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Almost all submaximal groups are paracompact and σ-discrete

by

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Abstract. We prove that any topological group of a non-measurable cardinality is hereditarily paracompact and strongly σ-discrete as soon as it is submaximal. Consequently, such a group is zero-dimensional. Examples of uncountable maximal separable spaces are constructed in ZFC.

0. Introduction. This paper has been motivated by several unsolved problems in general topology and topological algebra. The concepts we are mainly going to deal with are maximality and submaximality of general topological spaces introduced in [He] more than fifty years ago. Recall that a topological space X is called submaximal if every dense subset of X is open. In this paper we consider only submaximal spaces without isolated points, so "submaximal" is to be read "submaximal dense in itself". A dense-in-itself space X is called maximal if any strictly stronger topology on X has isolated points. Although it is not evident at first glance, every maximal space is submaximal. Let us mention that all definitions and formulations related to the topic will be given in Section 1 (Notation and terminology) or in the main text. A reader who has not got the hang of the subject can be referred to an excellent paper of van Douwen [vD] which combines detailed and transparent proofs with covering practically everything important in the theory of maximal and submaximal spaces up to the year 1990.

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The concepts of maximality and submaximality, although technical at first sight, can serve as an important tool in the study of extreme cases one comes across when studying the family of topologies without isolated points on a given set. The first attempt to provide an insight into the matter was made by E. Hewitt [He]. He was the first to discover a general way of constructing maximal topologies. The core of his method (which, by the way, is still in use) is the observation that any chain of topologies on the same set has an upper bound with the same separation axioms. This makes it possible to construct maximal and submaximal topologies by transfinite induction, taking care of the desired properties at successor steps.

There were some obstacles to progress in understanding the nature of maximality and submaximality which were discovered quite early. For example, it was not clear until very recently whether there existed a ZFC example of a maximal Tikhonov space. The breakthrough here is due to van Douwen [vD], who established that there is a countable regular maximal space in ZFC. Another brilliant result is the one of V. I. Malykhin [Ma]: Under Martin's axiom there exists a non-discrete Hausdorff topological group whose underlying space is maximal.

It seems that the first systematic study of submaximal spaces was undertaken in the paper of A. V. Arkhangel'skiĭ and P. J. Collins [ArCo]. They gave various necessary and sufficient conditions for a space to be submaximal and showed that every submaximal space is left-separated. This led naturally to the question whether every submaximal dense-in-itself space is σ-discrete (Problem 1.12 of [ArCo]). J. Schröder [Sch] established that this is not so if there is a measurable cardinal. It is quite possible that measurable cardinals show up with good reason, because in this paper we prove that if G is a topological submaximal group of non-measurable cardinality, then G is paracompact and strongly σ-discrete. This implies that G is zero-dimensional if its cardinality is not measurable, thus giving a consistent answer to Problem 8.18 of [ArCo].

Another question of Arkhangel'skiĭ and Collins (Problem 8.6) is motivated by their result that every separable submaximal topological group is countable. They asked whether there exists a submaximal separable uncountable Hausdorff (Tikhonov) space. In this paper we construct an example of a Hausdorff separable maximal space of power 2^{2^ω} and a Tikhonov maximal separable space of cardinality 2^ω. We also prove that any submaximal group with the Suslin property is countable, strengthening the corresponding result of Arkhangel'skiĭ and Collins for separable submaximal groups.

1. Notation and terminology. All topological spaces are supposed to be Hausdorff and dense in themselves if the opposite is not stated explicitly.

If X is a space, then $\tau(X)$ is its topology and $\tau^*(X) = \tau(X)\setminus\{\emptyset\}$. The closure of a set A in X is denoted by $\overline{A}$ or by $\mathrm{cl}(A)$. A space is said to be *of first category* if it can be represented as a countable union of nowhere dense subsets. A (*strongly*) *σ-discrete space* is one which is a countable union of (closed) discrete subspaces. If X is a space and $x \in X$, then the *dispersion character* $\Delta(x, X)$ of X at the point x is the minimum of the cardinalities of open subsets of X containing x. The cardinal number $\Delta(X) = \min\{\Delta(x, X) : x \in X\}$ is called the *dispersion character* of X.

A space (X, τ) is called *maximal* if for any topology μ on X strictly finer than τ, the space (X, μ) has an isolated point. A space X is *submaximal* if any dense subset of X is open. A non-empty family $\mathcal{D}$ of dense subsets of a space X is called a *filter of dense subsets* of X if $\mathcal{D}$ is closed with respect to finite intersections and $D \in \mathcal{D}$, $D \subset D_1 \subset X$ implies $D_1 \in \mathcal{D}$. The family $\mathcal{D}$ is called an *ultrafilter of dense subsets* of X if there is no filter of dense subsets of X that properly contains $\mathcal{D}$.

If $\mathcal{D}$ is an ultrafilter of dense subsets of (X, τ), then the topology generated by the subbase $\tau \cup \mathcal{D}$ is known to be submaximal [GRS] and is called the *submaximal extension* of τ by $\mathcal{D}$.

A space is called *irresolvable* if it is not the union of two disjoint dense subsets. If G is a group and $A \subset G$, then $\langle A\rangle$ is the group generated by A, i.e. the smallest subgroup of G containing A.

A space X is called *collectionwise Hausdorff* if for any discrete subset A of X it is possible to choose an open $V_p \ni p$ for every $p \in A$ in such a way that the family $\{V_p : p \in A\}$ is discrete. A space X is *metacompact* if every open cover of X has a point-finite refinement. A space X is called a *Q-set* if every subset of X is of type G_δ. The symbol $\mathfrak{c}$, as well as 2^ω, denotes the power of continuum. All other notions are standard and can be found in [En].

