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The structure of atoms
(hereditarily indecomposable continua)

by

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Abstract. Let X be an atom (= hereditarily indecomposable continuum). Define a metric ϱ on X by letting $\varrho(x,y) = W(A_{xy})$ where $A_{x,y}$ is the (unique) minimal subcontinuum of X which contains x and y and W is a Whitney map on the set of subcontinua of X with $W(X) = 1$. We prove that ϱ is an ultrametric and the topology of (X,ϱ) is stronger than the original topology of X. The ϱ-closed balls $C(x,r) = \{y \in X : \varrho(x,y) \le r\}$ coincide with the subcontinua of X. ($C(x,r)$ is the unique subcontinuum of X which contains x and has Whitney value r.) It is proved that for any two (nontrivial) atoms and any Whitney maps on them, the corresponding ultrametric spaces are isometric. This implies in particular that the combinatorial structure of subcontinua is identical in all atoms.

The set $M(X)$ of all monotone upper semicontinuous decompositions of X is a lattice when ordered by refinement. It is proved that for two atoms X and Y, $M(X)$ is lattice isomorphic to $M(Y)$ if and only if X is homeomorphic to Y.

1. Introduction. A *continuum* is a compact metrizable connected space. A continuum X is *decomposable* if it is representable as $X = X_1 \cup X_2$ with X_i proper subcontinua of X. All common naturally described continua are decomposable. Brouwer constructed an indecomposable continuum, and Knaster [Kn] constructed a hereditarily indecomposable continuum, i.e., a continuum all of whose subcontinua are indecomposable. Following [Lev-St3] we call such continua *atoms*. Bing [Bi] proved the existence of atoms of all

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dimensions. Krasinkiewicz [Kra] (see also [Lev]) proved that if M is a manifold then for every compact metrizable space X the functions $f : X \to M$ such that for each $m \in M$ every component of $f^{-1}(m)$ is an atom, form a dense G_δ set in the function space $C(X, M)$. This indicates, in particular, that "most" subcontinua of X are atoms.

In recent years atoms have been applied in various cases to solve problems which were originally unrelated to atoms (see [Po1], [St], [Po2], [Lev-St3], [Lev-St4], [Lev]) and a comprehensive text on atoms [Lew] is in preparation.

The usefulness of atoms follows from the rather simple relationships between their subcontinua:

1.1. *If A and B are subcontinua of an atom and $A \cap B \neq \emptyset$, then $A \subset B$ or $B \subset A$.*

(This is immediate—if not, $A \cup B$ would be a decomposable continuum.)

In this article we study the combinatorial structures of the family of subcontinua of an atom X, and that of the upper semicontinuous decompositions of X into continua, denoted by $M(X)$.

It turns out that in spite of the large diversity of atoms, the combinatorial structure of their subcontinua is the same in all of them: given two nontrivial atoms X and Y there exists a one-to-one function f of X onto Y such that both f and f^{-1} carry continua to continua. (This result is due to Nikiel [Ni].) The construction of f and the exhibition of its properties depend on the following consequence of 1.1.

1.2. *Let X be an atom. Then*

(a) *any two points $x, y \in X$ are contained in a unique minimal subcontinuum A_{xy}; and*

(b) *if z is yet another point of X, then two of the continua A_{xy}, A_{xz} and A_{yz} coincide and contain the third.*

For the proof of (b) observe that (by 1.1) the family $\{A_{xy}, A_{xz}, A_{yz}\}$ is nested. Assume e.g. that $A_{xy} \subset A_{xz} \subset A_{yz}$. If $A_{xz} \neq A_{yz}$ then $A_{xz} = A_{xz} \cup A_{xy}$ is a continuum which contains both y and z and hence must contain A_{yz}, a contradiction.

For any Whitney map W (see §2 for a definition) on the set $C(X)$ of all subcontinua of an atom X we let

$$\varrho(x, y) = W(A_{xy}) \quad \text{for } x, y \in X. \tag{$*$}$$

It follows immediately from 1.2 and properties of Whitney maps that ϱ is a metric and

$$\varrho(x, y) \le \max\{\varrho(x, z), \varrho(y, z)\} \quad \text{for } x, y, z \in X. \tag{$**$}$$

We call the metric space (X, ϱ) an *ultrametric atom* associated with the Whitney map W; the name is motivated by the fact that a metric ϱ on a set X which satisfies the strong form $(**)$ of the triangle inequality is called an *ultrametric* ([Sch], [VR]). We prove (see Theorem 2.4) that any two nontrivial ultrametric atoms are isometric and this is the central result of §2; it implies, in particular, that the combinatorial structure of the set of all subcontinua of a nontrivial atom is independent of that atom. Clearly, the isometries between two ultrametric atoms (X,ϱ) and (Y,τ) are in general discontinuous in the original topologies of X and Y (and are of necessity such if X is not homeomorphic to Y).

In §3 we turn to the study of the collection $M(X)$ of upper semicontinuous decompositions of X into subcontinua. We show that, in contrast, the lattice $M(X)$ determines the atom X uniquely (see Theorem 3.11).

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2. The ultrametric atom. Let (X,ϱ) be a metric space. For $x \in X$ and $0 \le r \le 1$, we let $B(x,r) = \{y \in X : \varrho(x,y) < r\}$, $C(x,r) = \{y \in X : \varrho(x,y) \le r\}$ and $S(x,r) = \{y \in X : \varrho(x,y) = r\}$ denote (respectively) the open r-ball, the closed r-ball and r-sphere with center x and radius r. Note that if $r = 0$ then $B(x,r) = \emptyset$ and $C(x,r) = S(x,r) = \{x\}$.

LEMMA 2.1. *Let ϱ be an ultrametric on a set X. Then the following hold* (cf. [Sch], [VR]).

(1) *If $r > 0$ then $B(x,r)$, $C(x,r)$ and $S(x,r)$ are clopen subsets of the metric space (X,ϱ).*

(2) *If $y \in B(x,r)$ then $B(x,r) = B(y,r)$.*

(3) *If $y \in C(x,r)$ then $C(x,r) = C(y,r)$.*

(4) *If two closed balls $C(x,r)$ and $C(y,s)$ intersect and $r \ge s$, then $C(x,r) \supset C(y,s)$.*

(5) *Every family of mutually intersecting closed balls or open balls is nested.*

(6) *For any three points x, y, and z in X, the two largest of the three reals $\varrho(x,y)$, $\varrho(x,z)$ and $\varrho(y,z)$ coincide. That is, every triangle in (X,ϱ) is isosceles and the two equal sides dominate the third side.*

Before continuing, we define Whitney maps and present other general facts that will be applied in the sequel. Let X be a compact metrizable space. Let 2^X denote the space of all nonempty closed subsets of X equipped with the Hausdorff metric and let $C(X) \subset 2^X$ denote the set of its subcontinua.

2.2. A continuous function $W : C(X) \to [0,1]$ is called a *Whitney map* if $W(\{x\}) = 0$ for every $x \in X$, $W(X) = 1$ and $A \subsetneq B \in C(X)$ implies $W(A) < W(B)$. Whitney maps exist for every nontrivial X (see e.g. [Na], or [Lev-St4] for a simple construction).

