Bounds for maps of an interval
with one reflecting critical point. I

by

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Abstract. We prove real bounds for interval maps with one reflecting critical point.

In this paper we prove real bounds for maps as in the title (^{1}). More exactly, we will consider the following class of maps.

Let $g : \bigcup I_i \to I$ be a map, where I is an open interval around the point $c = 0$ and where I_i is a finite collection of disjoint open intervals inside the interval I for which the following conditions hold:

(i) For every i, the map $g : I_i \to I$ is a homeomorphism; moreover, it is a C^3-function which extends to a smooth map on the closure of each I_i, and the Schwarzian derivative of g, $Sg := g'''/g' - (3/2)(g''/g')^2$, is defined and negative for any $x \neq 0$.

(ii) For every $i \neq 0$, the map $g : I_i \to I$ is a diffeomorphism, while the map $g : I_0 \to I$ has a unique critical point (i.e., a zero of g') at the point $c = 0 \in I_0$, so that $g(x) = g(0) + (h(x))^l$, where $l \geq 3$ is an odd number, and h is a local diffeomorphism, $h(0) = 0$. We call $g : I_0 \to I$ the *central branch*, and I_0 the *central interval* of g.

(ii) All iterates of the critical point $c = 0$ under g are in $\bigcup I_i$ (and, hence, well defined), c is a recurrent point, and the ω-limit set of the critical point $\omega(c)$ is a proper subset of $\bigcup I_i$, i.e., $\omega(c)$ stays away from the end points of the intervals I_i.

We always assume that g has no attracting or neutral periodic orbits.

EXAMPLES. 1. Consider a polynomial P with real coefficients and with all critical point escaping to infinity and real except for one reflecting critical

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(^{1}) For *complex bounds* and applications, see the second part of this paper, [LS1].

point. If we restrict P to an interval of the real axis, it will satisfy conditions (i)–(iii) (see [LS2] for details).

2. Consider a map

$$f(z) = \lambda z^d \frac{z - \frac{d+1}{d-1}}{1 - \frac{d+1}{d-1}z},$$

where $|\lambda| = 1$, and $d \geq 3$. Then f restricted to the circle $|z| = 1$ and written in the natural coordinate of the circle satisfies conditions (i)–(ii) above (the Schwarzian is negative), with $l = 3$ and with $d - 1$ intervals I_i. Note that $\lambda = f(1)$ is the critical value of f on the circle. Choosing λ, we can satisfy (iii) as well.

Define a sequence of *first return maps*

$$g^{s_n} : U_n \to U_{n-1}$$

inductively as follows. We set $s_1 = 1$, $U_1 = I_0$, and $U_0 = I$. If U_{n-1} is defined, then $g^{s_n} : U_n \to U_{n-1}$ is the central branch of the first return map to U_{n-1}.

Let us call the map $g^{s_n} : U_n \to U_{n-1}$ *central* if $g^{s_n}(c) \in U_n$, and *non-central* otherwise. Note that if some $g^{s_n} : U_n \to U_{n-1}$ is central, then by pulling back the central domain U_n by the map g^{s_n} several times, we always come to a non-central $g^{s_m} : U_m \to U_{m-1}$, where $m > n$ and where the function is the same, i.e., $s_m = s_n$.

Denote by $|U|$ the length of the interval U.

THEOREM 1. *There exist a positive number C, which depends on l only, and an index N so that, for every $n \geq N$, the length of every component of $U_{n-1} \setminus U_n$ is equal to at least $C|U_n|$, whenever the previous map $g^{s_{n-1}} : U_{n-1} \to U_{n-2}$ is non-central. In particular, this holds for a sequence $n_i \to \infty$.*

REMARK 1. The corresponding real bounds for maps with one folding critical point have been proved by Martens [Ma] (for non-renormalizable maps) and by Sullivan [S] (in the renormalizable case).

REMARK 2. As in the case of a folding critical point [S], [LS2], [LS3], the real bounds of Theorem 1 are the first step towards proving the complex bounds [LS1].

REMARK 3. As in the case of a folding critical point [MS], the real bounds of Theorem 1 yield: *if $\omega(c)$ is minimal, then the map is ergodic with respect to Lebesgue measure and $\omega(c)$ is of measure zero* (progress in this direction for several reflecting critical points has recently been made by Edson Vargas).

Theorem 1 will be a simple consequence of Propositions 1–4 below.