2. Submaximal and maximal spaces. For the reader who is not a specialist in this area, we formulate without proofs some well-known simple facts about submaximal and maximal spaces. For details see [ArCo], [vD] and [GRS].

2.0. Facts. (1) *Every maximal space is submaximal*;

(2) *a space X is submaximal iff every subset of X is the intersection of a closed and an open subset of X*;

(3) *if X is submaximal, then any subset of X with empty interior is closed and discrete*;

(4) *a space is maximal iff it is submaximal and extremally disconnected*;

(5) *every submaximal space is irresolvable*;

(6) *if (X, τ) is a Hausdorff space, then there exists a maximal Hausdorff*

topology $\mu \supset \tau$ *on* X *without isolated points*, *and then* μ *is a maximal topology*;

(7) *if* (X, τ) *is a submaximal space*, *and* μ *is a topology on* X *finer than* τ, *then* (X, μ) *is submaximal.*

2.1. DEFINITION. Call a space X *weakly collectionwise Hausdorff* if for any closed discrete $A = \{x_s : s \in S\} \subset X$ there exists a disjoint family $\gamma = \{U_s : s \in S\} \subset \tau(X)$ such that $x_s \in U_s$ for every $s \in S$. The family γ will be called *separating* for A.

V. I. Malykhin [Ma] proved that every collectionwise Hausdorff maximal space is hereditarily paracompact. The theorem that follows improves this result.

2.2. THEOREM. *A submaximal weakly collectionwise Hausdorff space is hereditarily paracompact.*

Proof. Suppose that X is submaximal and weakly collectionwise Hausdorff. We first establish that X is hereditarily normal.

2.3. LEMMA. *A submaximal space* Z *is hereditarily normal if and only if any two disjoint closed discrete subspaces of* Z *have disjoint open neighbourhoods.*

Proof. It suffices to prove the "if" part. Let Y be an arbitrary subspace of Z. If P, Q are disjoint closed subspaces of Y, then $P = U \cup A$ and $Q = V \cup B$, where $U = \mathrm{Int}(P), V = \mathrm{Int}(Q)$ (the interiors are taken in Z) and the sets A and B are closed and discrete in Z. By the hypothesis of the lemma, there exist disjoint open sets U_1 and V_1 such that $A \subset U_1$ and $B \subset V_1$. Let $U_0 = U_1 \setminus \overline{V}$ and $V_0 = V_1 \setminus \overline{U}$. It can easily be checked that the sets $U_P = U \cup U_0$ and $U_Q = V \cup V_0$ are disjoint open neighbourhoods of P and Q respectively. ∎

Returning to the proof of Theorem 2.2, we can conclude that X is hereditarily normal. Indeed, if A and B are disjoint closed and discrete subsets of X, let $\gamma \subset \tau(X)$ be a separating family for $A \cup B$. Then the sets $U_A = \bigcup\{U : U \in \gamma \text{ and } U \cap A \neq \emptyset\}$ and $U_B = \bigcup\{U : U \in \gamma \text{ and } U \cap B \neq \emptyset\}$ are open disjoint neighbourhoods of A and B so that by Lemma 2.3 the space X is hereditarily normal.

Let us show that X is hereditarily collectionwise normal. By the hereditary normality of X it suffices to check that for every $Y \subset X$ and for any discrete (in Y) family $\mathcal{F} = \{F_s : s \in S\}$ of closed subsets of Y there is a disjoint family $\mathcal{U} = \{U_s : s \in S\} \subset \tau(Y)$ with $F_s \subset U_s$ for all $s \in S$.

Let $O_s = \mathrm{Int}(F_s)$ (the interiors in Y) and $A_s = F_s \setminus O_s$ for any $s \in S$. The sets A_s are closed and discrete in Y and the family $\{A_s : s \in S\}$ is discrete in Y. Therefore the set $A = \bigcup\{A_s : s \in S\}$ is closed and discrete in

Y and hence in X (here we use 2.0(3)). Let $\gamma \subset \tau(Y)$ be a separating family for A obtained in an obvious way from a separating family for A in X. For every $s \in S$ consider the set $V_s = \bigcup\{U : U \in \gamma,\ U \cap A_s \neq \emptyset\}$. Now if we put $W_s = V_s \setminus \bigcup\{\overline{O}_t : t \in S \setminus \{s\}\}$ (the closures in Y) and $U_s = O_s \cup W_s$, then U_s is open in Y (because the family $\{\overline{O}_s : s \in S\}$ is discrete in Y), and contains F_s since $\bigcup\{\overline{O}_t : t \in S \setminus \{s\}\} \cap F_s = \emptyset$.

We claim that the family $\{U_s : s \in S\}$ is disjoint. Indeed, if $s \neq t$, then

$$\begin{aligned} U_s \cap U_t &= (O_s \cup W_s) \cap (O_t \cup W_t) \\ &= (O_s \cap O_t) \cup (O_s \cap W_t) \cup (W_s \cap O_t) \cup (W_s \cap W_t), \end{aligned}$$

and all intersections in the above union are empty. Thus we have established collectionwise normality of Y.

Recall that in the presence of collectionwise normality paracompactness is equivalent to metacompactness [En, Theorem 5.3.3], so let us prove that any subspace Y of X is metacompact.

