrm{End}(G) \text{ with } \varphi \circ S \overset{T}{\sim} w \circ \varphi\}.$$

When G is Abelian and T_φ is ergodic, there is a surjective semigroup homomorphism $\pi_\varphi : \mathcal{L}_\varphi \to \mathcal{E}_\varphi$ such that if $S \in \mathcal{L}_\varphi$, and Q is a commutor of T_φ of the form $(*)$ with $S = S_Q$, then $w_Q = \pi_\varphi(S)$. This result (called the *semigroup embedding lemma*) is proved at the end of this introduction.

It implies that $\mathcal{E}_\varphi$ is Abelian whenever the commutors of T form an Abelian semigroup, for instance when T is a Kronecker transformation.

It is shown in [A-L-V] that the restriction of π_φ to $L_\varphi(T)=\{S_Q : Q\in C(T_\varphi)$ of the form $(*)\}$ is continuous with respect to the relevant Polish topologies (cf. [G-L-S] for the case where G is compact).

The question arises when a homomorphism π from a sub-semigroup $\mathcal{S}$ of commutors of T into $\mathrm{End}(G)$ occurs in this manner. That is, when does there exist a measurable function $\varphi : X\to G$ such that T_φ is ergodic, $\mathcal{S}\subset\mathcal{L}_\varphi$, and $\pi=\pi_\varphi|_{\mathcal{S}}$?

In [L-L-T] it is shown that for an invertible, ergodic probability preserving transformation T with some invertible commutor S so that $\{S^mT^n : m,n\in\mathbb{Z}\}$ acts freely, and $G=\mathbb{T}$, there is $\varphi : X\to\mathbb{T}$ such that $S\in\mathcal{L}_\varphi$, $\mathcal{E}_\varphi\ni[x\mapsto 2x \bmod 1]$, and indeed, $\pi_\varphi(S)=[x\mapsto 2x \bmod 1]$. This includes the first example of a non-coalescent Anzai skew product (i.e. $\mathbb{T}$-extension of a rotation of $\mathbb{T}$).

The main results. We generalise this to all Abelian, locally compact, second countable G:

THEOREM 1. *Suppose that T is an ergodic probability preserving transformation, $d\le\infty$, and $S_1,\ldots,S_d\in C(T)$ $(d\le\infty)$ are such that $(T,S_1,\ldots,S_d)$ generate a free $\mathbb{Z}^{d+1}$ action of probability preserving transformations of X. If $w_1,\ldots,w_d\in\mathrm{End}(G)$ commute* (*i.e. $w_i\circ w_j=w_j\circ w_i$ for all $1\le i,j\le d$*), *then there is a measurable function $\varphi : X\to G$ such that T_φ is ergodic, and*

$$\varphi\circ S_i\overset{T}{\sim} w_i\circ\varphi\qquad(1\le i\le d)$$

(*in other words, $S_1,\ldots,S_d\in\mathcal{L}_\varphi$, $w_1,\ldots,w_d\in\mathcal{E}_\varphi$, and $\pi_\varphi(S_i)=w_i$ $(1\le i\le d)$*).

Theorem 1 can be applied to any Kronecker transformation T of an uncountable compact group.

THEOREM 2. *Suppose that T is an ergodic probability preserving transformation, and $\{S_t : t\in\mathbb{R}\}\subset C(T)$ are such that T and $\{S_t : t\in\mathbb{R}\}$ generate a free $\mathbb{Z}\times\mathbb{R}$ action of probability preserving transformations of X. There is a measurable function $\varphi : X\to\mathbb{R}$ such that T_φ is ergodic, and there is $g:\mathbb{R}\times X\to\mathbb{R}$ measurable* (*with respect to $m_{\mathbb{R}}\times m$*) *such that*

$$\varphi\circ S_t(x)-e^t\varphi(x)=g(t,Tx)-g(t,x),\tag{1}$$

$$g(t+u,x)=g(t,S_ux)+e^tg(u,x).\tag{2}$$

REMARKS. 1) If, under the conditions of Theorem 2, $Q_t(x,y):=(S_tx,e^ty+g(t,x))$, then $\{Q_t : t\in\mathbb{R}\}$ is a flow by (2), and $\{Q_t : t\in\mathbb{R}\}\subset C(T_\varphi)$ by (1). Indeed, $S_t\in\mathcal{L}_\varphi$, $w_t\in\mathcal{E}_\varphi$ where $w_t(y)=e^ty$, and $\pi_\varphi(S_t)=w_t$ for all $t\in\mathbb{R}$.

2) Theorem 1 can be extended (with analogous proof) to enable "realisation" of a semigroup homomorphism defined on a discrete, amenable

sub-semigroup of the centraliser which has Følner sets which tile (see [O-W]).

We show in §5 that the transformations T_φ constructed in Theorem 2 are isomorphic to Maharam transformations (Proposition 5.1), and we obtain $\mathbb{Z}$-extensions of Bernoulli transformations which are Maharam transformations (see the remarks after Proposition 5.1).

In §2 we give an application of Theorem 1 to infinite ergodic theory showing existence of pathological behaviour concerning laws of large numbers. We also show that ergodic $\mathbb{R}$-valued cocycles with $\mathcal{E}_\varphi \neq \{\mathrm{Id}\}$ are aperiodic.

The proofs of the main results are in §§3, 4.

Recall from [S] that the *essential values* of φ are defined by

$$E(\varphi) = \{a \in G : \forall A \in \mathcal{B}_+,\ a \in U \in \mathcal{T},\ \exists n \geq 1,\\ m(A \cap T^{-n}A \cap [\varphi_n \in U]) > 0\},$$

which is a closed subgroup of G. It is shown in [S] that T_φ is ergodic iff $E(\varphi) = G$.

The (more specific) conditions for ergodicity of skew products discussed in [A-L-M-N] and [L-V] are unsuitable for our constructions as they eliminate squashability. We need new conditions for the ergodicity of a measurable function $\varphi : X \to G$ which are flexible enough to allow $\mathcal{E}_\varphi \neq \{\mathrm{Id}\}$.

Such conditions, called *essential value conditions*, are introduced in §3.

The proofs of Theorems 1 and 2 are in §4. Cocycles are constructed as infinite sums of coboundaries. Each coboundary "contributes" a particular essential value condition, which the subsequent coboundaries are "too small" to destroy. The essential value conditions remaining for the infinite sum give its ergodicity.

This paper is a partial version of [A-L-V]. There is some overlap with the subsequent [D].

To conclude this introduction, we prove the

Semigroup Embedding Lemma. *Suppose that G is Abelian, and that $\varphi : X \to G$ is such that T_φ is ergodic. There is a surjective semigroup homomorphism*

$$\pi_\varphi : \mathcal{L}_\varphi \to \mathcal{E}_\varphi$$

such that if $Q(x, y) = (Sx, f(x) + w(y))$ defines a commutor of T_φ, then $w = \pi_\varphi(S)$.

Proof. We must show that if $S \in \mathcal{L}_\varphi$, $w_1, w_2 \in \mathcal{E}(G)$, $f_i : X \to G$ $(i = 1, 2)$ are measurable, and $Q_i(x, y) = (Sx, f_i(x) + w_i(y))$ are such that $Q_i \circ T_\varphi = T_\varphi \circ Q_i$ $(i = 1, 2)$, then $w_1 = w_2$.

