FUNDAMENTA
MATHEMATICAE
157 (1998)

A factorization theorem for the transfinite kernel dimension of metrizable spaces

by

M. G. Charalambous (Karlovassi)

Abstract. We prove a factorization theorem for transfinite kernel dimension in the class of metrizable spaces. Our result in conjunction with Pasynkov's technique implies the existence of a universal element in the class of metrizable spaces of given weight and transfinite kernel dimension, a result known from the work of Luxemburg and Olszewski.

1. Introduction and definitions. In this paper, all spaces are metrizable, I denotes the unit interval $[0,1]$, $\mathbb{N}$ the set of natural numbers, wX the weight of a space X, and $|A|$ the cardinality of a set A. For an ordinal α, $\lambda(\alpha)$ denotes the unique limit ordinal and $n(\alpha)$ the unique finite ordinal such that $\alpha = \lambda(\alpha) + n(\alpha)$. It is convenient to adjoin -1 and ∞ to the class of all ordinals and treat them as the least and greatest elements of the augmented class, respectively. For the standard results and terminology in dimension theory, we refer to Engelking's book [2].

For any space X, we set $D_{-1}(X) = \emptyset$ and $D_\infty(X) = X$. For an ordinal α, we define $E_\alpha(X)$ and $D_\alpha(X)$ inductively by $E_\alpha(X) = X - \bigcup\{D_\beta(X) : \beta < \alpha\}$ and

$$D_\alpha(X) = \bigcup\{U : U \text{ an open subset of } E_{\lambda(\alpha)}(X) \text{ with } \dim U \le n(\alpha)\}.$$

The *transfinite kernel dimension* of X, $\operatorname{trker} X$, is defined to be the first extended ordinal α for which $X = \bigcup\{D_\beta(X) : \beta \le \alpha\}$. Note that each $E_\alpha(X)$ is a closed subset of X and, if $\lambda = \operatorname{trker} X$ is an ordinal, then $|\lambda| \le wX$ [2, Theorem 7.3.5].

The main result of this paper, Theorem 2 of Section 3, is a factorization theorem for trker in the class of metrizable spaces. From this we deduce using Pasynkov's method [6] that the class of metrizable spaces with $\operatorname{trker} \le \lambda$ and

1991 *Mathematics Subject Classification*: Primary 54F45.

Key words and phrases: covering dimension, transfinite kernel dimension, D-dimension, metrizable spaces.

weight $\leq \tau$ has a universal element. Note that Henderson's [3] D-dimension on metrizable spaces coincides with trker [2, Theorem 7.3.18] and Olszewski [5] proved the existence of a universal space for the class of all metrizable spaces with D-dimension $\leq \lambda$ and weight $\leq \tau$, Luxemburg [4] having proved the corresponding result in the class of compact metrizable spaces as well as in the class of separable metrizable spaces.

2. Preliminary results. We start with the key construction that is employed in the sequel. Let $\{H_\alpha : \alpha \in A\}$ be a collection of open subsets of a space Y, and $H = \bigcup\{H_\alpha : \alpha \in A\}$. For each α in A, let $h_\alpha : Z_\alpha \to Y$ be a continuous function, and $\tau = \sup\{|A|, wY, w(h_\alpha^{-1}(H_\alpha)) : \alpha \in A\}$. Let Z be the set $(Y - H) \cup \bigcup\{h_\alpha^{-1}(H_\alpha) \times \{\alpha\} : \alpha \in A\}$, and define $h : Z \to Y$ to be the identity on $Y - H$ and, outside $Y - H$, by $h(x, \alpha) = h_\alpha(x)$. We let Z have the smallest topology that makes h continuous and $G \times \{\alpha\}$ open for each open set G of $h_\alpha^{-1}(H_\alpha)$. It is easy to see that Z is T_1 and regular. Let $\{U_{\lambda,n} : \lambda < \tau,\ n \in \mathbb{N}\}$ be a σ-locally finite open base of Y. Let $\{U_{\alpha,\lambda,n} : \lambda < \tau,\ n \in \mathbb{N}\}$ be a σ-locally finite open base of $h_\alpha^{-1}(H_\alpha)$. Let $H = \bigcup\{H_n : n \in \mathbb{N}\}$, where H_n is open and its closure is contained in H_{n+1}. Then it is easily verified that

$$\{h^{-1}(U_{\lambda,n}) : \lambda < \tau,\ n \in \mathbb{N}\} \cup\{(U_{\alpha,\lambda,n} \times \{\alpha\}) \cap h^{-1}(H_m) : \alpha \in A,\ \lambda < \tau,\ m, n \in \mathbb{N}\}$$

is a σ-locally finite base of Z of cardinality $\leq \tau$. Hence Z is metrizable and $wZ \leq \tau$. We will refer to Z as the *space*, and h as the *projection*, *determined by the pairs of maps and open sets* (h_α, H_α), $\alpha \in A$.