Let X be an atom and let $x \in X$. The *composant* of x in X is the union of all proper subcontinua of X which contain x. Every composant is the union of countably many continua. (Take, e.g., $C = \bigcup\{A_n : n \geq 1\}$ where A_n is the continuum with $x \in A_n$ and $W(A_n) = 1 - 1/n$, where W is some Whitney map.) As proper subcontinua of atoms have an empty interior (see e.g. [Ho-Yo]) composants are sets of first category in X. The Baire Category theorem thus implies that every atom contains uncountably many composants. Mazurkiewicz [Ma] improved this observation; he proved that every atom X contains a perfect G_δ subset Y which intersects a composant in one point at most. Y must contain a Cantor set and it follows that the cardinality of the set of composants in an atom is $c = 2^{\aleph_0}$.

Proposition 2.3. *Every ultrametric atom* (X, ϱ) *satisfies the following*:

(1) *The identity map* $\mathrm{id} : (X, \varrho) \to X$ *is continuous.*

(2) *For* $0 \leq r \leq 1$ *and* $x \in X$, $C(x, r)$ *is the unique subcontinuum* A *of* X (*with respect to the original topology on* X) *which contains* x *and with* $W(A) = r$. *In particular*, $C(x, 1) = X$.

(3) *For* $r > 0$ *and* $x \in X$, $B(x, r)$ *is the composant of* x *relative to the continuum* $C(x, r)$.

(4) *Every family of mutually intersecting closed balls in* (X, ϱ) *has a nonempty intersection. Moreover*, *if* $\{C(x_\alpha, r_\alpha)\}$ *are mutually intersecting balls*, *then* $\bigcap\{C(x_\alpha, r_\alpha)\} = C(x, r)$ *where* x *is any point in the intersection and* $r = \inf r_a$. (This property is referred to in [Sch] and [VR] as "spherical completeness;" a related property, called "hyperconvexity" was introduced by [Ar-Pa]; in [Li] and [Li-Tz] it is called "the 2-∞ intersection property.")

(5) *For every* $x \in X$ *the range of the function* $\varrho(x, \cdot)$ *is* $[0, 1]$.

(6) *The weight* $\mathrm{wt}(X, \varrho)$ (=*minimal cardinality of a dense set*) *of* (X, ϱ) *is* $c = 2^{\aleph_0}$.

(7) *In every closed ball* $C(x, r)$ *for* $r > 0$ *in* (X, ϱ) *the cardinality of the family of distinct open balls* $\{B(y, r) : y \in C(x, r)\}$ *is* $c = 2^{\aleph_0}$. (Note that by 2.1(5) two such balls are either disjoint or coincide.)

Proof. (1) Let $x_n \to x_0$ in (X, ϱ), i.e., $\varrho(x_0, x_n) = W(A_{x_0 x_n}) \to W(A_{x_0 x_0}) = 0$. Since $x_0 \in A_{x_0 x_n}$ for every n, it follows from 1.1 that the family $\{A_{x_0 x_n}\}$ is nested. Since $W(A_{x_0 x_n}) \to 0$, it also follows that $\bigcap_{n \geq 1} A_{x_0 x_n} = \{x_0\}$. Hence, $\mathrm{diam}(A_{x_0 x_n}) \to 0$. (Here, diam refers to diameter with respect to the original metric d on X.) That is, $d(x_0, x_n) \to 0$.

(5) Let $x \in X$. Then $L(x) = \{A \in C(X) : x \in A\}$ is an arc in $C(X)$ with end points $\{x\}$ and X (see [Ku, p. 186]). Also $W : L(x) \to [0, 1]$ is an embedding. It follows that for each $0 \leq r \leq 1$ there exists a unique $A \in L(x)$ such that $W(A) = r$. Let E denote the composant of x in A. If $r > 0$ then $E \subsetneq A$ (by 2.2). Let $y \in A \setminus E$. It follows that $A = A_{xy}$, and hence that $\varrho(x, y) = r$.

(2) $C(x,r)=\{y\in X:\varrho(x,y)\le r\}=\{y\in X:W(A_{xy})\le r\}$. Let $y\in X$ be such that $\varrho(x,y)=r$. Such a y exists by (5). Then for $A=A_{xy}$ we have $x\in A$ and $W(A)=r$. If some other subcontinuum A' satisfies $x\in A'$ and $W(A')=r$, then, since $A'\subset A$ or $A\subset A'$, we must have $A=A'$ since W is strictly monotone. Now it is clear that $A_{xy}\subset C(x,r)$ and so $W(C(x,r))\ge r$. If $W(C(x,r))>r$, then by (5) there is a $z\in C(x,r)$ with $W(A_{xz})>r$, i.e. $\varrho(x,z)>r$, an impossibility. Thus, if $\varrho(x,y)=r$, then $C(x,r)=A_{xy}$.

(3) $B(x,r)=\{y\in X:\varrho(x,y)<r\}=\bigcup\{C(x,s):s<r\}$. By (2), $\{C(x,s):s<r\}$ ranges over all subcontinua of X which contain x and have Whitney value $<r$, i.e., all proper subcontinua of $C(x,r)$ which contain x. This is precisely the composant of x in $C(x,r)$.

(4) Let $\{C(x_\alpha,r_\alpha)\}$ be a family of mutually intersecting closed balls in (X,ϱ). By 2.1(5) and 2.3(2), $\{A_\alpha=C(x_\alpha,r_\alpha)\}$ is a nested family of continua in X with $W(A_\alpha)=r_\alpha$. By compactness and continuity of W, $A=\bigcap A_\alpha\ne\emptyset$, $W(A)=\inf r_a=r$ and $A=C(x,r)$ for each $x\in A$.

(7) By (2) and (4), $\{B(y,r):y\in C(x,r)\}$ is the set of composants of the continuum $C(x,r)$. By [Ma] (see also [Ku, p. 213, Remark (i)]) the cardinality of this set is $c=2^{\aleph_0}$.

(6) This follows immediately since by (7), (X,ϱ) contains c mutually distinct open balls. But here is a simpler and more elementary argument: for $0<r<1$ the collection $\{C(x,r):x\in X\}$ is a decomposition of X into disjoint continua. It is easy to verify that this decomposition is continuous (with respect to the original atom topology on X). It follows that the quotient map $q:X\to Y$ maps X onto a nontrivial atom Y, and for each $y\in Y$, $q^{-1}(y)=C(x,r)$ for some $x\in X$. As the cardinality of Y is c and the balls $q^{-1}(y)$ are distinct, it follows that $\mathrm{wt}(X,\varrho)\ge c$. (Clearly, $\mathrm{wt}(X,\varrho)\le \operatorname{card} X=c$.) ∎

THEOREM 2.4. *Any two nontrivial ultrametric atoms (X,ϱ) and (Y,τ) are isometric. Moreover, given a subset $X'\subset X$ with $\mathrm{wt}_\varrho(X')<c$ (i.e., the weight of X' as a subspace of (X,ϱ)) and an isometry $f:X'\to(Y,\tau)$, f is extendable to an isometry of (X,ϱ) onto (Y,τ).*

REMARK 2.5. The restriction $\mathrm{wt}_\varrho(X')<c$ is essential.

We demonstrate this by an example. Let X be an atom. Let $E\subset X$ be a subset which consists of a single representative of each composant of X, i.e., for every composant B of X, $B\cap E$ consists of a single point. Let (X,ϱ) be the associated ultrametric atom. Then for $x\ne y\in E$, $\varrho(x,y)=1$. (If $\varrho(x,y)<1$ then x and y are in the same composant of X.) By 2.3(7), $\operatorname{card}(E)=c$ and hence $\mathrm{wt}_\varrho(E)=c$. Let $x_0\in E$ and let $f:E\setminus\{x_0\}\to E$ be one-to-one and onto. Then f is an isometry of $E\setminus\{x_0\}$ onto E. But f is not extendable to an isometry of (X,ϱ) onto itself. As a matter of fact, f is not even extendable over E. Indeed, in order to extend f to x_0, $f(x_0)$

must satisfy $\varrho(f(x), f(x_0)) = 1$ for all $x \in E \setminus \{x_0\}$. But $\varrho(f(x), f(x_0)) = 1$ if and only if $f(x)$ and $f(x_0)$ belong to different composants of X, and since $f(E\setminus\{x_0\}) = E$ and since E meets each composant, there are no composants left in which to put $f(x_0)$.

