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Inverse limit of M-cocycles and applications

by

Jan Kwiatkowski (Toruń)

Abstract. For any m, $2 \le m < \infty$, we construct an ergodic dynamical system having spectral multiplicity m and infinite rank. Given $r > 1$, $0 < b < 1$ such that $rb > 1$ we construct a dynamical system $(X, \mathcal{B}, \mu, T)$ with simple spectrum such that $r(T) = r$, $F^*(T) = b$, and $\#C(T)/\mathrm{wcl}\{T^n : n \in \mathbb{Z}\} = \infty$.

1. Introduction. It was conjectured in [M1] that for any pair (m, r) of integers or ∞, with $m \le r$, there exists an ergodic dynamical system (X, μ, T) with rank $r(T) = r$ and spectral multiplicity $m(T) = m$. Partial solutions of this question were obtained by several authors: [Ch] (the pair $(1,1)$), [dJ] $(1,2)$, [M1] $(1,r)$, [GoLe] $(2,r)$, [R1,2] (r,r), [M2] $(r,2r)$, [FeKw] $(p-1,p)$, p prime, and [Fe1] $(1,\infty)$, [FeKwMa] (given m, the set of r such that $m(T) = m$ and $r(T) = r$ has density 1). The latest result of this series [KwLa1] says that for any pair (m,r) with $2 \le m \le r < \infty$ there is an ergodic automorphism T with $r(T) = r$ and $m(T) = m$. Thus, together with [M1], every finite pair (m,r) with $m \le r$ is obtainable.

The solution of the (multiplicity, rank) problem will be complete if for any finite $m \ge 1$ and $r = \infty$ we can find an ergodic automorphism realizing (m, ∞). The pair $(1, \infty)$ is realized by the Gaussian–Kronecker system [dlR]. In this note we construct an ergodic automorphism realizing the pairs (m, ∞) for every $m \ge 2$.

We denote by $C(T)$ the set of all measure-preserving automorphisms of $(X, \mathcal{B}, \mu)$ wich commute with T. We say that a sequence $\{S_n\} \subset C(T)$ tends weakly to $S \in C(T)$ if for every $A \in \mathcal{B}$,

$$\mu(S_n A \triangle SA) \to 0.$$

With this topology, $C(T)$ is a Polish group. We denote by $\mathrm{wcl}\{T^n : n \in \mathbb{Z}\}$ the weak closure of the set $\{T^n : n \in \mathbb{Z}\}$. The weak closure theorem

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[Kin] says that $C(T) = \mathrm{wcl}\{T^n : n \in \mathbb{Z}\}$ if $r(T) = 1$. It turns out that it is the only relation between rank and the cardinality of the quotient group $C(T)/\mathrm{wcl}\{T^n : n \in \mathbb{Z}\}$ in the class of ergodic dynamical systems. In [KwLa2] examples of ergodic automorphisms T are constructed such that $r(T) = r \geq 2$ and $\#C(T)/\mathrm{wcl}\{T^n : n \in \mathbb{Z}\} = m \geq 1$, where r, m are given. We construct an example of an ergodic automorphism T such that T has simple spectrum, $r(T) = r$, $F^*(T) = b$ and $\#C(T)/\mathrm{wcl}\{T^n : n \in \mathbb{Z}\} = \infty$, where r, b are given and $r \geq 2,\ 0 < b < 1,\ br > 1$.

In [KwLa1] we used Morse automorphisms over finite abelian groups. Now, we use the class of inverse limits of Morse automorphisms over compact metric abelian groups. There are positive aspects of examining such dynamical systems. Any Morse automorphism is a group extension T_φ of an adding machine (X, T) defined by a special cocycle $\varphi : X \to G$, where G is a compact abelian group (the details follow).

The cocycle φ is determined by a sequence $\{b^t\},\ t \geq 0$, of blocks over G. Each group homomorphism $\pi : G \to H$ defines a natural factor T_ψ, where $\psi = \pi \circ \varphi$. The cocycle ψ is determined by the sequence $\{\pi(b^t)\},\ t \geq 0$, of blocks over H. Now, let $G = \varprojlim(G_t, \pi_t)$ be the inverse limit of finite groups G_t with homomorphisms $\pi_t : G_{t+1} \to G_t,\ \pi_t(G_{t+1}) = G_t,\ t \geq 0$.

Assume that $\{b^s\}_{s=0}^{\infty}$ is a sequence of blocks over G_s and that there are mappings $\tau_s : G_s \to G_{s+1}$ such that $\pi_s \circ \tau_s = \mathrm{id},\ s \geq 0$. This allows us to define an inverse limit T_φ of Morse automorphisms over G_s (see 3.2 and Sections 4 and 5). The spectral multiplicity $m(T_\varphi)$ and the rank $r(T_\varphi)$ of T_φ are the limits of $m(T_{\varphi_s})$ and $r(T_{\varphi_s})$. In Section 4 we construct an example of a Morse automorphism T_φ such that $m(T_{\varphi_s})$ is constant while $r(T_{\varphi_s}) \to \infty$. To compute $m(T_{\varphi_s})$ and $r(T_{\varphi_s})$ we use the same methods as in [GoKwLeLi] and in [KwLa1].

Similarly to [KwLa1] the automorphisms we construct here can be obtained within the class of weakly mixing transformations.

2. Preliminaries. Let $(X, \mathcal{B}, \mu, T)$ be an ergodic dynamical system. We can look at the associated spectral operator $U_T : L_0^2(X, \mu) \to L_0^2(X, \mu),\ U_T f = f \circ T,\ f \in L_0^2(X, \mu)$, where $L_0^2(X, \mu)$ consists of those functions of $L^2(X, \mu)$ such that $\int_X f\, d\mu = 0$. By the *spectral multiplicity* $m(T)$ of T we mean the supremum of all essential spectral multiplicities of T on $L_0^2(X, \mu)$. We refer the reader to [Fe2] for the definition of the rank $r(T)$ and the covering number $F^*(T)$ of T and for more information on those notions.

Now let $T : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ be the (p_t)-*adic adding machine*, i.e. $p_t \mid p_{t+1},\ \lambda_{t+1} = p_{t+1}/p_t \geq 2$ for $t \geq 0,\ p_0 = \lambda_0 \geq 2$,

$$X = \Big\{x = \sum_{t=0}^{\infty} q_t p_{t-1} : 0 \leq q_t \leq \lambda_t - 1,\ p_{-1} = 1\Big\}$$

is the group of (p_t)-adic integers and $Tx = x+\widehat{1}$, $\widehat{1} = (1,0,0,\ldots)$. The space X has a standard sequence (ξ_t) of T-towers. Namely

$$\xi_t = (D_0^t, D_1^t, \ldots, D_{p_t-1}^t),$$

where $D_0^t = \{x \in X : q_0 = \ldots = q_t = 0\}$, $D_j^t = T^j(D_0^t)$, $j = 0,\ldots,p_t-1$, $X = \bigcup_{j=0}^{p_t-1} D^t$.

