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Strongly meager sets and subsets of the plane

by

Janusz Pawlikowski (Wrocław)

Abstract. Let $X \subseteq 2^\omega$. Consider the class of all Borel $F \subseteq X \times 2^\omega$ with null vertical sections F_x, $x \in X$. We show that if for all such F and all null $Z \subseteq X$, $\bigcup_{x\in Z} F_x$ is null, then for all such F, $\bigcup_{x\in X} F_x \neq 2^\omega$. The theorem generalizes the fact that every Sierpiński set is strongly meager and was announced in [P].

A *Sierpiński set* is an uncountable subset of 2^ω which meets every null (i.e., measure zero) set in a countable set. Such sets may not exist, but they do, e.g., under the Continuum Hypothesis.

A *strongly meager* set is a subset of 2^ω whose complex algebraic sum with null sets cannot give 2^ω.

Answering a question of Galvin I proved in [P] that every Sierpiński set is strongly meager (see [M] and [P] for more about Galvin's question). Since Sierpiński sets may not exist, this result is somewhat defective. Here we prove its "absolute" version, which seems to be of independent interest. The paper is an elaboration of the Note given at the end of [P]. A different elaboration, using "small sets" of [B] and closely following my lecture at Cantor's Set Theory meeting, Berlin 1993, is given in [BJ] in Section 5 (repeated in [BJ1]). There is, however, a major gap in the exposition in [BJ], namely, in the proof of Lemma 5.5, where one really needs a sort of Kunugui–Novikov theorem (see the proof of Lemma 4 below). Also it seems reasonable to avoid "small sets" because they do not form an ideal.

Let $X \subseteq 2^\omega$. Consider the class of all Borel $F \subseteq X \times 2^\omega$ with null vertical sections F_x, $x \in X$. If for all such F, $\bigcup_{x\in X} F_x$ is null, resp. $\neq 2^\omega$, we say that $X \in \mathtt{Add}$, resp. $X \in \mathtt{Cov}$. (See [PR] for an explanation of this notation.)

Here Borel means relatively Borel. It is useful to remember that for a Borel $F \subseteq X \times 2^\omega$, the function $x \mapsto \mu(F_x)$ is Borel. In particular, any Borel

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subset of $Y \times 2^\omega$, $Y \subseteq X$, with all vertical sections null, extends to a Borel subset of $X \times 2^\omega$ with all vertical sections null.

THEOREM. *Suppose that every null subset of $X \subseteq 2^\omega$ is in* `Add`. *Then $X \in$* `Cov`.

The theorem implies that every Sierpiński set is strongly meager as follows. Suppose $X \subseteq 2^\omega$ is a Sierpiński set. Let $D \subseteq 2^\omega$ be null. We want to see that $D + X \neq 2^\omega$. Consider

$$F = \bigcup_{x \in X} \{x\} \times (D + x).$$

Clearly F is a Borel subset of $X \times 2^\omega$ and all its vertical sections are null. Also, every null subset of X, being countable, is in `Add`. So, by the theorem, $\bigcup_{x\in X} F_x \neq 2^\omega$. But $D + X = \bigcup_{x \in X} F_x$.

NOTATION. Given a set K and $A \subseteq 2^K$, let $\mu(A)$ be the measure of A in the product measure arising from assigning to each point in $\{0, 1\}$ weight $1/2$. Note that if K is finite, then $\mu(A) = |A| \cdot 2^{-|K|}$.

For $K \subseteq L \subseteq \omega$ and $A \subseteq 2^L$ let $[A] = \{t \in 2^\omega : t|L \in A\}$ and let $A|K = \{t|K : t \in A\}$. Likewise, for $A \subseteq (2^L)^k$ let $[A] = \{\langle t_1, \ldots, t_k\rangle \in (2^\omega)^k : \langle t_1|L, \ldots, t_k|L\rangle \in A\}$.