Before proving Theorem 2.4 we present some corollaries.

COROLLARY 2.6. *Let* (X, ϱ) *and* (Y, τ) *be ultrametric atoms*, *let* $x \in X$, $y \in Y$ *and* $0 \le r \le 1$. *Then* $B_X(x, r)$ *is isometric to* $B_Y(y, r)$, $C_X(x, r)$ *is isometric to* $C_Y(y, r)$ *and* $S_X(x, r)$ *is isometric to* $S_Y(y, r)$.

Proof. Let f be the isometry of (X, ϱ) onto (Y, τ) with $f(x) = y$. Then f maps $B_X(x, r)$ and $C_X(x, r)$ onto $B_Y(y, r)$ and $C_Y(y, r)$, respectively. ∎

COROLLARY 2.7. *Let* $x_0 \in X$ *and* $1 \ge r > 0$. *Then* $S(x_0, r) = C(x_0, r) \setminus B(x_0, r)$ *is isometric to* $C(x_0, r)$. *In particular*, $X \setminus B(x_0, 1) = C(x_0, 1) \setminus B(x_0, 1)$ *is isometric to* (X, ϱ).

Proof. As in 2.5, let $E \subset C(x_0, r)$ be a set which contains x_0 and which intersects every composant of $C(x_0, r)$ (i.e., every ball $B(y, r)$, $y \in C(x_0, r)$) in exactly one point. Let $f : E_0 = E \setminus \{x_0\} \to E$ be one-to-one and onto. Then f is an isometry since for $y \ne z \in E$, $\varrho(y, z) = r$. Extend f to an isometry $F : C(x_0, r) \setminus B(x_0, r) \to C(x_0, r)$ as follows. Let $x \in S(x_0, r)$. Then x belongs to a unique ball $B(y, r)$ for some $y \in E \setminus \{x_0\}$. By 2.6 there exists an isometry $f_y : B(y, r) \to B(f(y), r)$ and we define $F(x) = f_y(x)$, i.e., $F|_{B(y,r)} = f_y$ for $y \in E \setminus \{x_0\}$. F is an isometry since for $y_1 \ne y_2 \in E \setminus \{x_0\}$, $w \in B(y_1, r)$ and $z \in B(y_2, r)$, we have $\varrho(w, z) = r$, and $F(S(x_0, r)) = C(x_0, r)$, since $S(x_0, r) = \bigcup\{B(y, r) : y \in E \setminus \{x_0\}\}$ and $C(x_0, r) = \bigcup\{B(y, r) : y \in E\}$. ∎

A similar argument can be applied to prove the following.

COROLLARY 2.8. *Let* (X, ϱ) *and* (Y, τ) *be ultrametric atoms*, *let* $x_0 \in X$, $y_0 \in Y$, *and let* $0 < r \le 1$. *Let* $D \subset C_X(x_0, r)$ *and* $E \subset C_Y(y_0, r)$ *be subsets with the same cardinality such that the balls* $B_X(x, r)$, $x \in D$, *and* $B_Y(y, r)$, $y \in E$, *are mutually disjoint*, *i.e.*, D (*respectively* E) *intersects every composant of* $C_X(x_0, r)$ (*respectively* $C_Y(y_0, r)$) *in at most one point. Then* $\bigcup\{B_X(x, r) : x \in D\}$ *and* $\bigcup\{B_Y(y, r) : y \in E\}$ *are isometric.*

The following lemma will be applied in our proof of Theorem 2.4.

LEMMA 2.9. *Let* (X, ϱ) *and* (Y, τ) *be ultrametric atoms. Let* $E \subset X$ *be closed in* (X, ϱ) *with* $\mathrm{wt}_\varrho(E) < c$, *and let* $f : E \to (Y, \tau)$ *be an isometry. Let* $e \in X \setminus E$. *Then* f *is extendable over* $E \cup \{e\}$.

Proof. We need to find a point $v = f(e)$ in Y such that $\tau(f(x), v) = \varrho(x, e)$ for every $x \in E$. Evidently, v has this property if and only if

$$v \in \bigcap_{x\in E} S_Y(f(x), \varrho(x,e)) = \bigcap_{x\in E} C_Y(f(x), \varrho(x,e)) \setminus B_Y(f(x), \varrho(x,e)) = \bigcap_{x\in E} C_Y(f(x), \varrho(x,e)) \setminus \bigcup_{x\in E} B_Y(f(x), \varrho(x,e)), \tag{2.10}$$

and we shall show that the intersection in (2.10) is not empty.

Because E is closed in (X,ϱ) and $e \notin E$, $r = \mathrm{dist}_\varrho(e,E) > 0$. Consider the following two cases:

Case (i): The distance $r = \mathrm{dist}_\varrho(e,E)$ is not attained. In this case, let $(x_n)_{n\ge1}$ be points in E such that the sequence $(\varrho(x_n,e))$ is strictly decreasing to r. Let $x \in E$. Then for some n, it follows that $\varrho(x_n,e) < \varrho(x,e)$ and by 2.1(6), $\varrho(x_n,x) = \varrho(x,e)$. Hence also $\tau(f(x), f(x_n)) = \varrho(x,e)$ and it follows that $f(x_n) \in C_Y(f(x), \varrho(x,e))$. By 2.1(3) and 2.1(4), we conclude that

$$C_Y(f(x), \varrho(x,e)) = C_Y(f(x_n), \varrho(x,e)) \supset C_Y(f(x_n), \varrho(x_n,e)).$$

It follows that $A = \bigcap\{C_Y(f(x), \varrho(x,e)) : x \in E\} = \bigcap\{C_Y(f(x_n), \varrho(x_n,e)) : n \ge 1\}$. But $C_Y(f(x_n), \varrho(x_n,e))$ is a decreasing sequence of closed balls in (Y,τ) and by spherical completeness (2.3(4)), $A \ne \emptyset$; in fact, $A = C_Y(y,r)$ for some (every) $y \in A$.

Let us return to the arbitrary element $x \in E$ which satisfies $\varrho(x_n,e) < \varrho(x,e) = \varrho(x,x_n)$. Then $B_Y(f(x), \varrho(x,e))$ does not meet $C_Y(f(x_n), \varrho(x_n,e))$. To see this, note that $\tau(f(x_n), f(x)) = \varrho(x_n,x) = \varrho(x,e)$, which implies that $f(x_n) \notin B_Y(f(x), \varrho(x,e))$ and $C_Y(f(x_n), \varrho(x_n,e))$ is a ball of smaller radius.

In particular, $B_Y(f(x), \varrho(x,e)) \cap A = \emptyset$. Hence

$$\bigcup\{B_Y(f(x), \varrho(x,e)) : x \in E\} \cap A = \emptyset$$

and the intersection in (2.10) agrees with A. This completes case (i).

