FUNDAMENTA
MATHEMATICAE
157 (1998)

Entropy of Gaussian actions for countable Abelian groups

by

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Abstract. We prove that if a countable Abelian group A satisfies Thouvenot's conjecture then for any of its Gaussian actions on a standard Borel space the entropy is either zero or infinity, and moreover, the former case happens iff the spectral measure of the Gaussian action is singular with respect to Haar measure on the dual of A.

Introduction. In this note we extend a classical result about the entropy of Gaussian automorphisms ($\mathbb{Z}$-actions) to some weakly mixing Gaussian actions of countable Abelian groups. The groups we have in mind are those for which any of their positive entropy actions $\underline{T}$ has the property that its spectrum is Haar countable in the orthocomplement of $L^2(\pi(\underline{T}))$, where $\pi(\underline{T})$ stands for the Pinsker factor of $\underline{T}$. We recall that a conjecture—attributed to J.-P. Thouvenot—says that in fact any positive entropy action of a countable Abelian group has this property. Some particular cases have been proved in [4], [6], [7].

The fact that the entropy of a Gaussian automorphism T is zero or infinity (the latter happens iff the spectral measure of T is not singular with respect to Lebesgue measure) is commonly attributed to Pinsker, although it seems that the first "written" proof appears in [15]. De la Rue in [15] generalizes this classical Pinsker result to Gaussian $\mathbb{Z}^d$-actions, $d \geq 1$. His proof, however, does not seem to extend to general countable Abelian group Gaussian actions.

Here, we use some ideas from [10] and [11] about the existence of common group factors for some Gaussian automorphisms which allow us to prove the result formulated in the abstract. In case of $\mathbb{Z}^d$-actions, our proof is different from de la Rue's.

For a definition and basic properties of entropy of group actions we refer to [8] (see also [1]). It is also assumed that the reader is familiar with basic

1991 *Mathematics Subject Classification*: Primary 28D05.

Part of this work has been done when the author was visiting the E. Schrödinger Institute in Vienna in February/March 1997.

spectral theory for unitary representations of countable Abelian groups (e.g. a straightforward extension of the Appendix in [13]).

The author would like to thank Bruno Kamiński for fruitful discussions on the subject and the referee for a careful reading of the first version of this note.

1. The entropy of a Gaussian A-action. For a general theory of Gaussian automorphisms (i.e. of Gaussian $\mathbb{Z}$-actions) we refer to [2]. For its extension to Gaussian A-actions we refer to [9].

We say that a weakly mixing A-action $\underline{T} = (T_a)_{a\in A} : (X,\mathcal{A},\mu) \to (X,\mathcal{A},\mu)$ is a *Gaussian A-action* if there exists an invariant subspace $H \subset L^2_{0,\mathbb{R}}(X,\mu)$ (the space of real square-integrable functions with zero mean) such that

(i) the smallest sub-σ-algebra $\mathcal{A}(H)$ for which all elements of H are measurable is $\mathcal{A}$;

(ii) each variable of H is a Gaussian variable;

(iii) $H = \operatorname{span}\{fT_a : a \in A\}$ for some $f \in H$.

By the *spectral measure* of a Gaussian action we mean the measure σ on $\widehat{A}$ determined by

$$\widehat{\sigma}(a) = \int_{\widehat{A}} \chi(a)\, d\sigma(\chi) = \int_X fT_a f\, d\mu.$$

This measure is symmetric ($\sigma(D) = \sigma(\bar{D})$) and it is well known that given a symmetric finite Borel measure σ on $\widehat{A}$ there is a unique Gaussian A-action whose spectral measure is σ; it lives on $\mathbb{R}^A$ and $f = X_0$ corresponds to the projection onto the "zero coordinate" while T_a is the shift by a. Throughout, σ is assumed to be continuous (which is equivalent to weak mixing of the corresponding Gaussian action).