Clearly $\mu(A) = \mu([A])$. Note that any clopen subset of 2^ω can be written as $[A]$ for some $A \subseteq 2^n$. Also for $A \subseteq 2^n$ and $m > n$, $[A] = [B]$, where $B = \{\tau \in 2^m : \tau|n \in A\}$.

For $\sigma \in \omega^{<\omega}$ of length $n + 1$, let σ^* be $\sigma|n$.

We use the following abbreviations:

$\exists^\infty$ — there exist infinitely many,
$\forall^\infty$ — for all but finitely many,
$\bigvee_n$ — $\bigcap_m \bigcup_{n>m}$,
$\bigwedge_n$ — $\bigcup_m \bigcap_{n>m}$.

For $F \subseteq X \times T$ and $x \in X$ let $F_x = \{t \in T : \langle x, t\rangle \in F\}$. Likewise, for $F \subseteq X \times S \times T$, $x \in X$, $s \in S$, let $F_{xs} = \{t \in T : \langle x, s, t\rangle \in F\}$, etc. In particular, if $F \subseteq X \times (2^\omega)^\omega$, $t_0, \ldots, t_n \in 2^\omega$, then

$$F_{xt_0\ldots t_n} = \{\langle t_{n+1}, \ldots\rangle \in (2^\omega)^\omega : \langle x, t_0, \ldots, t_n, t_{n+1}, \ldots\rangle \in F\}.$$

Let $F[X] = \bigcup_{x\in X} F_x$. If F has all sections F_x, $x \in X$, null, we say that F is X-*null*.

The following simple lemma is crucial.

LEMMA 1. *Let every null subset of X be in* `Add`. *Suppose $F \subseteq X \times (2^\omega)^\omega$ is Borel and X-null. Then, given null $Y \subseteq X$, there exist $t \in 2^\omega$ and Borel null $Z \subseteq X \setminus Y$ such that F_{xt}, $x \notin Z$, are null.*

P r o o f. Let
$$G = \{\langle x,t\rangle \in X \times 2^\omega : \mu(F_{xt}) > 0\}.$$
Then G is Borel and X-null. By Fubini's theorem find $t \in 2^\omega \setminus G[Y]$ such that
$$Z = \{x : \langle x,t\rangle \in G\}$$
is null. (This is possible because $G[Y]$ is null.) ■

We shall need the following property $\mathtt{B}$, which is a Borel version of property H of Hurewicz (for more see [FM], [PR]):

$X \subseteq 2^\omega$ has *property* $\mathtt{B}$ if, given for each $n \in \omega$ a Borel cover $\{U^n_k\}_{k\in\omega}$ of X, there exist k_n's such that $X \subseteq \bigwedge_n \bigcup_{k\le k_n} U^n_k$.

It is not hard to see that we can use increasing covers in this definition and write $X \subseteq \bigwedge_n U^n_{k_n}$. Also, easily, X has property $\mathtt{B}$ iff for any Borel function $f : X \to \omega^\omega$, $f[X]$ is *dominated*. ($Y \subseteq \omega^\omega$ is *dominated* if there exist $z \in \omega^\omega$ such that $\forall^\infty n\ y(n) < z(n)$, for all $y \in Y$.) Moreover, it is enough to consider only f for which all $f(x)$ are increasing.

The following lemmas are well known.

LEMMA 2. *If all null subsets of* $X \subseteq 2^\omega$ *have property* $\mathtt{B}$, *then* X *has property* $\mathtt{B}$.

P r o o f. Let $\{U^n_k\}_{k\in\omega}$, $n \in \omega$, be increasing Borel covers of X. Find k_n's with $\mu^*(X \setminus U^n_{k_n}) < 2^{-n}$. Let $Z = X \setminus \bigwedge_n U^n_{k_n}$. Then Z is null, so it has property $\mathtt{B}$. Thus $Z \subseteq \bigwedge_n U^n_{l_n}$ for some l_n's. It follows that $X \subseteq \bigwedge_n U^n_{\max(k_n,l_n)}$. ■

LEMMA 3. (1) *If* $A \subseteq 2^\omega$ *is null then for any sequence* $\{\varepsilon_n\}$ *of positive reals there exists an increasing sequence* $\{a_n\} \in \omega^\omega$ *together with sets* $B_n \subseteq 2^{a_n}$ *of measure* $\le \varepsilon_n$, $n \in \omega$, *such that* $A \subseteq \bigvee_n [B_n]$.

