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Hamiltonian systems with linear potential and elastic constraints

by

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Abstract. We consider a class of Hamiltonian systems with linear potential, elastic constraints and arbitrary number of degrees of freedom. We establish sufficient conditions for complete hyperbolicity of the system.

0. Introduction. We study a class of Hamiltonian systems with linear potential and arbitrary number of degrees of freedom. The Hamiltonian is given by

$$H = \frac{1}{2}\sum_{i,j=1}^{n} k_{ij}\xi_i\xi_j + \sum_{l=1}^{n} c_l\eta_l,$$

where $(\eta,\xi) \in \mathbb{R}^n \times \mathbb{R}^n$ are "positions" and "momenta", and $K = \{k_{ij}\}$ is a constant symmetric positive definite matrix giving the kinetic energy. The equations of motions are

$$\frac{d^2\eta_i}{dt^2} = -\sum_{j=1}^{n} k_{ij}c_j = \text{const.}$$

We close the system and couple different degrees of freedom by restricting it to the positive cone

$$\eta_1 \ge 0,\ \ldots,\ \eta_n \ge 0.$$

When one of the η coordinates vanishes the velocity is changed instantaneously by the rules of elastic collisions, i.e., the component of the velocity

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parallel to the face of the cone is preserved and the component orthogonal to the face is reversed. Orthogonality is taken with respect to the scalar product defined by the kinetic energy.

With these elastic constraints the system is closed provided that all the coefficients $c_1, \ldots, c_n$ are positive. The restriction of this system to a level set of the Hamiltonian (i.e., we fix the total energy) has a finite Liouville measure which is preserved by the dynamics. There are singular trajectories in the system (hitting the lower dimensional faces of the cone or having zero velocity on a face of the cone) which are defined for finite time only but they form a subset of zero measure. Dynamics is well defined almost everywhere. Moreover, the derivative of the flow is also defined almost everywhere and Lyapunov exponents are well defined for our system (cf. [O], [R]).

MAIN THEOREM. *If all the off-diagonal entries of the positive definite matrix K^{-1} are negative then the Hamiltonian system with elastic constraints restricted to one energy level is completely hyperbolic*, *i.e.*, *all Lyapunov exponents but one are nonzero almost everywhere*,

By the structural theory of hyperbolic systems with singularities developed by Katok and Strelcyn [K-S] we can conclude that our system has at most countably many ergodic components. The mixing properties of the flow are as usual not readily accessible. But if we consider the natural Poincaré section map (from a face of the positive cone to another face) we can apply the results of Chernov and Hasskel [Ch-H] and Ornstein and Weiss [O-W] to get the Bernoulli property on ergodic components. We are unable to make rigorous claims about ergodicity because the singularities of the system are not properly aligned (except for $n = 2$), which does not allow the implementation of the Sinai–Chernov methods. This point is discussed in detail in [L-W]. At the same time there is little doubt that the system is actually ergodic.

There are concrete systems of interacting particles that fall into the category described in the Main Theorem. One such system is a variation of the system of parallel sheets interacting by gravitational forces, studied recently by Reidl and Miller [R-M].

Let us consider the system of $n+1$ point particles in a line with positions $q_0, q_1, \ldots, q_n$ and masses $m_0, \ldots, m_n$. Their interaction is defined by a linear translation invariant potential $U(q) = \sum_{i=1}^{n} c_i(q_i - q_0)$. The Hamiltonian of the system is

$$H = \sum_{i=0}^{n} \frac{p_i^2}{2m_i} + \sum_{i=1}^{n} c_i(q_i - q_0). \tag{0.1}$$

We introduce the elastic constraints

$$q_1 - q_0 \geq 0, \quad q_2 - q_0 \geq 0, \ \ldots, \ q_n - q_0 \geq 0, \tag{0.2}$$

i.e., the particles go through each other freely except for the q_0-particle which collides elastically with every other particle.

A convenient interpretation of the system is that of a horizontal floor of finite mass m_0 and n particles of masses $m_1, \ldots, m_n$. The floor and the particles can move only in the vertical direction and their positions are q_0 and $q_1, \ldots, q_n$, respectively. There is a constant force of attraction between any of the particles and the floor. Moreover, the particles collide elastically with the floor and there are no collisions between the particles (they move along different parallel lines perpendicular to the floor). Hence the particles "communicate" with each other only through collisions with the floor, which is a rather weak interaction.

Introducing symplectic coordinates (η, ξ),

$$\begin{aligned}
\eta_0 &= m_0 q_0 + m_1 q_1 + \ldots + m_n q_n, \\
\eta_i &= q_i - q_0, \\
p_0 &= m_0 \xi_0 - \xi_1 - \ldots - \xi_n, \\
p_i &= m_i \xi_0 + \xi_i, \qquad i = 1, \ldots, n,
\end{aligned} \tag{0.3}$$

and setting the total momentum and the center of mass at zero, $\eta_0 = 0$, $\xi_0 = 0$, we obtain the Hamiltonian

$$H = \frac{(\xi_1 + \ldots + \xi_n)^2}{2m_0} + \sum_{i=1}^{n} \frac{\xi_i^2}{2m_i} + \sum_{i=1}^{n} c_i \eta_i.$$

This system satisfies the assumptions of the Main Theorem and hence it is completely hyperbolic. Note that no conditions on the masses are required.

We can introduce additional interactions between particles by stacking groups of them on vertical lines. The particles on the same vertical line will collide elastically with each other and only the bottom particle collides with the floor. Mathematically this corresponds to adding more constraints to (0.2). We establish that such systems are also completely hyperbolic if the masses satisfy certain inequalities. We must assume though that the accelerations of all the particles in one stack are equal, they can be different for different stacks.

As we add more constraints our conditions on the masses which guarantee complete hyperbolicity become more stringent. This seems somewhat paradoxical: as the interactions of the particles become richer the ergodicity of the system (equipartition of energy) is more likely to fail.

This behavior becomes more intuitive when we modify the original system of noninteracting particles falling to the floor by splitting each mass into two or more masses that are stacked on one vertical line. In the original system the particles have to freely "share" their energy with the floor and hence with other particles. In the modified system the stack of particles acts as "internal" degrees of freedom which may store energy for extended periods of time. One would expect that the energy transfer between stacks is less vigorous than in the case when all the masses in one stack are glued into one particle.

The extremal case is that of one stack, i.e., where we introduce the constraints

$$q_0 \le q_1 \le \ldots \le q_n \tag{0.4}$$

and $c_i = \alpha m_i,\ i = 1, \ldots, n$. If $m_1 = \ldots = m_n$ then the resulting system is a factor of the system with the constraints (0.2) and in particular it is completely hyperbolic. In general, complete hyperbolicity occurs when the masses satisfy special inequalities. More precisely, if the sequence

$$a_i = \frac{m_0 + m_1 + \ldots + m_{i-1}}{m_i} + i, \quad i = 1, \ldots, n,$$

satisfies $a_1 < a_2 \le \ldots \le a_n$, then the system is completely hyperbolic. These conditions are substantial and not merely technical since the system is completely integrable if for some constant $a > n$,

$$\frac{m_i}{m_0} = \frac{a}{(a-i)(a-i+1)}, \quad i = 1, \ldots, n.$$

Let us end this introduction with the outline of the content of the paper. In Section 1 we review the notion of flows with collisions ([W1]), a mixture of differential equations and discrete time dynamical systems (mappings). We define hyperbolicity (complete and partial) for flows with collisions and formulate the criterion of hyperbolicity from [W3].

In Section 2 we study the geometry of simplicial cones, which we call wedges. We introduce a special class of wedges, called simple, and discuss their geometric invariants. As a byproduct we obtain a dual characterization of positive definite tridiagonal matrices which is of independent interest.

In Section 3 we introduce a Hamiltonian system with linear potential and elastic constraints which we call a PW system (Particle in a Wedge). It is defined by a wedge and an acceleration direction (from the dual wedge). A point particle is confined to the wedge and accelerated in the chosen direction (falling down). We establish that the system of falling particles in a line (PFL system), introduced and studied in [W1], is equivalent to a PW system in a simple wedge with acceleration parallel to the first (or last) generator of the wedge. We recast the conditions of partial hyperbolicity from [W1] in terms of the geometry of the simple wedge. In a recent paper Simányi [S] showed that these conditions guarantee complete hyperbolicity.

In Section 4 we give a new edition of the results of [W1], on monotonicity of PFL and PW systems, in a more geometric language appropriate for the present work. The new formulations are necessary for the proof of the Main Theorem.

In Section 5 we consider two special classes of Hamiltonian systems, the system (0.1) with the constraints (0.4) and another class. Both classes reduce straightforwardly to PW systems in simple wedges. We apply the criteria of complete hyperbolicity and complete integrability and get in particular the result formulated above.

In Section 6 we introduce wide wedges and we prove the Main Theorem.

In Section 7 we study the system (0.1) with arbitrary "stacking rules" added to the constraints (0.2). We derive conditions on the masses which guarantee complete hyperbolicity of the system, in terms of the graph of constraints.

Section 8 contains remarks and open problems.

1. Hamiltonian flows with collisions. A *flow with collisions* is a concatenation of a flow defined by a vector field on a manifold and mappings defined on submanifolds (*collision manifolds*) of codimension one. Trajectories of a flow with collisions follow the trajectories of the flow until they reach one of the collision manifolds where they are glued with another trajectory by the *collision map*. A more precise description of this simple concept is somewhat lengthy. We will do it for Hamiltonian flows only. A more detailed discussion can be found in [W1] and [W3].

Let (N,ω) be a smooth $2n$-dimensional symplectic manifold with symplectic form ω and H be a smooth function on N. We denote by ∇H the Hamiltonian vector field defined by the Hamiltonian function H. Let further M be a $2n$-dimensional closed submanifold of N with piecewise smooth boundary ∂M. For simplicity we assume that ∇H does not vanish in M. Let $N^h = \{x \in N \mid H(x) = h\}$ be a smooth level set of the Hamiltonian. The Hamiltonian vector field ∇H is tangent to N^h. We do not require that M is compact, but we do assume that the restricted level sets of the Hamiltonian, $M \cap N^h$, are compact for all values of h.

In the boundary we distinguish the *regular* part, ∂M_{r}, consisting of points which do not belong to more than one smooth piece and where the vector field ∇H is transversal to ∂M. The remaining part of the boundary is called *singular*. We assume that the singular part of the boundary has zero Lebesgue measure in ∂M. The regular part of the boundary is further divided into ∂M^-, the "outgoing" part, where ∇H points outside of the domain M, and ∂M^+, the "incoming" part, where ∇H points inside of M.

We assume that a mapping $\Phi : \partial M^- \to \partial M^+$, the *collision mapping*, is given and that it preserves the Hamiltonian, $H \circ \Phi = H$. Any codimension one submanifold of N^h transversal to ∇H inherits a canonical symplectic structure, the restriction of the symplectic form ω. Hence $\partial M_{\mathrm{r}} \cap N^h$ has a symplectic structure and we require that the collision map restricted to $\partial M^- \cap N^h$ preserves this symplectic structure. The Liouville measure (the symplectic volume element) defined by the symplectic structure is thus preserved.

In such a setup we define the *Hamiltonian flow with collisions*

$$\Psi^t : M \to M, \qquad t \in \mathbb{R},$$

by describing the trajectories of the flow. So $\Psi^t(x),\ t \geq 0$, coincides with the trajectory of the original Hamiltonian flow (defined by ∇H) until we get to the boundary of M at time $t_{\mathrm{c}}(x)$, the *collision time*. If $\Psi^{t_{\mathrm{c}}}(x)$ belongs to the singular part of the boundary then the flow is not defined for $t > t_{\mathrm{c}}$ (the trajectory "dies" there). Otherwise the trajectory is continued at the point $\Phi(\Psi^{t_{\mathrm{c}}}(x))$ until the next collision time, i.e.,

$$\Psi^{t_{\mathrm{c}}+t}(x) = \Psi^t \Phi \Psi^{t_{\mathrm{c}}}(x).$$

This flow with collisions may be badly discontinuous but thanks to the preservation of the Liouville measures by the Hamiltonian flow and the collision map, the flow Ψ^t is a well defined measurable flow in the sense of Ergodic Theory (cf. [C-F-S]). Let $\nu = \nu_h$ denote the Liouville measure on the level set $N^h \cap M$ of the Hamiltonian. By the compactness assumption ν_h is finite for all smooth level sets N^h. We can now study the ergodic properties of the flow Ψ^t restricted to one level set.