The unitary action of $\underline{T}$ restricted to H is isomorphic to the unitary representation $\underline{V} = (V_a)_{a\in A}$ of A acting on $L^2(\widehat{A},\sigma)$ by the formula $(V_a f)(\chi) = \chi(a)f(\chi)$. It is also classical that given a unitary action $(W_a)_{a\in A}$ there is a unique way to extend it to a measure-preserving Gaussian A-action on $(X,\mathcal{A},\mu)$ (indeed, given $U \in \mathcal{U}(L^2(\widehat{A},\sigma))$ on $\mathbb{R}^A$ consider the process $(UX_a)_{a\in A}$, note that it is Gaussian and its spectral measure is still σ; define then $S : \mathbb{R}^A \to \mathbb{R}^A$ by

$$X_a(S\overline{x}) = (UX_0)(T_a\overline{x}),$$

where $\overline{x} \in \mathbb{R}^A$). A similar argument can be used to prove that if σ and σ' are equivalent then the corresponding Gaussian actions are isomorphic. Note finally that the group of unitary actions on $L^2(\widehat{A},\sigma)$ endowed with strong topology and the group of their (unique) measure-preserving extensions endowed with the weak topology are isomorphic as topological groups (it is clear that there is a group isomorphism from the latter to the former which is continuous; its inverse is then continuous by a general open map theorem).

Assume that $\underline{T} = (T_a)_{a\in A}$ is a Gaussian A-action given by σ. Denote by H its Gaussian space. Assume that σ_1 is a Borel symmetric measure absolutely continuous with respect to σ. Put

$$H_1 = \{f \in H : \sigma_f \ll \sigma_1\}.$$

Then H_1 is a closed invariant subspace on which $\underline{T}$ still has simple spectrum, hence $\mathcal{A}(H_1)$ gives rise to a Gaussian factor (whose spectral measure is σ_1) of $\underline{T}$.

Let λ denote Haar measure on $\widehat{A}$. The key lemma to prove our entropy result is the following.

Lemma 1. *Assume that $D, E \subset \widehat{A}$ are symmetric Borel sets with*

$$\lambda(E) = \lambda(D) > 0.$$

Denote by $\underline{T}^D = (T_a^D)_{a\in A}$, $\underline{T}^E = (T_a^E)_{a\in A}$ the Gaussian A-actions corresponding to $\lambda|_D$ and $\lambda|_E$ respectively. Then $\underline{T}^D$ and $\underline{T}^E$ have the same entropy.

The proof of the lemma will be given in Section 2.

Let $\underline{T}$ be a Gaussian A-action corresponding to a symmetric finite Borel measure σ on $\widehat{A}$. Write

$$\sigma = \sigma_1 + \sigma_2, \quad \text{where} \quad \sigma_1 \ll \lambda,\ \sigma_2 \perp \lambda.$$

To the above decomposition there corresponds a decomposition of the Gaussian space H of $\underline{T}$ into $H_1 \oplus H_2$ which in turn leads to a representation of $\underline{T}$ as a direct product

$$\underline{T} = \underline{T}_1 \times \underline{T}_2,$$

where $\underline{T}_i = (T_{i,a})_{a\in A}$ is a Gaussian A-action corresponding to σ_i, $i = 1, 2$. We will now repeat the classical argument (see for example [12] or [15]) that the entropy of $\underline{T}_2$ is zero. Indeed, put

$$L^2(\mathcal{A}(H_2)) = L^2(\pi(\underline{T}_2)) \oplus F$$

and suppose that $h(\underline{T}_2) > 0$. Since A satisfies Thouvenot's conjecture, on F the spectrum of $\underline{T}_2$ is Haar. But the spectral type on H_2 is σ_2, so $H_2 \subset L^2(\pi(\underline{T}_2))$. Therefore $\mathcal{A}(H_2) \subset \pi(\underline{T}_2)$ and hence we obtain a contradiction.