To this end, let $U = w_1 - w_2$. Then $T_{U\circ\varphi}$ is an ergodic transformation of $X \times U(G)$ (being a factor of T_φ via $(x, y) \mapsto (x, U(y))$). The condition

$Q_i \circ T_\varphi = T_\varphi \circ Q_i$ means that

$$\varphi \circ S = w_i \circ \varphi + f_i \circ T - f_i \qquad (i = 1, 2),$$

whence

$$U \circ \varphi = g \circ T - g$$

where $g = f_1 - f_2$. Define $\widetilde{g} : X \to G/U(G)$ by $\widetilde{g}(x) = g(x) + U(G)$. It follows that $\widetilde{g} \circ T = \widetilde{g}$, whence by ergodicity of T, there is $\gamma \in G$ such that $\widetilde{g} = \gamma + U(G)$ a.e. Therefore $h := g - \gamma : X \to U(G)$ is measurable and satisfies

$$U \circ \varphi = h \circ T - h.$$

The ergodicity of $T_{U\circ\varphi}$ on $X \times U(G)$ now implies $U(G) = \{0\}$, i.e. $U \equiv 0$, or $w_1 = w_2$.

We have shown that for every $S \in \mathcal{L}_\varphi$, there is a unique $w =: \pi_\varphi(S) \in \mathcal{E}_\varphi$ such that there exists $f_S : X \to G$ measurable so that $Q(x, y) = (Sx, f_S(x) + \pi_\varphi(S)(y))$ defines a commutor of T_φ. The rest of the lemma follows easily from this. ■

2. Properties of some skew products T_φ with $\mathcal{E}_\varphi \neq \{\mathrm{Id}\}$

Laws of large numbers. Let $(X, \mathcal{B}, m, T)$ be a conservative, ergodic measure preserving transformation of the σ-finite measure space $(X, \mathcal{B}, m)$.

A *law of large numbers* for T with respect to $\mathcal{C} \subseteq \mathcal{B}$ is a function $L : \{0,1\}^{\mathbb{N}} \to [0, \infty]$ such that

$$L(1_A, 1_A \circ T, \ldots) = m(A) \qquad \text{a.e. for all } A \in \mathcal{C}.$$

Here, the intention is that $\mathcal{C}$ is either $\mathcal{B}$ or $\mathcal{F} := \{B \in \mathcal{B} : m(B) < \infty\}$.

PROPOSITION 2.1. *There exists a conservative*, *ergodic measure preserving transformation* $(X, \mathcal{B}, m, T)$ *which has a law of large numbers with respect to* $\mathcal{F}$, *but does not have a law of large numbers with respect to* $\mathcal{B}$.

P r o o f. Let $G = \mathbb{Z}^\infty = \{(n_1, n_2, \ldots) \in \mathbb{Z}^{\mathbb{N}} : n_k \to 0\}$ and let $w \in \mathrm{End}(G)$ be the shift $w((n_1, n_2, \ldots)) = (n_2, n_3, \ldots)$. Let T be a Kronecker transformation. Then there is $S \in C(T)$ so that $\{S, T\}$ generate a free $\mathbb{Z}^2$ action.

By Theorem 1, there exists $\varphi : X \to G$ such that T_φ is ergodic and $\varphi \circ S \overset{T}{\sim} w \circ \varphi$, whence there is a commutor Q of T_φ of the form $Q(x, y) = (Sx, f(x) + w(y))$ where $f : X \to G$ is measurable. Note that $m(Q^{-1}A) = |\mathrm{Ker}\, w| m(A) = \infty$ whenever $m(A) > 0$.

It follows that T_φ has no law of large numbers with respect to $\mathcal{B}$. To see this suppose otherwise that $L : \{0,1\}^{\mathbb{N}} \to [0, \infty]$ is such a law of large numbers and let $A \in \mathcal{B}$, $m(A) = 1$. Then $L(1_A(x), 1_A(Tx), \ldots) = m(A) = 1$

for a.e. $x \in X$, whence since Q is non-singular, for a.e. $x \in X$,

$$\begin{aligned}1 &= L(1_A(Qx), 1_A(TQx), \ldots) = L(1_{Q^{-1}A}(x), 1_{Q^{-1}A}(Tx), \ldots) \\ &= m(Q^{-1}A) = \infty.\end{aligned}$$

On the other hand, G does not have any finite subgroup other than $\{0\}$ whence by Corollary 2.3 and Theorem 3.4 of [A2], T_φ has a law of large numbers with respect to $\mathcal{F}$. ■

Eigenvalues. Recall that the measurable function $\varphi : X \to G$ is called *aperiodic* if all eigenfunctions for the skew product T_φ are eigenfunctions for T; that is, if $f : X \times G \to S^1 := \{\lambda \in \mathbb{C} : |\lambda| = 1\}$ is measurable and $f \circ T_\varphi = \lambda f$ where $\lambda \in S^1$, then there is $g : X \to S^1$ measurable such that $f(x,y) = g(x)$ a.e.

We prove

Proposition 2.2. *If $G = \mathbb{R}$ or $\mathbb{T}$, T_φ is ergodic, and $\mathcal{E}_\varphi \neq \{\mathrm{Id}\}$, then φ is aperiodic.*

Lemma 2.2. *Suppose that T_φ is ergodic and $f : X \times G \to S^1$ is measurable such that $f \circ T_\varphi = \lambda_0 f$ where $\lambda_0 \in S^1$. Then there is $f_0 : X \to S^1$ measurable and there is a unique $\gamma \in \widehat{G}$ such that $f = f_0 \otimes \gamma$ (that is, $f(x,y) = f_0(x)\gamma(y)$).*

Proof. For $Q \in C(T_\varphi)$, we have

$$(f \circ Q) \circ T_\varphi = f \circ T_\varphi \circ Q = \lambda_0 f \circ Q,$$

whence, by ergodicity of T_φ, there exists $\lambda(Q) \in S^1$ such that $f \circ Q = \lambda(Q) f$ (note that $\lambda(T_\varphi) = \lambda_0$). The mapping $\lambda(Q) : C(T_\varphi) \to S^1$ is a continuous homomorphism with respect to the natural Polish topologies.

Since $G \subset C(T_\varphi)$, we obtain $\gamma \in \widehat{G}$ by setting $\gamma(g) := \lambda(\sigma_g)$ where $\sigma_g(x,y) := (x, yg)$. Thus

$$f \circ \sigma_g = \gamma(g) f \qquad \forall g \in G.$$

Set $F(x,y) = \gamma(y)^{-1} f(x,y)$. Then $F \circ \sigma_g = F$ for all $g \in G$, whence (by ergodicity of right translation of G on itself) for a.e. fixed $x \in X$, $F(x,\cdot)$ is constant.

The unicity of γ follows from the ergodicity of T_φ: if $\gamma_i \in \widehat{G}$, $g_i : X \to G$ are measurable ($i = 1,2$) and $\lambda \in S^1$ is such that $g_i \otimes \gamma_i \circ T_\varphi = \lambda g_i \otimes \gamma_i$ ($i = 1,2$), then $\gamma(\varphi) = \overline{g} \circ Tg$ where $\gamma = \overline{\gamma}_1 \gamma_2$ and $g = \overline{g}_1 g_2$. It follows that $g \otimes \gamma \circ T_\varphi = g \otimes \gamma$, whence by ergodicity of T_φ, $g \otimes \gamma$ is constant and $\gamma \equiv 1$. ■

Remarks. 1) It follows from Lemma 2.2 that λ is an eigenvalue of the ergodic T_φ iff there is $\gamma \in \widehat{G}$ such that $\gamma(\varphi) \overset{T}{\sim} \lambda$ in S^1.

2) If T_φ is ergodic, then φ is aperiodic iff $\gamma(\varphi) \overset{T}{\sim} \lambda$ in S^1 implies $\gamma \equiv 1$.

LEMMA 2.3. *Suppose that T_φ is ergodic, $f = f_0 \otimes \gamma$ where $f_0 : X \to S^1$ is measurable, $\gamma \in \widehat{G}$, and $f \circ T_\varphi = \lambda_0 f$ for some $\lambda_0 \in S^1$. Then*

$$\gamma \circ w = \gamma \quad \forall w \in \mathcal{E}_\varphi.$$

Proof. By ergodicity of T_φ, for every $Q \in C(T_\varphi)$ there is $\lambda(Q) \in S^1$ such that $f \circ Q = \lambda(Q) f$.