Case (ii): The distance $r = \mathrm{dist}_\varrho(e,E)$ is attained. Let $E \supset E_1 = \{z \in E : \varrho(z,e) = r\}$. For $z,w \in E_1$, $\varrho(z,e) = \varrho(w,e) = r$ and by 2.1(2), $r \ge \varrho(z,w)$. Let $E_2 \subset E_1$ be a nonempty subset with $\varrho(z,w) = r$ for $z,w \in E_2$, $z \ne w$, and such that E_2 is maximal with respect to this property. Since E_2 is r-discrete, $\mathrm{card}(E_2) \le \mathrm{wt}(E) < c$. (Note that E_2 may be finite or even consist of a single point. Recall that in the example in Remark 2.5 we had $E = E_1 = E_2$ with $r = 1$ and $\mathrm{card}(E_2) = c$, and the conclusion of the lemma failed.)

Consider the intersection

$$D = \bigcap_{z\in E_2} S_Y(f(z), r) = \bigcap_{z\in E_2} C_Y(f(z), r) \setminus \bigcup_{z\in E_2} B_Y(f(z), r). \tag{2.11}$$

Since for $w, z \in E_2$, $\tau(f(w), f(z)) = \varrho(w, z) = r$, all the closed balls $C_Y(f(z), r)$ coincide. So $\bigcap\{C_Y(f(z), r) : z \in E_2\} = C_Y(f(z_0), r)$ for some $z_0 \in E_2$. By 2.3(7), $C_Y(f(z_0), r)$ contains c mutually disjoint open balls $B_Y(y, r)$ for $y \in C_Y(f(z_0), r)$, while $\bigcup\{B_Y(f(z), r) : z \in E_2\}$ is the union of at most $\mathrm{card}(E_2) < c$ such balls. It follows that $\bigcup\{B_Y(f(z), r) : z \in E_2\}$ does not exhaust all of $C_Y(f(z_0), r)$ and hence $D \neq \emptyset$. We claim that $D \subset \bigcap\{S_Y(f(x), \varrho(x, e)) : x \in E\}$, i.e., if $v \in D$ then for all $x \in E$, $\tau(f(x), v) = \varrho(x, e)$.

To establish this claim, first note that if $x \in E_2$ this follows from 2.11. If $x \in E_1 \backslash E_2$, then by the maximality of E_2 there is some $z \in E_2$ with $\varrho(x, z) < r = \varrho(z, e)$ and by 2.1(6), $\varrho(z, e) = \varrho(x, e) = r$. Now $\tau(f(x), f(z)) = \varrho(x, z)$ and since $v \in D$, $\tau(f(x), v) = r > \tau(f(x), f(z))$. Applying 2.1(6) once again, it follows that $\tau(f(x), v)$ must equal $\tau(f(z), v) = r = \varrho(x, e)$, and we are done.

Finally, if $x \in E \setminus E_1$, again let $z \in E_2$. Then $\varrho(z, e) = r < \varrho(x, e)$ and hence by 2.1(6), $\varrho(x, e) = \varrho(x, z) = \tau(f(x), f(z))$. Thus, $r = \tau(f(z), v) < \tau(f(x), f(z))$ and another application of 2.1(6) yields

$$\tau(f(x), v) = \tau(f(x), f(z)) = \varrho(x, e). \ \blacksquare$$

Proof of Theorem 2.4. Let (X, ϱ) and (Y, τ) be ultrametric atoms, let $X' \subset X$ with $\mathrm{wt}_\varrho(X') < c$, and let $f : X' \to Y$ be an isometry. We must show that f is extendable to X.

Note first that since (X, ϱ) and (Y, τ) are complete metric spaces (which follows from spherical completeness), f is extendable over the closure of X'. Hence we may assume that X' and $Y' = f(X')$ are closed. Let $A \subset X \setminus X'$ and $B \subset Y \setminus Y'$ be dense in $X \setminus X'$ and $Y \setminus Y'$ respectively, with $\mathrm{card}(A) = \mathrm{card}(B) = c$. Well order A and B by indexing their elements with ordinals $< c$. Let $A = \{x_\alpha : \alpha < c\}$ and $B = \{y_\alpha : \alpha < c\}$. By transfinite induction we construct for each ordinal $\alpha < c$ subsets $X_\alpha \supset X'$ of X and $Y_\alpha \supset Y'$ of Y with (i) $\{x_\beta \in A : \beta \le \alpha\} \subset X_\alpha$, (ii) $\{y_\beta \in B : \beta \le \alpha\} \subset Y_\alpha$, (iii) $\mathrm{wt}_\varrho(X_\alpha) < c$ and $\mathrm{wt}_\tau(Y_\alpha) < c$ and an isometric extension $f_\alpha : X_\alpha \to Y_\alpha$ of f.

This is done by a routine back and forth argument. Set $X_0 = X'$ and $Y_0 = Y'$. Apply Lemma 2.9 twice, first to extend f over $X_0 \cup \{x_0\}$ to $Y_0 \cup \{f(x_0)\}$, and then (if necessary) to extend f^{-1} over $Y_1 = Y_0 \cup \{f(x_0)\} \cup \{y_0\}$ to $X_1 = X_0 \cup \{x_0\} \cup \{f^{-1}(y_0)\}$. Let $\alpha < c$ be an ordinal. Assume that f_β, X_β and Y_β have been constructed for all ordinals $\beta < \alpha$. If $\alpha = \beta + 1$ for some β, then we repeat the above argument to construct $f_{\beta+1}$, $X_{\beta+1} = X_\beta \cup \{x_\beta\} \cup \{f^{-1}(y_\beta)\}$ and $Y_{\beta+1} = Y_\beta \cup \{f(x_\beta)\} \cup \{y_\beta\}$. Note that if $x_\beta \in X_\beta$ already then we take $X_{\beta+1} = X_\beta$ and similarly if $y_\beta \in Y_\beta$. If α is a limit ordinal then first we apply completeness to extend f to $f'_\alpha : \overline{\bigcup_{\beta<\alpha} X_\beta} \to \overline{\bigcup_{\beta<\alpha} Y_\beta}$ (with closure in (X, ϱ) and (Y, τ) respectively). Now apply Lemma 2.9 to add $x_\alpha \in A$ and

$y_\alpha \in B$. Note that the weights of both X_α and Y_α remain $< c$ during the whole procedure. Eventually we obtain an isometric extension of f over a closed set containing A into a closed set containing B, and as A and B are dense, this function maps X onto Y. ∎

Theorem 2.4 shows that there is a unique atomic ultrametric space up to an isometry. We conclude this section with a concrete model for this space. It turns out that only the following properties of an atomic ultrametric space (X, ϱ) were applied in the proof of Theorem 2.4.

2.12. (1) (X, ϱ) is an ultrametric space.

(2) Spherical completeness (2.3(4)).

(3) In every closed ball $C(x, r)$, $r > 0$, the cardinality of the family of disjoint open balls $B(y, r)$ for $y \in C(x, r)$ is c (2.3(7)).

Note that (3) also implies that $\mathrm{wt}_\varrho(X, \varrho) \geq c$ and that the sphere $S(x, r) = C(x, r) \setminus B(x, r)$ is not empty (2.3(7)). Also, spherical completeness implies completeness. It follows that every metric space which satisfies 2.12 is isometric to the ultrametric atom. The following space (J, ϱ) is such an example.

Example 2.13. (i) J consists of all real-valued continuous functions f on $[0, 1]$ with $f(1) = 0$.

(ii) For $f, g \in J$, $\varrho(f, g) \leq r$ if $f(x) = g(x)$ for all $r \leq x \leq 1$, i.e., if f and g agree on $[r, 1]$.