The tower ξ_{t+1} refines ξ_t and the sequence (ξ_t) of partitions converges to the point partition. Let G be an abelian compact metric group and let m_G be normalized Haar measure of G. A *cocycle* is a measurable function $\varphi : X \to G$. A cocycle φ defines an automorphism T_φ on $(X\times G, \widetilde{\mathcal{B}}, \mu\times m_G)$,

$$T_\varphi(x,y) = (Tx, g+\varphi(x)), \quad x \in X,\ g \in G,$$

where $\widetilde{\mathcal{B}}$ is the product of the σ-algebra $\mathcal{B}$ and the σ-algebra of borelian subsets of G.

Then $T_\varphi^n(x,y) = (T^n x, g+\varphi^{(n)}(x))$, $n = 0,\pm1,\ldots$, where

$$\varphi^{(n)}(x) = \begin{cases} \varphi(x)+\varphi(Tx)+\ldots+\varphi(T^{n-1}x), & n \ge 1,\\ 0, & n = 0,\\ -\varphi(T^{-1}x)-\ldots-\varphi(T^n x), & n \le -1.\end{cases} \tag{1}$$

The dynamical system $(X\times G, \widetilde{\mathcal{B}}, \mu\times m_G, T_\varphi)$ is called a *group extension* of $(X, \mathcal{B}, \mu, T)$.

T_φ is ergodic iff for every non-trivial $\gamma \in \widehat{G}$ ($\widehat{G}$ is the dual group), there is no measurable solution $f : X \to S^1$ (the unit complex circle) to the functional equation

$$\gamma(\varphi(x)) = \frac{f(Tx)}{f(x)}, \quad x \in X \text{ [Pa]}. \tag{2}$$

We say that $\varphi : X \to G$ is an *M-cocycle* if for every $t \ge 1$, φ is constant on each level D_i^t, $i = 0,\ldots,p_t-2$ (except on the top $D_{p_t-1}^t$). Such a cocycle is defined by a sequence a blocks $b^{(0)}, b^{(1)},\ldots$ over G. By a *block* B over G we mean a finite sequence

$$B = B[0]\ldots B[k-1],$$

where $k \ge 1$ and $B[i] \in G$, $i = 0,\ldots,k-1$. The number k is called the *length* of B and denoted by $|B|$. If $C = C[0]\ldots C[m-1]$ is another block then the *concatenation* of B and C is the block

$$BC = B[0]\ldots B[k-1]C[0]\ldots C[m-1].$$

We can concatenate more than two blocks in the obvious way. If $v : G \to G$ is a continuous group automorphism then we let $v(B)$ be the block

$$v(B) = v(B[0])\ldots v(B[k-1]).$$

We denote by $B(g),\ g \in G$, the block

$$B(g) = (B[0] + g) \dots (B[k-1] + g)$$

and by $\check{B}$ the block $\check{B} = (B[1] - B[0]) \dots (B[k-1] - B[k-2]),\ k \geq 2$. Now, we can define the product $B \times C$ of B and C as follows:

$$B \times C = B([C[0]) \dots B(C[m-1]).$$

Clearly,

$$|B \times C| = |B||C| \quad \text{and} \quad v(B \times C) = v(B) \times v(C).$$

This multiplication operation "×" is associative so it can be extended to more than two blocks. If $|B| = |C| = k$ then we define

$$\bar{d}(B, C) = k^{-1} \# \{0 \leq i \leq k-1 : B[i] \neq C[i]\}.$$

Now we describe Morse sequences (M-sequences). Let $b^{(0)}, b^{(1)}, \dots$ be finite blocks over G with $|b^{(t)}| = \lambda_t,\ b^{(t)}[0] = 0,\ t \geq 0$. Then we define a one-sided sequence over G by

$$\omega = b^{(0)} \times b^{(1)} \times \dots$$

Such a sequence ω allows one to define an M-cocycle $\varphi = \varphi_\omega$ on X as follows: let

$$B_t = b^{(0)} \times \dots \times b^{(t)}, \qquad t \geq 0.$$

Then $|B_t| = p_t$ and $|\check{B}_t| = p_t - 1$. We finally put

$$\varphi(x) = \check{B}_t[j] \quad \text{ if } x \in D_j^t,\ j = 0, \dots, p_t - 2.$$

Clearly, φ is an M-cocycle. It is easy to observe that each M-cocycle can be obtained as described above. As a consequence of the definition of φ and (1) we get

$$\varphi^{(n)}(x) = B_t[j+n] - B_t[j] \tag{3}$$

if $x \in D_j^t$ and $j = 0, \dots, p_t - n - 1$. If we examine $\varphi^{(kp_t)}(x),\ 1 \leq k \leq \lambda_{t+1} - 1$, on the tower ξ_{t+1} then (3) implies

$$\varphi^{(kp_t)}(x) = b^{(t+1)}[q+k] - b^{(t+1)}[q] \tag{4}$$

if $x \in D_{qp_{t+j}}^{(t+1)},\ 0 \leq q \leq \lambda_{t+1} - k - 1,\ j = 0, \dots, p_t - 1$.

3. Spectral analysis of M-cocycles and their inverse limit

3.1. *Spectral calculations.* It is known that

$$L^2(X \times G, \mu \times m_G) = \bigoplus_{\gamma \in \widehat{G}} L_\gamma, \tag{5}$$

where

$$L_\gamma = \{f \otimes \gamma \in L^2(X \times G, \mu \times m_G) : f \in L^2(X, \mu)\}.$$

Moreover, the subspaces L_γ are U_{T_φ}-invariant and using the same arguments as in [KwSi] we see that U_{T_φ} on L_γ has simple spectrum.

Let μ_γ be the spectral measure of U_{T_φ} on L_γ. The subspace L_e (e is the trivial character) is generated by the eigenfunctions of T_φ (in fact of T) corresponding to all p_t-roots of unity. An M-cocycle $\varphi = \varphi_\omega$ is called *continuous* if L_e contains all eigenfunctions of T_φ, or equivalently if each measure $\mu_\gamma, \gamma \neq e$, is continuous. We shall use the following criteria to find whether two measures μ_γ, $\mu_{\gamma'}$, $\gamma, \gamma' \in \widehat{G}$, $\gamma \neq \gamma'$, are orthogonal or equivalent.