We say that, for two intervals $U \subset V$, a component of $V \setminus U$ is *big compared with U* if the length of this component is at least $C|U|$ for some constant $C > 0$ which does not depend on U and V (C may depend on l). We say that U is *well inside V* if both components of $V \setminus U$ are big compared with U.

PROPOSITION 1. *Assume that $g^{s_n} : U_n \rightarrow U_{n-1}$ is central, while the previous first return map is non-central. Let $m > n$ be minimal so that $g^{s_m} : U_m \rightarrow U_{m-1}$ is non-central and $s_m = s_n$. Then for the next map, the interval U_{m+1} is well inside U_m.*

Define $c_i := g^i(c)$.

PROPOSITION 2. *Assume that all maps $g^{s_n} : U_n \rightarrow U_{n-1}$ are non-central. Let, for some n_0, the points $c_{s_{n_0}}$ and $c_{s_{n_0+1}}$ be on opposite sides of c. Then either U_{n_0+2} is well inside U_{n_0+1}, or U_{n_0+3} is well inside U_{n_0+2}.*

PROPOSITION 3. *Assume that all maps $g^{s_n} : U_n \rightarrow U_{n-1}$ are non-central, and all points c_{s_n} are on the same side of c (say, always $c_{s_n} > c$). Let, for some n_0, the map $g^{s_{n_0}} : U_{n_0} \rightarrow U_{n_0-1}$ reverse orientation. Then U_{n_0+4} is well inside U_{n_0+3}.*

PROPOSITION 4. *Assume that all maps $g^{s_n} : U_n \rightarrow U_{n-1}$ are non-central and preserve orientation, and all points c_{s_n} are on the same side of c (say, always $c_{s_n} > c$). Then, for every n, the interval U_n is well inside U_{n-1}.*

Our proofs of Propositions 1–3 are based on combinatorial arguments, the Koebe Principle, and the Interval Adding Procedure (see below). The proof of Proposition 4 is analytic and uses a very simple geometric condition, under which the cross-ratio "does not feel" the reflecting critical point (Proposition 5). It will allow us to pass through the critical point without decreasing the cross-ratio.

As usual, if $J \subset T$ are two intervals and L, R are the components of $T \setminus J$ then we define $C(T, J)$ to be the cross-ratio of this pair of intervals:

$$C(T, J) = \frac{|J| \cdot |T|}{|L| \cdot |R|}.$$

Cross-ratios play a crucial role in all recent results in real interval dynamics. Often, it suffices to use some qualitative estimates based on the so-called Koebe Principle. In our analysis, we shall sometimes need sharper estimates, which are based on direct use of the cross-ratio. For example, we shall often use the inequality

$$|L|/|J| \geq C^{-1}(T, J).$$

Note also that J is well inside T if and only if $C^{-1}(T, J)$ is greater than a positive constant (which does not depend on the intervals).

If f is a map which is a diffeomorphism of T onto its image and $Sf < 0$ then (see [MS])

$$C^{-1}(T, J) \geq C^{-1}(f(T), f(J)).$$

In our case we shall apply this to maps f of the form g^n. Since $Sg < 0$ one also has $Sg^n < 0$ so the previous inequality applies when we take $f = g^n$ and $g^n|T$ is a diffeomorphism. We will use the following amusing extension of this classical fact:

PROPOSITION 5 (^{2}). *Let L be the left component of $T \setminus J$, so that if we write $L = (x_1, x_2)$, then $x_1 < 0 < x_2$, and, moreover, $x_2 > |x_1|$. Fix a real number $l > 1$, and introduce a map*

$$F(x) = \frac{x}{|x|} \cdot |x|^l, \quad x \neq 0, \quad F(0) = 0.$$

Then

$$C^{-1}(T, J) \geq C^{-1}(F(T), F(J)).$$

Proof. First of all, we can assume that $x_1 = -1$, and then $x_2 = 1 + \delta < x_3 < x_4$, where $R = (x_3, x_4)$ and $\delta > 0$. To calculate the cross-ratios, we will use the following well known equality:

$$\ln \frac{1 + C^{-1}(F(T), F(J))}{1 + C^{-1}(T, J)} = \int\limits_{x_3}^{x_4} dy \int\limits_{x_1}^{x_2} U(x, y) \, dx,$$

