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# Parabolic perturbations of Hamilton–Jacobi equations

by

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*Dedicated to the memory of my friend W. Szlenk*

**Abstract.** We consider a parabolic perturbation of the Hamilton–Jacobi equation where the potential is periodic in space and time. We show that any solution converges to a limit not depending on initial conditions.

Consider a Hamiltonian system whose configuration space is $\mathrm{Tor}^n$ and the phase space is $\mathrm{Tor}^n \times \mathbb{R}^n$. Assume that the Hamiltonian is $H(q,p,t) = p^2/2 - F(q,t)$ where $F$ is a periodic function of $q$ and $t$. Without any loss of generality we may assume that $F$ is $\mathbb{Z}^n$-periodic in $q$ and periodic in $t$ with period 1.

The search of invariant Lagrangian $(n+1)$-dimensional tori for such systems is reduced to finding solutions of Hamilton–Jacobi equations (see [JKM])

$$\frac{\partial S}{\partial t} + (\nabla S)^2 = F + \mathrm{const} \tag{1}$$

where $\nabla S$ is a periodic function of $q$ and $t$. Indeed, if such a solution is found then the invariant torus is projected in one-to-one way to the coordinate × time torus and is given by the equations

$$p = \nabla_q S.$$

For such solutions $S(q,t)$ can be written in the form

$$S(q,t) = aq + s(q,t)$$

where $a$ is an $n$-dimensional vector, $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$, and $s(q,t)$ is periodic in $q$ and $t$. The vector $a$ can also be found from the equality

$$\int_{\mathrm{Tor}^n} \nabla_q S \, dq = a.$$

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It is well known that invariant Lagrangian tori do not always exist. If the dynamical system is close to an integrable one then the existence of such tori is proven with the help of KAM-theory. With the increase of the distance from integrable systems the tori eventually become destroyed and are replaced by the so-called Aubry–Mather sets (see [A], [M]).

In this paper we consider the parabolic perturbation of (1):

$$\frac{\partial S}{\partial t} + (\nabla S)^2 = \nu \Delta S + F(q,t). \tag{2}$$

The const in (1) is absorbed into $F$, the parameter $\nu$ is positive. It turns out that the situation with periodic solutions of (2) is much simpler. In this paper we prove two theorems.

THEOREM 1. *Assume that* $F \in C^r(\mathrm{Tor}^{n+1})$, $r \geq 1$, *and* $a$ *are given. Then there exists a solution of* (2) *having the form*

$$S^{(0)}(q,t) = aq + \lambda t + s^{(0)}(q,t) \tag{3}$$

*where* $\lambda = \lambda(a)$ *is a constant*, $s^{(0)}$ *is periodic in* $q$ *and* $t, s^{(0)} \in C^r(\mathrm{Tor}^{n+1})$.

THEOREM 2. *Let the initial condition for* (2) *have the form*

$$S(q,0) = aq + s(q,0) \tag{4}$$

*where* $s(q,0) \in C^r(\mathrm{Tor}^n)$. *Then the solution* $S(q,t)$ *of* (2) *for* $t > 0$ *can be represented in the form*

$$S(q,t) = S^{(0)}(q,t) + \mathrm{const} + s^{(1)}(q,t) \tag{5}$$

*where* $s^{(1)}(q,t)$ *for each* $t > 0$ *is* $\mathbb{Z}^n$*-periodic in* $q$ *and* $\|s^{(0)}(q,t)\|_{C^r(\mathrm{Tor}^n)} \leq \mathrm{const} \cdot \alpha^t$ *for some* const $> 0$ *and* $\alpha < 1$.

Our proofs give also the existence of global solutions of (2) which is a well-known fact by itself. Theorem 2 shows that the non-linear parabolic equation (2) has no complicated invariant sets like strange attractors, etc. Our proof also shows that the family of tori given by the equations $p = \nabla S^{(0)}(q,t)$ is a smooth foliation of $\mathrm{Tor}^n \times \mathbb{R}^n \times \mathbb{R}$. It would be interesting to study the limiting behaviour of this foliation in the limit $\nu \to 0$. It seems that it is an open question even for $n = 1$.

Theorems 1 and 2 were proven for $a = 0$ in [S]. In [JKM] Theorem 1 was proven for $n = 1$ and more general Hamiltonians $H = H(t,q,p)$ which are strictly convex in $p$, i.e. $\delta \leq H_{pp} \leq \delta^{-1}$ for some $\delta > 0$.