PROPOSITION 1 ([KwRo], [FeKw], [GoKwLeLi]). *If $v : G \to G$ is a group automorphism and blocks $b^{(0)}, b^{(1)}, \ldots$ satisfy*

$$\text{(a)} \qquad \sum_{t=0}^{\infty} \overline{d}(b^{(t)}[k_t, \lambda_t - 1], v(b^{(t)})[0, \lambda_t - k_t - 1]) < \infty$$

for a sequence $(k_t)_{t=0}^{\infty}$, $0 \le k_t < \lambda_t$, for which

$$\text{(b)} \qquad \sum_{t=0}^{\infty} \frac{k_t}{\lambda_t} < \infty,$$

then $\mu_\gamma \simeq \mu_{\widehat{v}(\gamma)}$ for all γ in $\widehat{G}$, where $\widehat{v}$ is the dual automorphism.

PROPOSITION 2 [GoKwLeLi]. *If for given $\gamma, \gamma' \in \widehat{G}$,*

(6) $\lim_{t\in\overline{N}} \int_X \gamma(\varphi^{(a_t p_t)}(x))\, \mu(dx)$ *and* $\lim_{t\in\overline{N}} \int_X \gamma'(\varphi^{(a_t p_t)}(x))\, \mu(dx)$ *exist along a subsequence $\overline{N}$ and are different*

then $\mu_\gamma \perp \mu_{\gamma'}$ whenever $\sum_{t=1}^{\infty} a_t/\lambda_{t+1} < \infty$ (note that $T^{a_t p_t} \to \mathrm{Id}$ in the weak topology).

Let H_0 be a subgroup of G and $H = G/H_0$ be the quotient group. Let $\pi : G \to H$ be the quotient map and let m_H be Haar measure on H. We can define a map $P = \mathrm{Id}_X \times \pi$ of the dynamical system $(X \times G, T_\varphi, \mu \times m_G)$ onto $(X \times H, T_{\varphi,H}, \mu \times m_H)$, where $\varphi_H(x) = \pi(\varphi(x))$. The systems $(X \times H, T_{\varphi,H}, \mu \times m_H)$ are called the *natural factors* of $(X \times G, T_\varphi, \mu \times m_G)$. If B is a block over G then $\pi(B)$ denotes the block over H defined by

$$\pi(B) = \pi(B[0]) \ldots \pi(B[k-1]), \qquad k = |B|.$$

Using the obvious equality $\pi(B \times C) = \pi(B) \times \pi(C)$, it is not hard to see that if φ is the M-cocycle defined by the sequence of blocks $b^{(0)}, b^{(1)}, \ldots$ over G then φ_H is the M-cocycle determined by the blocks $\pi(b^{(0)}), \pi(b^{(1)}), \ldots$

It is known that $\widehat{H}$ can be identified with a subgroup of $\widehat{G}$, namely with the subgroup of those $\gamma \in \widehat{G}$ such that $\gamma(H_0) = 1$. Let

$$L_{\gamma,H} = \{f \otimes \gamma \in L^2(X \times H, \mu \times m_H) : f \in L^2(X, \mu)\}, \qquad \gamma \in \widehat{H}.$$

Then

$$L^2(X\times H,\mu\times m_H)=\bigoplus_{\gamma\in\widehat{H}}L_{\gamma,H} \tag{7}$$

and the unitary operator $U_{T_{\varphi,H}}$ on $L_{\gamma,H}$ is spectrally isomorphic to the unitary operator U_{T_φ} on L_γ. Thus $U_{T_{\varphi,H}}$ has simple spectrum on $L_{\gamma,H}$ and its spectral measure is μ_γ.

3.2. *Inverse limit of M-cocycles.* Let $(X,\mathcal{B},\mu,T)$ and $(X_s,\mathcal{B}_s,\mu_s,T_s)$, $s=0,1,\ldots$, be dynamical systems. We say that $(X,\mathcal{B},\mu,T)$ is an *inverse limit* of $(X_s,\mathcal{B}_s,\mu_s,T_s)$ if there exist homomorphisms $V_s:(X,\mathcal{B},\mu,T)\to(X_s,\mathcal{B}_s,\mu_s,T_s)$ such that $V_s^{-1}(\mathcal{B}_s)\subset V_{s+1}^{-1}(\mathcal{B}_{s+1})$ and the σ-algebras $V_s^{-1}(\mathcal{B}_s)$ generate $\mathcal{B}$. For each $s\ge 0$ we have a homomorphism $W_s:(X_{s+1},\mathcal{B}_{s+1},\mu_{s+1},T_{s+1})\to(X_s,\mathcal{B}_s,\mu_s,T_s)$ and $W_s\circ V_{s+1}=V_s$. We write $T=\varprojlim T_s$. It follows from the definition of the spectral multiplicity, rank and covering number that $m(T)=\lim m(T_s)$, $r(T)=\lim r(T_s)$, $F^*(T)=\lim F^*(T_s)$ and moreover $m(T_s)\le m(T_{s+1})$, $r(T_s)\le r(T_{s+1})$, $F^*(T_s)\ge F^*(T_{s+1})$.

It is clear that T is ergodic (weakly mixing, mixing) iff so is T_s for every $s\ge 0$. Consider an ergodic dynamical system $(X,\mathcal{B},\mu,T)$ and sequences $(G_s)_{s=0}^\infty$ of metric compact abelian groups and group homomorphisms $\pi_s:G_{s+1}\to G_s$ with $\pi(G_{s+1})=G_s$. The sequence (G_s,π_s), $s\ge 0$, defines the inverse limit $G=\varprojlim(G_s,\pi_s)$ and the homomorphisms $\psi_s:G\to G_s$ such that $\pi_s\circ\psi_{s+1}=\psi_s$. Note that G is a metric compact abelian group. Assume that $\varphi_s:X\to G_s$ are cocycles such that $\pi_s\circ\varphi_{s+1}=\varphi_s$. The cocycles φ_s define a unique cocycle $\varphi:X\to G$ satisfying $\psi_s\circ\varphi=\varphi_s$. Then $T_\varphi=\varprojlim T_{\varphi_s}$.

Now, let $(X,\mathcal{B},\mu,T)$ be a (p_t)-adic adding machine, $p_t=\lambda_0\ldots\lambda_t$, $t\ge 0$. We describe special inverse limits of group extensions T_{φ_s} determined by M-cocycles. To do this assume additionally that we have one-to-one measurable mappings $\tau_s:G_s\to G_{s+1}$ such that $\pi_s\circ\tau_s=\mathrm{id}$, $s\ge 0$. Set $H_s=\tau_s(G_s)$.

