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The regular open algebra of $\beta\mathbb{R} \setminus \mathbb{R}$ is not equal to the completion of $\mathcal{P}(\omega)/\mathrm{fin}$

by

Alan Dow (North York, Ont.)

Abstract. Two compact spaces are *co-absolute* if their respective regular open algebras are isomorphic (i.e. homeomorphic Gleason covers). We prove that it is consistent that $\beta\omega \setminus \omega$ and $\beta\mathbb{R} \setminus \mathbb{R}$ are not co-absolute.

1. Introduction. While it is rather apparent that $\beta\mathbb{N} \setminus \mathbb{N}$ and $\beta\mathbb{R} \setminus \mathbb{R}$ are quite distinct they also have many similarities. Comfort and Negrepontis showed that if the Continuum Hypothesis holds then they even have homeomorphic dense subsets. Following that result, the investigation of the property of two spaces being *co-absolute* was of some interest. The *absolute* of a regular space (the Gleason cover in the case of compact spaces) is the unique extremally disconnected space which maps onto the space by a perfect irreducible map.

The Boolean algebra of clopen subsets of the absolute is isomorphic to the regular open algebra, r.o.(X), of a space X (the absolute is the subspace of ultrafilters on r.o.(X) which converge to a point of X). Since the regular open algebra of a space is isomorphic to the regular open algebra of any dense subspace it is clear how this investigation grew out of the Comfort–Negrepontis result. In the case of a zero-dimensional space, the clopen algebra of the absolute is just the completion of the algebra of clopen sets of the original space, thus the completion of $\mathcal{P}(\mathbb{N})/\mathrm{fin}$ is isomorphic to r.o.$(\beta\mathbb{N} \setminus \mathbb{N})$.

The interest in these regular open algebras was heightened following the remarkable discovery of Balcar *et al.* ([1]) that many of them had a dense subset which formed a tree (by reverse inclusion). Topologically, a dense subset is called a π-*base* for the space. Thus two spaces are co-absolute if and only if they share isomorphic (under set-theoretic inclusion) π-bases. The question of whether the remainders of $\mathbb{N}$ and of $\mathbb{R}$ are co-absolute has been

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asked in several forms many times in the literature. Van Douwen observed in [12] that $\mathfrak{p} = \mathfrak{c}$ implied they were co-absolute.

Our representation of r.o.$(\beta\mathbb{R}\setminus\mathbb{R})$ will be as the completion of a quotient of r.o.$(\mathbb{R})$. It is a basic fact of Čech–Stone remainders that the relative interiors of the closures of open subsets of the original space form a base for the open sets in the remainder. From this it easily follows that r.o.$(\mathbb{R})/\overline{\mathrm{cpt}} =$ r.o.$(\mathbb{R})/$bounded is isomorphic to a dense subset of r.o.$(\beta\mathbb{R}\setminus\mathbb{R})$, where $\overline{\mathrm{cpt}} =$ bounded denotes the ideal of regular open sets which are bounded, or equivalently, have compact closure.

For convenience we will adopt, by analogy with the standard notation for subsets of $\mathbb{N}$, the convention that a * adorning a word or symbol will mean that it should be interpreted modulo the ideal $\overline{\mathrm{cpt}}$ in r.o.$(\mathbb{R})$, e.g. $W \subset^* O \subset \mathbb{R}$ will mean that $W \setminus O$ is bounded and has compact closure. When we speak of a tree* in r.o.$(\mathbb{R})$ we shall really mean a tree in the quotient algebra (and similarly for dense* and for $\mathcal{P}(\mathbb{N})$).

We shall show that in the standard Mathias model the algebra r.o.$(\mathbb{R}^*)$ has a dense* subtree of height ω_1; it is well known that r.o.$(\mathbb{N}^*)$ does not have such a tree in this model. The latter fact is essentially due to Mathias who showed that if r is a Mathias real then $\{x \subset \mathbb{N} : r \subset^* x\}$ is a generic filter on $\mathcal{P}(\mathbb{N})/\mathrm{fin}$—for this implies that intermediate Mathias reals ensure that dense* subtrees of r.o.$(\mathbb{N}^*)$ must have branches longer than ω_1.