Suppose that $w \in \mathcal{E}_\varphi$, and let Q be a commutor of T_φ with $Q(x, y) = (Sx, h(x)w(y))$. Then

$$\begin{aligned}\lambda(Q) f_0 \otimes \gamma(x, y) &= \lambda(Q) f(x, y) = f \circ Q(x, y)\\ &= f_0(Sx)\gamma(h(x))\gamma(w(y))\\ &= [(f_0 \circ S) \cdot (\gamma \circ h)] \otimes \gamma \circ w(x, y),\end{aligned}$$

and since the character $\gamma \in \widehat{G}$ appearing in the eigenfunction $f_0 \otimes \gamma$ is unique, we get $\gamma \circ w = \gamma$. ∎

Proof of Proposition 2.2. This now follows from Lemma 2.3, because if $G = \mathbb{T}$, $\mathbb{R}$, and $\gamma \in \widehat{G}$, $w \in \mathrm{End}(G)$, then $\gamma \circ w = \gamma$ iff either $\gamma \equiv 1$ or $w = \mathrm{Id}$. ∎

3. Essential value conditions. Let T be an invertible, ergodic probability preserving transformation of the standard probability space $(X, \mathcal{B}, m)$, let G be a locally compact, second countable Abelian group, and let $\varphi : X \to G$ be measurable. We develop here a countable condition for ergodicity of T_φ. The EVC's to be defined are best understood in terms of orbit cocycles, and the *groupoid* of T (see [F-M]).

A *partial probability preserving transformation* of X is a pair (R, A) where $A \in \mathcal{B}$ and $R : A \to RA$ is invertible and $m|_{RA} \circ R^{-1} = m|_A$. The set A is called the *domain* of (R, A). We sometimes abuse this notation by writing $R = (R, A)$ and $A = \mathcal{D}(R)$. Similarly, the *image* of (R, A) is the set $\Im(R) = RA$.

The *equivalence relation* generated by T is

$$\mathcal{R} = \{(x, T^n x) : x \in X,\ n \in \mathbb{Z}\}.$$

For $A \in \mathcal{B}(X)$ and $\phi : A \to \mathbb{Z}$, define $T^\phi : A \to X$ by $T^\phi(x) := T^{\phi(x)}x$. The *groupoid* of T is

$$[T] = \{T^\phi :\ T^\phi \text{ is a partial probability preserving transformation}\}.$$

It is not hard to see that $[T] = \{R : R$ is a partial probability preserving transformation with $(x, Rx) \in \mathcal{R}$ a.e.$\}$. For $R = T^\phi \in [T]$, write $\phi^{(R)} := \phi$. Let

$$[T]_+ = \{R \in [T] :\ \phi^{(R)} \ge 1 \text{ a.e.}\}.$$

Recall from [H]:

E. Hopf's Equivalence Lemma. *If T is an ergodic measure preserving transformation of $(X,\mathcal{B},m)$ and $A,B\in\mathcal{B}$ with $m(A)=m(B)$, then there is $R\in[T]_+$ such that $\mathcal{D}(R)=A$ and $\Im(R)=B$.*

We also need a quantitative version of this lemma when $A=B$.

Lemma 3.1. *Suppose that T is an ergodic probability preserving transformation of $(X,\mathcal{B},m)$, $A\in\mathcal{B}_+$, and $c,\varepsilon>0$. Then for all $p,q\in\mathbb{N}$ large enough, there is $R\in[T]_+$ such that*

$$\mathcal{D}(R),\Im(R)\subset A,\qquad m(A\setminus\mathcal{D}(R))<\varepsilon,\qquad \phi^{(R)}=cpq(1\pm\varepsilon).$$

The proof of Lemma 3.1 will be given at the end of this section.

Let $\mathcal{R}$ be the equivalence relation generated by T. An *orbit cocycle* is a measurable function $\widetilde{\varphi}:\mathcal{R}\to G$ such that if $(x,y),(y,z)\in\mathcal{R}$, then

$$\widetilde{\varphi}(x,z)=\widetilde{\varphi}(x,y)+\widetilde{\varphi}(y,z).$$

Let $\varphi:X\to G$ be measurable, and let φ_n $(n\in\mathbb{Z})$ denote the cocycle generated by φ under T. The orbit cocycle $\widetilde{\varphi}:\mathcal{R}\to G$ corresponding to φ is defined by

$$\widetilde{\varphi}(x,T^nx)=\varphi_n(x).$$

For $R\in[T]$, the function $\varphi_R:\mathcal{D}(R)\to G$ is defined by

$$\varphi_R(x)=\widetilde{\varphi}(x,Rx).$$

Clearly $\varphi(R\circ S,x)=\varphi(S,x)+\varphi(R,Sx)$ on $\mathcal{D}(R\circ S)=\mathcal{D}(S)\cap S^{-1}\mathcal{D}(R)$.

Definition. Let α be a measurable partition of X, U a subset of G, and $\varepsilon>0$. We say that the measurable cocycle $\varphi:X\to\Gamma$ satisfies $\mathrm{EVC}_T(U,\varepsilon,\alpha)$ if for ε-almost every $a\in\alpha$, there is $R=R_a\in[T]_+$ such that

$$\mathcal{D}(R),\Im(R)\subset a,\qquad \varphi_R\in U\quad\text{on } \mathcal{D}(R_a),\qquad m(\mathcal{D}(R))>(1-\varepsilon)m(a).$$

Definition. We say that the partitions $\{\alpha_k:k\ge1\}$ *approximately generate* $\mathcal{B}$ if

$$\forall B\in\mathcal{B}(X),\ \varepsilon>0\ \exists k_0\ge1,\ \forall k\ge k_0,\ \exists A_k\in\mathcal{A}(\alpha_k),\quad m(B\bigtriangleup A_k)<\varepsilon.$$

Here $\mathcal{A}(\alpha)$ denotes the algebra generated by α. It is not hard to see that the partitions $\{\alpha_k:k\ge1\}$ approximately generate $\mathcal{B}$ if and only if $E(1_B|\mathcal{A}(\alpha_k))\to 1_B$ in probability for all $B\in\mathcal{B}$, and in this case,

$$\forall\varepsilon>0,\ B\in\mathcal{B},\ \exists k_0,\ \forall k\ge k_0,\qquad \sum_{a\in\alpha_k,\,1-m(B|a)\le\varepsilon} m(a)\ge(1-\varepsilon)m(B).$$

Proposition 3.1. *Suppose that the partitions $\{\alpha_k:k\ge1\}$ approximately generate $\mathcal{B}$, and let $\varepsilon_k\downarrow0$, $\gamma\in\Gamma$, and $U_k\subset G$ satisfy $U_n\downarrow\{\gamma\}$ and* $\mathrm{diam}U_n\downarrow0$. *If φ satisfies $\mathrm{EVC}_T(U_k,\varepsilon_k,\alpha_k)$ for all $k\ge1$, then $\gamma\in E(\varphi)$.*

P r o o f. Suppose that $B \in \mathcal{B}_+$ and $V \subset G$ is an open neighbourhood of γ. We show that

$$\exists n \geq 1, \quad m(B \cap T^{-n}B \cap [\varphi_n \in V]) > 0.$$

Evidently, $V \supset U_k$ for all k sufficiently large. It follows from the definitions that for all k sufficiently large, there exists $a \in \alpha_k$ such that

$$m(a \setminus B) < 0.1m(a),$$

and there is $R = R_a \in [T]_+$ such that

$$\mathcal{D}(R), \Im(R) \subset a, \quad \varphi_R \in U_k \quad \text{on } \mathcal{D}(R), \quad m(a \setminus \mathcal{D}(R)) < 0.1m(a).$$

It follows that

$$m(B \setminus \mathcal{D}(R)) < 0.2m(a).$$

Let $R = T^{\phi}$, where $\phi : \mathcal{D}(R) \to \mathbb{Z}$. We have

$$\sum_{n\in\mathbb{Z}} m(B \cap [\phi = n] \cap T^{-n}B \cap [\varphi_n \in U_k]) \\ \geq m(B \cap \mathcal{D}(R) \cap R^{-1}(B \cap \Im(R)) \cap [\varphi_R \in U_k]) \geq 0.6m(a),$$

whence there is $n \in \mathbb{Z}$ such that

$$m(B \cap T^{-n}B \cap [\varphi_n \in V]) \geq m(B \cap [\phi = n] \cap T^{-n}B \cap [\varphi_n \in U_k]) > 0. \ \blacksquare$$

COROLLARY 3.2. *Suppose that the partitions $\{\alpha_k : k \geq 1\}$ approximately generate $\mathcal{B}$, let $\{U_k : k \geq 1\}$ be a basis of neighbourhoods for the topology of G, and let $\varepsilon_k \downarrow 0$. If φ satisfies* $\mathrm{EVC}_T(U_k, \varepsilon_k, \alpha_k)$ *for all $k \geq 1$, then T_φ is ergodic.*

This sufficient condition for ergodicity is actually necessary.