Take an open cover $\mathcal{U} = \{U_s : s \in S\} \subset \tau(Y)$ of Y, where S is considered to be well-ordered. For each $s \in S$, let $V_s = U_s \setminus \overline{\bigcup\{U_t : t < s\}}$ (the closure in Y). It is clear that the family $\mathcal{V} = \{V_s : s \in S\}$ is disjoint and $\bigcup \mathcal{V}$ is dense in Y. Thus the set $D = Y \setminus \bigcup \mathcal{V}$ is closed and discrete in Y and hence in X. For every $d \in D$ fix a $U_d \in \mathcal{U}$ such that $d \in U_d$. Pick a separating family γ for D in Y and let $V_d = U_d \cap U$, where U is the unique element of γ which contains d. It is clear that the family $\mathcal{V} \cup \{V_d : d \in D\}$ is a point-finite refinement of $\mathcal{U}$ and hence Y is paracompact. ∎

2.4. Proposition. *Suppose that (X,τ) is a submaximal space. If μ is a maximal topology on X stronger than τ, then*

(1) *(X,τ) and (X,μ) have the same Suslin numbers*;
(2) *(X,τ) and (X,μ) have the same densities*;
(3) *a subset $D \subset X$ is τ-discrete iff it is μ-discrete*;
(4) *if (X,τ) is regular, then (X,μ) is Tikhonov*;
(5) *if (X,τ) is normal, then so is (X,μ)*;
(6) *if (X,τ) is collectionwise Hausdorff, then so is (X,μ)*;
(7) *if (X,τ) is paracompact, then so is (X,μ).*

Proof. To prove (1) it suffices to show that $c(X,\mu) \leq c(X,\tau)$. Let $\gamma \subset \mu^*(X)$ be disjoint. Then each $U \in \gamma$ is μ-dense in itself and hence τ-dense in itself. Thus, the τ-interior of U is non-empty so that the family of τ-interiors of the elements of γ does not exceed $c(X,\tau)$. Therefore $|\gamma| \leq c(X,\tau)$ and (1) is proved.

Let D be τ-dense in X. If it were not μ-dense, then there would exist a μ-open set $U \subset X$ with $U \cap D = \emptyset$. But U is μ-dense in itself and hence τ-dense in itself. This implies that the τ-interior of U is not empty, which gives a contradiction since D is τ-dense. Thus, we have proved (2).

To show that (3) holds it suffices to verify that if D is μ-discrete, then it is τ-discrete. But if it were not so, then D would have non-empty interior in (X,τ) and hence in (X,μ), giving a contradiction.

Assume that (X,τ) is regular. Take any point $x \in X$ and a μ-closed subset $F \subset X$ with $x \notin F$. Let O be the τ-interior of F. The set $F \setminus O$ is closed and discrete in (X,τ) so that there is a τ-open neighbourhood V_x of x and $U \in \tau$ such that $U \supset F \setminus O$ and $V_x \cap U = \emptyset$.

Now let $W_x = V_x \setminus \overline{O}$ (the closure in μ) and $W_F = O \cup U$. It is immediate that W_x and W_F are μ-open disjoint neighbourhoods of x and F respectively, and hence μ is regular. To finish the proof of (4) observe that any regular extremally disconnected space is Tikhonov.

By Lemma 2.3, to settle (5) we only have to prove that any two closed and discrete subsets A and B of (X,μ) can be separated by disjoint open sets. But it follows from (3) that A and B are also τ-discrete (and thus closed). Therefore they can be separated in (X,τ) and the same open sets will separate them in (X,μ).

If (X,τ) is collectionwise Hausdorff and D is closed and discrete in (X,μ), then D is closed and discrete in (X,τ) by (3). Now the same open sets which separate the points of D in (X,τ) will separate them in (X,μ). This settles (6). Condition (7) is an immediate consequence of (6) and Theorem 2.2. ∎

Arkhangel'skiĭ and Collins [ArCo] proved that every submaximal separable topological group is countable and asked whether there exists a submaximal separable uncountable Hausdorff or Tikhonov space ([ArCo, Problem 8.6]). We answer this question by constructing examples of separable maximal uncountable spaces.

2.5. Example. *There is a maximal Hausdorff separable space of cardinality* 2^{2^ω}.

Proof. E. K. van Douwen [vD] constructed a ZFC example of a countable Tikhonov maximal space Y. Every maximal space has to be extremally disconnected [vD, Theorem 2.2], so that $|\beta Y| = 2^{2^\omega}$. Consider the set βY with the topology τ such that

(1) any open subset of Y belongs to τ;

(2) if $z \in \beta Y \setminus Y$, then the family $\mathcal{B}_z = \{\{z\} \cup ((U \cap Y) \setminus N) : U$ runs over all open neighbourhoods of z in βY and N over closed discrete subsets of $Y\}$ is a base of z in τ.

Denote by X the space βY with the topology introduced in (1) and (2). It is clear that X is separable, Hausdorff and $|X| = 2^{2^\omega}$.

Let us prove that X is submaximal. Take any dense subset D of X. As Y is dense and open in X, the set $D \cap Y$ is dense in Y and hence open in Y.

It follows from the maximality of Y that $E = Y \setminus D$ is closed and discrete in Y. Applying (2) we can conclude that E is closed and discrete in X as well.

The set $X \setminus D = E \cup ((X \setminus Y) \setminus D)$ is closed and discrete because so are E and $(X \setminus Y) \setminus D$. Thus, the complement to D in X is closed and D is open, which proves submaximality of X.

There exists a Hausdorff maximal topology μ on X stronger than τ. By Proposition 2.4(2) the space (X, μ) is separable. ∎

2.6. Example. *There is a maximal Tikhonov separable space X of power 2^ω.*

Proof. We will use once more van Douwen's space Y mentioned in the proof of 2.5. Take any infinite closed discrete subset $A = \{a_n : n \in \omega\} \subset Y$ with $a_i \neq a_j$ if $i \neq j$. Denote by $\mathcal{M}$ a quasi-disjoint family of infinite subsets of ω of power 2^ω.

We construct a topology on the set $X = (Y \setminus A) \cup \mathcal{M}$. If $y \in Y \setminus A$, then we define the base of X at y to be the family of all open neighbourhoods of y in $Y \setminus A$.