Let $\overline{H}_s$ be the set of all sequences $\{g_t\}_{t=0}^\infty\in G$ such that g_s is an arbitrary element of G_s and $g_{s+1}=\tau_s(g_s)$, $g_{s+2}=\tau_{s+1}\tau_s(g_s)$ and so on, $g_{s-1}=\pi_{s-1}(g_s),\ldots,g_0=\pi_0\circ\ldots\circ\pi_{s-1}(g_s)$. Given blocks $b^{(t)}$, $t\ge 0$, over G_t, we can treat them as blocks over G if we identify the members of $b^{(t)}$ with the corresponding elements of $\overline{H}_t$. The sequence $(b^{(t)})_{t=0}^\infty$ defines a cocycle $\varphi:X\to G$. Let m and m_s be normalized Haar measures of G and G_s respectively. The dynamical system $(X\times G,\mathcal{B},T_\varphi,\mu\times m_G)$ has natural factors

$$(X\times G_s,\mathcal{B}_s,T_{\varphi_s},\mu\times m_s),\qquad s\ge 0,$$

where $\varphi_s=\psi_s\circ\varphi$ and the mappings

$$W_s=\mathrm{Id}_X\times\psi_s:X\times G\to X\times G_s$$

are homomorphisms of those systems. Each cocycle φ_s is an M-cocycle determined by the blocks $(b_s^{(t)})_{t=0}^{\infty}$, where $b_s^{(t)} = \psi_s(b^{(t)})$ if $t \geq s$ and $b_s^{(t)} = \tau_t \circ \ldots \circ \tau_{s-1}(b^{(t)})$ if $t < s$.

4. Example 1. In this section we describe an example of an M-cocycle φ such that T_φ has infinite rank and spectral multiplicity $r \geq 1$.

4.1. *Definition of the cocycle.* Let $r_t = r2^t$, $t \geq 0$, and $n \geq 2$. Select a sequence $(l_t)_{t=0}^{\infty}$ of positive integers such that $n \mid l_t$, $l_t \nearrow \infty$ and

$$(1 - n/l_t)^{r_t} \to 1. \tag{8}$$

Let $\mathbb{Z}_n = \{0, 1, \ldots, n-1\} \simeq \mathbb{Z}/n\mathbb{Z}$, and

$$G_t = \overbrace{\mathbb{Z}_n \oplus \ldots \oplus \mathbb{Z}_n}^{r_t}$$

be the direct product of r_t copies of Z_n's, $t = 0, 1, \ldots$ For $g \in G_t$ we write $g = (g_0, g_1, \ldots, g_{r_t-1})$, $g_i \in \mathbb{Z}_n$.

We let

$$e_i^{(t)} = e_i = (\underbrace{0, \ldots, 0}_{i-1}, 1, 0, \ldots, 0), \qquad i = 1, \ldots, r_t - 1.$$

Define homomorphisms $\pi_t : G_{t+1} \to G_t$ by $\pi_t(e_j^{(t+1)}) = e_i^{(t)}$, where $j = 0, 1, \ldots r_{t+1} - 1$, $i = 0, 1, \ldots, r_t - 1$ and $i \equiv j \pmod{r_t}$. We have the natural mappings $\tau_t : G_t \to G_{t+1}$ defined by

$$\tau_t\Big(\sum_{i=0}^{r_t-1} g_i e_i^{(t)}\Big) = \sum_{i=0}^{r_t-1} g_i e_i^{(t+1)}, \qquad g_0, \ldots, g_{r_t-1} = 0, 1, \ldots, n-1.$$

Then $\pi_t \circ \tau_t = \mathrm{id}$. Set

$$G = \varprojlim (G_t, \pi_t).$$

As above let $\psi_t : G \to G_t$ be continuous homomorphisms such that

$$\pi_t \circ \psi_{t+1} = \psi_t.$$

Now, we are in a position to describe M-cocycles φ_t as in part 3.2. To do this we define a sequence $\{b^{(t)}\}_{t=0}^{\infty}$ of blocks, each block $b^{(t)}$ over G_t. Put

$$F_i = F_i^{(t)} = 0(e_i)(2e_i)\ldots(l-1)(e_i), \qquad i = 0, 1, \ldots, r_t - 1,\ l = l_t,\ e_i = e_i^{(t)}. \tag{9}$$

Then define a block $\beta_{u,k}^{(t)} = \beta_{u,k}$, $u = 0, 1, \ldots, 2^t - 1$, $k = 0, \ldots, r - 1$, as follows:

(10) $\delta_{u,k} = F_{ur+k} \times F_{ur+(k\oplus 1)} \times \ldots \times F_{ur+(k\oplus r-1)}$ where $a \oplus b$ is $a+b$ taken mod r, $a, b = 0, 1, \ldots, r-1$, and $\beta_{u,k} = \delta_{u,k} \times \delta_{u\oplus 1,k} \times \ldots \times \delta_{u\oplus 2^t-1,k}$, and now $u \oplus \widetilde{u}$ is $u + \widetilde{u}$ taken mod 2^t.

Finally, define

$$\beta_u^{(t)} = \beta_u = \beta_{u,0}\beta_{u,1}\ldots\beta_{u,r-1}, \qquad u = 0,1,\ldots,2^t-1 \tag{11}$$

and

$$b^{(t)} = \overbrace{\beta_0\ldots\beta_0}^{q_{t,0}}\overbrace{\beta_1\ldots\beta_1}^{q_{t,1}}\ldots\overbrace{\beta_{2^t-1}\ldots\beta_{2^t-1}}^{q_{t,2^t-1}} \tag{12}$$

where $q_{t,u}$ are positive integers such that

$$\sum_{t=0}^{\infty}\frac{1}{q_t} < \infty, \qquad q_t = \min(q_{t,0}, q_{t,1}, \ldots, q_{t,2^t-1}). \tag{13}$$

Some additional conditions on $q_{t,u}$'s will be specified later.