Theorem 1. *The entropy of a Gaussian A-action $\underline{T}$ is either zero or infinity. The former case holds iff $\sigma \perp \lambda$.*

Proof. All we have to prove is that if σ is symmetric and $\sigma \ll \lambda$ then the entropy of $\underline{T}$ is infinite. This is clear if $\sigma = \lambda$ (indeed, the variables fT_a, $a \in A$, are independent and have continuous distribution, hence the entropy is infinite). In view of Lemma 1, we see that the entropy of a Gaussian A-action corresponding to $\lambda|_D$ depends only on the measure of D. It follows that the entropy depends monotonically on the measure of D. Assume now that $E \subset \widehat{A}$ is a symmetric Borel set satisfying $\lambda(E) > 0$. Then there exists a finite Borel partition $(E_1, \ldots, E_k)$ of $\widehat{A}$ such that

(i) $E_1 = E$,
(ii) E_i is symmetric,
(iii) $\lambda(E_i) \le \lambda(E)$, $i = 1, \ldots, k$.

Since $+\infty = h(\underline{T}^{\mathbb{T}}) = \sum_{i=1}^{k} h(\underline{T}^{E_i}) \le k h(\underline{T}^{E})$, the result follows. ∎

Remark 1. (1) The case of a non-ergodic Gaussian A-action can be reduced to the case of σ discrete. This means, however, that the A-action is compact and hence its entropy equals zero.

(2) If $(T_t)_{t\in\mathbb{R}}$ is a Gaussian flow and σ is its spectral measure (σ is defined on $\mathbb{R}$) then $T := T_1$ is a Gaussian automorphism whose maximal spectral type on its Gaussian space (in general, T has a nontrivial multiplicity on the Gaussian space) equals $\exp_*(\sigma)$ (here $\exp : \mathbb{R} \to \mathbb{T}$, $\exp(r) = e^{2\pi i r}$). Since $h((T_t)) = h(T)$ and exp is a nonsingular map, the entropy $h((T_t))$ equals zero or infinity; the former case happens iff σ is singular.

(3) It is well known that spectral methods can be applied to a more general case than the one of an Abelian group, namely to the case of type I groups. However, in the case of discrete topology a group is of type I iff it has a normal Abelian subgroup of finite index ([16]). It is also well known that an action has zero (infinite) entropy iff there exists a cocompact subaction with zero (infinite) entropy. Therefore (if Thouvenot's conjecture is true for any Abelian action) in the case of a Gaussian action of a type I countable group we still have a dichotomy: the entropy equals zero or infinity.

2. Proof of Lemma 1

2.1. *Ergodic theory preliminaries.* Let A be a countable Abelian group. Assume that A acts by measure-preserving maps on a standard probability Borel space $(X, \mathcal{A}, \mu)$, i.e. we have $\underline{T} = (T_a)_{a\in A} : (X, \mathcal{A}, \mu) \to (X, \mathcal{A}, \mu)$.

Proposition 1 (Hopf's equivalence lemma). *Assume that $\underline{T}$ is ergodic and let $D, E \in \mathcal{A}$. If $\mu(D) = \mu(E) > 0$ then there exist partitions*

$$D = \bigcup_{i\ge1} D_i, \quad E = \bigcup_{i\ge1} E_i \quad \textit{and} \quad a_i \in A,\ i \ge 1,$$

such that $T_{a_i} D_i = E_i$, $i \ge 1$.

Proof. Put $A = \{a_1 = 1, a_2, \ldots\}$ and let

$$D_1 = \{x \in D : T_{a_1}x \in E\},$$

$$D_n = \{x \in D \setminus (D_1 \cup \ldots \cup D_{n-1}) : T_{a_n}x \in E \setminus (T_{a_1}D_1 \cup \ldots \cup T_{a_{n-1}}D_{n-1})\}, \quad n \geq 2.$$

Define $F = \bigcup_{j\geq 1} D_j$ and suppose that $\mu(D \setminus F) > 0$. By ergodicity of $\underline{T}$, there exist $a_i \in A$ and $F' \subset D \setminus F$ of positive measure such that

$$T_{a_i}F' \subset E \setminus \bigcup_{j\geq 1} T_{a_j}D_j$$

and this clearly contradicts the definition of D_i. ∎

REMARK 2. Note that the map between D and E in Hopf's lemma is necessarily invertible.