*Proofs of Theorems 1 and 2.* We shall prove both theorems simultaneously. Write the Hopf–Cole substitution (see [S]) in the form

$$\varphi(q,t) = \exp\left\{\frac{1}{\nu}aq - \frac{1}{\nu}S(q,t)\right\}.$$

It follows from (4) that $\varphi(q,0)$ is a periodic function of $q$. One can check that

$$\frac{\partial S(q,t)}{\partial t}=-\nu\frac{1}{\varphi}\frac{\partial\varphi}{\partial t},\qquad \nabla S=a-\nu\frac{\nabla\varphi}{\varphi},$$
$$\Delta S=-\nu\frac{\Delta\varphi}{\varphi}+\nu\frac{(\nabla\varphi)^2}{\varphi^2}.$$

Putting all these expressions into (2) we get for $\varphi$ the equation

$$\frac{\partial\varphi}{\partial t}=\nu\Delta\varphi-2a\nabla\varphi+\frac{F-a^2}{\nu}\varphi.$$

We shall look for the solution of (6) which is periodic in $q$. It is given by the Feynman–Kac formula to be described below. Take a point $(q,t)\in\mathrm{Tor}^n\times\mathbb{R}^1$ and consider the space $\Omega_{(q,t)}^{(q',t')}$ of continuous maps $[t',t]\to\mathrm{Tor}^n$, $t'<t$, such that $\omega(t')=q'$, $\omega(t)=q$. On the Borel $\sigma$-algebra of its subsets introduce the (non-normalized) Wiener measure $W_{(q,t)}^{(q',t')}$ with drift $-2a$ and local diffusion matrix $2\nu I$.

One can construct this measure also in the following way. Consider the $n$-dimensional space $\mathbb{R}^n$ and the space $\widetilde{\Omega}_{(q,t)}$ of continuous functions $\widetilde{\omega}(\tau)$, $t'\le\tau\le t$, $\widetilde{\omega}(t)=q$, with values in $\mathbb{R}^n$. Let $\widetilde{W}_{(q,t)}$ be the Wiener measure on the Borel $\sigma$-algebra of subsets of $\widetilde{\Omega}_{(q,t)}$ for which the increments are independent, have Gaussian distributions with mean $-2a\Delta t$ and dispersion matrix $2\nu I\Delta t$. The natural projection $\mathbb{R}^n\to\mathrm{Tor}^n$ induces the Wiener measure $W_{(q,t)}$ on the Borel $\sigma$-algebra of subsets of the space $\Omega_{(q,t)}$ of continuous maps $\omega:[t',t]\to\mathrm{Tor}^n$, $\omega(t)=q$ with the end-point $\omega(t')$. Fixing $\omega(t')$ gives us the measure $W_{(q,t)}^{(q',t')}$.

In Theorem 2, $\varphi(q,0)=\exp\{-\frac{1}{\nu}s(q,0)\}$ is a $C^r$-function on $\mathrm{Tor}^n$. The Feynman–Kac formula gives a possibility to write the $q$-periodic solution of (6) as a functional integral

$$\varphi(q,t)=\int\exp\left\{-\frac{1}{\nu}s(q',0)\right\}dq' \times\int\exp\left\{\int_0^t\frac{F(\omega(\tau),\tau)-a^2}{\nu}\,d\tau\right\}dW_{(q,t)}^{(q',0)}(\omega). \tag{7}$$

The inner integral is a Markov cocycle and in principle we could use facts from the theory of such cocycles. However, we prefer the direct derivation of the needed results. Put

$$K(q'',q')=\int\exp\left\{\int_{m-1}^{m}\frac{1}{\nu}(F(\omega(\tau),\tau)-a^2)\,d\tau\right\}dW_{(q',m)}^{(q'',m-1)}(\omega).$$

Since $F$ is periodic in time this integral does not depend on $m \in \mathbb{Z}^1$. One can rewrite (7) as follows:

$$\varphi(q,t) = \int \exp\left\{-\frac{1}{\nu}s(q^{(0)},0)\right\} dq^{(0)} K(q^{(0)},q^{(1)})K(q^{(1)},q^{(2)}) \ldots K(q^{(T-1)},q^{(T)})\widetilde{K}_t(q^{(T)},q)\,dq^{(1)}\,dq^{(2)}\ldots dq^{(T)}. \tag{8}$$

Here $T=[t]$ and $\widetilde{K}_t(q^{(T)},q) = \int \exp\{\int_T^t (F(\omega(\tau),\tau)-a^2)\,d\tau\}\,dW_{(q,t)}^{(q^{(T)},T)}(\omega)$.