The derivative $D\Psi^t$ is also well defined almost everywhere in M and for all t, except the collision times. This allows the definition of Lyapunov exponents for our Hamiltonian flow with collisions, under the integrability assumption ([O], [R])

$$\int_{N^h} \ln^+ \|D\Psi^1\| \, d\nu_h < \infty.$$

In general, the Lyapunov exponents are defined almost everywhere and they depend on a trajectory of the flow. Due to the Hamiltonian character of the flow, two of the $2n$ Lyapunov exponents are automatically zero, and the other come in pairs of opposite numbers. Hence there is an equal number of positive and negative Lyapunov exponents.

Definition 1.1. A Hamiltonian flow with collisions is called (*nonuniformly*) *partially hyperbolic* if some of its Lyapunov exponents are nonzero almost everywhere, and it is called (*nonuniformly*) *completely hyperbolic* if all but two of its Lyapunov exponents are nonzero almost everywhere.

Definition 1.2. A Hamiltonian flow with collisions is called *completely integrable* if there are n functions $F_1, \ldots, F_n$ in involution, with linearly independent differentials almost everywhere, which are first integrals for both the flow and the collision map, i.e., $dF_i(\nabla H) = 0$ and $F_i \circ \Phi = F_i,\ i = 1, \ldots, n$.

As usual, completely integrable Hamiltonian flows with collisions have only zero Lyapunov exponents.

We will outline here a criterion for nonvanishing of Lyapunov exponents. Complete exposition can be found in [W3]. Note that we introduce some modifications in the formulations, to facilitate the applications of this criterion in the present paper.

We choose two transversal subbundles, $L_1(x)$ and $L_2(x)$, $x \in M$, of Lagrangian subspaces in the tangent bundle of M. We allow these bundles to be discontinuous and defined almost everywhere. The only requirement is their measurability.

An ordered pair of transversal Lagrangian subspaces, L_1 and L_2, defines a quadratic form Q by the formula

$$Q(v) = \omega(v_1, v_2), \qquad \text{where } v = v_1 + v_2,\ v_i \in L_i,\ i = 1, 2.$$

Further we define the sector $\mathcal{C}$ between L_1 and L_2 by $\mathcal{C} = \{v \mid Q(v) \geq 0\}$.

We assume that ∇H belongs to L_2 at almost all points (we could as well assume that it belongs to L_1). This assumption is very important for the Hamiltonian formalism, it allows us to project the quadratic form Q onto the factor of the tangent space to the level set of the Hamiltonian by the one-dimensional subspace spanned by ∇H. This factor space plays the role of the "transversal section" of the flow restricted to a smooth level set. Note that in general we do not have an invariant codimension one subspace transversal to the flow.

Definition 1.3. The Hamiltonian flow with collisions, Ψ^t, is called *monotone* (with respect to the bundle of sectors $\mathcal{C}(x)$, $x \in M$), if for almost all points in M,

$$Q(D\Psi^t v) \geq Q(v),$$

for all vectors v tangent to a smooth level set of the Hamiltonian, $M \cap N^h$, and all $t \geq 0$ for which the derivative is well defined.

The monotonicity of the flow does not imply nonvanishing of any Lyapunov exponents. Actually completely integrable Hamiltonian flows are typically monotone with respect to some bundle of sectors. To obtain hyperbolicity one needs to examine what happens to the "sides" L_1 and L_2 of the sector $\mathcal{C}$. Let $\widetilde{L}_1 = L_1 \cap \{v \mid dH(v) = 0\}$ be the intersection with the tangent space to the level set of the Hamiltonian (note that L_2 is always tangent to the level set because we assume that $\nabla H \in L_2$ and hence the dimension of $\widetilde{L}_1$ is always $n-1$). In a monotone system there are two possibilities for a vector from $\widetilde{L}_1$ (or from L_2): either it enters the interior of the sector $\mathcal{C}$ at some time $t > 0$ or it forever stays in $\widetilde{L}_1$ (or in L_2).

For a monotone flow we define the L_1-*exceptional subspace* $\mathcal{E}_1(x) \subset$

$\widetilde{L}_1(x)$ as

$$\mathcal{E}_1(x) = \widetilde{L}_1(x) \cap \bigcap_{t\geq 0} D\Psi^{-t}\widetilde{L}_1(\Psi^t x), \tag{1.1}$$

i.e., $\mathcal{E}_1(x)$ is the subspace of vectors from $\widetilde{L}_1(x)$ which do not ever enter the sector $\mathcal{C}$. Similarly we define the L_2-*exceptional subspace* $\mathcal{E}_2(x)$. The L_2-exceptional subspace always contains the Hamiltonian vector field ∇H. We call a point $x \in M$ L_1-*exceptional* if $\dim \mathcal{E}_1(x) \geq 1$, and L_2-*exceptional* if $\dim \mathcal{E}_2(x) \geq 2$.

The following theorem is essentially proven in [W3].

THEOREM 1.4. *If the Hamiltonian flow with collisions is monotone and the sets of L_1-exceptional points and L_2-exceptional points have measure zero then the flow is completely hyperbolic.*

A criterion for partial hyperbolicity is given by the following (cf. [W3])

THEOREM 1.5. *If the Hamiltonian flow with collisions is monotone then it is also partially hyperbolic, provided one of the following conditions is satisfied*:

(1) *the set of L_1-exceptional points has measure zero and* $\dim \mathcal{E}_2(x) \leq n-1$ *for almost all* $x \in M$,

(2) *the set of L_2-exceptional points has measure zero and* $\dim \mathcal{E}_1(x) \leq n-2$ *for almost all* $x \in M$.

2. Simple wedges. Consider the n-dimensional euclidean space E. We define a k-*dimensional wedge*, $k \leq n$, to be a convex cone in E generated by k linearly independent rays. Hence we have a k-dimensional wedge $W \subset E$ if there is a linearly independent set of k vectors, $\{e_1, \ldots, e_k\}$, such that

$$W = \{e \in E \mid e = \lambda_1 e_1 + \ldots + \lambda_k e_k,\ \lambda_i \geq 0,\ i = 1, \ldots, k\}.$$

We call the vectors $\{e_1, \ldots, e_k\}$ the *generators* of the wedge and we denote the wedge generated by them $W(e_1, \ldots, e_k)$. The generators are uniquely defined up to positive scalar factors.

We denote by $S(e_1, \ldots, e_k) \subset E$ the linear subspace spanned by the linearly independent vectors $\{e_1, \ldots, e_k\}$.

The dual space E^* can be naturally identified with E. Thus the cone W^* dual to the n-dimensional wedge W is itself an n-dimensional wedge in E.

Let $\{e_1, \ldots, e_n\}$ be an ordered basis in E and $\{f_1, \ldots, f_n\}$ be the dual basis, i.e., $\langle f_i, e_j\rangle = \delta_i^j$, the Kronecker delta. Clearly we have

$$(W(e_1, \ldots, e_n))^* = W(f_1, \ldots, f_n).$$

PROPOSITION 2.1. *The following properties of an ordered basis* $\{e_1, \ldots, e_n\}$ *of unit vectors and its dual basis* $\{f_1, \ldots, f_n\}$ *are equivalent*:

(1) *The orthogonal projection of* e_l *onto* $S(e_{l+1},\ldots,e_n)$ *is parallel to* e_{l+1}, *for* $l=1,\ldots,n-1$.

(2) $\langle e_i,e_j\rangle=\prod_{s=i}^{j-1}\langle e_s,e_{s+1}\rangle$ *for all* $1\le i\le j-1\le n-1$.

(3) *With the convention that* $e_0=e_{n+1}=0$ *we have*

$$f_i=b_{i-1}e_{i-1}+a_ie_i+b_ie_{i+1},\qquad i=1,\ldots,n,$$

where the coefficients are given by

$$a_i=\frac{1-\langle e_{i-1},e_i\rangle^2\langle e_i,e_{i+1}\rangle^2}{(1-\langle e_{i-1},e_i\rangle^2)(1-\langle e_i,e_{i+1}\rangle^2)},\qquad b_i=\frac{-\langle e_i,e_{i+1}\rangle}{1-\langle e_i,e_{i+1}\rangle^2}.$$

(4) $\langle f_i,f_j\rangle=0$ *for all* $1\le i,j\le n$, $|i-j|\ge 2$.

Proof. (1)⇔(2). Observe that since $\{e_1,\ldots,e_n\}$ are unit vectors, (1) can be reformulated as $\langle e_i,e_j\rangle=\langle e_i,e_{i+1}\rangle\langle e_{i+1},e_j\rangle$ for $i=1,\ldots,k-1$ and $j=i+2,\ldots,k$. We get (2) by induction on the distance between i and j. Clearly the converse is also true.

(2)⇒(3) and (4). In general for dual bases $e_1,\ldots,e_n$ and $f_1,\ldots,f_n$ we have

$$f_i=\sum_{j=1}^{n}\langle f_i,f_j\rangle e_j,$$

and the Gramm matrix $\{\langle f_i,f_j\rangle\}$ is the inverse of $\{\langle e_i,e_j\rangle\}$. One can check straightforwardly that the tridiagonal matrix of coefficients from (3) is the inverse of the matrix of coefficients from (2), which proves that (2) implies (3) and (4).

(4)⇒(1). Let P_l denote the orthogonal projection onto $S(e_{l+1},\ldots,e_n)$. We prove (1) by induction on l. We have $\langle f_1,f_1\rangle e_1=f_1-\langle f_1,f_2\rangle e_2$, which implies that $\langle f_1,f_1\rangle P_1e_1=-\langle f_1,f_2\rangle e_2$. Given that $P_{l-1}e_{l-1}=r_le_l$ we apply the projection P_{l-1} to both sides of

$$f_l=\langle f_{l-1},f_l\rangle e_{l-1}+\langle f_l,f_l\rangle e_l+\langle f_l,f_{l+1}\rangle e_{l+1}.$$

We get

$$P_{l-1}f_l=(\langle f_{l-1},f_l\rangle r_l+\langle f_l,f_l\rangle)e_l+\langle f_l,f_{l+1}\rangle e_{l+1}.$$

The coefficient of e_l cannot be zero since otherwise $P_{l-1}f_l=\langle f_l,f_{l+1}\rangle e_{l+1}$, which contradicts the orthogonality of f_l and e_{l+1}. Hence we can write $P_{l-1}f_l=s_le_l+\langle f_l,f_{l+1}\rangle e_{l+1}$ with $s_l\ne 0$. Applying P_l to both sides of the last equation we obtain $s_lP_le_l=-\langle f_l,f_{l+1}\rangle e_{l+1}$. ∎

We now introduce a special type of wedge.

Definition 2.2. A k-dimensional wedge $W\subset E$ is called *simple* if its generators $\{e_1,\ldots,e_k\}$ can be ordered in such a way that for any $i=1,\ldots,k-1$, the orthogonal projection of e_i onto the $(k-i)$-dimensional subspace $S(e_{i+1},\ldots,e_k)$ is a positive multiple of e_{i+1}. The ordering of the generators for which this property holds is called *distinguished*.

From Proposition 2.1 we obtain immediately

PROPOSITION 2.3. *Let $\{e_1, \ldots, e_k\}$ be a set of linearly independent unit vectors. The wedge $W(e_1, \ldots, e_k)$ is simple and the ordering of the generators is distinguished if and only if*

(1) $\langle e_i, e_{i+1} \rangle > 0$ *for* $i = 1, \ldots, n-1$ *and*
(2) $\langle e_i, e_j \rangle = \prod_{l=i}^{j-1} \langle e_l, e_{l+1} \rangle$ *for all* $1 \le i \le j-1 \le k-1$.

COROLLARY 2.4. *Any face of a simple wedge is a simple wedge. A simple wedge has exactly two distinguished orderings, one is the reversal of the other.*

Proof. It follows from Proposition 2.3 that any face of a simple wedge is simple and that the reversal of a distinguished ordering is distinguished.

It remains to show that there are no other distinguished orderings. This follows immediately from the following observation. Suppose $\{e_1, \ldots, e_k\}$ are unit generators of a simple wedge in a distinguished order. Then $\langle e_1, e_k \rangle < \langle e_i, e_j \rangle$ for any $1 \le i < j \le k$, $(i, j) \ne (1, k)$.