The possibility that, in the Mathias model, there is a dense* tree of height ω_1 in r.o.$(\mathbb{R}^*)$ is strongly suggested by previous results about dense* trees in $\mathcal{P}(\mathbb{N})$. To begin, there is Dordal's result (from [3]) that in the Mathias model there is a dense* tree in $\mathcal{P}(\mathbb{N})$ without branches of length ω_2. The key to this is the fact that single stage Mathias forcing does not fill towers (due to Baumgartner) [2].

When taken together, the results of Mathias and Baumgartner reveal an interesting subtlety when dealing with branches of dense* trees and Mathias forcing: Mathias forcing does not fill "old" branches through such trees but it creates and fills a new one. Analyzing which new towers are (and which are not) filled is a major component of Shelah and Spinas' paper [10], where it is shown that in the Mathias model the algebra r.o.$(\mathbb{N}^* \times \mathbb{N}^*)$ does have a dense* tree of height ω_1.

This result and the author's view that r.o.$(\mathbb{R}^*)$ has a certain two-dimensionality to it suggested that r.o.$(\mathbb{R}^*)$ should also have a dense* subtree of height ω_1. The two-dimensionality of r.o.$(\mathbb{R}^*)$ manifests itself in the way one can describe open sets U of $\mathbb{R}$. We regard $A_U = \{m \in \omega : U \cap (m, m+1) \neq \emptyset\}$ as one "coordinate" and, a little vaguely, the second coordinate is which rational open intervals of $(m, m+1)$ U contains (for $m \in A_U$).

In our proof we show that (iterated) Mathias forcing does not fill *remote* filters. We then use what we feel is a quite interesting enumeration principle on ω_2 to show that we can construct a dense* tree in r.o.$(\mathbb{R}^*)$ in which every ω_1-branch generates a remote filter in certain critical intermediate models.

There are two interesting questions we have not been able to resolve. Following [1] and [13], for a space X, let $\mathfrak{h}(X)$ denote the minimum number of dense open subsets of X whose intersection has empty interior (this is just the distributivity degree of the forcing Boolean algebra r.o.(X)). We also let $\mathfrak{n}(X)$ (for Novak number) denote the minimum number of dense open sets whose intersection is empty. It is shown in [1] that if r.o.(X) has a dense tree, then $\mathfrak{h}(X)$ is the minimum height of such a tree. Shelah and Spinas were answering (affirmatively) the question of whether $\mathfrak{h}(\mathbb{N}^*)$ can be larger than $\mathfrak{h}(\mathbb{N}^*\times\mathbb{N}^*)$. We conjecture that $\mathfrak{h}(\mathbb{R}^*)\le\mathfrak{h}(\mathbb{N}^*\times\mathbb{N}^*)$ and that $\mathfrak{h}(\mathbb{R}^*)=\mathfrak{h}(\mathbb{R}^*\times\mathbb{R}^*)$. Van Douwen has asked if $\mathfrak{n}(\mathbb{R}^*)=\mathfrak{n}(\mathbb{N}^*)$—we do not know. It may be worth reminding the reader that both inequalities, $\mathfrak{h}(\mathbb{R}^*)\le\mathfrak{h}(\mathbb{N}^*)$ and $\mathfrak{n}(\mathbb{R}^*)\le\mathfrak{n}(\mathbb{N}^*)$, are easily established (and are essentially due to van Douwen). To see this simply note that if $F\subset\mathbb{N}^*$ is nowhere dense in $\mathbb{N}^*$, then so is $\widehat{F}=\bigcap\{\mathrm{cl}\bigcup_{n\in A}[n,n+1]:A^*\supset F\}$ in $\mathbb{R}^*$.

Finally, let us remark that Dordal's result above implies that $\mathfrak{h}(\mathbb{N}^*)=\mathfrak{n}(\mathbb{N}^*)=\omega_2$ in the Mathias model. However, it is shown in [1] that $\mathfrak{n}(\mathbb{R}^*)$ is always at least ω_2, hence, in the Mathias model, $\mathfrak{h}(\mathbb{R}^*)<\mathfrak{n}(\mathbb{R}^*)$. However, $\mathfrak{h}(\mathbb{N}^*)=\mathfrak{n}(\mathbb{N}^*)$ was established in a different model in [4], and the corresponding result for $\mathbb{R}^*$ also holds there.