By the *centralizer* $C(\underline{T})$ we mean the group of all invertible $S : (X, \mathcal{A}, \mu) \to (X, \mathcal{A}, \mu)$ such that $ST_a = T_aS,\ a \in A$. The centralizer endowed with the weak topology becomes a Polish group. Assume that $\mathcal{P} \subset C(\underline{T})$ is a subgroup. Then put

$$\mathcal{A}(\mathcal{P}) = \{B \in \mathcal{A} : SB = B \text{ for each } S \in \mathcal{P}\}.$$

Clearly, $\mathcal{A}(\mathcal{P})$ is a $\underline{T}$-invariant σ-algebra (a factor of $\underline{T}$). If in addition $\mathcal{P}$ is compact then such a factor will be called a *group* factor.

The proof of the following proposition has been suggested to me by B. Kamiński.

PROPOSITION 2. *If $\mathcal{P}$ is compact then the entropies of $\underline{T}$ and its compact factor $\underline{T}|_{\mathcal{A}(\mathcal{P})}$ are equal.*

Proof. According to [18] (Prop. 4.3) and [3], all we need to show is that the relative entropy of $\underline{T}$ with respect to its relative distal factor $\mathcal{B}$ is zero. Suppose that this is not the case. Then similarly to [4], by adapting the methods from [17] we deduce that there is a Bernoulli factor $\mathcal{C}$ of $\underline{T}$ such that $\mathcal{B}$ and $\mathcal{C}$ are independent. Then on the one hand we see that $\mathcal{B}$ is a relative distal factor of $\mathcal{B} \vee \mathcal{C}$, the join of the two factors, but on the other hand $\mathcal{B}$ is clearly a relatively weakly mixing factor of $\mathcal{B} \vee \mathcal{C}$, a contradiction. ∎

We will also need the following simple lemma about the existence of isomorphic factors.

LEMMA 2. *Assume that $\underline{T} = (T_a)_{a\in A}$ and $\underline{T}' = (T'_a)_{a\in A}$ are weakly mixing A-actions defined on the same space $(X, \mathcal{A}, \mu)$ and let $\underline{P} = (P_a)_{a\in A}$ be another A-action on the same space. Assume moreover that*

$$T'_a = T_aP_a, \quad \text{with } P_a \in C(\underline{T}) \text{ for all } a \in A.$$

Suppose that the group $\{P_a : a \in A\}$ is relatively compact. Then $\underline{T}$ and $\underline{T}'$ have a common (isomorphic) nontrivial group factor.

Proof. Let $\mathcal{P}$ be the closure of $\{P_a : a \in A\}$ in $C(\underline{T})$. Now $\mathcal{P}$ is compact and clearly $\mathcal{A}(\mathcal{P}) = \mathcal{A}(\{P_a : a \in A\})$. It is obvious that $\mathcal{A}(\mathcal{P})$ is both $\underline{T}$- and $\underline{T}'$-invariant. Moreover, the actions of $\underline{T}$ and $\underline{T}'$ restricted to $\mathcal{A}(\mathcal{P})$ are isomorphic (via the identity map). Finally, the common factor is a group factor, hence is nontrivial by weak mixing of $\underline{T}$. ∎