(2) *If* $a_n \in \omega$ *and* $B_n \subseteq 2^{a_n}$, $n \in \omega$, *are such that* $\sum_n \mu(B_n) < \infty$, *then* $A = \bigvee_n [B_n]$ *is null. If moreover* $K \subseteq \omega$ *is such that* $\sum_n \mu(B_n)\cdot 2^{|a_n\cap K|} < \infty$, *then also* $A|(\omega \setminus K)$ *is null* (*in* $2^{\omega\setminus K}$).

P r o o f. We prove the first part, the second is straightforward. Given null $A \subseteq 2^\omega$ and $\varepsilon > 0$, we can cover A by an open set of measure $< \varepsilon/2^n$, which next can be split into disjoint clopen sets. In this way we can find clopens C_i, $i \in \omega$, such that $A \subseteq \bigvee_i C_i$ and $\sum_i \mu(C_i) < \varepsilon$.

Suppose now that $\{\varepsilon_n\}$ is a sequence of positive reals. Use the above to find clopens C_i, $i \in \omega$, such that $A \subseteq \bigvee_i C_i$ and $\sum_i \mu(C_i) < \varepsilon_0$. Next find an increasing sequence $\{i_n\}$ such that $\sum_{i\ge i_n} \mu(C_i) < \varepsilon_{n+1}$. Finally, let $A_0 = \bigcup_{i<i_0} C_i$ and for $n > 0$ let $A_n = \bigcup_{i_{n-1}\le i<i_n} C_i$. Then $\mu(A_n) < \varepsilon_n$ and $A \subseteq \bigvee_n A_n$. Each A_n, being clopen, is of the form $[B_n]$ for some $B_n \subseteq 2^{a_n}$. We can easily arrange that $a_{n+1} > a_n$. ■

It follows from Lemma 3 that, given $\{b_n\} \in \omega^\omega$ and null $A \subseteq 2^\omega$, there is an increasing $\{a_n\} \in \omega^\omega$ such that

$$\forall^\infty n\ |a_n \cap K| \le b_n \ \Rightarrow\ A|(\omega \setminus K) \text{ is null.}$$

LEMMA 4. *Suppose that* $F \subseteq X \times 2^\omega$ *is Borel* X*-null. Let* $\{\varepsilon_n\}$ *be a sequence of positive reals. Then there exist for each* n *a countable Borel partition* $\mathcal{U}_n$ *of* X *together with integers* a_U^n *and sets* $A_U^n \subseteq 2^{a_U^n}$ *of measure* $\le \varepsilon_n$, $U \in \mathcal{U}_n$, *such that*

$$F \subseteq \bigvee_n \bigcup_{U \in \mathcal{U}_n} U \times [A_U^n].$$

If additionally X *has property* $\mathtt{B}$, *we can require that for some increasing* $\{a_n\} \in \omega^\omega$ *all* a_U^n, $U \in \mathcal{U}_n$, *equal* a_n.

Proof. This is a parametrized version of the first part of Lemma 3. We indicate the main steps.

For any $\varepsilon > 0$ there are Borel sets W_i and clopens C_i, $i \in \omega$, such that $F \subseteq \bigvee_i W_i \times C_i$ and for all $x \in X$, $\sum_{i \in K(x)} \mu(C_i) < \varepsilon$, where $K(x) = \{i : x \in W_i\}$.