PROPOSITION 3.3. *If T_φ is ergodic, then for all $A \in \mathcal{B}_+$ and $U \neq \emptyset$ open in G, there is $R \in [T]_+$ such that*

$$\mathcal{D}(R) = \Im(R) = A, \quad \varphi_R \in U \quad \textit{a.e. on } A,$$

and hence, φ satisfies $\mathrm{EVC}_T(U, \varepsilon, \alpha)$ *for any measurable partition α of X, U open in G, and $\varepsilon > 0$.*

P r o o f. Let U be open in G. Choose $g \in U$; then $V := U - g$ is a neighbourhood of $0 \in G$. Choose W open in G such that $W + W \subset V$. By ergodicity of T_φ, for every $A, B \in \mathcal{B}_+$, there is $n \in \mathbb{N}$ such that $\mu((A \times W) \cap T_\varphi^{-n}(B \times (W + g))) > 0$, whence $m(A \cap T^{-n}B \cap [\varphi_n \in U]) > 0$. The proposition follows from this via a standard exhaustion argument. $\blacksquare$

We need a finite version of EVC better suited to sequential constructions.

DEFINITION. Let α be a measurable partition of X, U open in G, $\varepsilon > 0$, and $N \geq 1$. We say that the measurable cocycle $\varphi : X \to G$ satisfies

$\mathrm{EVC}^T(U,\varepsilon,\alpha,N)$ if for ε-almost every $a\in\alpha$, there is $R=R_a\in[T]_+$ with $\phi^{(R)}\le N$ such that

$$\mathcal{D}(R),\Im(R)\subset a,\qquad \varphi_R\in U\quad\text{on }\mathcal{D}(R),\qquad m(a\setminus\mathcal{D}(R))<\varepsilon m(a).$$

PROPOSITION 3.4. *Let α be a measurable partition of X, U open in G, and $\varepsilon>0$. The measurable cocycle $\varphi:X\to G$ satisfies $\mathrm{EVC}_T(U,\varepsilon,\alpha)$ iff it satisfies $\mathrm{EVC}^T(U,\varepsilon,\alpha,N)$ for some $N\ge 1$.*

The next lemma shows that addition of a sufficiently small cocycle does not affect EVC^T conditions too much.

LEMMA 3.5. *Let α be a partition, $\varepsilon,\delta>0$, $N\in\mathbb{N}$, $V\subset G$, and $\phi:X\to G$ be a cocycle satisfying $\mathrm{EVC}^T(U,\varepsilon,\alpha,N)$ where $U\subset G$. If $\varphi:X\to G$ is measurable, and*

$$m([\varphi\notin V])<\delta^2/N,$$

then $\phi+\varphi$ satisfies $\mathrm{EVC}^T(U+V,\varepsilon+\delta,\alpha,N)$.

Proof. Let $B=[\varphi\circ T^j\in V\text{ for }0\le j\le N-1]$. Then since $\varphi_n\in V$ on B for all $1\le n\le N$, it follows that $\varphi_R\in V$ on $B\cap\mathcal{D}(R)$ for all $R\in[T]_+$ with $\phi^{(R)}\le N$. Let α_1 consist of those $a\in\alpha$ such that there is $R=R_a\in[T]_+$ with $\phi^{(R)}\le N$ such that

$$\mathcal{D}(R),\Im(R)\subset a,\qquad \varphi_R\in U\quad\text{on }\mathcal{D}(R),\qquad m(a\setminus\mathcal{D}(R))<\varepsilon m(a).$$

We have

$$m\Big(\bigcup_{a\in\alpha_1}a\Big)>1-\varepsilon.$$

Let α_2 consist of those $a\in\alpha$ for which

$$m(B\cap a)>(1-\delta)m(a).$$

It follows from Chebyshev's inequality that

$$m\Big(\bigcup_{a\in\alpha_2}a\Big)>1-\frac{m(B)}{\delta}>1-\delta.$$

Therefore

$$m\Big(\bigcup_{a\in\alpha_1\cap\alpha_2}a\Big)>1-\varepsilon-\delta.$$

If $a\in\alpha_1\cap\alpha_2$, and $R'=R'_a:=(R_a,\mathcal{D}(R_a)\cap B)\in[T]_+$, then

$$\mathcal{D}(R'),\Im(R')\subset a,\qquad (\phi+\varphi)_{R'}\in U+V\quad\text{on }\mathcal{D}(R'),$$
$$m(a\setminus\mathcal{D}(R'))<(\varepsilon+\delta)m(a).\ \blacksquare$$

Our main result in this section is a sufficient condition for a group element to be an essential value of a sum of coboundaries.

THEOREM 3.6. *Suppose that $g \in G$, the partitions $\{\alpha_j\}$ approximately generate $\mathcal{B}$, $N_k \in \mathbb{N}$, $N_k \uparrow \infty$, and $\varepsilon_k > 0$, $\sum_{k\ge 1}\varepsilon_k < \infty$. If for $k \in \mathbb{N}$, $f_k : X \to G$ is measurable and*

$$\sum_{j=1}^{k}(f_j \circ T - f_j) \text{ satisfies } \mathrm{EVC}^T(N(g,\varepsilon_k),\varepsilon_k,\alpha_k,N_k),$$
$$m([\|f_k \circ T - f_k\| \ge \varepsilon_{k-1}/N_{k-1}]) \le \varepsilon_{k-1}^2/N_{k-1},$$

then

$$\sum_{k=1}^{\infty}\|f_k \circ T - f_k\| < \infty \quad \text{a.e.,} \quad \text{and} \quad g \in E\Big(\sum_{k=1}^{\infty}(f_k \circ T - f_k)\Big).$$

P r o o f. By the Borel–Cantelli lemma, $\sum_{k=1}^{\infty}\|f_k \circ T - f_k\| < \infty$ a.e. Write

$$\phi := \sum_{k=1}^{\infty}(f_k \circ T - f_k), \qquad \widetilde{\phi}_k = \sum_{j=1}^{k}(f_j \circ T - f_j), \qquad \widehat{\phi}_k = \sum_{j=k+1}^{\infty}(f_j \circ T - f_j).$$