Note that $\varrho(f, g) = r$ if $\varrho(f, g) \leq r$ but f and g do not agree on $[s, 1]$ for any $s < r$. We leave it to the reader to verify that (J, ϱ) satisfies 2.12. It follows that every ultrametric atom is isometric to (J, ϱ).

3. The tree of subcontinua and the lattice of monotone upper semicontinuous decompositions of an atom. The fact that any two ultrametric atoms are isometric, as proved in §2, indicates that the combinatorial structure of subcontinua is the same in all nondegenerate atoms. In this section we study a richer structure which distinguishes between different atoms.

Let X be a compact metrizable space. Define an equivalence relation on the family of all continuous functions (= maps) of X into any Hausdorff space by

Definition 3.1. $f : X \to f(X)$ and $g : X \to g(X)$ are *equivalent* ($f \sim g$) if there exists a homeomorphism $h : f(X) \to g(X)$ such that $g = hf$.

For a map $f : X \to f(X)$ and $x \in X$ set $(x)f = f^{-1}f(x)$, the fiber of f at x. One checks easily that $f \sim g$ if and only if $(x)f = (x)g$ for all $x \in X$, i.e., if f and g induce the same decomposition on X.

Let $D(X)$ denote that set of all maps of X into a Hausdorff space modulo the equivalence relation $\sim$. Recall that a decomposition F of X into mutually disjoint closed sets is called *upper semicontinuous* if whenever a sequence $(A_n)_{n\geq 1}$ of elements of F converges to some closed subset A of X in 2^X then A is contained in some element of F. If every such limit A is itself an element of F then F is called a *continuous* decomposition.

A map $f : X \to f(X)$ has a Hausdorff range if and only if the decomposition $\{(x)f : x \in X\}$ is upper semicontinuous (see [Ku, p. 66]). It follows that an element of $D(X)$ represents an upper semicontinuous decomposition of X, and we call $D(X)$ the *space of upper semicontinuous decompositions*. Note that an element of $D(X)$ is a continuous decomposition if and only if it represents an open mapping.

Although an element of $D(X)$ is a decomposition of X we shall take some liberties and also use functional notation. Thus, for example, for $f \in D(X)$,

$$f = \{(x)f : x \in X\} = \{f^{-1}(y) : y \in f(X)\},$$

and $A \in f$ if $A = (x)f$ for some $x \in X$. The class of the identity map is denoted by id, while const denotes the class of the constant maps.

3.2. $D(X)$ carries a natural order relation: $f \leq g$ if g refines f, i.e., if the decomposition g of X refines the decomposition f, or, equivalently, if for all $x \in X$, $(x)g \subset (x)f$. With this order, $D(X)$ is a complete lattice. Indeed, for $E \subset D(X)$, $\bigvee E = \sup E = g$ is defined by

$$(x)g = \bigcap\{(x)f : f \in E\}. \tag{3.3}$$

That is, $g = \sup E$ is the class of the product map $: X \to \prod\{f(X) : f \in E\}$ whose f coordinate, $f \in E$, is f itself.

In particular, for $f, g \in D(X)$, $(x)(f \vee g) = (x)f \cap (x)g$ and $f \vee g$ is the class of the product map $(f, g) : X \to f(X) \times g(X)$, where $(f, g)(x) = (f(x), g(x))$.

Also,

$$\bigwedge E = \bigvee\{g : g \in D(X),\ g \leq f \text{ for all } f \in E\}. \tag{3.4}$$

Note that for all $f \in D(X)$, const $\leq f \leq$ id. We use $f < g$ to indicate that $f \leq g$ and $f \neq g$.

The sets $D(X)$ (and $M(X)$, which will be defined below) were introduced and studied in [St], [Lev-St1], [Lev-St2] and [Lev-St3]. Note, however, that in these papers $f \leq g$ if f refines g and not as in this article.

Recall that a map $f : X \to Y$ is *monotone* if $f^{-1}(y)$ is connected for all $y \in Y$. Let

$$M(X) = \{f \in D(X) : f \text{ is monotone}\}.$$

Thus, $f \in M(X)$ if and only if $(x)f$ is a continuum for all $x \in X$. In general $M(X)$ is not a sublattice of $D(X)$ since the intersection of continua need not be connected. But if X is an atom then $M(X)$ is a complete sublattice of $D(X)$. Indeed, if $E \subset M(X)$ then

$$(x)\bigvee E = \bigcap\{(x)f : f \in E\} \tag{3.5}$$

is an intersection in X of continua all of which contain the point x and hence are nested. This implies that the intersection is a nonempty continuum.

Moreover, for $f, g \in M(X)$, we have

$$(x)(f \wedge g) = (x)f \cup (x)g. \tag{3.6}$$

The reader may verify (or else check in [St]) that if X is an atom, then $\{(x)f \cup (x)g : x \in X\}$ is indeed an upper semicontinuous monotone decomposition of X which agrees with the earlier definition of $f \wedge g$ in (3.4). Note though that (3.6) is valid only for a finite subset $E \subset M(X)$. Also, $\bigcup\{(x)(f) : f \in E\}$ is always a decomposition of X but is not necessarily upper semicontinuous when $E \subset M(X)$ is an infinite set.

It should also be noted that atoms with monotone maps form a category; the quotient $f(X)$ of a monotone upper semicontinuous decomposition f of an atom X is also an atom and one may think of the elements of $M(X)$ as of atomic quotients of X, where $f \le g$ means that f factors through g. Thus, $M(X)$ is a natural object to study when X is an atom.

For the remainder of the article we assume that X is a nontrivial atom. Let A be a nontrivial subcontinuum of X, i.e., A contains more than one point. Let $f_A \in M(X)$ be defined by

$$(x)f_A = \begin{cases} A & \text{if } x \in A, \\ \{x\} & \text{if } x \notin A, \end{cases} \tag{3.7}$$

i.e., f_A is the decomposition of X whose only nontrivial element is A. Clearly f_A is upper semicontinuous and monotone.

Set

$$T(X) = \{f_A : A \text{ is a nontrivial subcontinuum of } X\}. \tag{3.8}$$

Clearly $f_A \le f_B$ in $M(X)$ if and only if $B \subset A$. Hence the map $A \to f_A$ is an order reversing isomorphism of the nontrivial subcontinua of X (ordered by inclusion) into $T(X) \subset M(X)$.

REMARK 3.9. The singletons of X are not represented explicitly in $T(X)$ but they can be easily identified there. Let $x \in X$. Set $L(x) = \{A \in C(X) : x \in A\}$. Then $L(x)$ is a maximal chain (= linearly ordered set) in $C(X)$ with respect to inclusion and is order isomorphic to $[0, 1]$; indeed, every Whitney map is strictly monotone on $L(x)$ with $W(\{x\}) = 0$ and $W(X) = 1$. Conversely, every maximal chain $L \subset C(X)$ is of the form $L(x)$ for some $x \in$

X, since if $\bigcap L$ contains more than one point then $A = \bigcap L$ is a nontrivial continuum and for $x \in A$, L can be extended by $\{B \in C(X) : x \in B \subset A\}$, and if $X \notin L$ then L can also be extended.

Hence, $l(x) = \{f_A \in T(X) : x \in A\}$ is a maximal chain in $T(X)$. Note that $l(x)$ has a minimal end point at const but no largest element and every maximal chain in $T(X)$ is of the form $l(x)$ for some $x \in X$. Thus the singletons of X can be identified with the maximal chains of $T(X)$.