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2. Laver property and remote filters. The Mathias poset, $\mathbb{M}$, is very well known. It consists of ordered pairs $(a,A)\in[\omega]^{<\omega}\times[\omega]^{\omega}$ (where $a\cap A=\emptyset$) and a condition (a,A) is below (or stronger) than (b,B) providing $b\subset a$, $A\subset B$, and $a\subset b\cup B$. Because of the Ramsey-theoretic properties of $[\omega]^{\omega}$, $\mathbb{M}$ has the Laver property and so does its countable support iteration.

Let us recall that a forcing notion P has *the Laver property* if it is (f,g)-bounding for every increasing function $f\in{}^{\omega}\omega$, where $g(n)=n$ (in fact, g may be replaced by any other fixed increasing function in ${}^{\omega}\omega$). A proper forcing notion P is (f,g)-*bounding* (for some increasing $f,g\in{}^{\omega}\omega$) if

$$\Vdash_P\Big(\forall x\in\prod_{i\in\omega}f(i)\Big)\Big(\exists S\in V\cap\prod_{i\in\omega}[\omega]^{<\omega}\Big)(\forall i\in\omega)\qquad (|S(i)|\le g(i)\ \&\ x(i)\in S(i)).$$

LEMMA 2.1 (see [8]). *The countable support iteration of the Mathias poset has the Laver property.*

It is quite well known that many forcings have the Laver property (e.g. Sacks forcing, Laver forcing, and Mathias forcing). In addition, Goldstern

[7, 6.33] has shown that the Laver property is preserved by countable support proper iterations. Our main result, Theorem 3.5, depends primarily on the Laver property of the forcing poset and so holds in any model obtained by a length ω_2 countable support iteration of these Laver property proper posets (of cardinality at most $\mathfrak{c}$).

As discussed earlier, our main idea is to consider remote filters.

Definition 2.2. A filter or filter base $\mathcal{F}$ on a space X is said to be a *remote* filter if for each dense open $U \subset X$, there is an $F \in \mathcal{F}$ such that the closure of F is contained* in U. If V is a set (or even class), we will say that $\mathcal{F}$ is V-*remote* if for each dense open $U \subset X$ which is in V, there is an F in $\mathcal{F}$ as above.

Theorem 2.3. *If a partial order P has the Laver property then it does not diagonalize any V-remote filter.*

First we prove a lemma.

Lemma 2.4. *Suppose that P is a poset that has the Laver property and that G is P-generic. Let $W \in V[G]$ be an unbounded subset of $\mathbb{R}$. Then there is a dense open subset $U \in V$ of $\mathbb{R}$ such that $W \setminus U$ is unbounded.*

Proof. By shrinking W we may assume that for each n, there is at most one point $w_n \in W \cap (n, n+1)$ (and there is no loss in assuming that $W \cap \mathbb{N}$ is empty). For each n such that $W \cap (n, n+1)$ is empty, choose any $w_n \in (n, n+1)$.

Now define $x \in \prod_i (\prod_{n<i} \{(n + j/2^i, n + (j+1)/2^i) : j < 2^i\})$ so that for $n < i$, $w_n \in x(i)(n)$.

Apply the Laver property to find a sequence $S = \{S(i) : i \in \omega\} \in V$ of sets, $|S(i)| \le i$, so that for each i, $x(i) \in S(i)$. We can assume that for each $y \in S(i)$, y is a member of $\prod_{n<i} \{(n + j/2^i, n + (j+1)/2^i) : j < 2^i\})$.

From S we can define a dense open set $U \in V$. Simply, U is the union of all intervals of the form $(n + l/2^i, n + (l+1)/2^i)$ (with $n < i$ and $l < 2^i$) such that it is disjoint from $s(n)$ for each $s \in S(i)$. It is easy to see that the density of U follows from the small branching size of S, i.e., $|S(i)| \le i$ and for each $s \in S(i)$ and each n, $s(n)$ is an interval of length at most $1/2^i$. To see this, let $J = (j/2^k, (j+1)/2^k)$ be arbitrary. Fix an integer i so large that $(j+1)/2^k < i$ and $i < 2^{i-k}$. Clearly, one of the 2^{i-k} dyadic subintervals of length $1/2^i$ is disjoint from the at most i such subintervals which are contained in $\bigcup\{s(n) : s \in S(i)\}$.