This follows from the following facts:

• any Borel subset of $X \times 2^\omega$ with open vertical sections can be written as a union of countably many disjoint sets of the form $W \times C$, W Borel, C clopen (a theorem of Kunugui and Novikov, see [K]);

• for any Borel $B \subseteq X \times 2^\omega$ and $\varepsilon > 0$, there exists a Borel $A \subseteq X \times 2^\omega$, $B \subseteq A$, such that vertical sections of A are open and $\mu(A_x \setminus B_x) < \varepsilon$ (the sets with such a covering property form a monotone family that includes all finite unions of Borel rectangles).

Note that the function $x \mapsto \sum_{i \in K(x) \setminus j} \mu(C_i)$ is Borel. (Because the function $\mu(C_i)\mathbf{1}_{W_i}$ that takes $\mu(C_i)$ on W_i and 0 outside is Borel and $\sum_{i \in K(x) \setminus j} \mu(C_i) = \sum_{i \ge j} \mu(C_i)\mathbf{1}_{W_i}(x)$.) It follows that for any $\delta > 0$ we can find a countable Borel partition $\mathcal{U}$ of X and numbers $j_U \in \omega$, $U \in \mathcal{U}$, such that on each U the mapping $x \mapsto K(x) \cap j_U$ is constant and $\sum_{i \in K(x) \setminus j_U} \mu(C_i) < \delta$.

Using this find Borel sets U_σ and integers j_σ, $\sigma \in \omega^{<\omega}$, such that

• $U_\emptyset = X$, $j_\emptyset = 0$,
• U_σ is partitioned into $U_{\sigma^\frown k}$'s, $k \in \omega$,
• if $|\sigma| > 0$, then $j_{\sigma^*} < j_\sigma$ and $x \mapsto K(x) \cap [j_{\sigma^*}, j_\sigma)$ is constant on U_σ,
• if $|\sigma| > 0$, then on U_σ,

$$\sum_{i \in K(x) \cap [j_{\sigma^*}, j_\sigma)} \mu(C_i) < \varepsilon_{|\sigma|-1}.$$

If $|\sigma| > 0$, let

$$B_\sigma = \bigcup_{i \in K(x) \cap [j_{\sigma^*}, j_\sigma)} C_i, \quad x \in U_\sigma.$$

This is a clopen set of measure $< \varepsilon_{|\sigma|-1}$, so we can find $a_\sigma \in \omega$ and $A_\sigma \subseteq 2^{a_\sigma}$ of measure $< \varepsilon_{|\sigma|-1}$ such that $B_\sigma = [A_\sigma]$.

Now just note that

$$\bigvee_i W_i \times C_i \subseteq \bigvee_{n>0} \bigcup \{U_\sigma \times [A_\sigma] : \sigma \in \omega^n\}.$$

Up to some enumeration, we are done. ∎

The following lemma is a version of Miller's [M1] result that additivity of measure is below number **b**. (See also [PR].)

Lemma 5. $\mathtt{Add} \subseteq \mathtt{B}$.

Proof. Let $Y \in \mathtt{Add}$. Let $Y \ni y \mapsto \overline{y} \in \omega^\omega$ be Borel with all $\overline{y}$'s increasing. Define $F \subseteq Y \times 2^\omega$ by

$$t \in F_y \Leftrightarrow \exists^\infty n \ \forall i < n \ t(\overline{y}(n) + i) = 0.$$

Then F is Borel and Y-null, so $A = F[Y]$ is null. Use Lemma 3 to find an increasing sequence $\{a_n\}$ such that

$$\forall n \ |K \cap a_n| \le n(n-1)/2 \Rightarrow A|(\omega \setminus K) \text{ is null.}$$

We claim that $\{a_n\}$ dominates all $\overline{y}$'s. Indeed, suppose that $\exists^\infty n \ \overline{y}(n) \ge a_n$. Consider

$$K = \bigcup \{[\overline{y}(n), \overline{y}(n) + n) : \overline{y}(n) \ge a_n\}.$$

Then $\forall n \ |K \cap a_n| \le n(n-1)/2$ (we take to K below a_n at most $n-1$ intervals).