A dual characterization of a simple wedge is given by

PROPOSITION 2.5. *Let $W(e_1, \ldots, e_n)$ be a wedge in an n-dimensional Euclidean space E and $\{f_1, \ldots, f_n\}$ be the dual basis. $W(e_1, \ldots, e_n)$ is a simple wedge and the order of the generators is distinguished if and only if*

(1) $\langle f_i, f_{i+1} \rangle < 0$ *for* $i = 1, \ldots, n-1$ *and*
(2) $\langle f_i, f_j \rangle = 0$ *for all* $1 \le i, j \le n$, $|i-j| \ge 2$.

Proof. Assuming without loss of generality that $\{e_1, \ldots, e_n\}$ are unit vectors, we deduce from Proposition 2.1(3) that

$$\langle f_i, f_{i+1} \rangle = \frac{-\langle e_i, e_{i+1} \rangle}{1 - \langle e_i, e_{i+1} \rangle^2} \quad \text{for } i = 1, \ldots, n-1.$$

Hence indeed (1) is equivalent to the property (1) of Proposition 2.3. ■

The geometry of a k-dimensional simple wedge is completely determined by the angles $0 < \alpha_i < \pi/2$, $i = 1, \ldots, k-1$, that the vectors e_i make with e_{i+1} (or equivalently with the subspace $S(e_{i+1}, \ldots, e_k)$). Assuming that the generators $\{e_1, \ldots, e_k\}$ are unit vectors we have

$$\cos \alpha_i = \langle e_i, e_{i+1} \rangle, \quad i = 1, \ldots, k-1. \tag{2.1}$$

We choose to characterize the geometry of a simple wedge by another set of angles, $0 < \beta_i < \pi/2$, $i = 1, \ldots, k-1$, where β_i is the angle between two $(k-i)$-dimensional faces, $S(e_{i+1}, e_{i+2}, \ldots, e_k)$ and $S(e_i, e_{i+2}, e_{i+3}, \ldots, e_k)$, of the simple $(k-i+1)$-dimensional wedge $W(e_i, e_{i+1}, \ldots, e_k)$. In particular, the angle β_1 and the angle between f_1 and f_2 (from the dual basis

$\{f_1,\ldots,f_k\}$) add up to π. Hence we get, using Proposition 2.1(3),

$$\cos\beta_1 = \frac{\cos\alpha_1 \sin\alpha_2}{\sqrt{1-\cos^2\alpha_1\cos^2\alpha_2}}.$$

It follows immediately that for any $i=1,\ldots,k-2$, we also get

$$\cos\beta_i = \frac{\cos\alpha_i \sin\alpha_{i+1}}{\sqrt{1-\cos^2\alpha_i\cos^2\alpha_{i+1}}}.$$

This can be transformed into

$$\tan\beta_i = \frac{\tan\alpha_i}{\sin\alpha_{i+1}}, \qquad i=1,\ldots,k-2. \tag{2.2}$$

We also have obviously $\beta_{k-1}=\alpha_{k-1}$. Hence indeed the information contained in the set of β-angles determines the simple wedge completely (up to isometry).

3. Particle falling in a wedge (PW system) and the system of falling particles in a line (PFL system). Given an n-dimensional wedge W in an n-dimensional Euclidean space E and a vector $a\in\operatorname{int}W^*$, we consider the system of a point particle falling in W with constant acceleration $-a$ and bouncing off elastically from the $(n-1)$-dimensional faces of the wedge W (a *PW system*). In an elastic collision with a face the velocity vector is instantaneously changed: the component orthogonal to the face is reversed and the component parallel to the face is preserved.

The condition that the acceleration vector is in the interior of the dual cone is equivalent to the system being closed (finite) under the energy constraint. One can change the acceleration vector by rescaling time, so that in studying the dynamical properties of such a system only the direction of the acceleration matters.

A PW system is in a natural way a Hamiltonian flow with collisions. If we choose the generators of an n-dimensional wedge W as a basis in E, we can identify E with $\mathbb{R}^n$ with coordinates $(\eta_1,\ldots,\eta_n)$. The wedge W becomes the positive cone $W=\{(\eta_1,\ldots,\eta_n)\in\mathbb{R}^n \mid \eta_i\ge 0,\ i=1,\ldots,n\}$. Let the scalar product be defined in these coordinates by a positive definite matrix L. Proposition 2.5 immediately yields

PROPOSITION 3.1. *The wedge W is simple if and only if the matrix $K=L^{-1}$ is tridiagonal with negative entries below and above the diagonal.* ∎

The PW system in the wedge W with acceleration vector $a\in\operatorname{int}W^*$ has the Hamiltonian

$$H=\tfrac{1}{2}\langle K\xi,\xi\rangle+\langle c,\eta\rangle, \tag{3.2}$$

where $\langle\cdot,\cdot\rangle$ denotes the arithmetic scalar product in $\mathbb{R}^n$, $\xi\in\mathbb{R}^n$ is the momentum of the particle and $c\in\mathbb{R}^n$ is a vector with all entries positive, so that the acceleration vector $a = Kc$ is in the interior of the dual wedge. This representation of the wedge and the PW system will be referred to later on as the *standard representation* of the PW system.

Consider the system of n point particles (or rods) in a line with positions $0 \le q_1 \le \ldots \le q_n$ and masses $m_1, \ldots, m_n$, falling with constant acceleration (equal to 1) towards the floor (at 0). The particles collide elastically with each other and the floor. This system of falling particles in a line will be referred to as a *PFL system*.

Hence between collisions the motion of the particles is governed by the Hamiltonian

$$H = \sum_{i=1}^{n} \left(\frac{p_i^2}{2m_i} + m_i q_i \right).$$

The configuration space of the system, $W = \{q \in \mathbb{R}^n \mid 0 \le q_1 \le \ldots \le q_n\}$, with scalar product determined by the kinetic energy is a simple n-dimensional wedge. To see this we introduce symplectic coordinates (x, v) in which the scalar product (and the kinetic energy) have the standard form,

$$x_i = \sqrt{m_i}\, q_i, \qquad v_i = \frac{p_i}{\sqrt{m_i}}, \qquad i = 1, \ldots, n. \tag{3.3}$$

In these coordinates the Hamiltonian of the system changes to

$$H = \sum_{i=1}^{n} \left(\frac{v_i^2}{2} + \sqrt{m_i}\, x_i \right)$$

and we can consider x and v as vectors in the same standard Euclidean space $\mathbb{R}^n$. The elastic collisions of the particles are translated into elastic reflections in the faces of the wedge. In these coordinates the wedge W is generated by the unit vectors $\{e_1, \ldots, e_n\}$,

$$\sqrt{M_i}\, e_i = (0, \ldots, 0, \sqrt{m_i}, \ldots, \sqrt{m_n}),$$

where $M_i = m_i + \ldots + m_n$, $i = 1, \ldots, n$. We see that for $1 \le i < j \le n$,

$$\langle e_i, e_j \rangle = \sqrt{M_j}/\sqrt{M_i},$$

which immediately yields the properties (1) and (2) of Proposition 2.3. Further using (2.1) and (2.2) we get for this simple wedge

$$\cos^2 \alpha_i = \frac{M_{i+1}}{M_i}, \qquad \sin^2 \alpha_i = \frac{m_i}{M_i}, \qquad \tan^2 \beta_i = \frac{m_i}{m_{i+1}}. \tag{3.4}$$

It follows from (3.4) that every simple wedge can appear as the configuration space of a PFL system with appropriate masses, depending on the geometry of the wedge. (Note that the formulas (3.4) provide clear justification for the introduction of the β-angles in the geometric description of a simple wedge.) The acceleration vector for a PFL system has the direction of the first generator of the simple wedge, more precisely the acceleration vector is $\sqrt{M_1}\, e_1$.

We arrived at the important conclusion that a PW system in a simple n-dimensional wedge with acceleration along the first (or the last) generator of the wedge is equivalent to a PFL system with appropriate masses of the n particles.

Finally, we introduce yet another system of symplectic coordinates (η, ξ) for the PFL system in which the configuration wedge becomes the positive cone (*standard representation*). Let

$$\eta_1 = q_1, \quad \eta_{i+1} = q_{i+1} - q_i, \\ p_i = \xi_i - \xi_{i+1}, \quad p_n = \xi_n, \quad i = 1, \ldots, n-1.$$

The Hamiltonian of the system becomes

$$H = \sum_{i=1}^{n-1} \frac{(\xi_i - \xi_{i+1})^2}{2m_i} + \frac{\xi_n^2}{2m_n} + \sum_{i=1}^{n} M_i \eta_i.$$

We get a tridiagonal matrix with negative off-diagonal entries in the kinetic energy, as required by Proposition 3.1.

4. Monotonicity of PFL systems. We now recall the results about the monotonicity and hyperbolicity of PFL systems. These systems were introduced and studied in [W1], where the reader can find more details. When the masses of the particles are equal the system is completely integrable. Indeed, if we allow the particles to pass through each other then the n individual energies of the particles are preserved and provide us with n independent integrals in involution. In the case of elastic collisions of the particles we need to use symmetric functions of the n individual energies as first integrals in involution. It was established in [W1] that if the masses are nonincreasing, $m_1 \geq \ldots \geq m_n$, and are not all equal then the system is partially hyperbolic. In a recent paper Simányi [S] showed that if $m_1 > m_2 \geq \ldots \geq m_n$ then the system is completely hyperbolic.

We will give here a detailed and modified proof that PFL systems are monotone under the above condition, which will be the basis for the proof of our Main Theorem.

In the phase space of a PFL system we introduce the Euclidean coordinates (x, v) given by (3.3). We choose two bundles of Lagrangian subspaces L_1 and L_2,

$$L_1 = \{dv_1 = \ldots = dv_n = 0\}, \quad L_2 = \left\{dx_i = -\frac{v_i}{\sqrt{m_i}} dv_i,\ i = 1, \ldots, n\right\}.$$

The Hamiltonian vector field ∇H belongs to L_2. The quadratic form Q is given by

$$Q = \sum_{i=1}^{n} dx_i dv_i + \sum_{i=1}^{n} \frac{v_i}{\sqrt{m_i}} dv_i^2 .$$

THEOREM 4.1. *If* $m_1 \geq \ldots \geq m_n$, *then the PFL system is monotone* (*with respect to the bundle of sectors between* L_1 *and* L_2).

Between collisions the form Q is constant. Indeed, we have

$$\begin{aligned} \frac{d}{dt} x_i &= v_i, & \frac{d}{dt} dx_i &= dv_i, \\ \frac{d}{dt} v_i &= -\sqrt{m_i}, & \frac{d}{dt} dv_i &= 0, \quad i = 1, \ldots, n, \end{aligned}$$

which yields $dQ/dt = 0$.

The effect of a collision between different particles on the form Q will be obtained with the help of the following important construction which will play a crucial role in the future. We represent our system as a PW system in a simple n-dimensional wedge W, with geometry determined by the masses (cf. (3.4)), and with acceleration vector parallel to the first generator. A collision of two particles becomes a collision with an $(n-1)$-dimensional face of the wedge, containing the first generator. Consider the wedge $\widetilde{W}$ obtained by reflection in the face. Instead of reflecting the velocity in the face we can allow the particle to pass through the face to the reflected wedge $\widetilde{W}$. Note that the acceleration vector stays the same (since it lies in the face). What changes is the quadratic form, it experiences a jump discontinuity. Let $\widetilde{Q}$ be the quadratic form associated with the PW system in the reflected wedge $\widetilde{W}$. We want to examine the difference of Q and $\widetilde{Q}$ at the common face. Actually, if we identify all the tangent spaces to the common phase space of the two PW systems (in W and $\widetilde{W}$) the forms become functions of tangent vectors from that common space that depend only on velocities (but not on positions).

For the purpose of future applications we will consider a generalization of this construction, namely we will not assume that the two wedges are symmetric, but only that they share a common $(n-1)$-dimensional face.