Obviously, $F_i^{(t)}$, $\beta_{u,k}^{(t)}$, $\beta_u^{(t)}$, $b^{(t)}$ are blocks over G_t and we have

$$|F_i| = l_t, \qquad |\beta_{u,k}| = l_t^{r_t}, \qquad |\beta_u| = rl_t^{r_t}, \qquad |b^{(t)}| = rl_t^{r_t}Q_t$$

where

$$Q_t = \sum_{u=0}^{2^t-1} q_{t,u}.$$

Let $v = v_t : G_t \to G_t$ be the group automorphisms defined by

$$v(e_{ur+k}) = e_{ur+(k\oplus 1)},$$
$$u = 0,1,\ldots,2^t-1,\ k = 0,1,\ldots,r-1,\ e_{ur+k} = e_{ur+k}^{(t)}.$$

Then we have

$$v(F_{ur+k}) = F_{ur+(k\oplus 1)}, \qquad v(\beta_{u,k}) = \beta_{u,k\oplus 1}. \tag{14}$$

Now, let $(X, \mathcal{B}, \mu, T)$ be the (p_t)-adic adding machine, where $p_t = \lambda_0\ldots\lambda_t$, $\lambda_t = |b^{(t)}| = rl_t^{r_t}Q_t$, $t \ge 0$. The sequence $\{b^{(t)}\}_{t=0}^{\infty}$ determines the sequences of blocks $\{b_s^{(t)}\}_{t=0}^{\infty}$, $s \ge 0$, and in consequence M-cocycles $\varphi : X \to G$ and $\varphi_s : X \to G_s$ described in part 3.2.

We have a sequence of dynamical systems

$$(X\times G_0, T_{\varphi_0}) \xleftarrow{W_0} (X\times G_1, T_{\varphi_1}) \xleftarrow{W_1} (X\times G_2, T_{\varphi_2}) \xleftarrow{W_2} \ldots \tag{15}$$

determined by the homomorphisms π_t, the mappings τ_t (in this case τ_t are homomorphisms) and by the blocks (12).

4.2. *Additional conditions.* The blocks $b_s^{(t)}$, $t, s \ge 0$, can be obtained by a procedure similar to that for b_t's. If $t \le s$ then $b_s^{(t)} = b^{(t)}$ (with $e_i^{(s)}$ instead of $e_i^{(t)}$, $i = 0,\ldots,r_t-1$). If $t > s$, we define the blocks $F_{i,s}^{(t)}$ by (9) for $i = 0,1,\ldots,r_s-1$ and $l = l_t$. We have

$$\pi_s\circ\ldots\circ\pi_{t-1}(F_j^{(t)}) = F_{i,s}^{(t)}, \qquad |F_{i,s}^{(t)}| = l_t \tag{16}$$

for $j = 0,1,\ldots,r_t-1$, $i = 0,1,\ldots,r_s-1$ and $j \equiv i \pmod{r_s}$.

Then we define $\beta_{u,k}^{(t,s)},\ \beta_u^{(t,s)},\ u=0,1,\ldots,2^s-1,\ k=0,1,\ldots,r-1$, by (10) and (11) using the blocks $F_{ur+k,s}^{(t)}$. Let

$$\bar{\delta}_a = \overbrace{\beta_0\ldots\beta_0}^{q_{t,a2^s}}\overbrace{\beta_1\ldots\beta_1}^{q_{t,a2^s+1}}\ldots\overbrace{\beta_{2^s-1}\ldots\beta_{2^s-1}}^{q_{t,a2^s+2^s-1}} \tag{17}$$

for $a=0,1,\ldots,2^{t-s}-1$, $\beta_u=\beta_u^{(t,s)}$, $u=0,1,\ldots,2^s-1$. Then (16) implies $\beta_u^{(t,s)}=\psi_s(\beta_{a2^s+u}^{(t)})$ for $u=0,1,\ldots,2^s-1$ and $a=0,1,\ldots,2^{t-s}-1$.

Now, comparing the blocks (12) and (17) we get

$$b_s^{(t)}=\bar{\delta}_0\bar{\delta}_1\ldots\bar{\delta}_{2^{t-s}-1}.$$

To finish the definition of φ we must give conditions for the numbers $q_{t,u},\ u=0,1,\ldots,2^t-1,\ t\ge 0$. To do this consider the dual group $\widehat{G}$. We have $\widehat{G}=\bigcup_{s=0}^{\infty}\widehat{G}_s$. The group automorphisms $v_s:G_s\to G_s$ satisfy $v_s\circ\pi_s=\pi_s\circ v_{s+1}$ and they determine a continuous group automorphism $v:G\to G$ such that $v_s\circ\psi_s=\psi_s\circ v$. The dual group automorphism $\widehat{v}:\widehat{G}\to\widehat{G}$ satisfies $\widehat{v}(\widehat{G}_s)=\widehat{G}_s$. It is not hard to see that every $\widehat{v}$-trajectory of $\widehat{G}$ has length $\le r$ and there are $\widehat{v}$-trajectories having length r. Consider all possible pairs $(\gamma,\gamma'),\ \gamma,\gamma'\in\widehat{G}$, such that γ,γ' are from different $\widehat{v}$-trajectories. Divide the set $\mathbb{N}=\{0,1,\ldots\}$ into disjoint infinite subsets $N(\gamma,\gamma')$. For every such pair (γ,γ') we choose $s=s(\gamma,\gamma')\ge 0$ such that $\gamma,\gamma'\in\widehat{G}_s$. The functions

$$A_\gamma=\frac{1}{r}\sum_{p=0}^{r-1}\widehat{v}^{\,p}(\gamma),\qquad A_{\gamma'}=\frac{1}{r}\sum_{p=0}^{r-1}\widehat{v}^{\,p}(\gamma')$$

are orthogonal in $L^2(G_s,m_s)$ so we can find $g=g(\gamma,\gamma')\in G_s$ such that

$$A_\gamma(g)\ne A_{\gamma'}(g). \tag{18}$$

Choose $c=c(\gamma,\gamma')$ in such a way that

$$\tfrac{1}{2}<c<1\quad\text{and}\quad 2(1-c)<\tfrac{1}{2}c|A_\gamma(g)-A_{\gamma'}(g)|. \tag{19}$$

To find the numbers $q_{t,u}$ we need probability vectors $\overline{\omega}^{(t,s)}=\overline{\omega}=\langle\omega_z^{(t,s)}\rangle$ where $s<t$ and $z=0,1,\ldots,2^s-1$, defined as follows:

$$\omega_z=\sum_{a=0}^{2^{t-s}-1}\frac{q_{t,z+a2^s}}{Q_t},\qquad Q_t=\sum_{u=0}^{2^t-1}q_{t,u}. \tag{20}$$

Take $t\in N(\gamma,\gamma')$ and $t>s=s(\gamma,\gamma')$. Choose $q_{t,u}, u=0,1,\ldots,2^t-1$, in such a way that

$$\omega_0^{(t,s)}\ge c(\gamma,\gamma'), \tag{21}$$

$$\lim_{\substack{t\to\infty\\ t\in N(\gamma,\gamma')}}\omega_0^{(t,s)}=c(\gamma,\gamma'), \tag{22}$$

$$\omega_z^{(t,s)} = \omega_{z'}^{(t,s)} \quad \text{for } z, z' = 1, \ldots, 2^s - 1. \tag{23}$$

If $t \in N(\gamma, \gamma')$ and $t \le s(\gamma, \gamma')$ then we pick $q_{t,u}$ satisfying (23) for every $z, z' = 0, 1, \ldots, 2^s - 1$.