It follows that $A|(\omega \setminus K)$ is null. This is a contradiction because $F_y \subseteq A$ and $F_y|(\omega \setminus K)$ is $2^{\omega \setminus K}$. (Any element of $2^{\omega \setminus K}$ can be extended to an element of 2^ω which on infinitely many intervals $[\overline{y}(n), \overline{y}(n)+n)$ is constantly zero.) ∎

Proof of theorem. By Lemmas 2 and 5, $X \in \mathtt{B}$. Let $F \subseteq X \times 2^\omega$ be Borel X-null. We seek a point outside $F[X]$. Let $\mathbf{Q} = \{t \in 2^\omega : \forall^\infty n \ t(n) = 0\}$. Enlarging F if necessary we can assume that for all x, $F_x = F_x + \mathbf{Q}$. Use Lemma 4 to find an increasing $\{a_n\} \in \omega^\omega$ together with a sequence $\{\mathcal{U}_n\}$ of countable Borel partitions of X such that for some $A^n_U \subseteq 2^{a_n}$, $U \in \mathcal{U}_n$, of measure $\le 2^{-n}$,

$$F \subseteq \bigvee_n \bigcup_{U \in \mathcal{U}_n} U \times [A^n_U].$$

Let B^n_x be A^n_U for the unique $U \in \mathcal{U}_n$ that covers x.

Say that $\sigma_0, \ldots, \sigma_k \in 2^{a_n}$ *have a diagonal* in $A \subseteq 2^{a_n}$ if for some $n_0 \leq \ldots \leq n_{k-1} \leq n$,

$$\sigma_0|a_{n_0} \cup \sigma_1|[a_{n_0}, a_{n_1}) \cup \ldots \cup \sigma_k|[a_{n_{k-1}}, a_n) \in A.$$

Say that $t_0, \ldots, t_k \in 2^\omega$ have a diagonal in $A \subseteq 2^{a_n}$ if $t_0|a_n, \ldots, t_k|a_n$ do.

Define $E \subseteq X \times (2^\omega)^\omega$ by

$$\langle t_0, t_1, \ldots \rangle \in E_x \Leftrightarrow \exists k \; \exists^\infty n \;\; t_0, \ldots, t_k \text{ have a diagonal in } B_x^n.$$

Claim 1. *E is Borel and X-null.*

Proof. Let

$$B_x^n(k) = \{\langle \sigma_0, \ldots, \sigma_k \rangle \in (2^{a_n})^{k+1} : \sigma_0, \ldots, \sigma_k \text{ have a diagonal in } B_x^n\}.$$

Then $|B_x^n(k)| \leq 2^{a_n - n} 2^{a_n k} \cdot (1+n)^k$. Indeed, there are $\leq (1+n)^k$ possible sequences $n_0, \ldots, n_{k-1}$, and for each sequence we have $|B_x^n|$ times $2^{a_n k}$ possible choices for $\langle \sigma_0, \ldots, \sigma_k \rangle$.

So

$$\mu([B_x^n(k)]) = |B_x^n(k)|/2^{a_n(k+1)} \leq 2^{-n}(1+n)^k.$$

It follows that

$$\mu(E_x) \leq \sum_k \prod_m \sum_{n \geq m} 2^{-n}(1+n)^k = 0. \;\blacksquare$$

Claim 2. *There exist $\{t_i\} \subseteq 2^\omega$ and a Borel partition $\{X_i\}$ of X such that each*

$$E_{x t_i \ldots t_k}, \quad x \in X_i, \; k \geq i,$$

is null.