2.2. *Spectral background.* Let A be a countable Abelian group with discrete topology. The natural identification of A with its second dual is given by

$$a(\chi) := \chi(a) \tag{1}$$

for each $a \in A$, $\chi \in \widehat{A}$. Let σ be a Borel finite symmetric (i.e. $\sigma(D) = \sigma(\overline{D})$ for each Borel set $D \subset \widehat{A}$) measure on $\widehat{A}$. Then the linear span of the functions $a(\cdot)$, $a \in A$, is a dense subset of $L^2(\widehat{A}, \sigma)$. Let $\underline{V} = (V_a)_{a\in A}$ denote the natural unitary representation of A on $L^2(\widehat{A}, \sigma)$:

$$(V_a f)(\chi) = a(\chi) f(\chi) = \chi(a) f(\chi)$$

for each $f \in L^2(\widehat{A}, \sigma)$, $\chi \in \widehat{A}$. By the *spectral measure* of f (with respect to $\underline{V}$) we mean the measure σ_f on $\widehat{A}$ determined (in view of the Bochner–Herglotz theorem) by

$$\widehat{\sigma}_f[a] = \int_{\widehat{A}} \chi(a)\, d\sigma_f(\chi) = \langle V_a f, f\rangle.$$

Note that $\underline{V}$ has a simple spectrum, i.e. for some $f \in L^2(\widehat{A}, \sigma)$,

$$\overline{\operatorname{span}}\,\{V_a f : a \in A\} = L^2(\widehat{A}, \sigma)$$

(indeed, it is enough to take $f = 1$). Put

$$\mathcal{F}_\sigma = \{g \in L^2(\widehat{A}, \sigma) : |g| = 1\ \sigma\text{-a.e.}\}, \qquad \mathcal{F}_\sigma^{(\mathrm{r})} = \{g \in \mathcal{F}_\sigma : g(\overline{\chi}) = \overline{g(\chi)}\}.$$

Clearly, both these sets are closed subsets of $L^2(\widehat{A}, \sigma)$ and therefore with pointwise multiplication they become Polish groups. Let

$$C(\underline{V}) = \{W \in \mathcal{U}(L^2(\widehat{A}, \sigma)) : WV_a = V_a W \text{ for all } a \in A\}$$

denote the centralizer of $\underline{V}$. If $W \in C(\underline{V})$ then

$$a(\cdot) W(1)(\cdot) = V_a W(1)(\cdot) = W V_a(1)(\cdot) = W(a(\cdot)).$$

Since $\operatorname{span}\{a(\cdot) : a \in A\}$ is dense in $L^2(\widehat{A}, \sigma)$,

$$(Wf)(\chi) = W(1)(\chi) f(\chi)$$

for each $f \in L^2(\widehat{A}, \sigma)$. Moreover, since W is unitary, $|W(1)| = 1$ σ-a.e. Therefore, we can naturally identify $C(\underline{V})$ (with the strong operator topology on it) with $\mathcal{F}_\sigma$. Finally, note that under this isomorphism $\mathcal{F}_\sigma^{(\mathrm{r})}$ corresponds to

$C^{(\mathrm{r})}(\underline{V})$, the subset of $C(\underline{V})$ of those unitary operators that preserve the (real) subspace $L^2_{(\mathrm{r})}(\widehat{A},\sigma)$ of functions $f \in L^2(\widehat{A},\sigma)$ with $f(\overline{\chi}) = \overline{f(\chi)}$.

Let $j : \widehat{A} \to \widehat{A}$ be measurable (j is defined σ-a.e.). Then it gives rise to another A-action in $\mathcal{U}(L^2(\widehat{A},\sigma))$, namely

$$(W^j_a f)(\chi) := (j \circ \chi)(a) f(\chi). \tag{2}$$

This is indeed an A-action, since $j\chi \in \widehat{A}$. It is also clear that

$$V_b W^j_a = W^j_a V_b \tag{3}$$

for all $a, b \in A$, i.e. $W^j_a \in C(\underline{V})$ (and W^j_a corresponds to $j_a \in \mathcal{F}_\sigma$, where $j_a(\chi) = (j\chi)(a)$). Note also that if $(W_a)_{a\in A}$ is an A-action satisfying (3) then for each $a \in A$ there exists a function $j_a \in \mathcal{F}_\sigma$ such that the action of W_a is simply multiplication by j_a. Since $W_{a_1a_2} = W_{a_1}W_{a_2}$, given $\chi \in \widehat{A}$,

$$j_{a_1a_2}(\chi) = j_{a_1}(\chi) j_{a_2}(\chi)$$

for all $a_1, a_2 \in A$ and therefore $j_{\cdot}(\chi) \in \widehat{A}$, so we can identify the group of A-actions satisfying (3) with the group

$$\mathcal{G}_\sigma = \{ j : \widehat{A} \to \widehat{A} : j \text{ is measurable}\}.$$