4.3. *Propositions.* In the sequel let T_φ be the group extension of T defined by the cocycle φ described in 4.1 and 4.2.

PROPOSITION 3. *T_φ is ergodic and φ is continuous.*

Proof. Take $\gamma \in \widehat{G}_s$ and assume that

$$f(Tx)/f(x) = \gamma(\varphi_s(x))$$

for a.e. $x \in X$, where $f : X \to S^1$ is a measurable function (see (2)). Using the same arguments as in [FeKwMa] we get

$$\gamma(\varphi_s^{(p_t)}(x)) \xrightarrow{t} 1 \tag{24}$$

in measure. The definition of $b_s^{(t)}$, (4), (19) and (21)–(23) imply that $\varphi_s^{(p_t)}(x)$ is equal to $e_0^{(s)}, \ldots, e_{r_s-1}^{(s)}$ on a set $E_t \subset X$ with $\mu(E_t) \to 1$.

Moreover, if

$$E_{t,i} = \{x \in E_t : \varphi^{(p_t)}(x) = e_i^{(s)}\}, \quad i = 0, 1, \ldots, r_s - 1,$$

then

$$\mu(E_{t,i}) \ge \tfrac{1}{2} c(\gamma, \gamma')$$

if $t \in N(\gamma, \gamma')$ and γ' comes from a different $\widehat{v}$-trajectory than γ. It is obvious that the last inequality and (24) imply $\gamma = 1$. Thus T_{φ_s} is ergodic and then T_φ is ergodic because $T_\varphi = \varprojlim T_{\varphi_s}$.

To show the continuity of φ we must prove that the only eigenvalues of T_φ are p_t-roots of unity. Let $F(x, g)$ be an eigenfunction with eigenvalue λ. We have

$$F(x, g) = \sum_{\gamma \in \widehat{G}} f_\gamma(x) \gamma(g),$$

where $f_\gamma \in L^2(X, \mu)$. Then $f_\gamma(Tx)\gamma(\varphi(x)) = \lambda f_\gamma(x)$ for all $\gamma \in \widehat{G}$ and a.e. $x \in X$. Using again the same arguments as in [FeKwMa] we get

$$\gamma(\varphi^{(p_t)}(x))\lambda^{-p_t} \to 1 \quad \text{in measure} \tag{25}$$

for every $\gamma \in \widehat{G}$ such that $f_\gamma \ne 0$ in $L^2(X, \mu)$. Then $\gamma \in \widehat{G}_s$ for some $s \ge 0$ so (25) can be rewritten as

$$\gamma(\varphi_s^{(p_t)}(x))\lambda^{-p_t} \to 1.$$

Taking again γ' as before and $t \to \infty$, $t \in N(\gamma, \gamma')$ we find that $\gamma(e_i^{(s)})$ is constant for $i = 0, 1, \ldots, r_s - 1$. Thus $\gamma = 1$. This means that $F(x, y) = f_0(x)$ and λ is an eigenvalue of T, i.e. λ is a p_t-root of unity. We have proved the continuity of γ. ■

PROPOSITION 4. $m(T_\varphi) = r$.

Proof. Let μ_γ be the spectral measure defined in part 3.1, and $\gamma \in \widehat{G}$. We will show that

$$\mu_\gamma \simeq \mu_{\hat{v}(\gamma)}, \tag{26}$$

$$\mu_\gamma \perp \mu_{\gamma'} \quad \text{whenever } \gamma, \gamma' \text{ are in different } \widehat{v}\text{-trajectories.} \tag{27}$$

It follows from (14) that every fragment $\overbrace{\beta_u \beta_u \dots \beta_u}^{q_{t,u}}$, $u = 0, 1, \dots, 2^t - 1$, of $b^{(t)}$ is of the form $\beta_{u,0} v(\beta_{u,0}) \dots v^{r'}(\beta_{u,0})$, $r' = rq_{t,u} - 1$. Thus

$$\bar{d}(b^{(t)}[l_t^{r_t} - 1, \lambda_t - 1], v(b^{(t)})[0, \lambda_t - l_t^{r_t} - 1]) \le \frac{2^t |\beta_{u,0}|}{|\beta_{u,0}| r Q_t} \overset{(13)}{\le} \frac{1}{q_t}. \tag{28}$$

Choose $s \ge 0$ such that $\gamma \in \widehat{G}_s$. The inequality (28) is valid for the blocks $b_s^{(t)}$, because $\psi_s \circ v = v_s \circ \psi_s$. Thus the sequence $(b_s^{(t)})_{t=0}^\infty$ satisfies the conditions (a) and (b) of Proposition 1. In this manner (26) is proved.

Now we prove (27). Suppose γ, γ' do not belong to the same $\widehat{v}$-trajectory. Let $\gamma, \gamma' \in \widehat{G}_s$ and let $g = g(\gamma, \gamma')$ satisfy (18). Then

$$g = g_0 e_0^{(s)} + \dots + g_{r_s - 1} e_{r_s - 1}^{(s)},$$

with $g_0, \dots, g_{r_s - 1} = 0, 1, \dots, n - 1$. Define

$$a_t = g_0 + g_1 l_t + \dots + g_{r_s - 1} l_t^{r_s - 1}.$$

Then

$$\frac{a_t}{l_t^{r_t}} \le \frac{n r_s l_t^{r_s - 1}}{l_t^{r_t}} \le \frac{n r_s}{l_t} \xrightarrow{t} 0$$

and

$$\sum_{t=0}^\infty \frac{a_t}{\lambda_t} \le n r_s \sum_{t=0}^\infty \frac{1}{l_t Q_t} < \infty.$$

We now show that

$$\lim_{\substack{t \in N(\gamma, \gamma') \\ t \to \infty}} \Big[\int_X \gamma(\varphi_s^{(a_t p_t)}(x))\, \mu(dx) - \int_X \gamma'(\varphi_s^{(a_t p_t)}(x))\, \mu(dx) \Big] \ne 0.$$