Consider two simple n-dimensional wedges $W = W(e_1, \ldots, e_n)$ and $\widetilde{W} = W(\widetilde{e}_1, \ldots, \widetilde{e}_n)$ (we tacitly assume that the generators are always written in a chosen distinguished order). Assume that the two wedges have isometric $(n-1)$-dimensional faces, obtained when we drop e_{l+1} and $\widetilde{e}_{l+1}$, respectively, from the list of generators. We choose to glue the two wedges together along the isometric faces, i.e., we assume that $e_i = \widetilde{e}_i$ for $i \neq l+1$, and that the two wedges are on opposite sides of the hyperplane containing the isometric faces. Further we consider the PW systems in these wedges with common acceleration vector parallel to the first generator $e_1 = \widetilde{e}_1$. In each of the wedges the PW system is equivalent to a PFL system with appropriate masses of the particles, $(m_1, \ldots, m_n)$ and $(\widetilde{m}_1, \ldots, \widetilde{m}_n)$ respectively.

LEMMA 4.2. *We have*

$$m_i = \widetilde{m}_i \quad \text{for all } i \neq l, l+1,$$
$$m_l + m_{l+1} = \widetilde{m}_l + \widetilde{m}_{l+1}.$$

Proof. Since the two systems have acceleration vectors of the same length it follows that $M_1 = m_1 + \ldots + m_n = \widetilde{M}_1 = \widetilde{m}_1 + \ldots + \widetilde{m}_n$. Our claim now follows from the formulas (3.4) for the α-angles in a simple wedge, since the isometry of the faces implies $\alpha_i = \widetilde{\alpha}_i$ for $i \neq l, l+1$. ■

We introduce the standard Euclidean coordinates, (3.3), $x \in \mathbb{R}^n$ and $\widetilde{x} \in \mathbb{R}^n$ in W and $\widetilde{W}$ respectively, associated with the PFL systems. The common face of the two wedges is described by

$$x_l/\sin\beta_l = x_{l+1}/\cos\beta_l \quad \text{and} \quad \widetilde{x}_l/\sin\widetilde{\beta}_l = \widetilde{x}_{l+1}/\cos\widetilde{\beta}_l$$

in the respective coordinate systems. These coordinate systems in the configuration space give rise to the respective coordinates in the phase spaces, (x, v) in $W \times \mathbb{R}^n$ and $(\widetilde{x}, \widetilde{v})$ in $\widetilde{W} \times \mathbb{R}^n$. The tangent spaces of these phase spaces are naturally identified because the wedges are contained in the same Euclidean space.

The two coordinate systems are connected by the following "gluing" transformation:

$$\begin{aligned} \widetilde{x}_i &= x_i \quad \text{for all } i \neq l, l+1, \\ \widetilde{x}_l &= -\cos\Theta\, x_l + \sin\Theta\, x_{l+1}, \\ \widetilde{x}_{l+1} &= \sin\Theta\, x_l + \cos\Theta\, x_{l+1}, \end{aligned} \tag{4.2}$$

where $\Theta = \beta_l + \widetilde{\beta}_l$ is defined by the β-angles of the respective wedges, i.e.,

$$\tan^2\beta_l = m_l/m_{l+1}, \qquad \tan^2\widetilde{\beta}_l = \widetilde{m}_l/\widetilde{m}_{l+1}.$$

Consider the quadratic forms Q and $\widetilde{Q}$ associated with the respective PFL systems. These quadratic forms depend on velocities (but not on positions), and the space of velocities of the two models is the same Euclidean space. Hence we can compare the two quadratic forms as functions on the space of velocities cross the tangent to the phase space.

PROPOSITION 4.3. *For the velocities of trajectories leaving W and entering $\widetilde{W}$ we have*

$$\widetilde{Q} \geq Q \quad \text{if and only if} \quad \beta_l + \widetilde{\beta}_l \geq \pi/2.$$

More precisely, *we have*

$$\widetilde{Q} - Q = \frac{1}{\sqrt{m_l + m_{l+1}}} \cdot \frac{2\sin(2(\beta_l + \widetilde{\beta}_l))}{\sin 2\beta_l \sin 2\widetilde{\beta}_l} \times (-\cos\beta_l\, v_l + \sin\beta_l\, v_{l+1})(-\cos\beta_l\, dv_l + \sin\beta_l\, dv_{l+1})^2.$$

COROLLARY 4.4. $Q = \widetilde{Q}$ *if and only if* $\beta_l + \widetilde{\beta}_l = \pi/2$.

Proof. Let us examine the quadratic form

$$Q = \sum_{i=1}^{n} dx_i dv_i + \sum_{i=1}^{n} \frac{v_i}{\sqrt{m_i}} dv_i^2.$$

The first sum is invariant under any coordinate changes which respect the distinction between positions and velocities. In the second sum only two terms are affected by the gluing transformation. Hence we obtain

$$\widetilde{Q} - Q = \sum_{i=l}^{l+1} \frac{\widetilde{v}_i}{\sqrt{m_i}} d\widetilde{v}_i^2 - \sum_{i=l}^{l+1} \frac{v_i}{\sqrt{m_i}} dv_i^2.$$

Our claim now follows by straightforward calculations. To make them more transparent we introduce yet other coordinate systems in the planes (v_l, v_{l+1}) and $(\widetilde{v}_l, \widetilde{v}_{l+1})$,

$$\begin{aligned} z_1 &= \sin\beta_l\, v_l + \cos\beta_l\, v_{l+1}, \\ z_2 &= -\cos\beta_l\, v_l + \sin\beta_l\, v_{l+1}, \end{aligned}$$

and parallel formulas for $(\widetilde{z}_1, \widetilde{z}_2)$. We have

$$\sum_{i=l}^{l+1} \frac{v_i}{\sqrt{m_i}} dv_i^2 = \frac{1}{\sqrt{m_l + m_{l+1}}} (z_1 dz_1^2 + 2z_2 dz_1 dz_2 + (z_1 - 2\cot 2\beta_l\, z_2) dz_2^2)$$

and the gluing map (4.2) is given by

$$\widetilde{z}_1 = z_1, \qquad \widetilde{z}_2 = -z_2.$$

Now we get immediately

$$\widetilde{Q} - Q = \frac{2}{\sqrt{m_l + m_{l+1}}} (\cot 2\widetilde{\beta}_l + \cot 2\beta_l) z_2 dz_2^2.$$

It remains to observe that the crossing from W to $\widetilde{W}$ corresponds to $z_2 < 0$. ■

It follows from Proposition 4.3 that in a PW system in a simple wedge with acceleration along the first generator the value of the Q-form does not decrease in a collision with any face containing the acceleration vector provided that

$$\beta_i \ge \pi/4, \qquad i = 1, \ldots, n-1.$$

Equivalently, in a PFL system the Q-form does not decrease in a collision between two particles if only $m_1 \geq \ldots \geq m_n$, and this condition is necessary.

Let us further examine the change in the Q-form if in the time interval $[0, t]$ we have collision with the floor of the first particle at time $t_c, 0 < t_c < t$ (and no other collisions). It is clear that the calculation reduces to the variables (x_1, v_1) alone. Let $x = x_1(0), v = v_1(0), \widehat{x} = x_1(t), \widehat{v} = v_1(t), a = \sqrt{m_1}$ and $v_c = v_1(t_c^-) < 0$. We have

$$\widehat{x} = -(t - t_c)v_c - \tfrac{1}{2}a(t - t_c)^2, \quad \widehat{v} = -v_c - a(t - t_c),$$

where

$$at_c = v + \sqrt{v^2 + 2ax}, \quad v_c = -\sqrt{v^2 + 2ax}.$$

So $(\widehat{x}, \widehat{v})$ depends on (x, v) smoothly (unless $(x, v) = (0, 0)$) and we can calculate the derivative. We get $d\widehat{v} = dv - \frac{2}{v_c}(vdv + adx)$. From the preservation of the energy in a collision we conclude that $\widehat{v}d\widehat{v} + ad\widehat{x} = vdv + adx$. Now the difference of the quadratic forms at time t and at time 0 is

$$(\widehat{v}d\widehat{v} + ad\widehat{x})\frac{d\widehat{v}}{a} - (vdv + adx)\frac{dv}{a} = -\frac{2}{av_c}(vdv + adx)^2 \geq 0.$$

We conclude that in a collision with the floor the derivative of the Hamiltonian flow is monotone (the Q-form does not decrease). Note that no further conditions on the masses (on the β-angles in the PW system) are necessary to assure the monotonicity in a collision of the first particle with the floor (collision with the face which does not contain the acceleration vector). Theorem 4.1 is proven.

Let us now examine the L_1- and L_2-exceptional subspaces, and L_1- and L_2-exceptional points. First we inspect what happens to tangent vectors from the two Lagrangian subspaces in a collision with the floor. Using the formulas developed above we find that for a vector from L_1 either $dx_1 \neq 0$ and then the vector enters the interior of the sector $\mathcal{C} = \{Q > 0\}$, or $dx_1 = 0$ and then $d\widehat{x}_1 = 0$ and the vector stays in L_1, and under the identification of the tangent spaces the vector does not change.

For a vector from L_2 we have $d\widehat{v}_1 = dv_1$ and the vector stays in L_2. If we use $(dv_1, \ldots, dv_n)$ as coordinates in L_2 these Lagrangian subspaces become naturally identified and we conclude that in a collision with the floor a vector from L_2 stays in L_2 and is not changed at all.

By Proposition 4.3 in a collision of an lth particle with the $(l+1)$st lighter particle ($m_l > m_{l+1}$) a vector from L_2 either enters the interior of the sector $\mathcal{C}$ or $dv_l/\sqrt{m_l} = dv_{l+1}/\sqrt{m_{l+1}}$ and the v-components of the vector are not changed. (In the language of the PW system this last condition means that the velocity component of the vector is parallel to the face of the wedge in which the particle is reflected.) As a result, for vectors from L_2 we get one equation for each nondegenerate collision of two particles. Since also no collision with the floor can change the v-components of a vector from L_2, it follows immediately that if the masses of the particles decrease (strictly) every vector in L_2 enters eventually the interior of the sector $\mathcal{C}$ except when $dv_i/\sqrt{m_i}$ are all equal for $i = 1, \ldots, n$. This last condition means that the vector is parallel to the Hamiltonian vector field. It shows that if the masses decrease there are no L_2-exceptional trajectories of the flow (among regular trajectories).

None of the vectors from $L_1 \cap \{dH = 0\}$ can enter the interior of the sector as a result of a collision of two particles. They stay in L_1 but they are changed by the appropriate reflection in a face of the wedge. Only a collision with the floor can push vectors from L_1 into the interior of the sector $\mathcal{C}$. It happens if $dx_1 \neq 0$ immediately before the collision. Hence in principle there may be L_1-exceptional trajectories on which the collisions between particles always "prepare" some vectors before each collision with the floor so that $dx_1 = 0$.

In a recent paper Simányi [S] showed that the set of L_1-exceptional trajectories is at most a countable union of codimension 1 submanifolds.

THEOREM 4.5 (Simányi [S]). *If $m_1 > m_2 \geq \ldots \geq m_n$, then the PFL system is completely hyperbolic.*

5. Special examples

Capped system of particles. Let us explore the consequences of the property that a simple wedge has two distinguished orderings of generators. A PFL system is equivalent to a PW system with acceleration vector parallel to the first generator. Let us modify the PFL system so that the wedge stays the same but the acceleration becomes parallel to the last generator. This is accomplished by changing the potential energy and the resulting Hamiltonian is

$$H = \sum_{i=1}^{n} \frac{p_i^2}{2m_i} + m_n q_n.$$

As before the configuration space is $\{q \in \mathbb{R}^n \mid 0 \leq q_1 \leq \ldots \leq q_n\}$ and the particles collide with each other and the floor. We will call it the *capped system of particles in a line*. The new feature is that between collisions the particles move uniformly (with constant velocity) except for the last particle which is accelerated down (it falls down). It is this last particle ("the cap") that keeps the system closed, i.e., the energy surface $\{H = \text{const}\}$ is compact and it carries a finite Liouville measure.

The capped system of particles is equivalent to another PFL system with different masses. We will calculate these masses (or equivalently the β-angles) to establish conditions under which the capped system is completely hyperbolic or completely integrable. Note that the β-angles are complete

Euclidean invariants of a simple wedge with a chosen distinguished ordering of the generators, and they do change when we change the distinguished ordering and the last generator becomes the first.