Since $\phi = \widetilde{\phi}_k + \widehat{\phi}_k$ for all $k \ge 1$, $\widetilde{\phi}_k$ satisfies $\mathrm{EVC}^T(N(g,\varepsilon_k),\varepsilon_k,\alpha_k,N_k)$, and

$$\begin{aligned} m\Big(\Big[\|\widehat{\phi}_k\| \ge \frac{1}{N_k}\sum_{j=k+1}^{\infty}\varepsilon_j\Big]\Big) &\le \sum_{j=k+1}^{\infty} m([\|f_j \circ T - f_j\| \ge \varepsilon_j/N_k]) \\ &\le \sum_{j=k+1}^{\infty} m([\|f_j \circ T - f_j\| \ge \varepsilon_j/N_{j-1}]) \\ &< \sum_{j=k+1}^{\infty}\frac{\varepsilon_{j-1}^2}{N_{j-1}} \le \frac{1}{N_k}\sum_{j=k}^{\infty}\varepsilon_k^2, \end{aligned}$$

it follows from Lemma 3.5 that ϕ satisfies

$$\mathrm{EVC}^T\Big(N\Big(g,\sum_{j=k}^{\infty}\varepsilon_j\Big),2\Big(\sum_{j=k}^{\infty}\varepsilon_k^2\Big)^{1/2},\alpha_k,N_k\Big). \ \blacksquare$$

As promised above, we conclude this section with

Proof of Lemma 3.1. Let

$$A_n = \Big[\Big|\frac{1}{n}\sum_{k=0}^{n-1}1_A \circ T^k - m(A)\Big| < \varepsilon m(A)\Big].$$

By Birkhoff's ergodic theorem, there exists $p_0 \in \mathbb{N}$ such that $m(A_p^{\mathrm{c}}) < \varepsilon^4/2$ for all $p \ge p_0$. Fix $p \ge p_0$. Now fix $q \ge p/(c\varepsilon) =: q_0$. Set

$$B = A_p \cap T^{-[cq]p}A_p.$$

Evidently $m(B) > 1 - \varepsilon^2$.

By Birkhoff's ergodic theorem there is $N_0 \in \mathbb{N}$ such that
$$m(C_n^{\mathrm{c}}) < \frac{\varepsilon^2}{2p} \qquad \forall n \ge N_0$$
where
$$C_n = \Big[\frac{1}{n}\sum_{k=0}^{n-1} 1_B \circ T^{pk} \ge E(1_B|\mathcal{I}_{T^p}) - \varepsilon^2\Big].$$
Let $N > (pq/\varepsilon) \vee pN_0$. By Rokhlin's theorem, there exists $F \in \mathcal{B}$ such that $\{T^j F : 0 \le j \le N-1\}$ are disjoint, and
$$m\Big(X \setminus \bigcup_{j=0}^{N-1} T^j F\Big) < \frac{\varepsilon}{p}.$$
Note that since $E(1_B|\mathcal{I}_{T^p})$ is T^p-invariant, we have
$$\frac{N}{p}\sum_{k=0}^{p-1} \int_{T^k F} E(1_{B^{\mathrm{c}}}|\mathcal{I}_{T^p})\, dm \le \int_X E(1_{B^{\mathrm{c}}}|\mathcal{I}_{T^p})\, dm = m(B^{\mathrm{c}}) < \varepsilon^2,$$
whence there is $0 \le k \le p-1$ such that
$$\int_{T^k F} E(1_{B^{\mathrm{c}}}|\mathcal{I}_{T^p})\, dm < \varepsilon^2 m(F).$$
There is no loss of generality in assuming $k = 0$ as this merely involves taking $T^k F$ as the base for a slightly shorter Rokhlin tower, and adding $\bigcup_{j=0}^{k-1} T^j F$ to the "error set".

Set
$$X_0 = \bigcup_{j=0}^{N-pq} T^j F, \qquad J = X_0 \cap \bigcup_{j\ge 0,\, jp \le N} T^{jp} F.$$
Then $m(J) > 1/(2p)$ so
$$m(C_n^{\mathrm{c}} \cap J) \le \varepsilon^2 m(J) \qquad \forall n \ge N_0.$$
For $y \in J$, set $\kappa(y) = \#\{0 \le j \le p-1 : T^j y \in A\}$ and write
$$\{T^j y : 0 \le j \le p-1,\ T^j y \in A\} = \{T^{j_i(y)} y : 1 \le i \le \kappa(y)\}$$
in case $\kappa(y) \ge 1$, where $j_i(y) < j_{i+1}(y)$. Note that
$$\kappa = pm(A)(1 \pm \varepsilon) \qquad \text{on } J \cap A_p.$$
To estimate $m(J \cap B)$, note that
$$\begin{aligned}\sum_{0\le j\le N/p:\, m(C_{N/p}^{\mathrm{c}}|T^{jp}F)\ge\varepsilon} m(T^{jp}F) &\le \sum_{0\le j\le N/p} m(C_{N/p}^{\mathrm{c}} \cap T^{jp}F)/\varepsilon \\ &= m(C_{N/p}^{\mathrm{c}} \cap J)/\varepsilon \le m(C_{N/p}^{\mathrm{c}})/\varepsilon \\ &\le \frac{\varepsilon}{2p} \le \varepsilon m(J),\end{aligned}$$

whence, there is $i \le \varepsilon N/p$ such that

$$m(C_{N/p} \cap T^{ip} F) = m(T^{-ip} C_{N/p} \cap F) \ge (1-\varepsilon) m(F).$$

For $y \in T^{-ip} C_{N/p} \cap F$,

$$\begin{aligned}\#\{0 \le j \le N/p : T^{jp} y \in B\} &\ge \#\{0 \le j \le N/p : T^{(i+j)p} y \in B\} - \varepsilon N/p \\ &\ge \frac{N}{p}(E(1_B|\mathcal{I}_{T^p}) - 2\varepsilon).\end{aligned}$$

Therefore,

$$\begin{aligned} m(J \cap B) &= \sum_{k=0}^{N/p-1} m(T^{jp} F \cap B) = \int_F \Big(\sum_{k=0}^{N/p-1} 1_B \circ T^{jp} \Big)\, dm \\ &\ge \frac{N}{p} \int_{T^{-ip} C_{N/p} \cap F} (E(1_B|\mathcal{I}_{T^p}) - 2\varepsilon)\, dm \\ &\ge \frac{N}{p} \int_F (E(1_B|\mathcal{I}_{T^p}) - 3\varepsilon)\, dm \\ &\ge (1-4\varepsilon) m(F) N/p = (1-4\varepsilon) m(J). \end{aligned}$$

For $x \in \bigcup_{j=0}^{p-1} T^j J$, let $j(x)$ be such that $T^{-j(x)} x \in J$, and let $y(x) = T^{-j(x)} x$. Define $\psi : A \cap \bigcup_{j=0}^{p-1} T^j J \to \{1, \ldots, p\}$ by

$$\psi(x) = \sum_{k=0}^{j(x)} 1_A(T^{-k} x) = \sum_{k=0}^{j(x)} 1_A(T^k y(x)).$$

Note that

$$x = T^{j_{\psi(x)}(y(x))} y(x).$$

Now define $D \subset A \cap X_0$ by

$$D \cap \bigcup_{j=0}^{p-1} T^j J_0 = \{x \in A \cap J_0 : \psi(x) \le \kappa(y(T^{[cq]p} x))\},$$

and define $\phi : D \to \mathbb{N}$ by

$$\phi(x) = [cq]p + j_{\psi(x)}(y(T^{[cq]p} x)), \qquad x \in D \cap \bigcup_{j=0}^{p-1} T^j J.$$

We claim that if $R \in [T]_+$ is defined by $\mathcal{D}(R) = D$ and $\phi^{(R)} = \phi$, then ϕ is as desired. To see this, check that $\kappa \ge (1-\varepsilon) m(A) p$ on $J \cap B$, whence $m(D) \ge m(J_0 \cap B)(1-\varepsilon) m(A) p \ge (1-6\varepsilon) m(J) p m(A) \ge (1-7\varepsilon) m(A)$. ■

4. Proof of Theorems 1 and 2. In this section, we prove Theorems 1 and 2. The proofs are sequential using Theorem 3.6. The inductive steps are

Lemmas 4.1 and 4.2. Their proofs use the Rokhlin lemmas for Abelian group actions of Katznelson and Weiss [K-W], and Lind [L] respectively (see also [O-W] for a general Rokhlin lemma for amenable group actions implying these).