Repeating the same calculations as in [GoKwLeLi] and using (4) we get for $t > s$,

$$\begin{aligned} &\int_X \widetilde{\gamma}(\varphi_s^{(a_t p_t)}(x))\, \mu(dx) \\ &\qquad = \underbrace{\sum_{h \in G_t} \Big\{ \sum_{u=0}^{2^t - 1} \frac{q_{t,u}}{Q_t} \Big[\frac{1}{r} \sum \widehat{v}^p(\widetilde{\gamma})(\psi_s(h)) \Big] o_{t,u}(\psi_s(h)) \Big\}}_{I_1} + \varrho_t \end{aligned} \tag{29}$$

where

$$o_{t,u}(h)=\frac{1}{l_t^{r_t}}\#\{0\le j\le l_t^{r_t}-a_t-1:\beta_{u,0}[j+a_t]-\beta_{u,0}[j]=h\},$$

$$\widetilde{\gamma}=\gamma \text{ or } \gamma',\qquad \varrho_t\le\frac{a_t}{l_t^{r_t}}+\frac{2^t}{Q_t}\xrightarrow{t}0,\qquad \beta_{u,0}=\beta_{u,0}^{(t,s)}.$$

But $o_{t,u}(\psi_s(h))=o_{t,\overline{u}}(\psi_s(h))$ if $u\equiv\overline{u} \pmod{2^s}$. Thus

$$(30)\qquad I_1=\sum_{g\in G_s}\bigg[\frac{1}{r}\sum_{p=0}^{r-1}\widehat{v}^p(\widetilde{\gamma})(g)\bigg]\bigg\{\sum_{z=0}^{2^s-1}o_{t,z}(g)\bigg[\sum_{u\equiv z}\frac{q_{t,u}}{Q_t}\bigg]\bigg\}$$

$$\overset{(20)}{=}\sum_{g\in G_s}\bigg[\frac{1}{r}\sum_{p=0}^{r-1}\widehat{v}^p(\widetilde{\gamma})(g)\bigg]\bigg\{\sum_{z=0}^{2^s-1}o_{t,z}(g)\omega_z\bigg\}.$$

Take $j=0,1,\ldots,l_t^{r_t}-1$. We can represent it as

$$j=j_0+j_1l_t+\ldots+j_{r_t-1}l_t^{r_t-1},$$

where $j_0,j_1,\ldots,j_{r_t-1}=0,1,\ldots,l_t-1$. Let

$$K_t=\{0\le j\le l_t^{r_t}-1:0\le j_0,j_1,\ldots,j_{r_t-1}\le l_t-n-1\}.$$

We have

$$(31)\qquad \frac{\#K_t}{l_t^{r_t}}\ge\bigg(1-\frac{n}{l_t}\bigg)^{r_t}.$$

If $j\in K_t$ then it is easy to check that

$$(32)\qquad \beta_{u,0}[j+a_t]-\beta_{u,0}[j]=g_0e_{zr}^{(s)}+g_1e_{zr+1}^{(s)}+\ldots+g_{r_s-1}e_{zr+r_s-1}^{(s)}$$
$$=g_z^*,\qquad z=0,1,\ldots,2^s-1,\ u\equiv z \pmod{2^s}.$$

In particular, $g_0^*=g(\gamma,\gamma')$.

(31) and (32) imply

$$(33)\qquad o_{t,z}(g_0^*)\ge\bigg(1-\frac{n}{l_t}\bigg)^{r_t}.$$

Using (8) and (29)–(33) we obtain

$$\int_X\widetilde{\gamma}(\varphi_s^{(a_tp_t)}(x))\,\mu(dx)=\sum_{z=0}^{2^s-1}\omega_z\bigg[\frac{1}{\gamma}\sum\widehat{v}^p(\widetilde{\gamma})(g_z^*)\bigg]+\varrho_t+\varrho_t',$$

$$\varrho_t\to0,\ \varrho_t'\le1-\bigg(1-\frac{n}{l_t}\bigg)^{r_t}\xrightarrow{t}0.$$

Now, if $t\in N(\gamma,\gamma')$ then (18), (19) and (21)–(23) imply

$$\lim_{t\to\infty}\Big[\int_X\gamma(\varphi_s^{(a_tp_t)}(x))\,\mu(dx)-\int_X\gamma'(\varphi_s^{(a_tp_t)}(x))\,\mu(dx)\Big]$$
$$=c(\gamma,\gamma')[A_\gamma(g)-A_{\gamma'}(g)]+b,$$

and

$$|b| \le 2(1 - c(\gamma, \gamma')) < \tfrac{1}{2}c|A_\gamma(g) - A_{\gamma'}(g)|.$$

In this way

$$\lim_{\substack{t\in N(\gamma,\gamma')\\ t\to\infty}} \Big[\int_X \gamma(\varphi_s^{(a_t p_t)}(x))\,\mu(dx) - \int_X \gamma'(\varphi_s^{(a_t p_t)}(x))\,\mu(dx)\Big] \ne 0.$$

We have shown $\mu_{\gamma'} \perp \mu_\gamma$ by Proposition 2. It follows from (5) and from the simplicity of U_{T_φ} on L_γ, $\gamma \in \widehat{G}$, that

$$m(T_\varphi) = \max\{\text{lengths of } \widehat{v}\text{-trajectories of } \widehat{G}\} = r. \ \blacksquare$$

PROPOSITION 5. $r(T_\varphi) = \infty$.

Proof. We have $r(T_\varphi) = \lim_{s\to\infty} r(T_{\varphi_s})$. The blocks $b_s^{(t)}$, $t = 0, 1, \ldots$, defining the M-cocycle φ_s over G_s have a similar structure to those investigated in [KwLa1]. Repeating the same reasoning as in [KwLa1] we get $r(T_{\varphi_s}) = r_s$. In this manner $r(T_\varphi) = \lim_s r_s = \infty$. $\blacksquare$

5. Example 2. In this part we construct an M-cocycle φ such that T_φ has the properties announced in the second part of the abstract.