If $\xi \in \mathcal{M}$, then the base of X at ξ consists of the sets of the form $\{\xi\} \cup \bigcup\{U_k : k \in \xi \setminus P\}$, where the set $P \subset \xi$ is finite, and $U_k = V_k \cap (Y \setminus A)$ for some open (in Y) neighbourhood V_k of the point a_k, and the sets V_k are disjoint.

Let us check that the space X is Tikhonov and submaximal. Of course, it is evident that X is separable and its cardinality is 2^ω.

It suffices to establish complete regularity of X at each point ξ of $\mathcal{M}$. Let $U = \{\xi\} \cup \bigcup\{U_k : k \in \xi \setminus C\}$ be a basic open neighbourhood of ξ. Without loss of generality we may assume that U_k is clopen in $Y \setminus A$. Represent $\xi \setminus C$ as a strictly increasing subsequence $\{n_i : i \in \omega\}$ of ω. Let $f(\xi) = 0,\ f(x) = 1/(i+1)$ if $x \in U_{n_i}$ and $f(x) = 1$ if $x \notin U$. It is immediate to verify that f is a continuous function on X which witnesses complete regularity of X at x.

To establish submaximality, take a dense subset D of X. Then $D \cap (Y \setminus A)$ is open since it is open in X and $Y \setminus A$ is submaximal. Therefore $N = (Y \setminus A) \setminus D$ is closed and discrete in Y. Remembering the definition of basic neighbourhoods at all points of $\mathcal{M}$ we conclude that N is closed in X as well. Hence $X \setminus D = N \cup (\mathcal{M} \setminus D)$ is closed and discrete in X, whence D is open in X and we have proved that X is submaximal.

To finish the proof, consider a maximal Hausdorff topology μ on X which is stronger than $\tau(X)$. Using 2.0(6) we conclude that (X, μ) is a maximal space. It follows from 2.4(2) and 2.4(4) that the space (X, μ) is Tikhonov and separable. ∎

The following proposition shows that the cardinality of the previous example is the largest possible.

2.7. Proposition. *If X is a submaximal regular space, then $|X| \le 2^{d(X)}$. In particular, if X is separable, then $|X| \le 2^{\omega}$.*

Proof. Let $\kappa = d(X)$ and choose a dense subset D of X with $|D| = \kappa$. It is open together with every superset of D by submaximality of X. Hence $X \setminus D$ is a closed discrete subset of X. For each $x \in X \setminus D$ choose an open neighbourhood U_x of x such that $\overline{U}_x \cap (X \setminus D) = \{x\}$. It is easy to check that the correspondence $x \to U_x$ is an injection of $X \setminus D$ into the set of all subsets of D. Therefore $|X| = |D| + |X \setminus D| \le 2^{\kappa}$. ∎

Arkhangel'skiĭ and Collins [ArCo] asked whether every submaximal space is σ-discrete. Schröder [Sch] showed that this is equivalent to asking whether every maximal space is σ-discrete and gave an example of a maximal space of measurable cardinality which does not have countable pseudocharacter. But the question of whether every maximal space of non-measurable cardinality is σ-discrete remains open. We prove that this is the same as asking whether every submaximal space has countable pseudocharacter.

2.8. Proposition. *The following statements are equivalent*:

(1) *every submaximal space of non-measurable cardinality has countable pseudocharacter*;

(2) *every submaximal space of non-measurable cardinality is σ-discrete*;

(3) *every maximal space of non-measurable cardinality is σ-discrete*;

(4) *no submaximal space of non-measurable cardinality has a P-point.*

Proof. The equivalence of (2) and (3) was proved by Schröder [Sch]. It is evident that (2) implies (1). Therefore (3)⇒(1). Let us check that (1) implies (3).

Suppose that (X, τ) is a maximal space which is not σ-discrete and let D denote the discrete space of cardinality ω_1. Denote by Y the set $(X \times D) \cup \{p\}$, where $p \notin X \times D$, with the following topology:

(i) $X \times D$ has the product topology and

(ii) U is a neighbourhood of p if $p \in U$ and $U \cap (X \times \{d\})$ is dense in $X \times \{d\}$ for all but countably many $d \in D$.

It is straightforward to verify that Y is submaximal. Since X is not σ-discrete, a countable intersection of dense subsets of X is non-empty and hence $\psi(Y) = \omega_1$.

As it is trivial that (1)⇒(4) we only have to show that (4)⇒(1), and this is equivalent to (4)⇒(3).

If there exists a maximal space X, no open subset of which is σ-discrete, then any countable intersection of dense subsets of X is open and dense, and hence p is a P-point of Y, where Y is the space constructed in the proof of (1)$\Rightarrow$(3). However, if Z is any maximal space which is not σ-discrete, then we define $\mathcal{C}$ to be a maximal pairwise disjoint family of open σ-discrete subsets of Z. It follows that $\bigcup\mathcal{C}$ is open and σ-discrete and $\mathrm{cl}(\bigcup\mathcal{C})\setminus\bigcup\mathcal{C}$ is discrete. Thus $X = Z\setminus \mathrm{cl}(\bigcup\mathcal{C}) \neq \emptyset$ and each open set of X fails to be σ-discrete. Now that we have shown that (4)$\Rightarrow$(3) the proposition is proved. ∎

Let us prove that under some additional conditions submaximal spaces do have countable pseudocharacter.

2.9. THEOREM. *Let (X,τ) be a submaximal space such that $\chi(X)=\omega_1$. Then $\psi(X)=\omega$.*

Proof. Assume that for some $x\in X$ we have $\psi(x,X)=\omega_1$, and let $\mathcal{B}_x=\{V_\alpha:\alpha\in\omega_1\}$ be a local base at x. For each $\delta\in\omega_1$, observe that $\mathrm{Int}(\bigcap\{V_\alpha:\alpha\in\delta\})\neq\emptyset$ since otherwise this set would be closed and discrete and hence we could find an open set U with $U\cap\bigcap\{V_\alpha:\alpha\in\delta\}=\{x\}$ so that $\{x\}$ would be a G_δ.