THEOREM 5.1. *The capped system of particles in a line is completely integrable if*

$$m_k = \frac{n}{k(k+1)} m_n \qquad \textit{for } k = 1, \dots, n-1,$$

and completely hyperbolic if

$$\frac{m_1}{M_1} \geq \frac{1}{2} \quad \textit{and} \quad \frac{m_i}{M_i} \geq \frac{m_{i-1}}{M_{i-1}} \left(1 + \frac{m_{i-1}}{M_{i-1}}\right)^{-1} \quad \textit{for } i = 2, \dots, n-2$$

and

$$\frac{m_{n-1}}{M_{n-1}} > \frac{m_{n-2}}{M_{n-2}} \left(1 + \frac{m_{n-2}}{M_{n-2}}\right)^{-1}.$$

Proof. We need to calculate the α- and β-angles for the reversed ordering of the generators of the simple wedge. Denote these angles for the reversed ordering by $\widehat{\alpha}_k$ and $\widehat{\beta}_k$, $k = 1, \dots, n-1$, respectively. From (2.1), (2.2) and (3.4) we obtain for $k = 1, \dots, n-1$,

$$\cos^2 \widehat{\alpha}_k = \cos^2 \alpha_{n-k} = \frac{M_{n-k+1}}{M_{n-k}}, \qquad \sin^2 \widehat{\alpha}_k = \frac{m_{n-k}}{M_{n-k}},$$
$$\tan^2 \widehat{\beta}_{n-1} = \tan^2 \widehat{\alpha}_{n-1} = \frac{m_1}{M_2},$$

and for $k = 1, \dots, n-2$,

$$\tan^2 \widehat{\beta}_k = \frac{\tan^2 \widehat{\alpha}_k}{\sin^2 \widehat{\alpha}_{k+1}} = \frac{m_{n-k} M_{n-k-1}}{m_{n-k-1} M_{n-k+1}}.$$

Introducing $X_i = M_1/M_i$ for $i = 1, \dots, n$ and setting $X_0 = 0$ we can rewrite this as

$$\tan^2 \widehat{\beta}_{n-i} = \frac{X_{i+1} - X_i}{X_i - X_{i-1}}, \qquad i = 1, \dots, n-1.$$

By the results of Section 4, the condition of complete integrability is that $\tan^2 \widehat{\beta}_k = 1$ for $k = 1, \dots, n-1$. It is equivalent to the linearity condition

$$X_{i+1} - X_i = X_i - X_{i-1}, \qquad i = 1, \dots, n-1.$$

Since $X_1 - X_0 = 1$ we obtain $X_i = i$. The claim about complete integrability follows. We can apply Theorem 4.5 if $\tan^2 \widehat{\beta}_k \geq 1$ for $k = 1, \dots, n-1$, and $\tan^2 \widehat{\beta}_1 > 1$. This gives us the convexity condition

$$X_{i+1} - X_i \geq X_i - X_{i-1}, \qquad i = 1, \dots, n-1, \qquad X_n - X_{n-1} > X_{n-1} - X_{n-2},$$

which translates into the conditions in the theorem. ∎

System of attracting particles in a line. Consider a system of $n+1$ point particles in a line with positions $q_0 \leq q_1 \leq \dots \leq q_n$ and masses $m_0, \dots, m_n$.

They collide elastically with each other and their interaction is defined by a linear translation invariant potential $\sum_{i=1}^{n} m_i(q_i - q_0)$. Thus the Hamiltonian of the system is

$$H = \sum_{i=0}^{n} \frac{p_i^2}{2m_i} + \sum_{i=1}^{n} m_i(q_i - q_0).$$

The total momentum is preserved in this system. Setting the total momentum to zero and fixing the center of mass $m_0q_0 + m_1q_1 + \ldots + m_nq_n = 0$ we obtain a PW system in a simple wedge with acceleration parallel to the first generator (hence our system is also equivalent to a PFL system). Indeed, introducing symplectic coordinates (η, ξ),

$$\begin{aligned}
\eta_0 &= m_0q_0 + m_1q_1 + \ldots + m_nq_n,\\
\eta_i &= q_i - q_{i-1}, \qquad i = 1, \ldots, n,\\
p_0 &= m_0\xi_0 - \xi_1,\\
p_i &= m_i\xi_0 + \xi_i - \xi_{i+1}, \qquad i = 1, \ldots, n-1,\\
p_n &= m_n\xi_0 + \xi_n,
\end{aligned}$$

and setting the total momentum and the center of mass to zero, $\eta_0 = 0$, $\xi_0 = 0$, we obtain the Hamiltonian

$$H = \frac{\xi_1^2}{2m_0} + \sum_{i=1}^{n-1} \frac{(\xi_i - \xi_{i+1})^2}{2m_i} + \frac{\xi_n^2}{2m_n} + \sum_{i=1}^{n} M_i\eta_i,$$

where $M_i = m_i + \ldots + m_n$ for $i = 0, 1, \ldots, n$. By Proposition 3.1 the wedge $W = \{\eta_1 \geq 0, \ldots, \eta_n \geq 0\}$ is simple. It can also be checked that the acceleration is parallel to the first generator. (Acceleration parallel to the last generator corresponds to the potential $\sum_{i=0}^{n-1} m_i(q_n - q_i)$, which gives a symmetric system where the special role is played by q_n rather than q_0.)

Theorem 5.2. *The system of attracting particles is completely integrable if for some $a > n$,*

$$\frac{m_i}{m_0} = \frac{a}{(a-i)(a-i+1)}, \qquad i = 1, \ldots, n,$$

and it is completely hyperbolic if the sequence

$$a_i = \frac{m_0 + m_1 + \ldots + m_{i-1}}{m_i} + i, \qquad i = 1, \ldots, n,$$

satisfies $a_1 < a_2 \leq a_3 \leq \ldots \leq a_n$.

Proof. The n-dimensional wedge $W = \{\eta_1 \geq 0, \ldots, \eta_n \geq 0\}$ has in the original coordinates the unit generators $e_i = (e_i^0, e_i^1, \ldots, e_i^n,), i = 1, \ldots, n,$

where

$$\sqrt{M_0}\, e_i^k = \begin{cases} -\sqrt{\dfrac{M_i}{M_0 - M_i}} & \text{if } k < i, \\ \sqrt{\dfrac{M_0 - M_i}{M_i}} & \text{if } k \geq i. \end{cases}$$

The acceleration vector is parallel to e_1 and for $1 \leq i < j \leq n$ we get

$$\langle e_i, e_j \rangle = \sqrt{\frac{M_0 - M_i}{M_i}} \sqrt{\frac{M_j}{M_0 - M_j}}$$

(which implies immediately by Proposition 2.3 that the wedge is indeed simple). We calculate further that the β-angles are given by

$$\tan^2 \beta_i = \frac{m_i}{m_0 + m_1 + \ldots + m_{i-1}} \left(\frac{m_0 + m_1 + \ldots + m_i}{m_{i+1}} + 1 \right)$$

for $i = 1, \ldots, n-1$. If we introduce

$$X_i = \frac{m_0 + m_1 + \ldots + m_{i-1}}{m_i} + i, \qquad i = 1, \ldots, n,$$

then the condition for complete integrability is

$$X_{i+1} = X_i, \qquad i = 1, \ldots, n-1,$$

and the condition for complete hyperbolicity is

$$X_{i+1} \geq X_i, \qquad i = 2, \ldots, n-1, \qquad X_2 > X_1.$$

This leads to the conditions in the theorem. ∎

For example, in the case of equal masses $m_1 = \ldots = m_n$, we have complete hyperbolicity of the system. Note that we can rewrite the potential energy as

$$\sum_{i=1}^{n} M_i (q_i - q_{i-1}),$$

i.e., we can interpret the interaction of the particles as the attraction of nearest neighbors, but then the force of attraction decays for particles further to the right.

6. Wide wedges and the Main Theorem. Let $W = W(g_1, \ldots, g_k)$ be a k-dimensional wedge in a Euclidean n-dimensional space E.

DEFINITION 6.1. A k-dimensional wedge $W = W(g_1, \ldots, g_k)$ is called *wide* if the angles between the generators exceed $\pi/2$, i.e., $\langle g_i, g_j \rangle < 0$ for any $1 \leq i < j \leq k$.

Clearly every face of a wide wedge is a wide wedge of lower dimension.

PROPOSITION 6.2. *If an n-dimensional wedge W is wide then the dual wedge W^* is contained in W and the inclusion is strict, in the sense that the only point in the intersection of the boundaries of W and W^* is* 0.

Proof. This proof was communicated to us by Michał Rams. Consider a wide wedge $W = W(g_1, \ldots, g_n)$ and let $\{f_1, \ldots, f_n\}$ be the generators of the dual wedge W^* which also form a basis dual to the basis $\{g_1, \ldots, g_n\}$. We have

$$f_i = \sum_{j=1}^{n} \langle f_i, f_j \rangle g_j$$

for $i = 1, \ldots, n$. It is sufficient to show that all the coeffcients in these expansions are positive. Assume that this is not the case and for say f_1 the set $I = \{j \mid \langle f_1, f_j \rangle \le 0\}$ is nonempty. We define

$$f_1^+ = \sum_{j \notin I} \langle f_1, f_j \rangle g_j, \qquad f_1^- = -\sum_{k \in I} \langle f_1, f_k \rangle g_k,$$

so that $f_1 = f_1^+ - f_1^-$. Since f_1^- belongs to W we get

$$0 \le \langle f_1, f_1^- \rangle = \langle f_1^+, f_1^- \rangle - \langle f_1^-, f_1^- \rangle. \tag{6.1}$$

But since the wedge W is wide we also have

$$\langle f_1^+, f_1^- \rangle = -\sum_{j \notin I} \sum_{k \in I} \langle f_1, f_j \rangle \langle f_1, f_k \rangle \langle g_j, g_k \rangle \le 0. \tag{6.2}$$

Comparing (6.1) and (6.2) we conclude that $f_1^- = 0$ and $f_1 = f_1^+$. Finally, consider one of the vectors g_k missing from the expansion of f_1, i.e., $k \in I$ or $\langle f_1, f_k \rangle = 0$. We get

$$0 \le \langle f_1, g_k \rangle = \sum_{j \notin I} \langle f_1, f_j \rangle \langle g_j, g_k \rangle,$$

which is contradictory since all the terms in the last sum are negative. ∎

COROLLARY 6.3. *If the n-dimensional wedge $W(g_1, \ldots, g_n)$ is wide then the angle between any two of its codimension* 1 *faces exceeds* $\pi/2$.

COROLLARY 6.4. *If $W(g_1, \ldots, g_n)$ is wide and $\{f_1, \ldots, f_n\}$ is the basis dual to $\{g_1, \ldots, g_n\}$ then $\langle f_i, f_j \rangle > 0$ for any $1 \le i, j \le n$.*

Note that the converse of Proposition 6.2 (or of any of the two corollaries) does not hold for $n \ge 3$.

PROPOSITION 6.5. *If a wedge is wide, then the orthogonal projection of the interior of the dual wedge onto any of its faces is contained in the interior of that face.*

In particular, the orthogonal projections onto any face of the wedge of any vector from the interior of the dual wedge are nonzero.

Proof. Let $\{g_1, \ldots, g_n\}$ be generators of the wedge and let $\{f_1, \ldots, f_n\}$ be the dual basis (and hence also the generators of the dual wedge). The orthogonal projections of $f_1, \ldots, f_k$ onto the subspace $S(g_1, \ldots, g_k)$ form the basis dual to $\{g_1, \ldots, g_k\}$ in this subspace. It follows that the orthogonal projection of the dual wedge onto a face is the dual wedge of that face. Since all the faces are also wide, by Proposition 6.2 this projection is contained in the face and the inclusion is strict. ■

We can now formulate and prove our

Main Theorem 6.6. *The PW system in a wide wedge with arbitrary acceleration vector from the interior of the dual wedge is completely hyperbolic.*

In the standard representation, $W = \{(\eta_1, \ldots, \eta_n) \in \mathbb{R}^n \mid \eta_i \geq 0, i = 1, \ldots, n\}$ and the scalar product in the η coordinates is defined by a positive definite matrix $L = \{l_{ij}\}$. The assumption that W is wide translates into $l_{ij} < 0$ for $i \neq j$. By Corollary 6.4 it follows that the inverse matrix $K = L^{-1}$ has all entries positive. The Hamiltonian of the system is

$$H = \frac{1}{2}\langle K\xi, \xi\rangle + \langle c, \eta\rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the arithmetic scalar product in $\mathbb{R}^n$, $\xi \in \mathbb{R}^n$ is the momentum of the particle and $c \in \mathbb{R}^n$ is a vector with all entries positive, so that the acceleration vector equal to Kc is in the interior of the dual wedge. Now the Main Theorem can be reformulated as the Main Theorem from the Introduction.