where

$$U(x, y) = \frac{F'(x)F'(y)}{(F(x) - F(y))^2} - \frac{1}{(x - y)^2}.$$

Fix $y \in (x_3, x_4)$, and write the integral over x as a sum of two integrals I_1 and I_2, where x in I_1 varies from -1 to 1, and in I_2 from 1 to $1 + \delta$. Then $I_2 < 0$, since the function $x^l, x > 0$, has negative Schwarzian derivative. For the first integral, we have

$$I_1 = \int\limits_{-1}^{0} U(x, y) \, dx + \int\limits_{0}^{1} U(x, y) \, dx = \int\limits_{0}^{1} V(x, y) \, dx,$$

where

$$V(x, y) = \frac{l^2 x^{l-1} y^{l-1}}{(x^l + y^l)^2} + \frac{l^2 x^{l-1} y^{l-1}}{(x^l - y^l)^2} - \frac{1}{(x + y)^2} - \frac{1}{(x - y)^2}.$$

After computing the integral, we obtain

$$I_1 = \frac{2l}{y} \left\{ \frac{la^{l-1}}{1 - a^{2l}} - \frac{1}{1 - a^2} \right\},$$

(^{2}) This statement was established during the author's visit to Sebastian van Strien at Warwick and numerous conversations with him.

where the parameter $a = 1/y \in (0, 1)$. Since the expression in $\{\ldots\}$ is negative, we are done.

We make use of two principles.

KOEBE PRINCIPLE [MS]. *Let T be an interval from a component of the domain of definition of some iterate g^s, and so that $T = L \cup J \cup R$, where:* (a) *$g^s|J$ has a unique critical point at c, and* (b) *$g^s|L$ and $g^s|R$ have at most N critical points. Let $K > 0$ be so that*

$$|g^s(L)|/|g^s(J)|, |g^s(R)|/|g^s(J)| \geq K.$$

Then

$$|L|/|J|, |R|/|J| \geq C,$$

where $C > 0$ depends on K, N, and on the map g. Moreover, given g, there exists $\varepsilon > 0$ so that if $|g^i(T)| \leq \varepsilon$ for $i = 0, 1, \ldots, s$, then the constant C depends merely on K and N.

For any two intervals U, V, denote by $[U, V]$ the minimal interval which contains U and V.

INTERVAL ADDING PROCEDURE. Given $g^{s_n} : U_n \to U_{n-1}$, let t_n^+ be a maximal interval outside U_n and to the right of U_n with a common boundary point with U_n, so that $g^{s_n}|t_n^+$ is defined and a diffeomorphism. The interval t_n^- is defined in the same way, but to the left of U_n.

Let t be either t_n^+ or t_n^-. Note that one end point of $g^{s_n}(t)$ is in ∂U_{n-1}. If the other end point of $g^{s_n}(t)$ is an end point of the range I of the map g, then we set $k = 0$ and stop. Otherwise the boundary point of t which is different from ∂U_n is a critical point of g^{s_n}: there exists i, $1 \leq i \leq s_n - 1$, so that c lies at the boundary of the interval $g^i(t)$. In this case we will make an operation called "adding the interval U_{n-1}":

Since $g^i(U_n)$ is outside U_{n-1} and $g^i(t)$ has c in its boundary, the interval $g^i(t)$ also contains a boundary point of U_{n-1}. Hence, either $i + s_{n-1} > s_n$, or $g^{i+s_{n-1}}(t)$ contains a boundary point of U_{n-2}. Hence, again either $i + s_{n-1} + s_{n-2} > s_n$, or $g^{i+s_{n-1}+s_{n-2}}(t)$ contains a boundary point of U_{n-3}. We continue this process until we find $k \geq 1$ so that

$$S := i + s_{n-1} + \ldots + s_{n-k+1} < s_n,$$

while

$$S + s_{n-k} \geq s_n.$$

Then $g^S(t)$ contains a boundary point of U_{n-k}. Let us extend the interval $g^S(t)$ so that it turns into the interval $[U_{n-1}, g^S(t)]$, and call this operation "adding the interval U_{n-1}".

The boundary point of t different from ∂U_n lies in a component of $U_{m-1} \setminus U_m$ for some $m \leq n$. Since the points of ∂U_{m-1} are nice, there exists a unique

extension $\widehat{t}$ of t inside $U_{m-1} \setminus U_m$ so that

$$g^S(\widehat{t}) = [U_{n-1}, g^S(t)].$$

LEMMA 1. (i) $g^{s_n}(\widehat{t})$ covers an interval $g^j(U_{n-1})$ for some $1 \le j \le s_{n-1} - 1$.