Now we check that for each n, $w_n \notin U$. Suppose that $J = (n + l/2^i, n + (l+1)/2^i)$ (with $n < i$) is one of the intervals from the definition of U. There is an $s \in S(i)$ such that $x(i) = s$, hence $w_n \in s(n)$. Since J is disjoint from $s(n)$, it follows that $w_n \notin J$. ∎

Proof of Theorem 2.3. Suppose that P has the Laver property and that G is a P-generic filter over V. In $V[G]$, suppose that $\mathcal{F}\subset V\cap \mathrm{r.o.}(\mathbb{R})$ is a V-remote filter. Finally, assume that $W\in \mathrm{r.o.}(\mathbb{R})$ is unbounded. Apply Lemma 2.4 to find a dense open subset $U\in V$ of $\mathbb{R}$ such that $W\setminus U$ is unbounded. Since $\mathcal{F}$ is V-remote, there is an $F\in\mathcal{F}$ such that the closure, $\overline{F}$, of F is a subset* of U. Therefore $W\setminus\overline{F}$ is unbounded. ∎

3. Building the tree for $\mathbb{R}$**.** For each $\lambda\le\omega_2$, let P_λ denote the usual countable support iteration of the Mathias poset. It is most convenient to regard P_λ as a subset of P_μ for $\lambda\le\mu$. Therefore if G is a P_{ω_2}-generic filter we can just take $G_\lambda=G\cap P_\lambda$ and we have $V[G_\lambda]$ sitting as a natural submodel inside $V[G]$. We will need the following well-known consequence of properness (and the fact that each member of r.o.($\mathbb{R}$) is determined by a countable set).

PROPOSITION 3.1. *For each* $\lambda<\omega_2$ *of uncountable cofinality, each member of* $\mathrm{r.o.}(\mathbb{R})\cap V[G_\lambda]$ *is a member of* $V[G_\alpha]$ *for some* $\alpha<\lambda$.

Our plan is to build, in $V[G]$, a set T which is to be our dense* subtree of r.o.($\mathbb{R}$). However, it will be easier to build a tree which is simply shattering and refer to the following lemma which is proven exactly as the corresponding result for $\mathcal{P}(\mathbb{N})$ in [1]. A subfamily $T\subset\mathrm{r.o.}(\mathbb{R})$ is *shattering* if for each $w\in\mathrm{r.o.}(\mathbb{R})$, there are disjoint $t,t'\in T$, both of which meet* w.

LEMMA 3.2. *If there is a tree* $T\subset\mathrm{r.o.}(\mathbb{R})$ *which is shattering and has height at most* κ, *then there is a dense* tree of height at most* κ.

The main task in constructing our tree T is to ensure that ω_1-branches are nowhere dense. The key to ensuring this is to ensure that if λ is minimal such that some ω_1-branch is a subset of $V[G_\lambda]$, then the branch generates a $V[G_\lambda]$-remote filter. To do this we use a weak enumeration or $\diamondsuit$-like principle on ω_2.

Following [6], we will call the set $\mathcal{S}$ described next a *self-indexing*, *totally reflecting stationary set*. We will show how it can be usefully regarded as a weak $\diamondsuit$-like principle on ω_2. The "self-indexing" refers to property (1) and the "totally reflecting" refers to (2). It is easily established that $\mathcal{S}$ is a stationary subset of the familiar structure $([\omega_2]^\omega,\subset)$.

DEFINITION 3.3. Let S^2_0 denote the set of limit ordinals in ω_2 which have countable cofinality. A family $\mathcal{S}=\{S_\alpha:\alpha\in S^2_0\}$ is a *self-indexing*, *totally reflecting stationary set* if the following are satisfied:

(1) S_α is a countable cofinal subset of α,

(2) for each λ with uncountable cofinality, $\{S_\alpha:\alpha\in\lambda\cap S^2_0\}$ is stationary in $[\lambda]^\omega$.

Here is why we call this a *weak diamond principle*.