Proof. Apply Lemma 1 with $Y = \emptyset$ and $F = E$ to find $t_0 \in 2^\omega$ and Borel null $X_1 \subseteq X$ such that the following sets are null:

$$\begin{array}{ll} E_x, & x \in X_1, \\ E_{x t_0}, & x \notin X_1. \end{array}$$

Next apply Lemma 1 to $Y = X_1$ and

$$F = \bigcup_{x \in X_1} \{x\} \times E_x \;\cup\; \bigcup_{x \notin X_1} \{x\} \times E_{x t_0}$$

to get $t_1 \in 2^\omega$ and Borel null $X_2 \subseteq X \setminus X_1$ such that the following sets are null:

$$\begin{array}{ll} E_x, & x \in X_2, \\ E_{x t_1}, & x \in X_1, \\ E_{x t_0 t_1}, & x \notin X_1 \cup X_2. \end{array}$$

Similarly find $t_2 \in 2^\omega$ and Borel null $X_3 \subseteq X \setminus (X_1 \cup X_2)$ such that the following sets are null:

$$\begin{array}{ll} E_x, & x \in X_3, \\ E_{xt_2}, & x \in X_2, \\ E_{xt_1t_2}, & x \in X_1, \\ E_{xt_0t_1t_2}, & x \notin X_1 \cup X_2 \cup X_3, \end{array}$$

etc. Finally, set $X_0 = X \setminus \bigcup_{i>0} X_i$. ∎

It follows from Claim 2 that for $x \in X_i$,

$$\langle t_i, t_{i+1}, \ldots \rangle \notin E_x.$$

Otherwise we would have $E_{xt_i\ldots t_k} = (2^\omega)^\omega$ for some k.

Thus for $x \in X_i$ and $k \geq i$,

$$\forall^\infty n \; t_i, \ldots, t_k \text{ have no diagonal in } B_x^n.$$

For all k and n let

$$V_n^k = \bigcup_{i>k} X_i \cup \bigcup_{i\leq k} \{x \in X_i : \forall m \geq n \; t_i, \ldots, t_k \text{ have no diagonal in } B_x^m\}.$$

Then for all k, V_n^k's form an increasing Borel cover of X. By $X \in \mathtt{B}$, there is an increasing sequence $\{n_k\}$ such that

$$X \subseteq \bigwedge_k V_{n_k}^k.$$

Let

$$t = t_0|a_{n_0} \cup t_1|[a_{n_0}, a_{n_1}) \cup t_2|[a_{n_1}, a_{n_2}) \cup \ldots.$$

Claim 3. $t \notin F[X]$.

Proof. Fix $x \in X_i$. Since $\forall^\infty k \; x \in V_{n_k}^k$, for all sufficiently large $k \geq i$ and $n \geq n_k$, $t_i, \ldots, t_k$ have no diagonal in B_x^n. Hence,

$$t_i|a_{n_i} \cup t_{i+1}|[a_{n_i}, a_{n_{i+1}}) \cup \ldots \cup t_k|[a_{n_{k-1}}, a_n) \notin B_x^n,$$

and thus

$$t_i|a_{n_i} \cup \bigcup_{k>i} t_k|[a_{n_{k-1}}, a_{n_k}) \notin \bigvee_n [B_x^n].$$

It follows that $t \notin F_x$. ∎

Note. We have really proved that if $X \in \mathtt{B}$ has all its null subsets in $\mathtt{Cov}$, then $X \in \mathtt{Cov}$. The crucial Lemma 1 goes through because if $Y \in \mathtt{Cov}$, then for all Borel Y-null $F \subseteq Y \times 2^\omega$, $\mu^*(2^\omega \setminus F[Y]) = 1$. (Otherwise we could find in $F[Y]$ a perfect set P of positive measure. Then $D = F \cap (Y \times P)$ would be a Borel Y-null subset of $Y \times P$ such that $D[Y] = P$. This would yield a similar subset of $Y \times 2^\omega$.)

Note also that if $X \in \mathtt{B} \cap \mathtt{Cov}$, then $2^\omega \setminus F[X]$ contains a perfect set for all X-null $F \subseteq X \times 2^\omega$. (It is enough to require in Lemma 4 that A^n_U's have measure $\le 2^{-2n}$ and consider $B^n_U = \{\sigma \in 2^{a_n} : \exists \tau \in A^n_U\ \sigma|K = \tau|K\}$, where K is a fixed co-infinite subset of ω such that $\forall n\ |a_n \setminus K| \le n$. Then B^n_U is a subset of 2^{a_n} of measure $\le 2^{-n}$. If $X \times \{t\}$ avoids $\bigvee_n \bigcup_U U \times [B^n_U]$, then $\{s \in 2^\omega : s|K = t|K\}$ is a perfect set disjoint from $F[X]$.)