PROPOSITION 1. *Let $f : X \to Y$ be a continuous function, $\{H_\alpha : \alpha \in A\}$ a disjoint collection of open subsets of Y, $H = \bigcup\{H_\alpha : \alpha \in A\}$ and, for each α in A, let $g_\alpha : X \to Z_\alpha$ and $h_\alpha : Z_\alpha \to Y$ be continuous functions such that $f = g_\alpha \circ h_\alpha$. Then there is a space Z and continuous functions $g : X \to Z$ and $h : Z \to Y$ such that $f = h \circ g$, the restriction of h to $h^{-1}(Y - H)$ is a homeomorphism, $wZ \leq \tau = \sup\{wY, w(h_\alpha^{-1}(H_\alpha)) : \alpha \in A\}$, and $\dim g(E \cap f^{-1}(H)) \leq n$ for each subset E of X that satisfies $\dim g_\alpha(E \cap f^{-1}(H_\alpha)) \leq n$ for each α in A.*

Proof. We can assume that each H_α is non-empty, so that $|A| \leq wY \leq \tau$. Let Z be the space, and h the projection, determined by the pairs (h_α, H_α), $\alpha \in A$. Clearly, the restriction of h to $h^{-1}(Y - H)$ is a homeomorphism and $wZ \leq \tau$. Define g by $g(x) = (g_\alpha(x), \alpha)$ if $f(x)$ is a point of H_α for some $\alpha \in A$, and $g(x) = f(x)$ otherwise. Evidently, $f = h \circ g$ and g is continuous. Finally, suppose that a subset E of X satisfies $\dim g_\alpha(E \cap f^{-1}(H_\alpha)) \leq n$ for each α in A. As $g(E \cap f^{-1}(H))$ is the direct sum of $g_\alpha(E \cap f^{-1}(H_\alpha))$, $\alpha \in A$, we have $\dim g(E \cap f^{-1}(H)) \leq n$.

PROPOSITION 2. *Let $f : X \to Y$ be a continuous function, H an open subset of Y and A a subset of $f^{-1}(H)$. Then there are continuous functions $g : X \to Z$ and $h : Z \to Y$ such that $f = h \circ g$, $\dim g(A) \le \dim A$, $wZ \le wY$ and the restriction of h to $h^{-1}(Y - H)$ is a homeomorphism* (cf. [1, Theorem 4 and Remark 2]).

Proof. By a factorization theorem due to Pasynkov [7, Theorem 1], there are continuous functions $g_1 : X \to Z_1$ and $h_1 : Z_1 \to Y$ such that $f = h_1 \circ g_1$, $\dim g_1(A) \le \dim A$ and $wZ_1 \le wY$. The result is now a straightforward application of Proposition 1.

A *tower* of a space X will mean a collection $\{G_\alpha : \alpha < \lambda\}$ of open subsets of X, where λ is an ordinal or -1, with $G_{-1} = \emptyset$ and $G_\alpha \subset G_\beta$ for $\alpha \le \beta < \lambda$.

PROPOSITION 3. *Let τ be an infinite cardinal and $\{G_\alpha : \alpha < \lambda\}$ a tower of a space X, where $|\lambda| \le \tau$. Then there exist an open collection $\{H_\alpha : \alpha < \lambda\}$ of a space Y with $wY \le \tau$ and a continuous function $f : X \to Y$ such that $G_\alpha = f^{-1}(H_\alpha)$ for all $\alpha < \lambda$.*

REMARK. Evidently, we can additionally stipulate that $\{H_\alpha : \alpha < \lambda\}$ is a tower of Y.

Proof (of Proposition 3). The proof is by induction on λ. The result holds for $\lambda = -1$. Assume that $\lambda > -1$ and the result holds for all ordinals $< \lambda$.