(ii) The map $g^{s_n}|\widehat{t}$ has at most k critical points.

(iii) Let $\widehat{t}_n^{\pm}$ be the intervals corresponding to $t_n^{\pm}$, and $k_n^{\pm}$ be the corresponding numbers. If $k_n^{\pm} \le N$, then $|t_n^{\pm}| \ge C|U_n|$, where $C > 0$ only depends on l and N.

P r o o f. Since $g^S(\widehat{t})$ contains U_{n-1} and $S + s_{n-k} \ge s_n$, the interval $g^{s_n}(\widehat{t})$ contains some $g^j(U_{n-1})$, where $j \le s_{n-k} \le s_{n-1} - 1$. To prove the second statement, first note that the map $g^i|\widehat{t}$ has no critical points, since $g^i(\widehat{t} \setminus t) \subset U_{n-1}$ and $i < s_n$. On the other hand, the interval $g^i(\widehat{t})$ covers c. Hence, the map $g^{s_{n-1}}|g^i(\widehat{t})$ has one critical point. Since $g^{s_{n-1}}(g^i(\widehat{t})) \setminus g^{s_{n-1}}(g^i(t))$ is part of U_{n-2}, the map $g^{s_{n-2}}|g^{i+s_{n-1}}(\widehat{t})$ can have at most one critical point, and so on until $g^S(\widehat{t})$ (each time at most one critical point can be added). The iterate of g from the latter interval to the interval $g^{s_n}(\widehat{t})$ does not have critical points, by definition of k. The last statement follows from the previous ones and from the Koebe Principle (here we use the smallest interval argument).

REMARK 4. In what follows, if $g^{s_n}(\widehat{t}_n^{\pm})$ covers more than one interval $g^j(U_{n-1})$, we shorten $\widehat{t}_n^{\pm}$ so that $g^{s_n}(\widehat{t}_n^{\pm})$ covers exactly one interval of this form.

Let us call the numbers $k_n^{\pm}$ the *parameters*, or the *numbers of steps* of the procedure.

To start with, let us prove the following

LEMMA 2. (i) Let $g^{j_1}(U_n)$ and $g^{j_2}(U_n)$ be two neighbours of U_n from the collection $g^j(U_n)$, $j = 1, \ldots, s_n - 1$. Then U_n is well inside $[g^{j_1}(U_n), g^{j_2}(U_n)]$.

(ii) Let the map $g^{s_{n-1}} : U_{n-1} \to U_{n-2}$ be non-central. Then, at least for one component K of $U_{n-1} \setminus U_n$, the iterate $g^{s_n}(K)$ covers $g^j(U_n)$ for some $1 \le j \le s_n - 1$. If, for example, $c_{s_{n-1}} > c$, then K is the left (resp. right) component of $U_{n-1} \setminus U_n$ depending on whether $g^{s_{n-1}} : U_{n-1} \to U_{n-2}$ preserves (resp. reverses) orientation.

(iii) Let the map $g^{s_{n-1}} : U_{n-1} \to U_{n-2}$ be non-central. Then one of the two components of $U_{n-1} \setminus U_n$ is big compared with U_n.

P r o o f. (i) follows from the shortest interval argument and from the Koebe Principle.

(ii) Let $c_{s_{n-1}} > c$. Then the end points of $g^{s_{n-1}}(K)$ already lie on opposite sides of the critical point c.

(iii) Denote by t the interval around U_n which is mapped by g^{s_n} onto $[g^{j_1}(U_n), g^{j_2}(U_n)]$. The map $g^{s_n}|t$ has at most 3 critical points, because $[g^{j_1}(U_n), g^{j_2}(U_n)]$ contains precisely 3 critical values of this map. Since $g^{s_{n-1}} : U_{n-1} \to U_{n-2}$ is non-central, by (ii) at least one of the ends of t (say, the left one) lies inside $U_{n-1} \setminus U_n$. Assume now the contrary: every component of $U_{n-1} \setminus U_n$ is not big compared with U_n. Then, by (i) the interval U_{n-1} is well inside $[g^{j_1}(U_n), g^{j_2}(U_n)]$. Pulling $[g^{j_1}(U_n), g^{j_2}(U_n)]$ back by g^{s_n} to t (so that U_{n-1} is pulled back to U_n) we then conclude that the left component of $U_{n-1} \setminus U_n$ is big compared with U_n. A contradiction.