Denote by H_δ the set $\bigcap\{V_\alpha:\alpha\in\delta\}$ and note that $x\in\mathrm{cl}(\mathrm{Int}(H_\delta))$, since otherwise $H_\delta\setminus\mathrm{cl}(\mathrm{Int}(H_\delta))=\{x\}\cup(H_\delta\setminus\mathrm{cl}(\mathrm{Int}(H_\delta)))$ has empty interior and hence is closed and discrete. A repeat of the above argument would again make $\{x\}$ a G_δ.

Now choose $x_0\in V_0\setminus\{x\}$ and recursively

$$x_\alpha\in\bigcap\{V_\beta:\beta\in\alpha\}\setminus(\{x\}\cup\{x_\beta:\beta\in\alpha\}).$$

To show that this choice can be made for all $\alpha\in\omega_1$, suppose to the contrary that for some $\delta\in\omega_1$,

$$\bigcap\{V_\alpha:\alpha\in\delta\}\subset\{x_\alpha:\alpha\in\delta\}\cup\{x\}.$$

Then

$$\bigcap\{V_\alpha:\alpha\in\delta\}\cap\bigcap\{X\setminus\{x_\gamma\}:\gamma\in\delta\}$$

is a G_δ equal to $\{x\}$. Thus we can choose x_α for each $\alpha\in\omega_1$; let $M=\{x_\alpha:\alpha\in\omega_1\}$.

Note first that $\mathrm{Int}(M)\neq\emptyset$, since otherwise M would be closed and discrete and hence $X\setminus M$ would be a neighbourhood of x, contradicting the fact that the well-ordered net $\{x_\alpha:\alpha\in\omega_1\}$ converges to x.

Secondly, we claim that $\mathrm{Int}(M)$ cannot be countable, for otherwise $M\setminus\mathrm{Int}(M)$ is closed and discrete and for some $\gamma\in\omega_1$, contains $\{x_\alpha:\gamma\in\alpha\}$, again giving a contradiction.

However, since the net M converges to x, it follows that each point $x_\alpha\in\mathrm{Int}(M)$ must have a countable neighbourhood contained in $\mathrm{Int}(M)$. Let $\{C_\alpha:\alpha\in\omega_1\}$ be a maximal disjoint family of countable open sets contained in $\mathrm{Int}(M)$, and for each $\alpha\in\omega_1$, choose $y_\alpha\in C_\alpha$. The set $D=\{y_\alpha:\alpha\in\omega_1\}$

is closed and discrete and $x \notin D$. Thus for some $\nu \in \omega_1$, $X \setminus D \supset V_\nu$, again giving a contradiction. ∎

2.10. PROPOSITION. *Let X be a submaximal Hausdorff space with $\pi w(X) \le \kappa$. Then the set $L = \{x \in X : \Delta(x,X) < \kappa\}$ is dense in X, and X is the union of less than κ closed discrete subspaces.*

Proof. If $\Delta(x,X) = \kappa$ for all $x \in X$, then choosing recursively two distinct points in each element of the π-base, in such a way that all the points chosen are distinct, gives two disjoint dense subsets of X, contradicting the submaximality of X. Thus $L \ne \emptyset$. If L is not dense, then $X \setminus \mathrm{cl}(L)$ would be a submaximal Hausdorff space with π-weight less than or equal to κ, but with $\Delta(X) = \kappa$, again giving a contradiction.

To prove the second assertion, let $\mathcal{C}$ be a maximal disjoint family of open sets, each of cardinality less than κ. Clearly $\bigcup\mathcal{C} \subset L$ and $\bigcup\mathcal{C}$ is dense in X, for otherwise $L \setminus \mathrm{cl}(\bigcup\mathcal{C}) \ne \emptyset$, contradicting the maximality of $\mathcal{C}$. It follows that $X \setminus \bigcup\mathcal{C}$ is closed and discrete. Furthermore, if for each $C \in \mathcal{C}$, we choose $x_C \in C$, then the set $\{x_C : C \in \mathcal{C}\}$ has empty interior and hence is closed and discrete. Thus, since each element of $\mathcal{C}$ is of cardinality less than κ, it follows that $\bigcup\mathcal{C}$ is the union of less than κ closed discrete subsets. It follows that $X = (X \setminus \bigcup\mathcal{C}) \cup \bigcup\mathcal{C}$ and the first set is closed and discrete, while the second is the union of less than κ closed and discrete sets; the result follows. ∎

2.11. COROLLARY. *If X is a submaximal Hausdorff space with $\pi w(X) \le \aleph_1$, then X is strongly σ-discrete.*

2.12. COROLLARY. *If X is a submaximal Hausdorff space such that $c(X) < \mathrm{cf}(\pi w(X))$, then $d(X) < \pi w(X)$.*

Proof. Since $c(X) < \pi w(X)$, each element of the family $\mathcal{C}$ of Proposition 2.10 is of cardinality less than $\pi w(X)$ and $|\mathcal{C}| \le c(X)$. The result follows since $\bigcup\mathcal{C}$ is dense in X. ∎

2.13. COROLLARY. *If X is a submaximal Hausdorff space with $\pi w(X) = \aleph_1$, then $c(X) = d(X)$.*

However, for every cardinal κ, submaximal spaces with $c(X) = \omega < d(X) = \kappa$ can easily be constructed as the following example shows:

2.14. EXAMPLE. Let $\mathcal{D}$ denote the filter of subsets of I^{2^κ} whose complement is of cardinality less than κ, and let $\mathcal{U}$ be any ultrafilter of dense subsets of I^{2^κ} containing $\mathcal{D}$. If we denote by (X,τ) the submaximal extension of I^{2^κ} by $\mathcal{U}$, then clearly $d(X) \ge \kappa$, but it is well known that each open set in (X,τ) is dense in some open subset of I^{2^κ} and hence $c(X) = \omega$. It is easy to see that $\pi w(X) > |X|$.