Proof of the Main Theorem. Let the wide wedge be $W = W(g_1, \ldots, g_n)$. We will divide it into $n!$ simple wedges. For that purpose let us note that a simple wedge is uniquely defined by the choice of a first generator e_1 and a flag of subspaces

$$S_1 \supset \ldots \supset S_n$$

such that e_1 is not orthogonal to any of the subspaces. Indeed, given such a flag we define e_k as the orthogonal projection of e_1 onto S_k. Clearly the wedge $W(e_1, \ldots, e_n)$ is simple. Conversely, for a simple wedge $W(e_1, \ldots, e_n)$ we obtain the flag of subspaces by considering $S_k = S(e_k, \ldots, e_n)$, $k = 1, \ldots, n$.

The first generator of all our simple wedges will be the acceleration vector. We define a simple wedge W_σ, for any permutation σ of $\{1, \ldots, n\}$, by the flag

$$S_k = S(g_{\sigma(k)}, g_{\sigma(k+1)} \ldots, g_{\sigma(n)}), \qquad k = 1, \ldots, n.$$

By Proposition 6.5 the acceleration vector is not orthogonal to any of the faces of the wide wedge, so that these $n!$ flags define indeed simple wedges and moreover

$$\bigcup_{\sigma} W_\sigma = W(g_1, \ldots, g_n),$$

and the interiors of these simple wedges are mutually disjoint. The intersection of all these wedges is the ray spanned by their first generator e_1 (the acceleration vector).

Consider two adjacent wedges, W_{σ_0} and W_{σ_1}, i.e., two wedges which have a common $(n-1)$-dimensional face. Without loss of generality we can assume that σ_0 is the identity permutation. Then by necessity σ_1 is a transposition of two consecutive indices, say l and $l+1$. Consider the β-angles for W_{σ_0} and W_{σ_1}, $\beta_1, \ldots, \beta_{n-1}$ and $\widetilde{\beta}_1, \ldots, \widetilde{\beta}_{n-1}$, respectively.

LEMMA 6.7. *For the two adjacent wedges*, $\beta_l + \widetilde{\beta}_l > \pi/2$.

P r o o f. The two adjacent wedges have the same first l generators, $e_1, \ldots$ $\ldots, e_l$, and the same last $n-l-1$ generators, $e_{l+2}, \ldots, e_n$. Let $e_{l+1}, \widetilde{e}_{l+1}$ be the two different generators for the wedges W_{σ_0} and W_{σ_1}, respectively.

The angles β_l and $\widetilde{\beta}_l$ are equal to the angles between codimension 1 subspaces of $S_l = S(g_l, g_{l+1}, \ldots, g_n)$: β_l is the angle between $S(e_l, g_{l+2}, \ldots, g_n)$ and $S(e_{l+1}, g_{l+2}, \ldots, g_n) = S(g_{l+1}, g_{l+2}, \ldots, g_n)$, while $\widetilde{\beta}_l$ is the angle between $S(e_l, g_{l+2}, \ldots, g_n)$ and $S(\widetilde{e}_{l+1}, g_{l+2}, \ldots, g_n) = S(g_l, g_{l+2}, \ldots, g_n)$. Since these three subspaces of S_l have in common the codimension 2 subspace $S(g_{l+2}, \ldots, g_n)$ we conclude that $\beta_l + \widetilde{\beta}_l$ is the angle between two subspaces of S_l, $S(g_{l+1}, g_{l+2}, \ldots, g_n)$ and $S(g_l, g_{l+2}, \ldots, g_n)$. Observing that these are two faces of the wide wedge $W(g_l, g_{l+1}, \ldots, g_n)$ we obtain the lemma from Corollary 6.3. ∎

In each of the simple wedges we introduce the form Q furnished by the canonical isomorphism with a PFL system. We obtain a piecewise continuous Q-form in the tangent bundle of the phase space of our system. This form is defined by two Lagrangian bundles, L_1 and L_2. Note that L_1 is continuous (with the natural identification of the tangent spaces to the phase space $W \times \mathbb{R}^n$ it is actually constant) while L_2 experiences jump discontinuities when we cross from one simple wedge to another (cf. Corollary 4.4).

By Proposition 4.3 and Lemma 6.7 our system is Q-monotone. To apply Theorem 1.4 it remains to examine L_1- and L_2-exceptional trajectories.

Consider Euclidean coordinates $(x, v) \in \mathbb{R}^n \times \mathbb{R}^n$ in which the kinetic energy has the standard form $\langle v, v\rangle/2$, where $x \in \mathbb{R}^n$ represents the position of the particle in the wedge W and $v \in \mathbb{R}^n$ represents its velocity. In these coordinates $L_1 = \{dv_1 = \ldots = dv_n = 0\}$ and the Q-form is

$$Q = \sum_{k=1}^{n} dx_k dv_k + \sum_{i,j=1}^{n} z_{ij} dv_i dv_j,$$

where the symmetric matrix $\{z_{ij}\}$ depends on the simple wedge W_σ in which x is located and the velocity v. Vectors from L_1 are not changed by the derivative of the flow unless there is a collision with one of the faces of the wide wedge. This collision considered in the corresponding PFL system becomes a collision with the floor. The effect of such a collision on vectors from $\widetilde{L}_1 = L_1 \cap \{dH = 0\}$ was discussed in Section 4. In our present language the conclusion is the following. Either a vector from $\widetilde{L}_1$ is parallel to the face of the wedge where the collision is occurring and then it is not changed by the derivative of the flow, or it is transversal to the face and then it enters the interior of the sector $\mathcal{C} = \{Q \geq 0\}$ as a result of the collision. Hence the only vectors from $\widetilde{L}_1$ which do not ever enter the interior of $\mathcal{C}$ are the vectors parallel to all the faces with which the particle collides in the future. There are no vectors parallel to all the $(n-1)$-dimensional faces of the wide wedge. It follows that the only L_1-exceptional trajectories could be those for which the particle does not collide in the future with one (or several) of the faces.

CLAIM 1. *There are no nondegenerate trajectories avoiding future collisions with one of the faces.*

A *degenerate trajectory* is one which hits two faces simultaneously or has velocity with zero orthogonal component to the face at the time of collision, i.e., it is a trajectory for which there is no natural continuation of the dynamics. To prove the claim note that, if the avoided face is $W(g_2, \ldots, g_n)$, then the component of the velocity in the direction of g_1 is preserved at all other collisions, since all the faces with which our orbit collides are parallel to g_1. Between collisions we have

$$\frac{d}{dt}\langle v, g_1 \rangle = -\langle a, g_1 \rangle < 0$$

because the acceleration vector a is taken from the interior of the dual cone. We obtain the contradiction that on our orbit $\langle v, g_1 \rangle$ goes to $-\infty$. Hence there are no such orbits. (Note that we did not need to use Simányi's method [S] and we have established more. It is quite plausible that also in the case of a simple wedge there are no L_1-exceptional orbits. It is indeed so in the case obtained from the symmetric wide wedge, when every pair of adjacent simple wedges, W_{σ_1} and W_{σ_2}, is symmetric with respect to the common face, cf. Section 8.)

It remains to show that there are only few L_2-exceptional trajectories. We identify all the L_2 subspaces with the tangent velocity space by the natural projection. Further the tangent velocity space can be naturally identified with the tangent configuration space. Hence we can use $(d\eta_1, \ldots, d\eta_n)$ as coordinates in the subspaces L_2.

With this identification vectors from L_2 are not changed in a collision with the faces of the wide wedge. Neither they are changed between collisions. But when the trajectory crosses from one simple wedge to an adjacent one a tangent vector from L_2 is likely to get into the interior of the sector $\mathcal{C}$ because L_2 experiences a jump discontinuity. Again the results of Section 4 (Proposition 4.3) can be translated into our current language as the following alternative. When the trajectory crosses transversally from one simple wedge to another, a vector from L_2 either enters immediately the interior of the sector $\mathcal{C}$ or it is parallel to the common face.

The trajectories with velocities which are not transversal to the common face at the time of crossing can be dropped from consideration, they form a set of zero measure in the phase space. Let us also call such trajectories *degenerate*. We are going to prove

Claim 2. *There are no nondegenerate L_2-exceptional trajectories.*

We need to prove that along any nondegenerate trajectory the intersection of all subspaces of codimension 1 containing the faces of the wedges that the trajectory crosses is equal to the one-dimensional subspace spanned by the acceleration vector. The task of tracing these subspaces along a given trajectory is made cumbersome by the fact that they depend in general on the geometry of the wide wedge. We avoid this difficulty by focusing on very special common faces, which in particular lie in subspaces that do not depend on the geometry of the wedge.

As observed earlier, for a pair of adjacent simple wedges, W_{σ_1} and W_{σ_2}, the permutation $\sigma_2\sigma_1^{-1}$ is a transposition of two consecutive indices. We consider only those pairs of adjacent simple wedges for which this transposition is the transposition of 1 and 2. It is not hard to see that the common face of such a pair must be contained in the subspace of the form

$$S(a, g_{\sigma_1(3)}, g_{\sigma_1(4)} \ldots, g_{\sigma_1(n)}).$$

This subspace is given by the equation

$$\frac{\eta_k}{a_k} = \frac{\eta_l}{a_l},$$

where $\sigma_1(1) = k$ and $\sigma_1(2) = l$. Hence the crossing of such a common face forces the corresponding relation on the L_2-exceptional subspace:

$$\frac{d\eta_k}{a_k} = \frac{d\eta_l}{a_l}. \tag{6.3}$$

It remains to show that the relations (6.3) along every nondegenerate trajectory suffice to force

$$\frac{d\eta_1}{a_1} = \frac{d\eta_2}{a_2} = \ldots = \frac{d\eta_n}{a_n}.$$

This follows readily from Claim 1. Indeed, according to Claim 1, any trajectory will collide with the face $W(g_2, \dots, g_n)$ and then after some time it will collide with every other face of the wide wedge. Hence, for any $s = 2, \dots, n$, the trajectory must go from a simple wedge W_{σ_1} to W_{σ_2}, where $\sigma_1(1) = 1$ and $\sigma_2(1) = s$ (needless to say, these simple wedges are not adjacent in general). Let us trace the crossings from one simple wedge to the adjacent one, on the way from W_{σ_1} to W_{σ_2}, by the transpositions required to get from σ_1 to σ_2. Among these transpositions we must have enough transpositions of the first two indices to change 1 into s. Independent of how many times and when this special transposition occurs, the respective equalities (6.3) will force

$$\frac{d\eta_1}{a_1} = \frac{d\eta_s}{a_s},$$

which proves Claim 2, and ends the proof of the Main Theorem. ∎

7. Systems of attracting particles with arbitrary constraints. Consider the system of particles falling to the floor of finite mass described in the introduction, with Hamiltonian

$$H = \sum_{i=0}^{n} \frac{p_i^2}{2m_i} + \sum_{i=1}^{n} c_i(q_i - q_0) \tag{7.1}$$

and elastic constraints

$$q_1 - q_0 \geq 0, \quad q_2 - q_0 \geq 0, \ \dots, \ q_n - q_0 \geq 0. \tag{7.2}$$

It satisfies the conditions of the Main Theorem. Indeed, by the change of variables (0.3) and the Hamiltonian reduction $\eta_0 = 0$, $\xi_0 = 0$, we obtain the Hamiltonian

$$H = \frac{(\xi_1 + \ldots + \xi_n)^2}{2m_0} + \sum_{i=1}^{n} \frac{\xi_i^2}{2m_i} + \sum_{i=1}^{n} c_i\eta_i, \tag{7.3}$$

and the system is constrained to the wedge $W = \{\eta_1 \geq 0, \dots, \eta_n \geq 0\}$.

It is a straightforward calculation that the inverse of the matrix K giving the kinetic energy has all off-diagonal elements negative. Hence the wedge W is wide. We will denote by $\{g_1, \dots, g_n\}$ the generators of W. In the case $n = 3$ by taking different masses we can obtain all possible wide wedges. For $n \geq 4$ there are many more wide wedges than covered by these Hamiltonians. Geometrically the special property of the wide wedge can be described in the following way. The generators of the wedge after appropriate scaling form an orthocentric simplex, i.e., a simplex in which all the heights intersect at a unique point (see [Ro]). In such an *orthocentric wedge* there is a special ray, perpendicular to the opposite face of the orthocentric simplex. This property of a wedge is shared by the dual wedge and the *orthocentric ray* is the same for both.