The kernel $K$ is positive, periodic in $q'', q'$ and $C^r$-smooth. We shall need the following well-known statement which we formulate as a separate lemma.

LEMMA 1. *The kernel $K$ has unique positive eigenfunctions $e$, $e^{(a)} \in C^r(\mathrm{Tor}^n)$,*

$$\int_{\mathrm{Tor}^n} e(q')K(q',q)\,dq' = \Lambda e(q), \qquad \int_{\mathrm{Tor}^n} K(q',q)e^{(a)}(q)\,dq = \Lambda e^{(a)}(q')$$

*corresponding to the same eigenvalue $\Lambda > 0$.*

Using these eigenfunctions we can introduce a Markov chain whose phase space is $\mathrm{Tor}^n$ and the transition density of the distribution of $q''$ given $q'$ is

$$p(q''\,|\,q') = \frac{K(q'',q')e(q'')}{\Lambda e(q')}.$$

It is easy to check that the stationary distribution of this chain is given by the density $e(q)e^{(a)}(q)$. Now (8) can be rewritten as follows:

$$\begin{aligned}\varphi(q,t) &= \Lambda^T \int \frac{\exp\left\{\frac{1}{\nu}s(q^{(0)},0)\right\}}{e(q^{(0)})}\,dq^{(0)}\,p(q^{(0)}\,|\,q^{(1)})p(q^{(1)}\,|\,q^{(2)}) \\ &\quad \ldots p(q^{(T-1)}\,|\,q^{(T)})\widetilde{K}_t(q^{(T)},q)\prod_{i=1}^{[T]} dq^{(i)} \\ &= \Lambda^T \int \frac{\exp\left\{\frac{1}{\nu}s(q^{(0)},0)\right\}}{e(q^{(0)})} p^{(T)}(q^{(0)}\,|\,q^{(T)}) \\ &\quad \times e(q^{(T)})\widetilde{K}_t(q^{(T)},q)\,dq^{(0)}\,dq^{(T)}\end{aligned} \tag{9}$$

where $p^{(T)}$ is the transition density after $T$ steps.

To finish the proof of Theorem 1 put $\exp\left\{\frac{1}{\nu}s(q^{(0)},0)\right\} = e(q^{(0)})$ and denote the corresponding solution by $\varphi^{(0)}(q,t)$. It follows from (9) that

$$\varphi^{(0)}(q,t) = \Lambda^T \int e(q^{(T)})\widetilde{K}_t(q^{(T)},q)\,dq^{(T)}.$$

It is easy to see that the last integral tends to $e(q)$ as $t \to T$ since $\widetilde{K}_t$ converges to the $\delta$-function. On the other hand, $\widetilde{K}_t(q^{(T)},q) \to K(q^{(T)},q)$ as

$t \to T+1$ and the last integral converges to $\lambda e(q)$. This shows that $\varphi^{(0)}$ can be written in the form

$$\varphi^{(0)}(q,t) = e^{t\ln \Lambda + \frac{1}{\nu} s^{(0)}(q,t)}$$

and $S^{(0)}(q,t) = aq - \nu t \ln \Lambda - s^{(0)}(q,t)$, which gives the statement of Theorem 1 with $\lambda = -\nu \ln \Lambda$.

Theorem 2 follows from the ergodic theorem for Markov chains. Indeed, for the transition density $p^{(T)}$ we can write

$$\|p^{(T)}(q^{(0)} \mid q^{(T)}) - e(q^{(T)})e^{(a)}(q^{(T)})\|_{C^r} \le \alpha^T$$

for some $\alpha < 1$. Therefore from (9),

$$\varphi(q,T)\Lambda^{-T} = \int \frac{\exp\left\{\frac{1}{\nu}s(q^{(0)},0)\right\}}{e(q^{(0)})}\, dq^{(0)} \int e(q^{(T)})\widetilde{K}_t(q^{(T)},q)\, dq^{(T)} + O(\alpha^t)$$

and

$$\frac{\varphi(q,t)}{\varphi^{(0)}(q,t)} = \int \frac{\exp\left\{\frac{1}{\nu}s(q^{(0)},0)\right\}}{e(q^{(0)})}\, dq^{(0)} + O(\alpha^t).$$

This gives the statement of Theorem 2.

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