We cannot drop $\mathtt{B}$ in the above remark. If we add ω_2 random reals to a model of CH then the ground model reals constitute a counterexample. (Use the fact that a random real does not add a perfect set of random reals.)

We cannot require in the theorem that $X \in \mathtt{Add}$. It is enough to take for X a Sierpiński set and for F the diagonal in $X \times X$. There is however no ZFC example for this. Indeed, suppose the Dual Borel Conjecture holds, i.e. all strongly meager sets, hence also all $\mathtt{Cov}$ sets, are countable. Suppose also that every uncountable set has an uncountable null subset. (Both assumptions are true when ω_2 Cohen reals are added to a model of CH, see [C].) If all null subsets of X are in $\mathtt{Cov}$, then they are all countable by the first assumption. So X has no uncountable null subsets, and thus X itself is countable by the second assumption. It follows that $X \in \mathtt{Add}$.

Suppose that all null subsets of X are in $\mathtt{Cov}$. Does it follow that X is strongly meager? We have the following partial result:

PROPOSITION. *Let X have property* $\mathtt{B}$. *Let $D = \bigvee_k [B_k]$, $B_k \subseteq 2^{L_k}$, where $L_k \subseteq \omega$, $k \in \omega$, are pairwise disjoint. Suppose for every finite $F \subseteq 2^\omega$, $D + (X \cap (D + F)) \ne 2^\omega$. Then $D + X \ne 2^\omega$.*

Proof. Choose $t_0 \in 2^\omega$ and inductively $t_n \in 2^\omega$ so that

$$t_n \notin D + (X \cap (D + \{t_0, t_1, \ldots, t_{n-1}\})).$$

Then for all $x \in X$,

$$\forall^\infty n\ x \notin D + t_n.$$

Indeed, if $x \in D + t_n$ and $m > n$, then $t_m \notin D + x$, so $x \notin D + t_m$.

Let

$$U^n_k = \{x : x \notin D + t_n \Rightarrow \forall m \ge k\ x|L_m \notin B_m + t_n|L_m\}.$$

Then for all n, U^n_k's form an increasing Borel cover of X. So $X \subseteq \bigwedge_n U^n_{k_n}$ for some increasing sequence $\{k_n\}$. Then for all x,

$$\forall^\infty n\ \forall k \ge k_n\ x|L_k \notin B_k + t_n|L_k$$

(remember that $\forall^\infty n\ x \notin D + t_n$).

So for all x,

$$\forall^\infty n\ \forall k \ge k_n\ t_n|L_k \notin B_k + x|L_k.$$

Let $t \in 2^\omega$ be such that for all n,

$$\bigcup_{k_n \le k < k_{n+1}} t_n | L_k \subseteq t.$$

Then for all x,

$$\forall^\infty n \ \forall k \in [k_n, k_{n+1}) \ t|L_k \notin B_k + x|L_k.$$

It follows that $\forall x \in X \ t \notin D + x$. ■

NOTE. Let small mean small in the sense of Bartoszyński [B]. Suppose that every small subset of X has property $\mathtt{B}$. Suppose also that for every small $Y \subseteq X$ and small $D \subseteq 2^\omega$, $Y + D \neq 2^\omega$. Then for every small $D \subseteq 2^\omega$, $X + D \neq 2^\omega$. (Every null set is a union of two small sets. So, if all small subsets of X have property $\mathtt{B}$, then all null subsets of X have property $\mathtt{B}$, thus X itself has property $\mathtt{B}$. Also, a union of finitely many translates of a small set is small.)

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Department of Mathematics
University of Wrocław
Pl. Grunwaldzki 2/4
50-384 Wrocław, Poland
E-mail: pawlikow@math.uni.wroc.pl

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