Let G be a locally compact, second countable Abelian group with invariant metric d, and let T be an ergodic probability preserving transformation of the standard probability space $(X,\mathcal{B},m)$.

LEMMA 4.1. *Let $\phi : X \to G$ be a T-coboundary, let $S_1,\ldots,S_d$ be probability preserving transformations generating a free $\mathbb{Z}^{d+1}$ action together with T, and let $w_1,\ldots,w_d \in \mathrm{End}(G)$, $w_i \circ w_j = w_j \circ w_i$. If α is a finite, measurable partition of X, and $\varepsilon > 0$, then there is a measurable function $f : X \to G$ such that*

$$m([f\circ T - f \neq 0]) < \varepsilon, \tag{1}$$

$$m([f\circ S_j \neq w_j \circ f]) < \varepsilon \qquad (1\le j\le d) \tag{2}$$

and

$$\phi + f - f\circ T \quad \textit{satisfies} \quad \mathrm{EVC}_T(N(\gamma,\varepsilon),\varepsilon,\alpha). \tag{3}$$

Proof. Write $\phi = H - H\circ T$. Possibly refining α, we may assume that for $\varepsilon/2$-a.e. $a\in\alpha$, the oscillation of H on a is less than $\varepsilon/2$.

For $\underline{i} = (i_1,\ldots,i_d)\in\mathbb{Z}_+^d$, we write

$$S_{\underline{i}} := S_1^{i_1}\circ\ldots\circ S_d^{i_d}, \qquad w_{\underline{i}} := w_1^{i_1}\circ\ldots\circ w_d^{i_d}.$$

Then

$$S_{\underline{i}+\underline{j}} = S_{\underline{i}}\circ S_{\underline{j}}, \qquad w_{\underline{i}+\underline{j}} = w_{\underline{i}}\circ w_{\underline{j}}$$

since $S_i\circ S_j = S_j\circ S_i$ and $w_i\circ w_j = w_j\circ w_i$.

Given $\underline{i} = (i_1,\ldots,i_d)$, $\underline{k} = (k_1,\ldots,k_d)$ we write $\underline{i}\le\underline{k}$ (resp. $\underline{i}<\underline{k}$) if $i_j\le k_j$ (resp. $i_j<k_j$) for all $1\le j\le d$.

Fix $k>10/\varepsilon$. There is an ergodic cocycle $\varphi : X\to G$ such that

$$m([\varphi\neq 0]) < \frac{\varepsilon}{3k^d}.$$

It follows that $w_{\underline{i}}\circ\varphi\circ S_{-\underline{i}}$ is ergodic for $\underline{i}\ge\underline{0}$ (as $w_{\underline{i}}$ is surjective, and $S_{-\underline{i}}$ commutes with T for $\underline{i}\ge\underline{0}$), whence $\phi + w_{\underline{i}}\circ\varphi\circ S_{-\underline{i}}$ is ergodic for $\underline{i}\ge\underline{0}$ (as ϕ is a coboundary), and so satisfies $\mathrm{EVC}_T(N(\gamma,\varepsilon/4),\varepsilon/(4k^d),\alpha)$. Therefore (by Propositions 3.3 and 3.4), there exists $M\in\mathbb{N}$ such that

$$\phi + w_{\underline{i}}\circ\varphi\circ S_{-\underline{i}} \quad \text{satisfies} \quad \mathrm{EVC}^T\left(N\left(\gamma,\frac{\varepsilon}{4}\right),\frac{\varepsilon}{4k^d},\alpha,M\right)$$

for $\underline{0}\le\underline{i}\le\underline{k}$ where $\underline{k}=(k,\ldots,k)$ (d times).

Now choose $N\ge 1$ such that

$$\frac{M}{N} < \frac{\varepsilon\eta_\alpha}{5}$$

where $\eta_\alpha := \min\{m(a) : a \in \alpha\}$. By the Katznelson–Weiss Rokhlin lemma [K-W], there is $F \in \mathcal{B}(X)$ such that $\{T^j S_{\underline{i}} F : 0 \le j \le N-1,\ \underline{0} \le \underline{i} < \underline{k}\}$ are disjoint, and

$$m\Big(X \setminus \bigcup_{0\le j\le N-1,\, \underline{0}\le \underline{i}<\underline{k}} T^j S_{\underline{i}} F\Big) < \frac{\varepsilon\eta_\alpha}{6}.$$

Let

$$C = \bigcup_{j=0}^{N-1} T^j F, \qquad \widetilde{C} = \bigcup_{j=0}^{N-M} T^j F, \qquad \mathcal{T} = \bigcup_{\underline{0}\le \underline{i}<\underline{k}} S_{\underline{i}} C, \qquad \widetilde{\mathcal{T}} = \bigcup_{\underline{0}\le \underline{i}<\underline{k}} S_{\underline{i}} \widetilde{C}.$$

There is a measurable function $f_0 : X \to G$ such that

$$\varphi = f_0 - f_0 \circ T \qquad \text{on } \mathcal{T}.$$

Set $\varphi' = f_0 - f_0 \circ T$. Then $m([\varphi \ne \varphi']) < \varepsilon\eta_\alpha/6$.

Now define $f : \mathcal{T} \to G$ by

$$f = \begin{cases} w_{\underline{i}} \circ f_0 \circ S_{-\underline{i}} & \text{on } S_{\underline{i}} C\ (\underline{0} \le \underline{i} \le \underline{k}), \\ 0 & \text{elsewhere,} \end{cases}$$

and define

$$\psi = f - f \circ T.$$

To establish (1), note that

$$\begin{aligned} m([\psi \ne 0]) &< m([\psi \ne 0] \cap \widetilde{\mathcal{T}}) + m(X \setminus \widetilde{\mathcal{T}}) \\ &\le k^d m([\varphi \ne 0] \cap \widetilde{C}) + m(X \setminus \widetilde{\mathcal{T}}) \\ &< \varepsilon/3 + M/N < \varepsilon. \end{aligned}$$

Next, to prove (2), suppose that $\underline{0} \le \underline{i} < \underline{k},\ 1 \le j \le d$ and $i_j < k-1$. If $x \in S_{\underline{i}} C$, then

$$\begin{aligned} f(S_j x) &= w_{\underline{i}+\underline{e}_j} \circ f_0 \circ S_{-(\underline{i}+\underline{e}_j)}(S_j x) \\ &= w_j \circ w_{\underline{i}} \circ f_0 \circ S_{-\underline{i}}(x) = w_j \circ f(x), \end{aligned}$$

whence

$$\begin{aligned} m([f \circ S_j \ne w_j \circ f]) &< m\Big(\bigcup_{\underline{0}\le \underline{i}<\underline{k},\, i_j = k-1} S_{\underline{i}} C\Big) + m(X \setminus \mathcal{T}) \\ &< 1/k + \varepsilon\eta_\alpha/6 < \varepsilon. \end{aligned}$$

To complete the proof, we show (3). We know that $\phi + w_{\underline{i}} \circ \varphi \circ S_{-\underline{i}}$ satisfies $\mathrm{EVC}^T(N(\gamma, \varepsilon/4), \varepsilon/(4k^d), \alpha, M)$ for all $\underline{i}$, whence $\phi + w_{\underline{i}} \circ \varphi' \circ S_{-\underline{i}}$ satisfies $\mathrm{EVC}^T(N(\gamma, \varepsilon/4), \varepsilon/(3k^d), \alpha, M)$ for all $\underline{i}$.

It follows that for $\varepsilon/3$-a.e. $a \in \alpha$, and for each $\underline{0} \le \underline{i} < \underline{k}$, there is $R_{\underline{i}} = R_{a,\underline{i}} \in [R]_+$ such that $\mathcal{D}(R_{\underline{i}}), \Im(R_{\underline{i}}) \subset a$, $m(a \setminus \mathcal{D}(R_{\underline{i}})) < \frac{\varepsilon}{3k^d} m(a)$, and $(\phi + w_{\underline{i}} \circ \varphi \circ S_{-\underline{i}})_{R_{\underline{i}}} \in N(\gamma, \varepsilon/4)$ on $\mathcal{D}(R_{\underline{i}})$.