On $\mathcal{G}_\sigma$ we consider the topology given by the metric

$$d(j,j') := \sum_{n=2}^{\infty} \frac{1}{2^n} \Big(\int_{\widehat{A}} |(j\chi)(a_n) - (j'\chi)(a_n)|^2 \, d\sigma(\chi) \Big)^{1/2}$$

(i.e. $j_k \xrightarrow{d} j$ iff $(j_k)_a \to j_a$ in $L^2(\widehat{A},\sigma)$ for each $a \in A$), where $A = \{a_1 = 1, a_2, \ldots\}$, which corresponds to the strong operator topology of the group of A-actions. It is also clear that the subgroup of A-actions preserving $L^2_{(\mathrm{r})}(\widehat{A},\sigma)$ corresponds to

$$\mathcal{G}^{(\mathrm{r})}_\sigma := \{ j \in \mathcal{G}_\sigma : j(\overline{\chi}) = \overline{j(\chi)} \ \sigma\text{-a.e.}\}.$$

If the closed subgroup generated by j is compact, then j is called *compact*.

Lemma 3. *Suppose that*

$$P = (P_1, \bar{P}_1, P_2, \bar{P}_2, \ldots)$$

is a symmetric Borel partition of $\widehat{A}$. *Assume that* $j \in \mathcal{G}^{(\mathrm{r})}_\sigma$ *satisfies*

$$j|_{P_i} = \chi \ (= \chi(i))$$

for $i \geq 1$. *Then* j *is compact. Moreover*, *the subgroup* $\{j_a : a \in A\}$ *is a relatively compact subgroup of* $\mathcal{F}^{(\mathrm{r})}_\sigma$.

Proof. The group of all functions that are constant on each element of P is isomorphic to $\widehat{A}^{\operatorname{card} P}$, hence is compact. The same argument works for the second statement since each j_a is constant on any atom of P. ∎

Remark. By the methods of [11] it follows that for each element j which generates a compact subgroup of $\mathcal{G}_\sigma$ there exists a Borel partition so that j is constant on each atom of this partition. We will, however, make no use of this fact.

Two symmetric Borel subsets $D, E \subset \widehat{A}$ ($\bar{D} = D$, $\bar{E} = E$) are called *σ-compactly equivalent* if there exists a σ-a.e. 1-1 map $g \in \mathcal{G}_\sigma^{(\mathrm{r})}$ of the form

$$g(\chi) = \chi j(\chi), \quad j \in \mathcal{G}_\sigma^{(\mathrm{r})},$$

where j is compact, satisfying $g(D) = E$.

Proposition 3. *Any two symmetric Borel sets of equal positive Haar measure λ are λ-compactly equivalent.*

Proof. Fix a countable dense subgroup G of $\widehat{A}$. Consider the natural action of this group (by translations) on $(\widehat{A}, \lambda)$. This action is clearly ergodic. Assume that $\lambda(D) = \lambda(E)$ (with $\bar{D} = D$, $\bar{E} = E$). Represent $D = D' \cup \bar{D}'$, where $D' \cap \bar{D}' = \emptyset$, and do the same with E. By Proposition 1, there exist partitions

$$D' = \bigcup_{i \ge 1} D_i, \quad E' = \bigcup_{i \ge 1} E_i \quad \text{and} \quad \{\chi_1, \chi_2, \ldots\} \subset G$$

such that $\chi_i D_i = E_i$, $i \ge 1$. Define $j \in \mathcal{G}_\lambda^{(\mathrm{r})}$ by

$$j(\chi) = \begin{cases} \chi_i & \text{if } \chi \in D_i,\ i \ge 1, \\ 1 & \text{if } \chi \notin \widehat{A} \setminus D. \end{cases}$$