To do this choose a prime number $p > r$, set $G_t = \mathbb{Z}_{p^{t+1}}$, $t \ge 0$, and denote by $\pi_t : G_{t+1} \to G_t$ the natural homomorphisms. Next, let $\tau_t : G_t \to G_{t+1}$ be defined by $\tau_t(g) = g$, $g = 0, 1, \ldots, p^{t+1} - 1$. The groups G_t, the homomorphisms π_t and the mappings τ_t satisfy the conditions described in 3.2. Take a probability vector $\langle\omega(i)\rangle$, $i = 1, \ldots, r$, with $\omega(i) > 0$. Select positive integers $\lambda_t^{(1)}, \ldots, \lambda_t^{(r)}$ such that

$$\lambda_t^{(i)} = l_t^{(i)} p^t, \quad l_t^{(i)} \nearrow_t \infty, \tag{34}$$

$$\omega_t(i) = \lambda_t^{(i)}/\lambda_t \xrightarrow{t} \omega(i), \quad i = 1, \ldots, r, \ \lambda_t = \lambda_t^{(1)} + \ldots + \lambda_t^{(r)}. \tag{35}$$

Set

$$\beta_i^{(t)} = \beta_i = 0(i)(2i)\ldots((l-1)i), \quad l = \lambda_t^{(i)},$$

and

$$b^{(t)} = \beta_1^{(t)}\beta_2^{(t)}\ldots\beta_r^{(t)}.$$

The sequence $\{b^{(t)}\}$ of blocks determines an M-cocycle φ over the group $G = \varprojlim(G_t, \pi_t)$ (G is the group of p-adic integers) and M-cocycles φ_s over G_s according to the definitions in 3.2.

PROPOSITION 6. *There exists a probability vector $\langle\omega(i)\rangle$, $i = 1, \ldots, r$, with $\omega(1) > 1/r$, $0 < \omega(i) < \omega(1)$, $i = 2, \ldots, r$, such that $r(T_\varphi) = r$, $F^*(T_\varphi) = \omega(1)$, $\#C(T_\varphi)/\mathrm{wcl}\{T_\varphi^n : n \in \mathbb{Z}\} = \infty$ and T_φ has simple spectrum.*

Proof. It is proved in [FiKw] that for every $s \ge 0$, T_{φ_s} is ergodic and $r(T_{\varphi_s}) = r$, $F^*(T_{\varphi_s}) = \max(\omega(1), \ldots, \omega(r)) = \omega(1)$. Then $r(T_\varphi) =$

$\lim_s r(T_{\varphi_s})$ and $F^*(T_\varphi) = \lim_s F^*(T_{\varphi_s}) = \omega(1)$. To prove the next properties of T_φ let us remark that the set $\bigcup_{s=0}^\infty \overline{H}_s$ from 3.2 coincides with the set of all rational p-adic integers. For $g \in G$ let $\sigma_g : X \times G \to X \times G$ be defined by $\sigma_g(x,h) = (x, g+h), h \in G$. By this formula G acts as a group of measure-preserving transformations in $X \times G$. Moreover, $\sigma_g \in C(T_\varphi)$.

Consider σ_g, $g \in G_s \simeq \overline{H}_s$, $s \geq 0$. We show that $\sigma_g \notin \mathrm{wcl}\{T_\varphi^n : n \in \mathbb{Z}\}$. Assume that $(T_\varphi)^{u_t} \to \sigma_g$ in $C(T_\varphi)$. Then $(T_\varphi)^{u_t} \xrightarrow{t} \sigma_g$ for every $s \geq 0$, which implies

$$\mu\{x \in X : \varphi_s^{(u_t)}(x) \neq g\} = \varepsilon_{t,s} \xrightarrow{t} 0. \tag{36}$$

Fix $s \geq 0$. Choose $\tau(t) = \tau$ such that $u_t/p_\tau < \varepsilon_{t,s}/2$. It follows from (3) that

$$\varphi_s^{(u_t)}(x) = B_\tau[i + u_t] - B_\tau[i]$$

if $x \in D_i^\tau$, $i = 0, 1, \ldots, p_\tau - u_t - 1$. Then (36) implies

$$\frac{1}{p_\tau}\{0 \leq i \leq p_\tau - u_t - 1 : B_\tau[i+u_t] - B_\tau[i] = g\} \geq 1 - \varepsilon_{t,s}.$$

On the other hand, from [FiKw] we can deduce that

$$\frac{1}{p_\tau}\{0 \leq i \leq p_\tau - u - 1 : B_\tau[i+u] - B_\tau[i] \neq g\} \geq \varrho > 0$$

whenever $g \neq 0$ and $0 \leq u < p_\tau/2$.

In this way $\sigma_g \notin \mathrm{wcl}\{T_\varphi^n : n \in \mathbb{Z}\}$ for every $g \in \bigcup_{s=0}^\infty G_s$. To finish the proof it remains to select a probability vector $\langle \omega(i) \rangle$, $i = 1, \ldots, r$, for T_φ to have simple spectrum. It follows from [KwSi] that if the numbers $\omega(i)$ satisfy the condition

$$\sum_{i=1}^r [\gamma(i) - \gamma'(i)]\omega(i) \neq 0 \tag{37}$$

whenever $\gamma \neq \gamma'$, $\gamma, \gamma' \in \widehat{G}_s$ then T_{φ_s} has simple spectrum.

Fix $\omega(1)$ with $1/r < \omega(1) < 1$. If $r = 2$ then $F^*(T_\varphi) > 1/2$ and it is known [Fe2] that T_φ has simple spectrum.

Let $r \geq 3$. Consider the set

$$\Delta = \Big\{(\omega(2), \ldots, \omega(r)) \in \mathbb{R}^{r-2} : 0 \leq \omega(i) \leq \omega(1),\ \sum_{i=2}^r \omega(i) = 1 - \omega(1)\Big\}.$$

For distinct $\gamma, \gamma' \in \widehat{G} = \bigcup_{s=0}^\infty \widehat{G}_s$ we have an $(r-3)$-dimensional plane $D(\gamma, \gamma')$ in $\mathbb{R}^{r-2}$ described by

$$D(\gamma, \gamma') = \Big\{(\omega(2), \ldots, \omega(r)) : \sum_{i=2}^r [\gamma(i) - \gamma'(i)]\omega(i) = [\gamma'(1) - \gamma(1)]\omega(1)\Big\}.$$

The set $\Delta_0 = \bigcup_{\gamma\neq\gamma'} D(\gamma,\gamma')$ has Lebesgue measure 0 (in $\mathbb{R}^{r-2}$) so that we can find $\langle\omega(i)\rangle \in \Delta - \Delta_0$, $i = 2,\ldots,r$. Then the condition (37) is satisfied and T_{φ_s} has simple spectrum for $s \geq 0$. But $m(T_\varphi) = \sup_s m(T_{\varphi_s}) = 1$.

The proposition is proved. ∎

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Mathematics and Informatics Faculty
Nicholas Copernicus University
Chopina 12/18
87-100 Toruń, Poland
E-mail: jkwiat@mat.uni.torun.pl

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