That $e(X) = s(X) = |X| \leq w(X)$ for all submaximal spaces has been shown in [ArCo, Proposition 5.3]. It is a long-standing open question whether $w(X) \leq 2^{2^{d(X)}}$ for all Hausdorff spaces X, although since $|X| \leq 2^{2^{d(X)}}$, it is clear that $w(X) \leq 2^{2^{2^{d(X)}}}$. However, as we now show, the former inequality is valid for Hausdorff submaximal spaces and is a consequence of the following simple result.

2.15. Proposition. *If* (X, τ) *is a Hausdorff submaximal space*, *then* $w(X) \leq |X| \cdot 2^{d(X)}$.

Proof. Let D be a dense, hence open, subset of X of cardinality $d(X)$; hence $w(X) \leq w(D)|X \setminus D| \sup\{\chi(X,x) : x \in X \setminus D)\}$. Furthermore, since $X \setminus D$ is discrete, each point of $X \setminus D$ has a neighbourhood of the form $\{x\} \cup U$, where $U \subset D$. Thus for each $x \in X \setminus D$, $\chi(X,x) \leq 2^{d(X)}$ and the result follows since $|X \setminus D| \leq |X|$ and $w(D) \leq 2^{d(X)}$. ■

From Example 2.14 we infer that there are submaximal Hausdorff spaces with countable Suslin number and $w(X) > |X| \geq 2^{\omega}$. Hence $d(X)$ cannot be replaced by $c(X)$ in Proposition 2.15.

The following example shows that the bound for $w(X)$ obtained in 2.15 is the best possible.

2.16. Example. Suppose now that X is the submaximal extension of $\beta\mathbb{R}$ by some ultrafilter of dense sets which contains the rationals. Then it is easy to verify that $c(X) = d(X) = \omega$, while $\omega < \pi w(X) \leq \mathfrak{c} < w(X) = |X| = 2^{\mathfrak{c}}$. Furthermore, if Y is the discrete space of cardinality $\mathfrak{c}$ then $X \times Y$ is submaximal and $d(X \times Y) = \pi w(X \times Y) = \mathfrak{c}$.

The following proposition deals with weakly collectionwise Hausdorff spaces. Theorem 2.2 implies that every maximal weakly collectionwise Hausdorff space is hereditarily paracompact. However, not all maximal spaces are even Tikhonov: the example we constructed in 2.5 is not completely regular because otherwise it would have cardinality at most continuum by Proposition 2.7.

It is worth mentioning that under some set-theoretic assumptions V. I. Malykhin [Ma] constructed examples of Tikhonov non-normal maximal spaces as well as of normal non-paracompact maximal spaces.

2.17. Proposition. *Every weakly collectionwise Hausdorff maximal space of non-measurable cardinality is a* Q*-set.*

Proof. We need to show that every subset of X is a G_δ; however, every subset of a submaximal space is the intersection of an open set with a closed set (see [ArCo]) and hence we need only show that each closed subset of X is a G_δ. To this end, let C be a closed set in X; then $C = \mathrm{cl}(\mathrm{Int}(C)) \cup (C \setminus \mathrm{cl}(\mathrm{Int}(C)))$, which, since X is extremally disconnected and submaximal, is the union of an open set and a closed discrete set. Thus to show that every closed set is a G_δ, it suffices to show that every closed discrete subset of X is a G_δ. Now if D is a closed discrete subset of X then for any $x \in D$ there is an open $U_x \ni x$ such that the family $\{U_x : x \in D\}$ is disjoint. For each $x \in D$ choose open sets $F_n(x) \subset U_x$ such that $\bigcap\{F_n(x) : n \in \omega\} = \{x\}$—this is possible because $|X|$ is non-measurable and the open neighbourhoods of a point in a maximal space form an ultrafilter on the complement of the point [vD]. Clearly then $D = \bigcap\{\bigcup\{F_n(x) : x \in D\} : n \in \omega\}$. ∎

It is an old unsolved problem whether every regular submaximal space is disconnected. In any submaximal space the boundary of every open set is closed and discrete. We show that if a submaximal space has a base of open sets with finite boundaries, then it has a lot of clopen subsets.

2.18. THEOREM. *If X is a regular submaximal space which has a base consisting of sets with finite boundaries, then X has a π-base of clopen sets.*

Proof. Fix a base $\mathcal{B}$ in X such that the boundary $\mathrm{Bd}(U)$ is finite for every $U \in \mathcal{B}$. Let O be a non-empty open subset of X. Fix a point $p \in O$ and let γ be a maximal disjoint family of open subsets of X with the following properties:

(1) $\overline{U} \cap \overline{V} = \emptyset$ for distinct $U, V \in \gamma$;
(2) $p \notin \overline{U}$ for every $U \in \gamma$.

If $W = \bigcup \gamma$, then $A = \mathrm{Bd}(W)$ is closed and discrete in X. Hence there exists a neighbourhood $U \in \mathcal{B}$ of the point p such that $U \subset O$ and $\overline{U} \cap (A \setminus \{p\}) = \emptyset$.

No finite subfamily of γ can cover the set $U \setminus \{p\}$ because of (2). Therefore, there is a $V \in \gamma$ with $V \cap U \neq \emptyset$ and $V \cap \mathrm{Bd}(U) = \emptyset$.

The set $V \cap U$ is clopen in X. Indeed, the boundary of $U \cap V$ is contained in $\mathrm{Bd}(U) \cup \mathrm{Bd}(V)$. But $\overline{U \cap V} \cap (\mathrm{Bd}(U) \cup \mathrm{Bd}(V)) = \emptyset$ and we have found a non-empty clopen subset of X which lies in O. ∎

3. Nice properties of arbitrary submaximal groups. Topological groups often behave better than Tikhonov spaces. We confirm this once more for submaximal and similar groups.