Define $R = R_a \in [T]_+$ by

$$\mathcal{D}(R) = \bigcup_{\underline{0} \le \underline{i} < \underline{k}} \mathcal{D}(R_{\underline{i}}) \cap S_{\underline{i}} \widetilde{C},$$

$$R = R_{\underline{i}} \quad \text{ on } S_{\underline{i}} \widetilde{C} \ (\underline{0} \le \underline{i} < \underline{k}).$$

For $x \in \mathcal{D}(R)$, there is $\underline{i} = \underline{i}(x)$ such that $x \in \mathcal{D}(R_{\underline{i}}) \cap S_{\underline{i}} \widetilde{C}$, and we have

$$(\phi + \psi)_R(x) = (\phi + w_{\underline{i}} \circ \varphi' \circ S_{-\underline{i}})_{R_{\underline{i}}}(x) \in N(\gamma, \varepsilon/4).$$

Lastly,

$$\begin{aligned} m(a \setminus \mathcal{D}(R)) &= \sum_{0 \le \underline{i} < \underline{k}} m((a \setminus \mathcal{D}(R)) \cap S_{\underline{i}} \widetilde{C}) + m(\mathcal{T} \setminus \widetilde{\mathcal{T}}) + m(X \setminus \mathcal{T}) \\ &< \sum_{0 \le \underline{i} < \underline{k}} m(a \cap S_{\underline{i}} \widetilde{C} \setminus \mathcal{D}(R_{\underline{i}})) + \frac{M}{N} + m(X \setminus \mathcal{T}) \\ &\le \sum_{0 \le \underline{i} < \underline{k}} m(a \setminus \mathcal{D}(R_{\underline{i}})) + \frac{\varepsilon}{5} \eta_\alpha + \frac{\varepsilon}{6} \eta_\alpha \le \varepsilon m(a). \ \blacksquare \end{aligned}$$

Proof of Theorem 1. We only prove Theorem 1 for d finite. The proof in case d is infinite is analogous and left to the reader.

Choose a countable, dense subset Γ of G. Let $(\gamma_1, \gamma_2, \ldots) \in \Gamma^{\mathbb{N}}$ satisfy $\{\gamma_k : k \ge 1\} = \Gamma$, and

$$\forall \gamma \in \Gamma, \qquad \gamma_k = \gamma \text{ for infinitely many } k,$$

let the partitions $\{\alpha_j\}$ approximately generate $\mathcal{B}$, and let $\varepsilon_k = 2^{-k^2}$.

Using Lemma 4.1, construct (sequentially) a sequence of coboundaries $\phi_k = f_k - f_k \circ T$ such that

$$m([f_k \circ S_j \ne E \circ f_k]) \le \varepsilon_k \qquad (1 \le j \le d),$$

$\widetilde{\phi}_k := \sum_{j=1}^k \phi_j$ satisfies $\mathrm{EVC}^T(N(\gamma_k, \varepsilon_k), \varepsilon_k, \alpha_k, N_k)$ where $N_k \in \mathbb{N}$, $N_k \uparrow$, and

$$m([\phi_k \ne 0]) \le \varepsilon_k / N_{k-1}.$$

Clearly $\phi := \sum_{k=1}^\infty \phi_k$ converges a.e. Also

$$\psi_j := \sum_{k=1}^{\infty} (f_k \circ S_j - w_j \circ f_k) \qquad (1 \le j \le d)$$

converges a.e., whence

$$\phi \circ S_j - w_j \circ \phi = \psi_j - \psi_j \circ T \qquad (1 \le j \le d).$$

Theorem 3.6 now shows that $\Gamma \subset E(\phi)$, and the ergodicity of ϕ is established. $\blacksquare$

LEMMA 4.2. *Let $\phi : X \to \mathbb{R}$ be a T-coboundary, and let $\{S_t : t \in \mathbb{R}\}$ be probability preserving transformations generating a free $\mathbb{Z} \times \mathbb{R}$ action together* *with T. If α is a finite, measurable partition of X, $\varepsilon > 0$, and $J \subset \mathbb{R}_+$ is an open interval, then there is a measurable function $f : X \to \mathbb{R}$ such that*

$$m([|f \circ T - f| \geq \varepsilon]) < \varepsilon, \tag{1}$$

$$m([f \circ S_t \neq e^t f]) < \varepsilon \qquad (0 \leq t \leq 1), \tag{2}$$

and

$$\phi + f - f \circ T \quad \textit{satisfies} \quad \mathrm{EVC}_T(J, \varepsilon, \alpha). \tag{3}$$

Proof. Write $J = ((1-\delta)b, (1+\delta)b)$ where $b, \delta > 0$. We sometimes use the notation $x = (1 \pm \delta)b$ which means $x \in J$.

Write $\phi = \psi \circ T - \psi$ where $\psi : X \to \mathbb{R}$ is measurable. Choose a refinement α_1 of α with the property that

$$\forall a \in \alpha_1,\ \exists y_a \in \mathbb{R}, \qquad |\psi - y_a| < b\delta/2 \text{ a.e. on } a,$$

and set $\eta_\alpha := \min\{m(a) : a \in \alpha\}$. Fix $K = 10/\varepsilon$, and $0 = t_0 < t_1 < \ldots < t_M = K$ such that $e^{t_{i+1}} < (1 + \delta/3)e^{t_i}$.

By Lemma 3.1, there are $p, q \in \mathbb{N}$ such that $be^K/(pq) < \varepsilon$, and for all $a \in \alpha_1$ and $0 \leq k \leq M-1$, there is $R_{a,k} \in [T]_+$ such that

$$\mathcal{D}(R_{a,k}), \Im(R_{a,k}) \subset a, \qquad m(a \setminus \mathcal{D}(R_{a,k})) < \frac{\varepsilon}{7M} m(a),$$
$$\phi^{(R_{a,k})} = e^{-t_k} pq(1 \pm \delta/9).$$

Now choose $N \geq 1$ such that

$$\frac{e^K pq}{N} < \frac{\varepsilon \eta_\alpha}{5}.$$

By the Rokhlin theorem for continuous groups ([L], [O-W]) there is $F \in \mathcal{B}(X)$ such that $T^k S_t F$ are disjoint for $0 \leq k \leq N$, $0 \leq t \leq K$, and

$$m\Big(X \setminus \bigcup_{0 \leq k \leq N-1,\, 0 \leq t \leq K} T^k S_t F\Big) < \frac{\varepsilon \eta_\alpha}{6}.$$

Let

$$C = \bigcup_{j=0}^{N-1} T^j F, \qquad \widetilde{C} = \bigcup_{j=0}^{N-2} T^j F, \qquad \mathcal{T} = \bigcup_{0 \leq t \leq K} S_t C, \qquad \widetilde{\mathcal{T}} = \bigcup_{0 \leq t \leq K} S_t \widetilde{C}.$$

There is a measurable function $f : \mathcal{T} \to \mathbb{R}$ such that

$$f \circ T - f = \frac{b}{pq} e^t \qquad \text{on } S_t \widetilde{C}.$$

Complete the definition of $f : X \to \mathbb{R}$ by setting $f = 0$ on $\mathcal{T}^{\mathrm{c}}$.