Consider first the case when λ has an immediate predecessor μ. By the induction hypothesis, there is an open collection $\{U_\alpha : \alpha < \mu\}$ of a space Z with $wZ \le \tau$ and a continuous function $g : X \to Z$ such that $G_\alpha = g^{-1}(U_\alpha)$, $\alpha < \mu$. Let $h : X \to I$ be continuous with $h^{-1}(0,1] = G_\mu$. Finally, let $Y = Z \times I$, $f = g \triangle h$, $H_\alpha = \sigma^{-1}(U_\alpha)$, $\alpha < \mu$, and $H_\mu = \pi^{-1}(0,1]$, where σ and π denote the canonical projections of Y onto Z and I, respectively.

Consider next the case of λ being a non-zero limit ordinal. Let $\{V_{i,\mu} : i \in \mathbb{N},\ \mu \in M\}$ be a σ-discrete base of X. For each i in $\mathbb{N}$ and $\alpha < \lambda$, let

$$U_{i,\alpha} = \bigcup\{V_{i,\mu} : \alpha \text{ is the first extended ordinal with } V_{i,\mu} \subset G_\alpha\}.$$

Let $U_i = \bigcup\{U_{i,\alpha} : \alpha < \lambda\}$. Note that, for i in $\mathbb{N}$ and $\beta \le \alpha < \lambda$, we have $U_{i,\beta} \subset G_\beta \subset G_\alpha$ so that $G_\alpha \cap U_{i,\beta} = U_{i,\beta}$. By the induction hypothesis, we therefore have, for each $\beta < \lambda$, an open collection $\{H_{i,\alpha,\beta} : \alpha < \lambda\}$ in some space $Z_{i,\beta}$ with weight $\le \tau$ and a continuous function $g_{i,\beta} : X \to Z_{i,\beta}$ such that $G_\alpha \cap U_{i,\beta} = g_{i,\beta}^{-1}(H_{i,\alpha,\beta})$. Let $h_{i,\beta} : Z_{i,\beta} \to I$ be a continuous function such that $h_{i,\beta}^{-1}(0,1] = \bigcup\{H_{i,\alpha,\beta} : \alpha < \lambda\}$. Let Z_i be the space and $h_i : Z_i \to I$ the projection determined by pairs $(h_{i,\beta}, (0,1])$, $\beta < \lambda$. Then $wZ_i \le \tau$ and, because $\{U_{i,\beta} : \beta < \lambda\}$ is discrete in X, the function $f_i : X \to Z_i$ that sends $X - U_i$ to 0 and x of $U_{i,\beta}$ to $(g_{i,\beta}(x), \beta)$ is continuous.

Letting $Y = \prod\{Z_i : i \in \mathbb{N}\}$, $f = \Delta\{f_i : i \in \mathbb{N}\}$, $\pi_i : Y \to Z_i$ the canonical projection and $H_\alpha = \bigcup\{\pi_i^{-1}(H_{i,\alpha,\beta} \times \{\beta\}) : i \in \mathbb{N},\ \beta < \lambda\}$, one can check that the required properties are satisfied.

PROPOSITION 4. *Let $f : X \to Y$ be a continuous function, $\{G_\alpha : \alpha < \lambda\}$ a tower of X and τ a cardinal $\geq \max\{|\lambda|, wY\}$. Then there is a space Z with $wZ \leq \tau$, a tower $\{H_\alpha : \alpha < \lambda\}$ of Z and continuous functions $g : X \to Z$ and $h : Z \to Y$ such that $f = h \circ g$ and $g^{-1}(H_\alpha) = G_\alpha$ for each $\alpha < \lambda$.*

Proof. By Proposition 3, there exist a tower $\{U_\alpha : \alpha < \lambda\}$ of a space S with $wS \leq \tau$ and a continuous function $r : X \to S$ such that $G_\alpha = r^{-1}(U_\alpha)$. It suffices to let $Z = S \times Y$, $g = r \bigtriangleup f$, and q and h be the projections of Z onto S and Y, respectively, and $H_\alpha = q^{-1}(U_\alpha)$.

3. The main results

THEOREM 1. *Let $f : X \to Y$ be a continuous function, $\{G_\alpha : \alpha < \lambda\}$ a tower of X and τ a cardinal $\geq \max\{|\lambda|, wY\}$. For $\alpha < \lambda$, let $E_\alpha = G_\alpha - \bigcup\{G_\beta : \beta < \alpha\}$ and suppose that $n_\alpha = \dim E_\alpha$ is finite. Then there is a space Z and continuous functions $g : X \to Z$ and $h : Z \to Y$ such that $f = h \circ g$, $wZ \leq \tau$, $g(G_\alpha)$ is open in $g(X)$ and $\dim g(E_\alpha) \leq n_\alpha$ for each $\alpha < \lambda$.*