$T(X)$ also has a tree structure: its branches are the abovementioned chains $l(x)$ for $x \in X$, all of which emerge from the bottom element const.

Theorem 2.4 of §2 implies the following.

Theorem 3.10. *Let X and Y be nontrivial atoms. Then $T(X)$ and $T(Y)$ are order isomorphic*; *moreover*, *given Whitney maps W and V on X and Y respectively*, *there exists an order isomorphism $\mu : T(X) \to T(Y)$ such that $V(\mu f) = W(f)$ for all $f \in T(X)$.* (*Here*, *by $W(f_A)$ we mean*, *of course*, *$W(A)$.*)

Proof. Let $h : (X, \varrho) \to (Y, \tau)$ be an isometry of the ultrametric atoms. Let $A \subset X$ be a subcontinuum. Then $A = C_X(x, r)$ for some $x \in X$ and $1 \geq r \geq 0$. Since h is an isometry, $h(A) = C_Y(h(x), r) = B$ is a subcontinuum of Y and we let $\mu f_A = f_B$. ∎

Remark. The model (J, ϱ) for the ultrametric atom (2.13) can be adjusted to obtain a model T for $T(X)$: Let T be the set of all pairs (r, f) where $0 < r \leq 1$ and $f : [r, 1] \to \mathbb{R}$ is continuous and satisfies $f(1) = 0$. Let $(r, f) \leq (s, g)$ if $[s, 1] \supset [r, 1]$ (i.e., $s \leq r$) and $g|_{[r,1]} = f$. We leave it to the reader to verify that $(T, \leq)$ is order isomorphic to $T(X)$ for every nontrivial atom X.

The main result of §3 is that, unlike $T(X)$, the lattice structure of $M(X)$ determines X.

Theorem 3.11. *Let X and Y be nontrivial atoms. Then $M(X)$ and $M(Y)$ are lattice isomorphic if and only if X and Y are homeomorphic.*

To prove Theorem 3.11 we shall need the following proposition.

Proposition 3.12. *Let X and Y be atoms and let $\mu : M(X) \to M(Y)$ be a lattice isomorphism. Then $\mu(T(X)) = T(Y)$.*

Recall that $T(X)$ has been defined (3.7, 3.8) in terms of subcontinua of X. To prove Proposition 3.12 we shall need to show that $T(X)$ is determined by the lattice structure of $M(X)$.

Definition 3.13. An element f of a lattice M is *meet irreducible* if whenever $f = g \wedge h$, then either $f = g$ or $f = h$.

Observe that id $\in M(X)$ is meet irreducible but id $\notin T(X)$. The next result characterizes those meet irreducible elements in $M(X)$ different from id.

PROPOSITION 3.14. *An element* $f \in M(X)$, $f \neq \mathrm{id}$, *is in* $T(X)$ *if and only if it is meet irreducible*, *i.e.*, $T(X)$ *is the set of meet irreducible elements of* $M(X)$ *different from* id.

Proof. Let $f_A = g \wedge h \in T(X)$ for some $g, h \in M(X)$. For $x \in X \setminus A$, $(x)f_A = \{x\} \supset (x)g \cup (x)h$ and it follows that every nontrivial fiber of g or h is contained in A. For $a \in A$,

$$A = (a)f_A = (a)(g \wedge h) = (a)g \cup (a)h.$$

But $(a)g \cup (a)h$ is either $(a)g$ or $(a)h$ and hence f_A is either g or h.

Let $f \in M(X) \setminus T(X)$ and $f \neq \mathrm{id}$. We must find g, h in $M(X)$ such that $f < g$, $f < h$ and $f = g \wedge h$. Since $f \notin T(X)$, f has at least two nontrivial fibers A and B. Let W be a Whitney map on X such that $W(A) \neq W(B)$. (Such a Whitney map always exists: if $A \cap B = \emptyset$ and $W_1(A) = W_1(B)$ for some Whitney map W_1, let $f : X \to [0,1]$ with $f(A) = 0$ and $f(B) = [0,1]$, and let $W(E) = \frac{1}{2}W_1(E) + \frac{1}{2}\operatorname{diam} f(E)$.)

Let $W(A) < r < W(B)$. Define decompositions g and h of X as follows:

$$(x)g = \begin{cases} (x)f & \text{if } W((x)f) \le r, \\ E & \text{if } W((x)f) > r, \end{cases} \tag{3.15}$$

where E is the unique subcontinuum of X which contains x and satisfies $W(E) = r$, i.e., $E = C(x,r)$ (= the closed ultrametric ball for the ultrametric ϱ obtained from the Whitney map W.) Now g is a decomposition of X (recall that closed ultrametric balls are disjoint) and clearly, since $W(B) > r$ and B is a fiber of f, g refines f and $g \neq f$. We need to show that $g \in M(X)$, i.e., that g is upper semicontinuous. So, let $(x_n)g \to E$ in $C(X)$. Since $(x_n)g \subset (x_n)f$ and f is upper semicontinuous, $E \subset (x)f$ for some $x \in E$. Also, since $W((x_n)g) \le r$ for all n and W is continuous on $C(X)$, $W(E) \le r$. Hence by (3.15) if $W((x)f) \le r$ then $(x)g = (x)f \supset E$, while if $W((x)f) > r$ then $W((x)g) = r$ and since $W(E) \le r$ and both $(x)g$ and E contain x, $(x)g \supset E$. Next, set

$$(x)h = \begin{cases} (x)f & \text{if } W((x)f) \ge r, \\ \{x\} & \text{if } W((x)f) < r. \end{cases} \tag{3.16}$$

Evidently h is a decomposition of X, h refines f and $h \neq f$ (since for $x \in A$, $(x)f = A$ while $(x)h = \{x\}$ and A is a nontrivial continuum).

We claim that h is upper semicontinuous. To see this let $(x_n)h \to E$ in $C(X)$. We need to show that $E \subset (x)h$ for some $x \in X$. If for infinitely many values of n, $(x_n)h = \{x_n\}$, then E is a singleton and we are done. Hence we may assume without loss of generality that for all n, $(x_n)h$ is nontrivial, which by (3.16) implies that $W((x_n)h) \geq r$ and hence $(x_n)h = (x_n)f$. Since f is upper semicontinuous, $E \subset (x)f$ for some $x \in E$. From the continuity of W it follows that $r \leq W(E) \leq W((x)f)$. Then (3.16) implies that $(x)h = (x)f \supset E$ and we are done.

Thus $g, h \in M(X)$, $f < g$, $f < h$ and we claim that $f = g \wedge h$. Indeed, let $x \in X$. If $W((x)f) \leq r$ then $(x)g = (x)f$ by (3.15) while if $W((x)f) \geq r$ then $(x)h = (x)f$ by (3.15). It follows that $(x)(g \wedge h) = (x)g \cup (x)h \supset (x)f$, i.e., $g \wedge h \leq f$, so we have $f = g \wedge h$ and Proposition 3.14 is proved. ∎

Proof of Proposition 3.12. Clearly a lattice isomorphism preserves meet irreducibility and by Proposition 3.14 it carries $T(X)$ onto $T(Y)$. ∎

LEMMA 3.17. *Let X be an atom. Let $(x_n)_{n\geq 1}$ be a sequence in X such that $\lim x_n = y$. For each n, let $x_n \in A_n \in C(X)$, and let $g = \bigwedge\{g_{A_n} : n \geq 1\}$. Then for every $x \in X$ such that $(x)g \neq \{x\}$, either $(x)g = A_n$ for some n or $y \in (x)g$.*

Proof. Set $E = \{A_n : n \geq 1\} \subset C(X)$ and let $\overline{E}$ denote the closure of E in $C(X)$. Let $A \in \overline{E} \setminus E$. Then $A = \lim A_{n_k}$ for some subsequence (A_{n_k}) of (A_n). Since $x_{n_k} \in A_{n_k}$ and $x_{n_k} \to y$, $y \in A$. Now, by 1.1, the members of the family $\{A \in \overline{E} : y \in A\}$ are nested. A straightforward compactness argument shows that the set

$$B = \overline{\bigcup\{A \in \overline{E} : y \in A\}}$$

is also in $\overline{E}$. So either $E = \overline{E}$ or every $A \in \overline{E} \setminus E$ is contained in B.