LEMMA 3. *Let the map* $g^{s_{n-1}} : U_{n-1} \to U_{n-2}$ *be non-central, and let* t *be an interval around* U_n *such that* $g^{s_n}(t) = [g^{j_1}(U_n), g^{j_2}(U_n)]$ *(where, as in Lemma 2,* $g^{j_1}(U_n)$ *and* $g^{j_2}(U_n)$ *are two neighbours of* U_n *from the collection* $g^j(U_n), j = 1, \ldots, s_n - 1$*). Then* $t \subset U_{n-2}$. *In particular,* $t_{n+1}^{\pm} \subset U_{n-2} \setminus U_{n+1}$ *(see Remark 4).*

P r o o f. First, one can assume without loss of generality that $c_{s_{n-1}} > c$. Now, the map $g^{s_{n-1}} : U_{n-1} \to U_{n-2}$ is non-central, and hence $g^{s_{n-1}}(U_n)$ lies in the right component of $U_{n-2} \setminus U_{n-1}$. It follows immediately that the right end of t belongs to U_{n-2}. To prove the same for the left end of t one should consider different combinations of the following possibilities: $g^{s_{n-1}} : U_{n-1} \to U_{n-2}$ preserves or reverses orientation, $c_{s_{n-1}}$ and $c_{s_{n-2}}$ are on the same side of c, or on the opposite sides:

1) $g^{s_{n-1}} : U_{n-1} \to U_{n-2}$ preserves orientation. Then the component K defined in Lemma 2(ii) is the left component of $U_{n-1} \setminus U_n$, and hence the left end of t is inside K, and we are done in this case.

2) $g^{s_{n-1}} : U_{n-1} \to U_{n-2}$ reverses orientation. Consider also the map $g^{s_{n-2}} : U_{n-2} \to U_{n-3}$.

2a) Let $s_{n-2} = s_{n-1}$ (i.e., the return to U_{n-3} is central). Denote by $r(V \setminus U)$ (resp. $l(V \setminus U)$) the right (left) component of $V \setminus U$. Since we are in case 2, we have $g^{s_{n-1}}(r(U_{n-2} \setminus U_{n-1})) = l(U_{n-3} \setminus U_{n-2})$. But $g^{s_{n-1}}(U_n) \subset r(U_{n-2} \setminus U_{n-1})$. Hence, $g^{2s_{n-1}}(U_n) \subset l(U_{n-3} \setminus U_{n-2})$. This is enough, since g^{s_n} takes the end points of U_{n-2} outside U_{n-3}.

2b) Let $s_{n-2} < s_{n-1}$ (i.e., the return to U_{n-3} is non-central). If $c_{s_{n-2}} < c$, then $g^{s_{n-2}}(U_n) \subset l(U_{n-3} \setminus U_{n-2})$, and because g^{s_n} takes the end points of U_{n-2} outside U_{n-3}, we are done. Let $c_{s_{n-2}} > c$. Then $g^{s_{n-2}}(U_n) \subset r(U_{n-3} \setminus U_{n-2})$. Hence, already $g^{s_{n-1}}(l(U_{n-2} \setminus U_n))$ covers $g^{s_{n-2}}(U_n)$ (we are in case 2). Thus all cases have been considered.

Proof of Proposition 1. The idea of the proof is to use the space around U_n given by the first part of Lemma 2 and pull it back by g^{s_n}. Since the return is central, we again obtain a map from our sequence of the first return maps.

First, for the interval t from Lemma 3, we have $t \subset U_{n-2}$. It follows immediately from this observation and from Lemma 2(i) that U_{n+1} is well inside U_{n-2} (here we use the fact that $s_{n+1} = s_n$). Consider now two cases:

1. $m - n \geq 2$. Then we can pull U_{n+1} back by the same central branch one more time and find that U_{n+2} (and hence U_m) is well inside U_{n-1}. Since $g^{s_n} : U_m \to U_{m-1}$ is non-central, the next map, i.e., the first return to U_m will already have a bound which is determined by the space between U_m and U_{n-1}. Therefore, U_{m+1} will be well inside U_m.