PROPOSITION 3.4. *For each $\lambda \in \omega_2$ with uncountable cofinality and for each ξ in λ, the set $\{d \in \lambda : \xi \in S_d\}$ is stationary in λ.*

Proof. Let $\{Y_\beta : \beta \in \omega_1\}$ be any continuous increasing chain of countable sets whose union is λ and has $\xi \in Y_0$. Notice that $\{Y_\beta : \beta \in \omega_1\}$ is a cub in the structure $([\lambda]^\omega, \subset)$. Define, for $\beta \in \omega_1$, $g(\beta) = \sup(Y_\beta)$. Clearly, g is continuous and so there is a cub $C \subset \omega_1$ such that g restricted to C is strictly increasing. Now let $D \subset S^2_0$ be any cub in λ. Notice that $C(D) = \{\beta \in C : g(\beta) \in D\}$ is cub in ω_1 and therefore, $\{Y_\beta : \beta \in C(D)\}$ is also cub in $[\lambda]^\omega$. Finally then, since $\{S_\alpha : \alpha \in \lambda\}$ is stationary in $[\lambda]^\omega$, there is a $\beta \in C(D)$ and an α such that $Y_\beta = S_\alpha$. So, we claim that $\alpha = g(\beta)$ completing the proof. Since $\beta \in C(D)$, we know that $g(\beta)$ has cofinality ω and that Y_β is cofinal in $g(\beta)$. Therefore S_α is cofinal in $g(\beta)$, which means $g(\beta) = \alpha$ since S_α is, by definition, cofinal in α. ∎

The existence of such a stationary set is not a consequence of ZFC but it is an interesting and useful principle nonetheless. In the current situation we are using a proper countable support iteration of length ω_2 hence we can easily deduce the existence of our stationary set in the extension. Every such, non-trivial, iteration will contain a completely embedded copy of the usual countable condition Cohen poset $\mathrm{Fn}(\omega_2, 2, \omega_1)$. It is not difficult to see that $\mathrm{Fn}(\omega_2, 2, \omega_1)$ will add a self-indexing totally reflecting stationary set. In addition, the properties of a self-indexing totally reflecting stationary set are preserved by any poset which preserves ω_2 and stationary subsets of ω_1. However, we do not know if every finite support iteration of length ω_2 introduces one. In any case, we may assume that there is such a set in $V[G]$.

For the remainder we work in $V[G]$.

THEOREM 3.5. *There exists, in $V[G]$, a shattering tree*for* r.o.$(\mathbb{R})$ *of height ω_1.*

Proof. Let $\{x_\xi : \xi \in \omega_2\}$ be an enumeration (repetitions allowed) of r.o.$(\mathbb{R})$ such that r.o.$(\mathbb{R}) \cap V[G_\alpha]$ is contained in $\{x_\xi : \xi < \alpha + \omega_1\}$ for each α. This is easily done since r.o.$(\mathbb{R}) \cap V[G_\alpha]$ has cardinality $\aleph_1$ for each $\alpha < \omega_2$.

We fix a self-indexing totally reflecting stationary set, $\mathcal{S} = \{S_\alpha : \alpha \in \omega_2,\ \mathrm{cf}(\alpha) = \omega\}$, and we construct our tree* $T \subset$ r.o.$(\mathbb{R})$. Although it is not strictly needed, for technical convenience in the definition of T, let $S_\lambda = \emptyset$ for each $\lambda < \omega_2$ of uncountable cofinality. For each $t \in$ r.o.$(\mathbb{R})$, let β_t denote the minimum ordinal such that $t \in V[G_{\beta_t}]$.

The construction is remarkably simple. Let T be any maximal subset of r.o.$(\mathbb{R})$ with the following three properties:

1. $\mathbb{R}$ is a member of T,

2. for $t \in T$, β_t is greater than β_s for all $s \in T$ with $t \subset^* s$,

3. for $t \in T$, let $\gamma_t = \sup\{\beta_s : s \in T \text{ and } t \subset^* s\}$; then for each $\xi \in S_{\gamma_t}$, the closure of t is contained in one of $\{x_\xi, \mathbb{R} \setminus \overline{x}_\xi\}$.

The main point to the proof, then, is the following lemma.

LEMMA 3.6. *Each* ω_1*-branch of* T *is nowhere dense**.