Proof. By Proposition 4, we may assume that there is a tower $\{H_\alpha : \alpha < \lambda\}$ of Y such that $f^{-1}(H_\alpha) = G_\alpha$, $\alpha < \lambda$. This assures that $g(G_\alpha)$ will be open in $g(X)$. Let $H = \bigcup\{H_\alpha : \alpha < \lambda\}$. Note that, by Proposition 1, whenever the result holds, it holds with the additional requirement that the restriction of h to $h^{-1}(Y - H)$ is a homeomorphism. The proof is by induction on λ. The result holds for $\lambda = -1$. Assume that $\lambda > -1$ and the result holds for all ordinals $< \lambda$.

Consider first the case when λ has an immediate predecessor μ. Let $U = \bigcup\{H_\alpha : \alpha < \mu\}$. By Proposition 2, there is a metrizable space Z_1 with $wZ_1 \leq \tau$ and continuous functions $g_1 : X \to Z_1$ and $h_1 : Z_1 \to Y$ such that $f = h_1 \circ g_1$, $\dim g_1(E_\mu) \leq n_\mu$ and the restriction of h_1 to $h_1^{-1}(Y - H)$ is a homeomorphism. Next, by the induction hypothesis, there is a metrizable space Z with $wZ \leq \tau$ and continuous functions $g : X \to Z$ and $h_2 : Z \to Z_1$ such that $g_1 = h_2 \circ g$, $\dim g(E_\alpha) \leq n_\alpha$ for $\alpha < \mu$, and the restriction of h_2 to $(h_1 \circ h_2)^{-1}(Y - U)$ is a homeomorphism. It now suffices to set $h = h_1 \circ h_2$.

Consider now the case of λ being a non-zero limit ordinal. Let $\{H_{i,\beta} : i \in \mathbb{N},\ \beta \leq \tau\}$ be a σ-discrete in Y open cover of H that refines $\{H_\alpha : \alpha < \lambda\}$. Let $H_i = \bigcup\{H_{i,\beta} : \beta \leq \tau\}$. Note that, given i and β, there is $\mu < \lambda$ such that $E_\alpha \cap f^{-1}(H_{i,\beta}) = \emptyset$ for $\mu \leq \alpha$. By the induction hypothesis, we can apply the result to the tower $\{G_\alpha \cap f^{-1}(H_{1,\beta}) : \alpha < \lambda\}$ to get, for each $\beta \leq \tau$, a space Z_β and continuous functions $g_\beta : X \to Z_\beta$ and $h_\beta : Z_\beta \to Y$

such that $f = h_\beta \circ g_\beta$, $wZ_\beta \le \tau$, and $\dim g_\beta(E_\alpha \cap f^{-1}(H_{1,\beta})) \le n_\alpha$ for each $\alpha < \lambda$. Then, by Proposition 1, there is a space Z_1 and continuous functions $g_1 : X \to Z_1$ and $h_1 : Z_1 \to Y$ such that $f = h_1 \circ g_1$, $wZ_1 \le \tau$, and $\dim g_1(E_\alpha \cap f^{-1}(H_1)) \le n_\alpha$ for each $\alpha < \lambda$.

Let $N_1, N_2, N_3, \ldots,$ be a partition of $\mathbb{N}$ into infinite disjoint sets with N_1 containing 1. By the argument of the previous paragraph, for each n in $\mathbb{N}$ we can construct, by induction on n, a space Z_n with $wZ_n \le \tau$ and continuous functions $g_n : X \to Z_n$ and $h_n : Z_n \to Z_{n-1}$ such that $g_{n-1} = h_n \circ g_n$, where $Z_0 = Y$ and $g_0 = f$, and, if $n \in N_i$, then $\dim g_n(E_\alpha \cap f^{-1}(H_i)) \le n_\alpha$ for each $\alpha < \lambda$. Write $h_{m,n}$ for the composite of $h_{m+1}, h_{m+2}, \ldots, h_n$. Let Z be the limit of the inverse sequence $(Z_n, h_{m,n}; \mathbb{N} \cup \{0\})$, let $\pi_n : Z \to Z_n$ be the canonical projection and $h = \pi_0$. Evidently, $wZ \le \tau$ and we have a continuous function $g : X \to Z$ such that $g_n = \pi_n \circ g$. In particular, $f = h \circ g$.