2. $m - n = 1$. We apply in this case the Interval Adding Procedure to the map $g^{s_{m+1}} : U_{m+1} \to U_m$. Then either we finish this procedure after two steps, or we cover a component of $U_{n-2} \setminus U_{n+1}$, which, as we know, is big compared with U_{n+1}. In both cases, we again conclude that U_m is well inside U_{n-1}. (Note that $g^{s_{m+1}}(\partial U_m)$ is outside U_{n-2}.) Therefore, U_{m+1} is well inside U_m.

Proof of Proposition 2. According to Lemma 2, one of the two components of $U_{n-1} \setminus U_n$ is big compared with U_n.

One can assume that $c_{s_{n_0}} < c$ and $c_{s_{n_0+1}} > c$. Note that the interval $g^{s_{n_0}}(U_{n_0+1})$ is in $U_{n_0-1} \setminus U_{n_0}$. Therefore, by Lemma 2(i), the left component of $U_{n_0-1} \setminus U_{n_0+1}$ is big compared with U_{n_0+1}. If we now assume that the right component of $U_{n_0-1} \setminus U_{n_0}$ is big (compared with U_{n_0}), then we see that U_{n_0+2} is well inside U_{n_0+1} (we apply the Interval Adding Procedure to the map from U_{n_0+2} to U_{n_0+1}, and either we finish it with $k \leq 2$ and then we "add the interval" or otherwise without adding we cover a component of $U_{n_0-1} \setminus U_{n_0}$). Therefore, we should assume that the left component of $U_{n_0-1} \setminus U_{n_0}$ is big (compared with U_{n_0}). By the same reason, we should assume that the right component of $U_{n_0} \setminus U_{n_0+1}$ (which contains $c_{s_{n_0+1}}$) is also big compared with U_{n_0+1} (otherwise U_{n_0+3} would be well inside U_{n_0+2}). But then again U_{n_0+2} is well inside U_{n_0+1}.

Proof of Proposition 3. Note that the interval $g^{s_{n_0+1}}(U_{n_0+2})$ is in the right component of $U_{n_0} \setminus U_{n_0+1}$.

Let us first show that either

• $s_{n_0} + s_{n_0+1} < s_{n_0+2}$ and the interval $g^{s_{n_0}+s_{n_0+1}}(U_{n_0+2})$ is in the left component of $U_{n_0-1} \setminus U_{n_0}$, or
• U_{n_0+4} is well inside U_{n_0+3}.

Namely, consider the Interval Adding Procedure for the map $g^{s_{n_0+2}} : U_{n_0+2} \to U_{n_0+1}$, and let t be one of the two intervals there. Then either (a) we finish the procedure after two steps, or (b) $g^{i+s_{n_0+1}}(t)$ has $c_{s_{n_0+1}}$ as an end point, and contains an end point of U_{n_0}, and, moreover, $g^{s_{n_0}+1}$ is a diffeomorphism of this interval. In case (b), the interval $g^{s_{n_0}+s_{n_0+1}}(U_{n_0+2})$ is to the left of c. Indeed, the map $g^{s_{n_0}}$ takes the right end of U_{n_0} to the left

end of U_{n_0-1} (we use the fact that $g^{s_{n_0}} : U_{n_0} \to U_{n_0-1}$ reverses orientation). Hence, in case (b), this map takes $c_{s_{n_0+1}}$ to the left of c as well. So, at least part of the interval $g^{s_{n_0}+s_{n_0+1}}(U_{n_0+2})$ is to the left of c. If the latter interval covered c then $s_{n_0}+s_{n_0+1} = s_{n_0+2}$ and since all c_{s_n} are to the right, $g^{s_{n_0}}(g^{i+s_{n_0+1}}(t))$ would cover c.

Thus, either $g^{s_{n_0}+s_{n_0+1}}(U_{n_0+2})$ is to the left of c, or the parameter for each side of the procedure is ≤ 2. In the latter case, we use Lemmas 1 and 3 to conclude that U_{n_0+3} is well inside U_{n_0+1}, which implies that U_{n_0+4} is well inside U_{n_0+3}.

Finally, we can assume that $g^{s_{n_0+1}}(U_{n_0+2})$ and $g^{s_{n_0}+s_{n_0+1}}(U_{n_0+2})$ are on the opposite sides of c (but inside U_{n_0-1}). Then we apply the procedure to the map $g^{s_{n_0+3}} : U_{n_0+3} \to U_{n_0+2}$ this time, and either we finish it after 3 steps, or we cover one of these two intervals. Note that $g^{s_{n_0+3}}(\partial U_{n_0+1})$ is outside U_{n_0-1}. In the case considered, one can assume that $\widehat{t}_{n_0+3}^{\pm} \subset U_{n_0+1} \setminus U_{n_0+3}$. Therefore, in any case, U_{n_0+3} is well inside U_{n_0+1}, and then U_{n_0+4} is well inside U_{n_0+3} (the returns are non-central).