Proof. Suppose that $\{t_\alpha : \alpha \in \omega_1\}$ is an ω_1-branch of T. For each $\alpha \in \omega_1$, let $\beta_\alpha = \beta_{t_\alpha}$ and recall that this forms a strictly increasing sequence in ω_2. Let λ denote the supremum, and for each limit α, let γ_α denote the supremum of $\{\beta_\zeta : \zeta < \alpha\}$. Observe that $\{\gamma_\alpha : \lim(\alpha) \text{ and } \alpha \in \omega_1\}$ is a closed and unbounded subset of λ.

By Theorem 2.3, it suffices to show that $\{t_\alpha : \alpha \in \omega_1\}$ is a $V[G_\lambda]$-remote filter. So, let $U \subset \mathbb{R}$ be any dense open set which is a member of $V[G_\lambda]$. Clearly for each interval $(a, b) \subset \mathbb{R}$, there are disjoint open subsets of (a, b) each of which has both a and b as limit points. Therefore, there is a regular open set $x \in V[G_\lambda]$ such that $x \subset U$ and the boundary of x contains the boundary of U. By Proposition 3.1 and the assumption on our enumeration, there is a $\xi < \lambda$ such that $x = x_\xi$. By Proposition 3.4, there is a limit $\alpha < \omega_1$ such that $\xi \in S_{\gamma_\alpha}$. By the property of T, it follows that the closure of t_α is either contained in x_ξ or contained in $\mathbb{R} \setminus x_\xi$. Clearly then the closure of t_α does not hit the boundary of x_ξ nor the boundary of U. ∎

Clearly it follows from Lemma 3.6 that the height of T is at most ω_1. Now, to finish the proof of Theorem 3.5, we must show that T is shattering. Let $w \in \text{r.o.}(\mathbb{R})$ and assume that there is no pair of incomparable members of T which meet* w. Therefore the set $C = \{s \in T : s \cap w \neq^* \emptyset\}$ is a chain. Suppose there is some $s \in C$ such that $w \setminus s$ is unbounded. Let s be maximal; hence $w' \subset^* s'$ for each $s' \supset^* s$ and w' is disjoint* from each $t \in T$. Therefore, if we replace w by w' we can assume that the chain C is countable. On the other hand, if $w \setminus s$ is bounded for each $s \in C$, then w diagonalizes C. By Lemma 3.6, it again follows that C is countable. Also, in this case, by replacing w by any $w' \subset w$ such that $w' \subset^* (w \cap s)$ for each member, s, of the countable chain C, we can assume that $w \subset^* s$ for each $s \in C$.

Let γ be the supremum of $\{\beta_s : s \in C\}$ and again use the countable completeness of $\text{r.o.}(\mathbb{R})$ to deduce that there is an unbounded $w' \subset w$ such that w' refines $\{x_\xi, \mathbb{R}\setminus x_\xi\}$ for each $\xi \in S_\gamma$. Finally, choose $t \subset w'$ such that β_t is greater than γ. It follows easily that $T \cup \{t\}$ has all the properties required of T (except possibly the maximality), and this, of course, contradicts the maximality of T. Thus we conclude that T is shattering and this completes the proof of the theorem. ∎

4. The existence of self-indexing stationary sets. Recall that a $\square$-*sequence* is a sequence $\{C_\alpha : \alpha \in \omega_2\}$ such that for each limit α, C_α is a closed and unbounded subset of α and if α has cofinality ω, then C_α is countable. Furthermore, if α is a limit point in C_λ, then C_α is equal to $C_\lambda \cap \alpha$. It follows that for λ with uncountable cofinality, C_λ has order type ω_1.

Todorčević has constructed his well-known and important ϱ-functions ([11]) from $\square$ and as he informed the author, a self-indexing totally reflecting set can be constructed quite directly from such a ϱ-function.