3.1. PROPOSITION. *Let G be a group of cardinality $\kappa > \omega$ (without any topology). Then for each $\alpha < \kappa$ there exist subsets G_α and H_α of G with the following properties*:

(1) *G_α is a subgroup of G for all $\alpha < \kappa$;*
(2) *if $\alpha < \beta < \kappa$, then $G_\alpha \subset G_\beta$ and $G_\alpha \neq G_\beta$;*
(3) *$|G_\alpha| = |\alpha|$ for all $\alpha < \kappa$;*
(4) *$G_\alpha = \bigcup\{H_\nu : \nu \le \alpha\}$ for all $\alpha < \kappa$;*

(5) $\bigcup\{H_\alpha : \alpha < \kappa\} = G$ *and* $H_\alpha \cap H_\beta = \emptyset$ *if* $\alpha \neq \beta$;

(6) *if* $g \in H_\alpha$ *and* $\alpha < \beta$, *then* $g \cdot H_\beta = H_\beta \cdot g = H_\beta$;

(7) $H_\alpha = H_\alpha^{-1}$ *for all* $\alpha < \kappa$;

(8) *if* A *is a cofinal subset of* κ, *then the cardinality of* $\bigcup\{H_\alpha : \alpha \in A\}$ *is* κ.

We call the family $\{H_\alpha : \alpha < \kappa\}$ *a canonical decomposition of* G.

Proof. Let $G = \{g_\alpha : \alpha < \kappa\}$, where $g_\alpha \neq g_\beta$ if $\alpha \neq \beta$. Our first step is to set $G_0 = \langle\{g_0\}\rangle$. Suppose that $\beta < \kappa$ and that for every $\alpha < \beta$ we have constructed a subgroup G_α of G with the following properties:

(i) $G_\alpha \subset G_\gamma$ and $G_\alpha \neq G_\gamma$ if $\alpha < \gamma < \beta$;

(ii) $|G_\alpha| = |\alpha|$ for all $\alpha < \beta$;

(iii) $\{g_\gamma : \gamma \leq \alpha\} \subset G_\alpha$ for every $\alpha < \beta$.

Let $P_\beta = \bigcup\{G_\alpha : \alpha < \beta\}$. Then it follows from (ii) that $P_\beta \neq G$ and we can choose $\beta^* = \min\{\alpha < \kappa : g_\alpha \notin P_\beta\}$. Now we can construct the group $G_\beta = \langle P_\beta \cup \{g_{\beta^*}\}\rangle$. After we are through with the construction of G_α for all $\alpha < \kappa$, the family $\{G_\alpha : \alpha < \kappa\}$ satisfies (i)–(iii) as well as the property

(iv) $\bigcup\{G_\alpha : \alpha < \kappa\} = G$.

For every $\alpha < \kappa$ let $H_\alpha = G_\alpha \setminus \bigcup\{G_\beta : \beta < \alpha\}$.

It is evident that the sets G_α and H_α satisfy (1)–(5). To prove (6) observe that $g \in G_\alpha$ and G_α is a subgroup of G_γ for every γ such that $\alpha \leq \gamma \leq \beta$. This implies $g \cdot G_\gamma = G_\gamma \cdot g = G_\gamma$ for all γ and $g \cdot H_\beta = H_\beta \cdot g = H_\beta$ so that (6) also holds. The property (7) is evident.

To see that (8) holds, take any cofinal $A \subset \kappa$. Since $|H_{\alpha+1}| = |G_\alpha| = |\alpha|$ we have

$$\Big|\bigcup\{H_\alpha : \alpha \in A\}\Big| = \Big|\bigcup\{G_\alpha : \alpha \in A\}\Big| = |G| = \kappa. \ \blacksquare$$

3.2. Proposition. *Suppose that* G *is a group with a non-discrete irresolvable topology* τ *such that all left* (*or all right*) *translations in* G *are continuous and* $|G| = \Delta(G,\tau) = \kappa > \omega$. *Take any canonical decomposition* $\{H_\alpha : \alpha < \kappa\}$ *of* G. *For a subset* $A \subset \kappa$, *let* $H_A = \bigcup\{H_\alpha : \alpha \in A\}$. *Then the family* $\xi = \{A \subset \kappa : \mathrm{Int}(H_A) \neq \emptyset\}$ *is a free ultrafilter on* κ.

Proof. Let $A \subset \kappa$. The sets H_A and $H_{\kappa\setminus A}$ are disjoint and $G = H_A \cup H_{\kappa\setminus A}$. Consequently, one of the sets H_A or $H_{\kappa\setminus A}$ has non-empty interior as G is irresolvable. Hence $A \in \xi$ or $\kappa \setminus A \in \xi$.

We will show that both these sets cannot belong to ξ, because the one whose interior is not empty has to be dense in G.

Suppose, for example, that $U = \mathrm{Int}(H_A) \neq \emptyset$. If U is not dense in G, then there is a non-empty open $V \subset G$ with $V \cap U = \emptyset$. Pick $x \in U$ and $y \in V$. Then $W = (yx^{-1} \cdot U) \cap V \neq \emptyset$. The set W is open and non-empty, which implies $|W| = \kappa$. On the other hand, W is a subset of a translate of H_A.