Set $H = \bigcup\{A : A \in \overline{E}\}$. Then clearly H is a closed subset of X. Let $h \in M(X)$ be the decomposition which consists of the components of H and the singletons of $X \setminus H$. (By [Ku, p. 182], h is upper semicontinuous.) For each $n \geq 1$, $h \leq g_{A_n}$ since $A_n \subset H$ is contained in some component of H. Hence also $h \leq g = \bigwedge\{g_{A_n} : n \geq 1\}$. It follows that for $x \in X \setminus H$, $\{x\} = (x)h \supset (x)g$, i.e., $(x)g = \{x\}$ for $x \notin H$.

Now let $x \in X$ be such that $(x)g \neq \{x\}$ and $y \notin (x)g$. Then $x \in H$. We claim that $x \notin B$. Indeed, since $B \in \overline{E}$, one of these two cases occurs.

CASE (i): $B = A_n$ for some n. If $x \in B$, then $(x)g \supset A_n = B$, so $y \in (x)g$, a contradiction.

CASE (ii): $B = \lim A_{n_k}$ for some subsequence (A_{n_k}) of (A_n). Since $(x_{n_k})g \supset (x_{n_k})g_{A_{n_k}} = A_{n_k}$ and since g is upper semicontinuous, B must be contained in a fiber of g. If $x \in B$, then $(x)g \supset B$ and since $y \in B$, this is impossible.

So it follows in both cases that $x \in \bigcup\{A_n : n \geq 1\}$. But x can be a member only of finitely many A_n's, since if $x \in A_{n_k}$ for $k = 1, 2, \ldots$, then $(x)g \supset \overline{\bigcup A_{n_k}}$ and $y \in \overline{\bigcup A_{n_k}}$. Hence $(x)g = A_n$ for some n. ∎

REMARK. It follows from Lemma 3.17 and its proof that

$$(x)g = \begin{cases} \{x\} & \text{if } x \notin H = \bigcup \overline{E}, \\ B & \text{if } x \in B, \\ A_k & \text{if } x \in H \setminus B, \end{cases}$$

where, if $x \in H \setminus B$, $x \in A_n$ for only finitely many values of n, $x \in A_{n_1} \subset \ldots \subset A_{n_m}$, and $A_k = A_{n_m}$ is the union of all the A_n's which contain x.

PROPOSITION 3.18. *Let X be an atom. Let $(x_n)_{n\geq 1}$ be a sequence in X and let $x_0 \in X$. Then the following are equivalent*:

(1) $\lim x_n = x_0$ *in X.*

(2) *For every continuum $A_0 \subset X$ with $x_0 \in A_0$ and every subsequence (x'_n) of (x_n), there exist continua $A_n \subset X$ with $x'_n \in A_n$ for all n such that $(x_0)g = A_0$, where $g = \bigwedge\{g_{A_n} : n \geq 1\}$.*

Proof. (1)⇒(2). First, suppose that $x_n \to x_0$, let $x_0 \in A_0$ and let (x'_n) be a subsequence of (x_n). Set $A_n = C(x'_n, W(A_0))$, where W is a Whitney map on X, and let $g = \bigwedge\{g_{A_n} : n \geq 1\}$.

Let f denote the (continuous) decomposition of X into r-balls $C(x, r)$, where $r = W(A_0)$. Then for all $n \geq 1$, $f \leq g_{A_n}$. Hence $f \leq g$. In particular, $A_0 = C(x_0, r) = (x_0)f \supset (x_0)g$.

To obtain the converse inclusion let $(A_{n_k}) \subset (A_n)$ converge (in $C(X)$) to some $A \in C(X)$. Then $(x'_{n_k})g \supset A_{n_k} \to A$, as $x'_{n_k} \in A_{n_k}$, so $(x_0)g \supset A$. But $W(A_{n_k}) = r$ and from the continuity of W it follows that $W(A) = r$. Hence, since A and A_0 both contain x_0, we must have $A = A_0 = C(x_0, r)$ and $(x_0)g \supset A_0$.

(2)⇒(1). Assume that $x_n \nrightarrow x_0$. Then by compactness there is a subsequence (x'_n) such that $x'_n \to y \neq x_0$ and such that $x'_n \neq x_0$ for all n. Then

$$\operatorname{dist}(x_0, \{x'_n : n \geq 1\} \cup \{y\}) = 2\delta$$

is positive. Let A_0 be a continuum of diameter δ in X which contains x_0. We claim that (2) fails for these A_0 and (x'_n). Indeed, let $x'_n \in A_n \in C(X)$. Then by Lemma 3.17, $(x_0)g$ is either $\{x_0\} \neq A_0$, or $(x_0)g = A_n$ for some n. Now $A_n \neq A_0$ since this would imply $x'_n \in A_0$, and thus $\operatorname{diam}(A_0) \geq 2\delta$, while $\operatorname{diam}(A_0) = \delta$. Or, $y \in (x_0)g$, which again implies $\operatorname{diam}((x_0)g) \geq 2\delta$, and we are done. ∎

DEFINITION 3.19. Let $f \in M(X)$. Then

$$T_f = \{g \in T(X) : g \geq f \text{ and for all } h \in T(X),\ g \geq h \geq f \text{ implies } g = h\}.$$

PROPOSITION 3.20. *Let $f \in M(X)$. Then $g = g_A \in T_f$ if and only if for every $x \in A$, $(x)f = A$. Hence $T_f = \{g_{(x)f} : x \in X$ and $(x)f$ is nontrivial$\}$.*

Proof. Let $g = g_{(x)f}$ for some $f \in M(X)$ and $x \in X$ such that $(x)f$ is nontrivial. Then $g \geq f$. If $h = h_B \in T(X)$ is such that $g_{(x)f} \geq h_B \geq f$ then $(x)g_{(x)f} = (x)f \subset (x)h_B = B \subset (x)f$, i.e., $B = (x)f$ and $g = h_B$. Hence $g \in T_f$. Let $g_A = g \in T_f$. Then $g \geq f$. Thus for $x \in A$, $(x)g = A \subset (x)f$. If $A \subsetneq (x)f$, let $A \subsetneq B \subsetneq (x)f$. Then $h_B \in T(X)$ and $g > h > f$. Hence $A = (x)f$. ∎

We continue with the proof of Theorem 3.11. Let X and Y be atoms and let $\mu : M(X) \to M(Y)$ be a lattice isomorphism. Then by Proposition 3.12, $\mu(T(X)) = T(Y)$. Define a function $\widehat{\mu} : X \to Y$ as follows. Let $x \in X$. Then (see 3.9) $l(x) = \{f_A \in T(X) : x \in A\}$ is a maximal chain in $T(X)$. Hence $\mu l(x) = \{\mu f_A : f_A \in l(x)\}$ is a maximal chain in $T(Y)$, and, by 3.9 again, $\mu l(x) = l(y)$ for some $y \in Y$. Define $\widehat{\mu}(x) = y$. Clearly $\widehat{\mu}$ is one-to-one and onto and we shall prove that it is continuous.