Let us take the special potential function with $c_i = \alpha m_i$. The Hamiltonian equations become

$$\begin{aligned} \frac{d\eta_i}{dt} &= \frac{\xi_1 + \ldots + \xi_n}{m_0} + \frac{\xi_i}{m_i} = u_i, \\ \frac{d\xi_i}{dt} &= -\alpha m_i, \quad i = 1, \ldots, n. \end{aligned} \tag{7.4}$$

We have further

$$\frac{d^2\eta_i}{dt^2} = \frac{du_i}{dt} = -\alpha \frac{m_0 + m_1 + \ldots + m_n}{m_0} = -\alpha \frac{M}{m_0},$$

i.e., the accelerations of all η coordinates are equal. Geometrically this means that the acceleration vector spans the orthocentric ray of the wedge W. The fact that the wide wedge given by the kinetic energy in (7.3) is orthocentric and the choice of the acceleration vector along the orthocentric ray lead to the partition into simple wedges, introduced in the proof of the Main Theorem, given by

$$W_\sigma = \{\eta \in \mathbb{R}^n \mid 0 \le \eta_{\sigma(1)} \le \ldots \le \eta_{\sigma(n)}\}. \tag{7.5}$$

Note that in general the partition of a wide wedge into simple wedges is achieved by slicing the wedge with $n!(n-1)/2$ hyperplanes but in our special case only $n(n-1)/2$ hyperplanes $\eta_k = \eta_l$ are used.

We now consider a system obtained by adding more constraints of the form $q_k \le q_l$ to the wide constraints (7.2). These additional constraints constitute the "stacking rules" as explained in the introduction. They define a convex polyhedral cone T contained in the wide wedge W. In our list of constraints some constraints are consequences of others. We can naturally introduce a minimal set of constraints. Clearly the minimal set of constraints is in one-to-one correspondence with the faces of the cone T.

A convenient way of describing the minimal set of constraints is by an oriented graph $\mathcal{G}$ with $n+1$ vertices labeled by the masses $m_0, m_1, \ldots, m_n$. The graph contains an edge from m_k to m_l if the constraint $q_k \le q_l$ belongs to the minimal set of constraints. The resulting graph is connected. We will refer to m_0 as the *floor* of the graph. Every vertex can be reached by a path from the floor. We will call such a graph a *graph of constraints*. In Fig. 1 we give all possible graphs of constraints for 3 particles (up to permutations of the masses). If the graph of constraints is a tree, the cone T is a wedge which is in general neither simple nor wide. The leftmost graph corresponds to a wide wedge, and the rightmost graph to a simple wedge. For 4 particles there are 16 possible graphs of constraints, out of which 8 graphs define T which is a wedge.

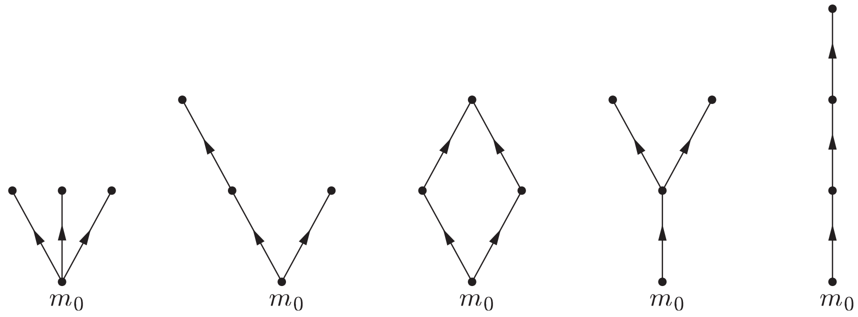

Fig. 1. Possible graphs of constraints for 3 particles

The edges starting at the floor of a graph of constraints correspond to possible collisions of particles with the floor. All the other edges correspond to possible collisions between two particles.

A graph of constraints defines naturally a partial order of the vertices (masses) which we denote by $\prec$.

Fix a graph of constraints $\mathcal{G}$. A vertex m_l is a *successor* of m_k if $m_k \prec m_l$, in particular m_k is its own successor. We call a vertex m_l an *immediate successor* of m_k if there is an edge in the graph from m_k to m_l. If m_l is an immediate successor of m_k, then m_k is an *immediate predecessor* of m_l. Only immediate successors of the floor can collide with it.

Let

$$\mathcal{N}(m_k) = \sum_{m_k \prec m_l} m_l$$

be the total mass of all successors of m_k. Let as before $M = \mathcal{N}(m_0) = \sum_{l=0}^{n} m_l$ be the total mass. We define $\mathcal{P}(m_k) = M - \mathcal{N}(m_k)$.

THEOREM 7.1. *The system with Hamiltonian*

$$H = \frac{(\xi_1 + \ldots + \xi_n)^2}{2m_0} + \sum_{i=1}^{n} \frac{\xi_i^2}{2m_i} + \sum_{i=1}^{n} \alpha_i m_i \eta_i,$$

with $\alpha_i > 0$, $i = 1, \ldots, n$, *and a given graph of constraints* $\mathcal{G}$ *is completely hyperbolic if for every edge in* $\mathcal{G}$ *from* m_k *to* m_l, $k > 0$, *we have*

$$\alpha_k = \alpha_l \tag{7.6}$$

and

$$\frac{m_l}{m_k} < 1 + \frac{m_k + m_l}{\mathcal{P}(m_k)}. \tag{7.7}$$

Note that in (7.6) we do not necessarily require that the α-coefficients are all equal (unless the graph obtained from $\mathcal{G}$ by the removal of the floor and all the edges starting at the floor is connected). The following proof is greatly simplified if all the α-coefficients are equal and hence the acceleration vector has the direction of the orthocentric ray. In the first reading of the proof it may be helpful to settle for this simplifying assumption.

Proof. We follow the proof of the Main Theorem. We split the wide wedge W into $n!$ simple wedges W_σ indexed by all permutations σ of $\{1,\ldots,n\}$. We first prove that the cone T is the union of some of these simple wedges, i.e.,

$$T = \bigcup_{W_\sigma\cap \operatorname{int} T\neq\emptyset} W_\sigma. \tag{7.8}$$

If the acceleration vector has the direction of the orthocentric ray then the partition into simple wedges is given by (7.5). We call a permutation σ *compatible* with the graph of constraints $\mathcal{G}$ if $\sigma(k)\le\sigma(l)$ whenever $m_k\prec m_l$. Clearly the configuration space T of our system is the union of simple wedges W_σ for all permutations σ compatible with the constraints.

In general the acceleration vector $a=(a_1,\ldots,a_n)$ is equal to

$$a_k = -\frac{d^2\eta_k}{dt^2} = \frac{\alpha_1 m_1+\ldots+\alpha_n m_n}{m_0}+\alpha_k,$$

and by (7.6) it is parallel to all the faces of T which are not the faces of the wide wedge W. Moreover, due to the special geometry of the wide wedge any such face is orthogonal to most of the faces of W. More precisely, the face of T which corresponds to a collision of m_k and m_l, $k,l>0$ (i.e., the face defined by the equation $\eta_k=\eta_l$) is orthogonal to all of the faces of W with the exception of $\eta_k=0$ and $\eta_l=0$.

The proof of (7.8) is now accomplished by induction on the dimension n. When $n=2$ the claim is obvious. (When $n=3$ we can convince ourselves about the validity of (7.8) by straightforward considerations.) Assume that (7.8) holds for $n\le N, N\ge 2$ and all possible graphs of constraints. We prove (7.8) for $n=N+1$. If $m_{\sigma(1)}$ is not an immediate successor of the floor then W_σ is disjoint from the interior of the cone T. Hence the simple wedges having nonempty intersections with the interior of T can be split according to $\sigma(1)$, and $m_{\sigma(1)}$ must be one of the immediate successors of the floor. Consider only the simple wedges W_σ with a fixed allowed value of $\sigma(1)$, say $\sigma(1)=N+1$. By intersecting T with $\{\eta_{N+1}=0\}$ we obtain a convex cone $\widehat{T}$ corresponding to the graph of constraints $\widehat{\mathcal{G}}$ obtained from $\mathcal{G}$ by collapsing the edge from the floor to m_{N+1}. Clearly the orthogonal projection $\widehat{a}$ of the acceleration vector a onto $\{\eta_{N+1}=0\}$ lies in all the faces of $\widehat{T}$ corresponding to the edges of $\widehat{\mathcal{G}}$ except the edges starting at the floor (note that there are in general many more edges starting at the floor in $\widehat{\mathcal{G}}$). Using the inductive assumption we conclude that the cone $\widehat{T}$ is the union of some N-dimensional simple wedges defined by the acceleration vector $\widehat{a}$. It now follows from the convexity of T that the simple wedges W_σ

with $\sigma(1) = N + 1$ are all contained in T, which proves (7.8).

As in the proof of the Main Theorem, in each of the simple wedges we introduce the quadratic form Q furnished by the canonical isomorphism with a PFL system. We are going to prove that with this choice of the quadratic form (or equivalently of the two fields of Lagrangian subspaces) our system is monotone. Indeed, the form Q is conserved as long as the trajectory stays in one simple wedge and as shown in the proof of the Main Theorem it does not decrease when the orbit crosses to an adjacent simple wedge or collides with one of the faces of the wide wedge.

It remains to study the conditions of monotonicity when the trajectory hits a face of T which does not lie in the face of the wide wedge. This corresponds to a collision of two masses, m_k and m_l, $k, l > 0$, and hence also to an edge of the graph which does not start at the floor. The appropriate conditions were calculated in Section 4, they are formulated in terms of β-angles of the simple wedge. The problem is to translate them into conditions on the masses in our system.

For clarity we first find appropriate β-angles under the assumption that all the coefficients α_i, $i = 1, \ldots, n$, are equal.

Consider the collision of two particles, $m_k \prec m_l$, occurring in the simple wedge W_σ. We put $k = \sigma(s)$; then by necessity $l = \sigma(s+1)$. The condition of monotonicity in such a collision is, according to the results of Section 4, that the angle β_s in the simple wedge W_σ is not less than $\pi/4$. It was established in the proof of Theorem 5.2 that

$$\tan^2 \beta_s = \frac{m_k}{m_l}\left(1 + \frac{m_k + m_l}{m_0 + m_{\sigma(1)} + \ldots + m_{\sigma(s-1)}}\right).$$

This is the angle in $S(g_{\sigma(s)}, g_{\sigma(s+1)}, \ldots, g_{\sigma(n)})$ between two subspaces of codimension 1, $S(g_{\sigma(s+1)}, \ldots, g_{\sigma(n)})$ and $\{\eta_k = \eta_l\}$.

Hence the condition of monotonicity reads

$$\frac{m_l}{m_k} \le 1 + \frac{m_k + m_l}{m_0 + m_{\sigma(1)} + \ldots + m_{\sigma(s-1)}}. \tag{7.9}$$

This is most restrictive when the denominator on the right hand side is the largest possible. After a moment's reflection it becomes apparent that this denominator does assume the value of $\mathcal{P}(m_k)$ in one of the simple wedges of our configuration space and it cannot be greater. This shows that (7.9) follows from the assumption (7.7).

We conclude that our system is Q-monotone at least in the case of the special acceleration vector. In the general case we observe that although the simple wedges are changed when the acceleration vector is changed, the β-angles that appear above remain the same. Indeed, β_s is equal to the angle in $S(g_{\sigma(s)}, g_{\sigma(s+1)}, \ldots, g_{\sigma(n)})$ between two subspaces of codimension 1,

$S(g_{\sigma(s+1)}, \ldots, g_{\sigma(n)})$ and the intersection of $S(g_{\sigma(s)}, g_{\sigma(s+1)}, \ldots, g_{\sigma(n)})$ with the face with which our trajectory collides. This face is given by the equation $\eta_k = \eta_l$ independent of the acceleration vector.

To apply Theorem 1.4 we still need to examine L_1- and L_2-exceptional trajectories.