It is immediate from this construction that f satisfies (1) and (2). We establish (3) by showing that $f \circ T - f$ satisfies $\mathrm{EVC}_T(J, \varepsilon, \alpha_1)$. Let

$$\widehat{C} = \bigcup_{j=0}^{N-pq} T^j F, \qquad \widehat{\mathcal{T}} = \bigcup_{0 \le t \le K} S_t \widehat{C}.$$

For $0 \le k \le M-1$, let

$$\widehat{\mathcal{T}}_k = \bigcup_{t_k \le t < t_{k+1}} S_t \widehat{C}.$$

Fix $a \in \alpha_1$, and define $R'_a \in [T]_+$ by $R'_a = R_{a,k}$ on $\mathcal{D}(R_{a,k}) \cap \widehat{\mathcal{T}}_k$. It follows that $\mathcal{D}(R'_a), \Im(R'_a) \subset a$ and

$$\begin{aligned} m(a \setminus \mathcal{D}(R'_a)) &= \sum_{k=0}^{M-1} m(\widehat{\mathcal{T}}_k \cap [a \setminus \mathcal{D}(R_{a,k})]) \\ &\le \sum_{k=0}^{M-1} m(a \setminus \mathcal{D}(R_{a,k})) \le \frac{\varepsilon}{7} m(a); \end{aligned}$$

moreover, on $\mathcal{D}(R'_a) \cap \widehat{\mathcal{T}}_k$,

$$|\psi \circ R'_a - \psi| < b\delta/2,$$

whence, on $S_t \widetilde{C}$ for $t \in [t_k, t_{k+1}]$,

$$\begin{aligned} \varphi_{R'_a} &= \frac{e^t b}{pq} \phi^{(R'_a)} \pm \frac{b\delta}{2} = e^{t-t_k} b\left(1 \pm \frac{\delta}{9}\right) \pm \frac{b\delta}{2} \\ &= b\left(1 \pm \frac{\delta}{9}\right)\left(1 \pm \frac{\delta}{3}\right)\left(1 \pm \frac{\delta}{2}\right) \in J. \ \blacksquare \end{aligned}$$

Proof of Theorem 2. Fix $(g_1, g_2, \ldots) = (1, \sqrt{2}, 1, \sqrt{2}, \ldots)$. Using Lemma 4.2, construct a sequence of coboundaries $f_k \circ T - f_k$ such that

$$m([f_k \circ S_t \ne e^t f_k]) \le 1/2^k \qquad (0 \le t \le 1),$$

$$\phi_k := \sum_{j=1}^{k} (f_j \circ T - f_j) \quad \text{satisfies} \quad \mathrm{EVC}^T\left(\left(\gamma_k - \frac{1}{2^k}, \gamma_k + \frac{1}{2^k}\right), \varepsilon_k, \alpha_k, N_k\right)$$

where $N_k \in \mathbb{N}$, $N_k \uparrow$, and

$$m\left(\left[|f_k \circ T - f_k| \ge \frac{1}{2^k N_{k-1}}\right]\right) \le \frac{1}{2^k N_{k-1}}.$$

The ergodicity of $\sum_{k=1}^{\infty} (f_k \circ T - f_k)$ follows from

$$1, \sqrt{2} \ \in \ E\Big(\sum_{k=1}^{\infty} (f_k \circ T - f_k)\Big),$$

which follows from Theorem 3.6. $\blacksquare$

5. Maharam transformations. For a non-singular conservative, ergodic transformation R of $(\Omega, \mathcal{A}, p)$, the transformation $T : X = \Omega \times \mathbb{R} \to X$ defined by

$$T(x, y) = \left(Rx, y - \log \frac{d(p \circ R)}{dp} \right)$$

preserves the measure $dm_T(x, y) = dp(x)e^y dy$, and is called the *Maharam transformation* of R; it was shown in [M] to be conservative. If $Q_t(x, y) = (x, y + t)$, then $Q_t \in C(T)$ and $D(Q_t) := d(m_T \circ Q^{-1})/dm_T = e^t$.

Conservative, ergodic Maharam transformations were constructed in [K].

In this section, we give conditions for a conservative, ergodic, measure preserving transformation to be isomorphic to a Maharam transformation showing that the transformations constructed in Theorem 2 are Maharam transformations. We conclude by showing that any Bernoulli transformation has a $\mathbb{Z}$-extension which is isomorphic to a Maharam transformation.

PROPOSITION 5.1. *A conservative*, *ergodic*, *measure preserving transformation T of the standard*, *non-atomic*, *σ-finite measure space* $(X, \mathcal{B}, m)$ *is isomorphic to a Maharam transformation if and only if there is a flow* $\{Q_t : t \in \mathbb{R}\} \subset C(T)$ *such that* $D(Q_t) = e^t$ *for all* $t \in \mathbb{R}$.

Proof. Suppose first that T is a Maharam transformation, i.e. $T : X = \Omega \times \mathbb{R} \to X$ is defined by

$$T(x, y) = \left(Rx, y - \log \frac{d(p \circ R)}{dp} \right)$$

and preserves the measure $dm(x, y) := dp(x)e^y dy$, where R is a non-singular conservative, ergodic transformation of the standard probability space $(\Omega, \mathcal{A}, p)$. Set $Q_t(x, y) = (x, y + t)$. Then $\{Q_t : t \in \mathbb{R}\} \subset C(T)$ is a flow, and $D(Q_t) = e^t$.

Conversely, suppose that there is a flow $\{Q_t : t \in \mathbb{R}\} \subset C(T)$ such that $D(Q_t) = e^t$ for all $t \in \mathbb{R}$. The flow $\{Q_t : t \in \mathbb{R}\}$ is dissipative on X. It is well known that up to measure-theoretic isomorphism, $X = \Omega \times \mathbb{R}$ where Ω is some probability space, $Q_t(x, y) = (x, y + t)$, and $dm(x, y) = e^y dp(x)dy$ where p is the probability on Ω.

Since $\{Q_t : t \in \mathbb{R}\} \subset C(T)$, there is a non-singular transformation $R : \Omega \to \Omega$ such that

$$T(x, y) = (Rx, Y(x, y)).$$

A calculation shows that indeed $Y(x, y) = y - \log R'(x)$ where $R' = d(\lambda \circ R)/d\lambda$, i.e. T is the Maharam transformation of R. The ergodicity of T implies that Ω is non-atomic, and hence standard. ∎

REMARK. By Proposition 5.1, the skew products constructed in Theorem 2 are isomorphic to Maharam transformations.

PROPOSITION 5.2. *If T is Bernoulli, then there is an ergodic $\mathbb{Z}$-extension of T which is isomorphic to a Maharam transformation.*

Proof. Let $(X, \mathcal{B}, m, T)$ be a Bernoulli probability preserving transformation. By Theorem 2 and the above remark, there is $\psi : X \to \mathbb{R}$ such that T_ψ is a conservative, ergodic Maharam transformation.

As in [M-S] and [H-O-O] let

$$H := \{t \in \mathbb{R} : e^{2\pi i t\psi} \text{ cohomologous to a constant in } S^1\},$$

a Borel subgroup of $\mathbb{R}$. We claim that there is $c > 0$ with $nc \notin H$ for all $n \ge 1$. This follows from H having Lebesgue measure zero.

To see that H indeed has Lebesgue measure zero, we note that otherwise $H = \mathbb{R}$ and (by [M-S] and [H-O-O]) ψ is cohomologous to a constant in $\mathbb{R}$, contradicting ergodicity of T_ψ.

Let $\varphi : X \to \mathbb{T} \cong [0, 1/c)$ be defined by $\varphi = \psi \bmod 1/c$. There is a measurable function $\phi : X \times \mathbb{T} \to \mathbb{Z}$ such that $T_\psi \cong (T_\varphi)_\phi$.

By construction of $c > 0$, there are no $n \ge 1$ and $g : X \to S^1$ measurable and non-constant such that $e^{2\pi i n\varphi} = \overline{g} \circ Tg$. It follows from §2 that T_φ is weakly mixing, whence by Theorem 1 of [R], T_φ is Bernoulli, and since $h(T_\varphi) = h(T)$, we see by [O] that $T_\varphi \cong T$. The conclusion is that $T_\psi \cong (T_\varphi)_\phi \cong T_{\phi'}$, a $\mathbb{Z}$-extension of T. ■

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