In order to prove Proposition 4, we first prove two lemmas. Set $U_n = (-l_n, r_n)$ and $\varrho_n = l_n/r_n$, and define

$$\delta_n^+ = \frac{r_{n-1} - r_n}{r_n + l_n}, \quad \delta_n^- = \frac{l_{n-1} - l_n}{r_n + l_n}.$$

By the assumption, $c_{s_n} \in (r_n, r_{n-1})$ for any n.

The next lemma is a direct consequence of Proposition 5 from the beginning of the paper.

LEMMA 4. *Under the conditions of Proposition* 4, *assume that*

(1) $$r_n > l_{n-1}$$

for all n. *Then* $C^{-1}(U_{n-1}, U_n)$ *tends to infinity as* n *tends to infinity.*

Proof. Fix n, and look at the map $g^{s_{n-1}} : U_{n-1} \to U_{n-2}$. We have

$$C^{-1}(g(U_{n-1}), g(U_n)) \geq C^{-1}(U_{n-2}, g^{s_{n-1}}(U_n)).$$

Since the return is non-central,

$$g^{s_{n-1}}(U_n) \subset (r_{n-1}, r_{n-2}).$$

By condition (1), $r_{n-1} > l_{n-2}$, so that we can apply Proposition 5:

$$C^{-1}(U_{n-2}, g^{s_{n-1}}(U_n)) \geq C^{-1}(g(U_{n-2}), g^{s_{n-1}+1}(U_n)).$$

Now we look at the map $g^{s_{n-2}} : U_{n-2} \to U_{n-3}$, and since the return is again non-central and *on the same* (*right*) *side*, we can continue, up to the level $U_0 = I$:

$$C^{-1}(g(U_{n-1}), g(U_n)) \geq C^{-1}(I, g^{S_n}(U_n)),$$

where $S_n = s_{n-1} + \ldots + s_1$. On the other hand,

$$|g^{S_n}(U_n)| \to 0$$

as $n \to \infty$ (because any two intervals $g^i(U_n)$, $g^j(U_m)$ are either disjoint or one contains the other and because no wandering interval exists [MS]), and this interval is away from the end points of I (condition (iii) of the definition of the class of maps g). Therefore, $C^{-1}(I, g^{S_n}(U_n)) \to \infty$.

REMARK 5. We have also proved the following: if condition (1) holds for all n between some n_0 and $n_1 > n_0$, then

$$C^{-1}(g(U_{n_1}), g(U_{n_1+1})) \geq C^{-1}_{n_0-1},$$

where $C^{-1}_{n_0-1}$ is the infimum of the cross-ratios $C^{-1}(U_{n_0-1}, K)$ over all those components K of the first return map to U_{n_0-1} which contain an iterate of the critical point.

LEMMA 5. *Under the conditions of Proposition* 5, *assume that*

(2) $$r_{m-1}(1 + \varepsilon) \leq l_{m-2}$$

for some m *and some small enough* $\varepsilon > 0$. *Then* U_{m+1} *is well inside* U_m.

Proof. For any n, (a), (b), and (c) below hold:

(a) By Lemma 2(i), there exists a positive constant δ_0, which depends only on l, such that each of the two gaps between U_n and the end points of the interval $[U'_n, U''_n]$ is at least $\delta|U_n|$, where U'_n, U''_n are the neighbours of U_n from the collection $g^j(U_n)$, $j = 1, \ldots, s_n - 1$.

(b) Since the returns are non-central and to the right of 0, the interval $g^{s_n-2}(U_{n-1})$ lies in (r_{n-2}, r_{n-3}), and $g^{s_n}((r_n, r_{n-1}))$ covers (r_{n-1}, r_{n-3}). Hence, by the Interval Adding Procedure and Lemma 1, there exists an interval $\widehat{t}_n^+ = (r_n, x_n)$, where $x_n \in (r_n, r_{n-1})$, such that $g^{s_n}|_{\widehat{t}_n^+}$ has at most one critical point and $g^{s_n}(\widehat{t}_n^+)$ covers some interval of the form $g^j(U_{n-1})$, $j = 1, \ldots, s_{n-1} - 1$. In particular, by (a), $|g^{s_n}(\widehat{t}_n^+)| \geq \delta_0|U_{n-1}|$.