Remark. We have seen that for any two atoms X and Y, $T(X)$ and $T(Y)$ are order isomorphic and every order isomorphism $\mu : T(X) \to T(Y)$ gives rise to a surjective injection $\widehat{\mu} : X \to Y$ as above. But in general $\widehat{\mu}$ is not continuous. Our proof will show that if $\widehat{\mu}$ originates from an isomorphism of $M(X)$ onto $M(Y)$ then it is continuous.

Claim 3.21. *Let $A \subset X$ be a nontrivial continuum. Then $\widehat{\mu}(A) = \{\widehat{\mu}(x) : x \in A\} \subset Y$ is a continuum. Moreover, if $\mu f_A = f_B$ for some $B \in C(Y)$, then $\widehat{\mu}(A) = B$.*

Proof. We know that $x \in A$ if and only if $A \in L(x) = \{B \in C(X) : x \in B\}$. Hence

$$A \in \bigcap \{L(x) : x \in A\} = L_A = \{E \in C(X) : A \subset E\}$$

and $A = \bigcap \{E : E \in L_A\}$. Thus

$$f_A = \bigvee \Big\{ f_E : E \in \bigcap \{L(x) : x \in A\} \Big\}.$$

By definition $\mu l(x) = l(\widehat{\mu}(x))$. Hence

$$\begin{aligned} \mu f_A &= \bigvee \Big\{ f_H : H \in \bigcap \{L(\widehat{\mu}x) : x \in A\} \Big\} \\ &= \bigvee \bigcap \{l(y) : y \in \widehat{\mu}A\} = f_B \end{aligned}$$

where $B = \widehat{\mu}A$. ∎

Claim 3.22. *For $f \in M(X)$ and $x \in X$, $\widehat{\mu}((x)f) = (\widehat{\mu}(x))\mu f$.*

Proof. Note that (by 3.19 and 3.20) if $A = (x)f$ is nontrivial then $f_A = T_f \cap l(x)$. As $\mu T_f = T_{\mu f}$ we obtain

$$\mu f_A = \mu(T_f \cap l(x)) = T_{\mu f} \cap l(\widehat{\mu}(x)) = f_B$$

where $B = (\widehat{\mu}(x))\mu f$, and by 3.21, $B = \widehat{\mu}A$. If $(x)f = \{x\}$ then $(\widehat{\mu}(x))\mu f = \widehat{\mu}(x)$ since if $(\widehat{\mu}(x))\mu f$ is nontrivial we can apply the above for μf and μ^{-1} and obtain a contradiction. ∎

We complete the proof of Theorem 3.11. Let $x_n \to x_0$ in X. We apply Proposition 3.18 to show that $y_n = \widehat{\mu}(x_n) \to \widehat{\mu}(x_0) = y_0$ in Y. So, let $y_0 \in A_0 \subset Y$ and let $(y'_n) = (\widehat{\mu}(x'_n)) \subset (y_n)$ be a subsequence. By 3.21 $x_0 \in \widehat{\mu}^{-1}A_0 = B_0$ and is a continuum in X. Applying 3.18(2) to B_0 and $(x'_n) \subset (x_n)$ we obtain continua $x'_n \in B_n \subset X$ such that $g = \bigwedge\{g_{B_n} : n \geq 1\}$ satisfies $(x_0)g = B_0$. Set $A_n = \widehat{\mu}B_n$. Then $y'_n \in A_n$, A_n are continua in Y, and for $h = \bigwedge\{g_{A_n} : n \geq 1\}$ we have (by 3.21)

$$\begin{aligned}\mu g = \mu\Big(\bigwedge\{g_{B_n} : n \geq 1\}\Big) &= \bigwedge\{\mu g_{B_n} : n \geq 1\} \\ &= \bigwedge\{g_{A_n} : n \geq 1\} = h.\end{aligned}$$

And 3.22 implies

$$(y_0)h = (\widehat{\mu}(x_0))\mu g = \widehat{\mu}((x_0)g) = \widehat{\mu}(B_0) = A_0,$$

which shows that 3.18(2) holds, and by 3.18, $y_n \to y_0$, i.e., $\widehat{\mu}$ is continuous. ∎

References

[Ar-Pa] N. Aronszajn and P. Panitchpakdi, *Extension of uniformly continuous transformations and hyperconvex metric spaces*, Pacific J. Math. 6 (1956), 405–439.

[Bi] R. H. Bing, *Higher dimensional hereditarily indecomposable continua*, Trans. Amer. Math. Soc. 71 (1951), 267–273.

[Ho-Yo] J. G. Hocking and G. S. Young, *Topology*, Addison-Wesley, 1961.

[Kn] B. Knaster, *Un continu dont tout sous-continu est indécomposable*, Fund. Math. 3 (1922), 247–286.

[Kra] J. Krasinkiewicz, *On mappings with hereditarily indecomposable fibers*, Bull. Polish Acad. Sci. Math. 44 (1996), 147–156.

[Ku] K. Kuratowski, *Topology*, Volume II, Academic Press and PWN, 1968.

[Lev] M. Levin, *Certain finite dimensional maps and their application to hyperspaces*, Israel J. Math., to appear.

[Lev-St1] M. Levin and Y. Sternfeld, *Mappings which are stable with respect to the property* $\dim f(X) \geq k$, Topology Appl. 52 (1993), 241–265.

[Lev-St2] —, —, *Monotone basic embeddings of hereditarily indecomposable continua*, ibid. 68 (1996), 241–249.

[Lev-St3] —, —, *Atomic maps and the Chogoshvili–Pontrjagin claim*, Trans. Amer. Math. Soc., to appear.

[Lev-St4] —, —, *The space of subcontinua of a two-dimensional continuum is infinite dimensional*, Proc. Amer. Math. Soc. 125 (1997), 2771–2775.

[Lew] W. Lewis, *The Pseudo-Arc*, Marcel-Dekker, in preparation.

[Li] J. Lindenstrauss, *Extension of compact operators*, Mem. Amer. Math. Soc. 48 (1964).

[Li-Tz] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces*, Lecture Notes in Math. 338, Springer, 1973.

[Ma] S. Mazurkiewicz, *Sur les continus indécomposables*, Fund. Math. 10 (1927), 305–310.

[Na] S. B. Nadler, Jr., *Hyperspaces of Sets*, Marcel Dekker, 1978.

[Ni] J. Nikiel, *Topologies on pseudo-trees and applications*, Mem. Amer. Math. Soc. 416 (1989).

[Po1] R. Pol, *A two-dimensional compactum in the product of two one-dimensional compacta which does not contain any rectangle*, Topology Proc. 16 (1991), 133–315.

[Po2] —, *On light mappings without perfect fibers on compacta*, Tsukuba Math. J. 20 (1996), 11–19.

[Sch] W. H. Schikhof, *Ultrametric Calculus. An introduction to p-adic analysis*, Cambridge Univ. Press, 1984.

[St] Y. Sternfeld, *Stability and dimension—a counterexample to a conjecture of Chogoshvili*, Trans. Amer. Math. Soc. 340 (1993), 243–251.

[VR] A. C. M. Van Rooij, *Non-Archimedean Functional Analysis*, Marcel Dekker, 1978.

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