Here is a simple direct construction from the square sequence. For each δ in ω_2, let $<_\delta$ denote a well-ordering of δ in order type at most ω_1. Define S_μ for $\mu < \omega_2$ with countable cofinality inductively as follows. If the limits of C_μ, denoted C'_μ, are cofinal in μ, then set $S_\mu = \bigcup\{S_\delta : \delta \in C'_\mu\}$. Otherwise, for each $\delta \in C'_\mu$, let $\xi(\delta)$ denote the $<_\delta$-minimal element of δ which is not in S_γ for any $\gamma \in C'_\mu$, and set

$$S_\mu = C_\mu \cup \bigcup\{S_\delta \cup \{\xi(\delta)\} : \delta \in C'_\mu\}.$$

For λ with uncountable cofinality, it is easy to see that $\{S_\delta : \delta \in C'_\lambda\}$ is a continuous increasing sequence. It is also easy to see that every member of λ is in the union. This means that this set is actually a cub in $[\lambda]^\omega$.

PROPOSITION 4.1. *Martin's Maximum implies that there is no self-indexing totally reflecting stationary set.*

Proof. Assume that S_μ is a countable subset of μ for each $\mu \in S^2_0$. For each $\gamma \in \omega_2$, let $T_\gamma = \{\mu : \gamma \in S_\mu\}$. Notice that $\{T_\gamma : \gamma \in \omega_2\}$ is a point-countable family. There are at most countably many of the T_γ which contain the intersection of some cub with S^2_0 since the intersection of ω_1 many cub's in ω_2 is again a cub. Fix any γ such that T_γ does not contain such a cub, i.e. $S^2_0 \setminus T_\gamma$ is stationary. Martin's Maximum implies that every stationary subset of S^2_0 contains a closed copy of ω_1, hence there is some λ with uncountable cofinality such that $S^2_0 \setminus T_\gamma$ contains a cub C in λ. It follows then that γ is not in S_μ for all $\mu \in C$, which implies that $\{S_\delta : \delta \in \lambda\}$ is not stationary in $[\lambda]^\omega$. ∎

It is not known if ZFC implies the weaker principle obtained by replacing "all λ with cofinality ω_1" by "a stationary set of λ with cofinality ω_1". This principle is discussed in [6] and such a set is called a self-indexing reflecting stationary set.

References

[1] B. Balcar, J. Pelant, and P. Simon, *The space of ultrafilters on N covered by nowhere dense sets*, Fund. Math. 110 (1980), 11–24.

[2] J. Baumgartner, *Iterated forcing*, in: Surveys in Set Theory, A. R. D. Mathias (ed.), Cambridge Univ. Press, 1983, 1–59.

[3] P. L. Dordal, *A model in which the base-matrix tree cannot have cofinal branches*, J. Symbolic Logic 52 (1987), 651–664.

[4] A. Dow, *Tree π-bases for $\beta N - N$ in various models*, Topology Appl. 33 (1989), 3–19.

[5] —, *Set theory in topology*, in: Recent Progress in General Topology, M. Hušek and J. van Mill (eds.), Elsevier, 1992, 169–197.

[6] A. Dow and J. Zhou, *On subspaces of pseudoradial spaces*, Proc. Amer. Math. Soc., to appear.

[7] M. Goldstern, *Tools for your forcing construction*, in: Set Theory of the Reals (Ramat Gan, 1991), Israel Math. Conf. Proc., 6, Bar-Ilan Univ., Ramat Gan, 1993, 305–360.

[8] S. Shelah, *Proper Forcing*, Lecture Notes in Math. 940, Springer, 1982.

[9] —, *On cardinal invariants of the continuum*, in: Axiomatic Set Theory, Contemp. Math. 31, Amer. Math. Soc., 1984, 183–207.

[10] O. Spinas and S. Shelah, *The distributivity numbers of* $\mathcal{P}(\omega)/\mathrm{fin}$ *and its squares*, Number 494 of Shelah Archives.

[11] S. Todorčević, *Partitioning pairs of countable ordinals*, Acta Math. 159 (1987), 261–294.

[12] E. K. van Douwen, *Transfer of information about $\beta\mathbf{N} - \mathbf{N}$ via open remainder maps*, Illinois J. Math. 34 (1990), 769–792.

[13] J. van Mill and S. W. Williams, *A compact F-space not co-absolute with $\beta\mathbb{N}-\mathbb{N}$*, Topology Appl. 15 (1983), 59–64.

Department of Mathematics
York University
North York, Ontario
Canada M3J 1P3
E-mail: dowa@mathstat.yorku.ca

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