In view of Lemma 3, j is compact. Moreover, $g(\chi) := \chi j(\chi)$ is 1-1 (λ-a.e.), $g(D) = E$, so the result follows. ■

Lemma 4. *Let $g \in \mathcal{G}_\sigma^{(\mathrm{r})}$ and $\underline{W} = (W_a^g)_{a \in A}$ be the corresponding A-action. Then the spectral measure of $\underline{W}$ is equal to $g_*\sigma$, the image of σ via g. If, moreover, g is* 1-1 *σ-a.e. then $\underline{W}$ has simple spectrum on $L^2(\widehat{A}, \sigma)$.*

Proof. Since

$$\int_{\widehat{A}} \chi(a)\, dg_*\sigma(\chi) = \int_{\widehat{A}} (g\chi)(a)\, d\sigma(\chi) = \langle W_a^g 1, 1 \rangle,$$

the first assertion follows. If in addition g is 1-1, then

$$g : (\widehat{A}, \sigma) \to (\widehat{A}, g_*\sigma)$$

is a measure-theoretic isomorphism. Let U_g stand for the corresponding unitary operator

$$U_g : L^2(\widehat{A}, g_*\sigma) \to L^2(\widehat{A}, \sigma), \quad U_g(f) := f \circ g.$$

All we need to show is that $\underline{W}$ is isomorphic to $(V_a)_{a\in A}$ defined on $L^2(\widehat{A}, g_*\sigma)$. This is, however, clear since

$$U_g V_a = W_a^g U_g$$

for each $a \in A$; the result follows. ∎

LEMMA 5. *Let $\lambda(D) > 0$ and put $\sigma = \lambda|_D$. Assume that $g(\chi) = \chi j(\chi) \in \mathcal{G}_\sigma^{(\mathrm{r})}$, where j is constant on elements of the Borel partition*

$$P = (P_1, \bar{P}_1, P_2, \bar{P}_2, \ldots).$$

Then $g_*\sigma$ is a finite measure equivalent to Haar measure concentrated on $\bigcup g(P_i \cup \bar{P}_i)$.

Proof. This is merely the fact that Haar measure is invariant under translations. ∎

REMARK 4. In fact, an image of Haar measure (via g above) is an absolutely continuous measure whenever j is compact.

As a corollary of Lemma 3, Lemma 4 (and its proof), Lemma 5 and Proposition 3, we obtain the following.

COROLLARY 1. *Let $D, E \subset \widehat{A}$ be symmetric Borel sets of positive λ-measure, $\lambda(D) = \lambda(E)$. Put $\sigma = \lambda|_D$, $\eta = \lambda|_E$. Then there exists an A-action $(Q_a)_{a\in A}$ in $\mathcal{U}(L^2(\widehat{A}, \sigma))$ such that*

(1) *$\{Q_a : a \in A\}$ is relatively compact in $\mathcal{U}(L^2(\widehat{A}, \sigma))$;*

(2) *$Q_b \in C((V_a)_{a\in A})$ for each $b \in A$;*

(3) *$W_a := V_a Q_a$, $a \in A$, is an A-action on $L^2(\widehat{A}, \sigma)$ unitarily equivalent to the action $(V_a)_{a\in A}$ on $L^2(\widehat{A}, \eta)$.*

2.3. *Proof of Lemma 1.* First we apply Corollary 1 and basic properties of Gaussian actions (explained in Section 1) to find that there exists an A-action $(S_a)_{a\in A}$ such that $T_a^D = S_a T_a^E$, where $S_a \in C(\underline{T}^E)$, $a \in A$, and $\{S_a : a \in A\}$ is relatively compact. Using Lemma 2 we infer that $\underline{T}^D$ and $\underline{T}^E$ have a common group factor. The result now follows directly from Proposition 2.

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Received 19 May 1997;
in revised form 9 October 1997