As in the proof of the Main Theorem, we consider the Euclidean coordinates $(x, v) \in \mathbb{R}^n \times \mathbb{R}^n$ in which the kinetic energy has the standard form $\langle v, v\rangle/2$, where $x \in \mathbb{R}^n$ represents position in the wedge W and $v \in \mathbb{R}^n$ represents velocity. In these coordinates $L_1 = \{dv_1 = \ldots = dv_n = 0\}$ and we can identify all of these Lagrangian subspaces with the tangent to the configuration space. Vectors from L_1 are not changed by the derivative of the flow if there are no collisions with the faces of T in the time interval. Collisions with the faces of T do change vectors in L_1. We need to distinguish between the faces of T which lie in the faces of the wide wedge W (collisions with the floor) and those which do not (collisions between particles). In a collision with a face of the wide wedge, say $\{\eta_1 = 0\}$, a vector from $L_1 \cap \{dH = 0\}$ will enter the interior of the sector $\mathcal{C}$ unless it is parallel to the face, i.e., $d\eta_1 = 0$; in which case the vector will not be changed in the collision. Furthermore, in a collision between particles no vector from L_1 can be pushed into the interior of the sector, but all of them are changed by the orthogonal reflection in the face. Thus we have to address the possible presence of L_1-exceptional trajectories on which the collisions between particles always "prepare" some vectors before each collision with a face $\{\eta_i = 0\}$, so that $d\eta_i = 0$. We now apply the method of Simányi [S] to prove

Claim 1. *The set of L_1-exceptional trajectories is contained in a countable union of submanifolds of codimension at least one.*

Proof. Consider an exceptional trajectory for which there are some vectors in $L_1 \cap \{dH = 0\}$ which never enter the interior of the sector. We denote the subspace of these vectors (the L_1-exceptional subspace) by $\mathcal{E}_1 \subset L_1 \cap \{dH = 0\}$. We will establish that the L_1-exceptional subspaces depend only on the combinatorics of finitely many collisions along the trajectory but not on the velocities. Indeed, let $R_1, R_2, \ldots$ be the sequence of orthogonal reflections in the faces of T corresponding to consecutive collisions of the particles. (In the graph of constraints these collisions are represented by the edges which do not start at the floor.) Let the consecutive collisions with the floor of the particles $m_{k_1}, m_{k_2}, \ldots$, etc., occur exactly after $t_1, t_2, \ldots$, etc., collisions between particles. We have (cf. (1.1))

$$\mathcal{E}_1 = \bigcap_{i=1}^{\infty} R_1^{-1} R_2^{-1} \ldots R_{t_i}^{-1} \{d\eta_{k_i} = 0\}.$$

Clearly this intersection must be finite, say t_N reflections determine $\mathcal{E}_1$. As a consequence there are at most countably many possible L_1-exceptional subspaces.

We will establish that the vectors in an L_1-exceptional subspace satisfy

$$\langle v, dx\rangle = v_1 dx_1 + \ldots + v_n dx_n = 0, \tag{7.10}$$

i.e., the velocity v must be orthogonal to $\mathcal{E}_1$.

Given (7.10) we obtain the claim by observing that for a fixed d-dimensional L_1-exceptional subspace the relation (7.10) describes a submanifold of codimension d in the phase space. We conclude that the set of L_1-exceptional points is contained in a countable union of submanifolds of codimension at least one.

To show (7.10) we observe that for vectors in L_1-exceptional subspaces, $\langle v, dx\rangle$ is constant in time. Indeed, it does not change in collisions because both the velocity v and the tangent vector are changed by the same orthogonal reflection. Between collisions we have

$$\frac{d^2}{dt^2}\langle v, dx\rangle = -\frac{d}{dt}\langle a, dx\rangle = 0,$$

where $a = -dv/dt$ is the acceleration vector. Hence between collisions $\langle v, dx\rangle$ could change linearly with constant rate $\langle a, dx\rangle$. This rate would not change in a collision. Hence it must be zero or else $|\langle v, dx\rangle|$ would grow unboundedly, which is impossible (velocity must be bounded due to energy conservation and the tangent vector is changed only by orthogonal reflections).

Further we observe that for vectors in an L_1-exceptional subspace also $\langle x, dx\rangle = x_1 dx_1 + \ldots + x_n dx_n$ is not changed in collisions and between collisions it has constant rate of change equal to $\langle v, dx\rangle$. We conclude again that this rate of change has to be zero or else $|\langle x, dx\rangle|$ would grow unboundedly, which is impossible.

The idea to use (7.10) and its proof belong to Simányi [S]. ∎

Finally, let us examine the L_2-exceptional subspace along a nondegenerate trajectory. As in the proof of the Main Theorem, we use projections of the L_2 subspaces onto the tangent velocity space as a way to identify all of these spaces. Moreover, the tangent velocity space can be naturally identified with the tangent configuration space. With this identification, by the results of Section 4, the action of the derivative of the flow on vectors from L_2 is the following. The vectors stay in L_2 and are unchanged except at crossings from one simple wedge to another or in collisions of particles. At a crossing from one simple wedge to another a vector from L_2 is pushed inside the sector $\mathcal{C}$ except for vectors parallel to the common face of the two simple wedges, which stay unchanged. Since we assumed the strict inequalities in (7.7), the corresponding β-angles are always strictly greater than $\pi/4$. As a consequence in a collision of two particles a vector from L_2 is pushed inside the sector $\mathcal{C}$ except for vectors parallel to the corresponding face of T, which are not changed.

Hence an L_2-exceptional subspace consists of vectors which are parallel to all the faces of the simple wedges which are crossed by the trajectory or in which the trajectory is reflected, with the exception of the faces of the wide wedge. By the assumption (7.6) the acceleration vector is parallel to all these faces. We now prove that there are no other vectors with this property.

Claim 2. *For a nondegenerate trajectory the L_2-exceptional subspace is spanned by the acceleration vector.*

Proof. Since the L_2 subspace is identified with the tangent configuration space we can use $(d\eta_1, \ldots, d\eta_n)$ as coordinates. Our goal is to show that there are enough collisions and crossings on every nondegenerate trajectory to insure that the intersection of all the faces is spanned by the acceleration vector. The task of bookkeeping is facilitated by the graph of collisions $\mathcal{G}$. Let $m_{s_1}, \ldots, m_{s_r}$ be the r immediate successors of the floor. Since every mass has to collide with one of its immediate predecessors (but not necessarily with any of its immediate successors) we conclude that for every mass there is a chain of collisions which connects it to one of the immediate successors of the floor. A collision between the particles m_l and m_p forces the equality $d\eta_l = d\eta_p$. Hence for every particle m_l there is m_{s_j}, an immediate successor of the floor, such that $d\eta_l = d\eta_{s_j}$ must hold for all vectors in the L_2-exceptional subspace.

Further, every immediate successor of the floor must have infinitely many collisions with the floor. The immediate successor m_{s_j} can collide with the floor only in the simple wedge W_σ for which $\sigma(1) = s_j$.

Consider the permutation σ_1 such that $\sigma_1(i) = s_i$, $i = 1, 2$, and $W_{\sigma_1} \subset T$, and the permutation σ_2 differing from σ_1 by the transposition of the first two elements, i.e.,

$$\sigma_2(1) = s_2, \quad \sigma_2(2) = s_1, \quad \sigma_2(i) = \sigma_1(i), \quad i \neq 1, 2.$$

Clearly $W_{\sigma_2} \subset T$ and the common face of W_{σ_1} and W_{σ_2} is (cf. the proof of Claim 2 in Section 6)

$$\frac{\eta_{s_1}}{a_{s_1}} = \frac{\eta_{s_2}}{a_{s_2}}.$$

The crossing of this common face forces

$$\frac{d\eta_{s_1}}{a_{s_1}} = \frac{d\eta_{s_2}}{a_{s_2}}.$$

As in the proof of Claim 2 in Section 6 we can conclude that there are enough of these crossings to force

$$\frac{d\eta_{s_1}}{a_{s_1}} = \frac{d\eta_{s_2}}{a_{s_2}} = \ldots = \frac{d\eta_{s_r}}{a_{s_r}}.$$

Combining with the equalities above we conclude that the L_2-exceptional subspace contains only vectors parallel to the acceleration. ■

Our Theorem is proved. ■

Let us apply Theorem 7.1 to the problem of "splitting and stacking" the masses described in the introduction. We start with the system (7.1) with elastic constraints (7.2). This system is completely hyperbolic. Further we split each of the masses $m_1, \ldots, m_n$ into two,

$$m_i = (1-\kappa_i)m_i + \kappa_i m_i \quad \text{for } 0 < \kappa_i < 1,$$

and we assume that $m_0 \prec (1-\kappa_i)m_i \prec \kappa_i m_i$, $i = 1, \ldots, n$, i.e., we have n stacks with two particles. By Theorem 7.1 this system is completely hyperbolic if we assume additionally that for $i = 1, \ldots, n$,

$$\frac{1}{\kappa_i} + \frac{m_i}{M} > 2, \tag{7.11}$$

where $M = m_0 + m_1 + \ldots + m_n$.

8. Remarks and open problems. 1. In the case $n = 2$ the Main Theorem was proven in the Appendix of [W1]. The billiard in a wedge symmetric with respect to the acceleration direction was studied by Miller and Lehtihet [L-M], and they discovered numerically the sharp transition from the mixed behavior to complete hyperbolicity as the angle increases past 90 degrees.

2. The system (7.1) with constraints (7.2) in the special case of equal masses $m_1 = \ldots = m_n$ and $c_1 = \ldots = c_n$ reduces to a PFL system. More precisely, it is a finite extension of the system with elastic constraints $q_0 \le q_1 \le \ldots \le q_n$, which was determined to be equivalent to a completely hyperbolic PFL system (Theorem 5.2). Hence in this special case the Main Theorem follows from [W1] and the work of Simányi [S] (Theorem 4.5).

3. When choosing the bundles of Lagrangian subspaces (the quadratic form Q) in the proof of the Main Theorem we relied on the canonical isomorphism of PW systems in simple wedges with PFL systems. In the general case we are unable to write down the quadratic form Q explicitly. We can do it for the system (7.1) if we take the special potential function $c_i = \alpha m_i$. For such a system the quadratic form Q is given in W_σ by

$$Q = \sum_{i=1}^{n}\left(d\eta_i d\xi_i + \frac{u_i}{\alpha m_i}(d\xi_i)^2\right) - \sum_{k<l} u_{\sigma(k)}\frac{m_{\sigma(k)}m_{\sigma(l)}}{\alpha M}\left(\frac{d\xi_{\sigma(k)}}{m_{\sigma(k)}} - \frac{d\xi_{\sigma(l)}}{m_{\sigma(l)}}\right)^2,$$

where u_i are defined in (7.4).

Independently of the isomorphism with a PFL system one can check that the form Q is constant in the absence of collisions. It is also straightforward to see that Q is not decreased when the trajectory crosses from one simple wedge to another (only one term in the second sum is changed). The monotonicity of reflections in the faces is also not hard to check. Indeed, for the reflection in the face $\{\eta_1 = 0\}$ we have

$$\xi_1^+ = -\xi_1^- - \frac{2m_1}{m_0+m_1}\sum_{i=2}^{n}\xi_i, \qquad \xi_k^+ = \xi_k^-,$$
$$u_1^+ = -u_1^-, \qquad u_k^+ = -\frac{2m_1}{m_0+m_1}u_1^- + u_k^-, \qquad k = 2,\ldots,n.$$

4. We expect that the Main Theorem remains valid if some of the entries in L (or K) are zero. When L (or K) is diagonal we get of course a completely integrable system. We conjecture that if the off-diagonal elements of L are all equal and positive then elliptic periodic orbits are present, excluding hyperbolicity of the system. This is suggested by the results of [Ch-W], where it was established for the PFL system that beyond the completely integrable case elliptic periodic points appear.

5. It is an interesting question if the "splitting and stacking" of the masses, with or without violation of the sufficient conditions (7.11), produces systems with slower mixing.

6. It is of considerable interest to find completely hyperbolic systems of an arbitrary number of particles with nonlinear potential of interaction. For PFL systems it was established in [W2] that the system is completely hyperbolic for the potential function from a large class of convex functions. Translation of this result into other classes of systems considered in this paper produces only "unnatural" interactions.

7. The setup in the Main Theorem allows introducing infinite-dimensional limits of our systems. It remains an open and intriguing question which limit should be taken and what are its properties.

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