Let $\alpha < \lambda$ and $i \in \mathbb{N}$. For each n in N_i, we have $\dim g_n(E_\alpha \cap f^{-1}(H_i)) \le n_\alpha$, and $g(E_\alpha \cap f^{-1}(H_i))$ is contained in the limit of the inverse sequence $(g_n(E_\alpha \cap f^{-1}(H_i)), h_{m,n}; N_i)$. By the inverse limit and the subset theorems, we therefore have $\dim g(E_\alpha \cap f^{-1}(H_i)) \le n_\alpha$. Now, the sets $g(E_\alpha \cap f^{-1}(H_i)) = g(E_\alpha) \cap h^{-1}(H_i)$, $i \in \mathbb{N}$, form an open cover of $g(E_\alpha)$. Hence, by the countable sum theorem, $\dim g(E_\alpha) \le n_\alpha$. This concludes the proof of the theorem.

LEMMA 1. *Let $\{G_\alpha : \alpha < \lambda\}$ be a tower of a space X and suppose that $\dim E_\alpha \le n(\alpha)$ for $\alpha < \lambda$, where $E_\alpha = G_\alpha - \bigcup\{G_\beta : \beta < \alpha\}$. Then, for each $\alpha < \lambda$,*

$$G_\alpha \subset \bigcup\{D_\beta(X) : \beta \le \alpha\}.$$

P r o o f. The proof is by induction on α. The result is true for $\alpha = -1$. Assume that $\alpha > 0$ and the result holds for all $\beta < \alpha$. Then $E_{\lambda(\alpha)}(X) \cap G_\alpha$ is contained in the union of the F_σ-subsets $E_{\lambda(\alpha)+i}$ of X, $0 \le i \le n(\alpha)$. The subset and the countable sum theorems assure that the open subset $E_{\lambda(\alpha)}(X) \cap G_\alpha$ of $E_{\lambda(\alpha)}(X)$ has $\dim \le n(\alpha)$. Hence $E_{\lambda(\alpha)}(X) \cap G_\alpha \subset D_\alpha(X)$ and $G_\alpha \subset \bigcup\{D_\beta(X) : \beta \le \alpha\}$.

THEOREM 2. *Let $f : X \to Y$ be a continuous function, $\mu = \operatorname{trker} X$ and suppose that τ is a cardinal $\ge \max\{|\mu|, wY\}$. Then there is a space Z and continuous functions $g : X \to Z$ and $h : Z \to Y$ such that $f = h \circ g$, $wZ \le \tau$ and $\operatorname{trker} Z \le \mu$.*

P r o o f. In Theorem 1, put $\lambda = \mu + 1$ and $G_\alpha = \bigcup\{D_\beta(X) : \beta \le \alpha\}$, $\alpha < \lambda$. Then there is a space Z and continuous functions $g : X \to Z$ and $h : Z \to Y$ such that $f = h \circ g$, $wZ \le \tau$, $g(G_\alpha)$ is open in $g(X)$ and $\dim g(G_\alpha - \bigcup\{G_\beta : \beta < \alpha\}) \le n(\alpha)$, $\alpha < \lambda$. We take g to be surjective so that $\{g(G_\alpha) : \alpha < \lambda\}$ is a tower of Z and, since $X = G_\mu$, we have $Z = g(G_\mu)$. Noting that the subset $g(G_\alpha) - \bigcup\{g(G_\beta) : \beta < \alpha\}$ of $g(G_\alpha - \bigcup\{G_\beta : \beta < \alpha\})$

has dim $\le n(\alpha)$, we deduce from Lemma 1 that $Z = g(G_\mu) \subset \bigcup\{D_\alpha(Z) : \alpha \le \mu\}$. This shows that $\operatorname{trker} Z \le \mu$ and completes the proof.

COROLLARY 1. *The class $\mathcal{C}$ of all metrizable spaces with* trker $\le \alpha$ *and* weight $\le \tau$ *contains a universal element* (cf. [4, 5]).

Proof. We can of course assume that $|\alpha| \le \tau$. Let Y be a universal space for the class of all metrizable spaces of weight $\le \tau$. Let X be the direct sum of all subspaces X_λ of Y with $\operatorname{trker} X_\lambda \le \alpha$. Then $\operatorname{trker} X \le \alpha$. Let $f : X \to Y$ be the map whose restriction to X_λ is its embedding into Y. Then the space Z supplied by Theorem 2 is a universal element of $\mathcal{C}$.

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Department of Mathematics
University of the Aegean
Karlovassi 83200, Samos
Greece
E-mail: mcha@aegean.gr

Received 14 August 1997;
in revised form 15 December 1997