(c) Since the returns are non-central, and to the right of 0, there exists an interval $\widehat{t}_n^- = (-y_n, -l_n)$, where $-y_n \in (-l_{n-1}, -l_n)$, such that $g^{s_n}|_{\widehat{t}_n^-}$ has at most one critical point and $g^{s_n}(\widehat{t}_n^-)$ covers some interval of the form $g^j(U_n)$, $j = 1, \ldots, s_n - 1$. In particular, by (a), $|g^{s_n}(\widehat{t}_n^-)| \geq \delta_0|U_n|$. (Note the difference between (b) and (c)). Moreover, by the same reason, there exists an interval $t \subset \widehat{t}_n^-$ such that the map $g^{s_n} : t \to (-l_{n-2}, -l_{n-1})$ is one-to-one.

Consider now two cases:

1. $\delta_{m-1}^- \geq \delta_0/10$. Then, by (b) and (c) (with $n = m$), U_m is well inside U_{m-1}, and, hence, U_{m+1} is well inside U_m.

2. $\delta_{m-1}^- \leq \delta_0/10$. Then

$$l_{m-2} \leq l_{m-1} + \frac{\delta_0}{10}(l_{m-1} + r_{m-1}).$$

Together with $l_{m-2} \geq r_{m-1}(1 + \varepsilon)$, this gives

$$(3) \qquad \varrho_{m-1} \geq \frac{10(1 + \varepsilon) - \delta_0}{10(1 + \varepsilon) + \delta_0}.$$

Consider 3 subcases:

2.1. $\delta_m^- \geq \delta_0/10$. Then U_{m+1} is well inside U_m (see case 1).

2.2. $\delta_m^- \leq \delta_0/10$ and $\delta_m^+ \leq 10$. Consider the two gaps between U_{m-1} and the end points of $g^{s_m}(\widehat{t}_m^- \cup U_m \cup \widehat{t}_m^+)$. The right gap is at least $\delta_0|U_{m-1}|$. The left gap is at least $9\delta_0|U_m|/10$. But

$$\frac{9\delta_0|U_m|/10}{|U_{m-1}|} = \frac{9\delta_0/10}{1 + (\delta_m^- + \delta_m^+)} \geq \frac{9\delta_0/10}{1 + \delta_0/10 + 10} =: \delta_1 > 0,$$

and therefore U_m is well inside U_{m-1}.

2.3. $\delta_m^- \leq \delta_0/10$ and $\delta_m^+ \geq 10$. Then

$$\varrho_{m-1} = \frac{l_{m-1}}{r_{m-1}} = \frac{l_m + \delta_m^-(l_m + r_m)}{r_m + \delta_m^+(l_m + r_m)} \leq \frac{1 + \delta_m^-}{\delta_m^+} \leq \frac{1}{10}\left(1 + \frac{\delta_0}{10}\right).$$

This contradicts (3) if ε and δ_0 are small enough (but fixed).

Since all the cases have been considered, the statement is proved.

Proof of Proposition 4. This is a simple application of Lemmas 4–5 above. Indeed, if the conditions of Lemma 4 are satisfied, we are done. Otherwise, there are m and $\varepsilon > 0$ small enough so that (2) holds. By Lemma 5, U_{m+1} is then well inside U_m, and therefore each component of the first return map to U_{m+1} is well inside U_{m+1}. In particular, the quantity C_{m+1}^{-1} which was defined in the remark after the proof of Lemma 4 is bounded away from zero by a constant which depends on l only. Now, if (1) holds for $n = m+2, m+3, m+4$, then U_{m+6} is well inside U_{m+5} by the remark after the proof of Lemma 4, and we start over again replacing m by $m+5$. If not, then we apply Lemma 5 to $m+2$, $m+3$, or $m+4$ instead of $m-1$, and start again as well.

Proof of Theorem 1. We repeat the proof of Proposition 4, replacing Lemma 5 by one of Propositions 1–3 if needed.

REMARK 6. Combining the proofs of the paper with some methods of the recent thesis by O. Kozlovski, one can get rid of the negative Schwarzian derivative condition to obtain the same real bounds for any C^2-map with one reflecting critical point (see [LS1] for details).

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