Claim. *For every $g \in G$, the set $(g\cdot H_A)\setminus H_A$ has cardinality less than κ.*

Proof. Pick an $\alpha < \kappa$ such that $g \in H_\alpha \subset G_\alpha = \bigcup\{H_\nu : \nu \le \alpha\}$ (see 3.1(4)). Now if $\alpha < \beta$, then $g\cdot H_\beta = H_\beta$, by 3.1(6). Thus, $(g\cdot H_A)\setminus H_A \subset G_\alpha$ and we can apply 3.1(3). ∎

It follows from the Claim that $|W \setminus H_A| < k$ and hence $W \setminus H_A$ has empty interior in G. But W lies outside $U = \mathrm{Int}(H_A)$, which implies that

$$W = (W \setminus H_A) \cup (W \cap H_A) \subset (W \setminus H_A) \cup (H_A \setminus U).$$

Thus W is covered by two disjoint subsets $W \setminus H_A$ and $H_A \setminus U$ with empty interior, and hence both are dense in W. As a consequence, W is resolvable. W. Comfort and Li Feng [CoLF] proved that a union of resolvable spaces is resolvable. Since all possible left translates of W cover the group G, we conclude that G is resolvable, which is a contradiction.

We have shown that if $U = \mathrm{Int}(H_A) \ne \emptyset$, then U is dense in G. Hence exactly one of the sets A and $\kappa \setminus A$ belongs to ξ and it is established that ξ is an ultrafilter on κ. The fact that ξ is a free ultrafilter follows from $H_\alpha \notin \xi$ for all $\alpha \in \kappa$ and 3.1(5). ∎

3.3. Theorem. *Suppose that G is a group of non-measurable cardinality $\kappa > \omega$ with a non-discrete topology τ such that all left* (*or all right*) *translations are continuous*, *$\Delta(G,\tau) = \kappa$ and (G,τ) is irresolvable. Take any canonical decomposition $\{H_\alpha : \alpha < \kappa\}$ of the group G. Then there exists a family $\{A_n : n \in \omega\}$ of subsets of κ such that*:

(1) $\bigcup\{A_n : n \in \omega\} = \kappa$;

(2) *each set $H_n = H_{A_n} = \bigcup\{H_\alpha : \alpha \in A_n\}$ is closed and nowhere dense in G*;

(3) $\bigcup\{H_n : n \in \omega\} = G$.

In particular, *(G,τ) is of first category.*

Proof. Use Proposition 3.2 to conclude that the family $\xi = \{A \subset \kappa : \mathrm{Int}(H_A) \ne \emptyset\}$ is a free ultrafilter on κ. As κ is a non-measurable cardinal, there exists a family $\{B_n : n \in \omega\}$ such that $B_n \in \xi$ for every n and $\bigcap\{B_n : n \in \omega\} = \emptyset$. It is easy to see that $\bigcap\{H_{B_n} : n \in \omega\} = \emptyset$ and therefore $\bigcup\{H_{\kappa\setminus B_n} : n \in \omega\} = G$. Let $A_n = \kappa \setminus B_n$. Every subset $H_n = H_{A_n}$ has empty interior, because $A_n \notin \xi$ for all $n \in \omega$. Therefore $\{H_n : n \in \omega\}$ is a family of nowhere dense sets whose union covers G, and this proves our theorem. ∎

3.4. Corollary. *Let G be a group of non-measurable cardinality. If τ is a non-discrete irresolvable topology on G such that all left* (*or all right*) *translations are continuous*, *then (G,τ) is of first category.*

Proof. If G is such a group, denote by κ its dispersion character. The case $\kappa = \omega$ is trivial, so we assume that $\kappa > \omega$. There is an open neighbourhood U of the identity with $|U| = \kappa$. Then the group $G_0 = \langle U \rangle$ is open in G and the dispersion character of G_0 coincides with its power. But then G_0 is of first category by Theorem 3.3. Now G contains an open set of first category, and hence is of first category. ■

3.5. Corollary. *Every irresolvable topological group of non-measurable cardinality is of first category.*

3.6. Corollary. *Every submaximal topological group of non-measurable cardinality is strongly σ-discrete.*

Proof. Indeed, in a submaximal space every nowhere dense subset is closed and discrete and every submaximal space is irresolvable, so we can apply 3.5. ■

3.7. Corollary. *If there is no measurable cardinal, then every irresolvable topological group is of first category.*

3.8. Corollary. *If there is no measurable cardinal, then every submaximal topological group is a countable union of its closed discrete subspaces.*

3.9. Remark. The result of Corollary 3.8 is true in ZFC for maximal topological groups, because every such group has an open countable subgroup by a result of Malykhin [Ma].

Following Guran [Gu] we say that a topological group G is *κ-bounded* for some cardinal κ if for every open neighbourhood U of the identity there exists a subset $A \subset G$ with $|A| \le \kappa$ such that $A \cdot U = G$. It is known that if G satisfies one of the conditions $c(G) \le \kappa$ or $l(G) \le \kappa$, then G is κ-bounded [Gu]. One can easily verify that a subgroup of a κ-bounded group is also κ-bounded. This fact will be used below.

3.10. Theorem. *Let κ be an infinite cardinal number. If G is a κ-bounded submaximal topological group, then $|G| \le \kappa$.*

Proof. Let λ be the dispersion character of G. Take an open subgroup N of G such that $|N| = \lambda$. Then G is a discrete union of translates of N and it is impossible to cover G by less than $|G/N|$ translates of N. Therefore $|G/N| \le \kappa$, so that if we show that $\lambda = |N| \le \kappa$, then the theorem will be proved.

Consider first the case when λ is a limit cardinal. We need the following simple result, which seems to be known as a part of mathematical folklore (see, for example, the proof of Theorem 1.1 of [CoRo]).

3.11. Lemma. *Let G be a topological group. Suppose that H is a closed and discrete subgroup of G. Take any open neighbourhood V of the identity in G such that $V \cap H = \{e\}$.*

(1) *If U is an open symmetric neighbourhood of the identity with $U^2 \subset V$, then $H \not\subset A \cdot U$ for any $A \subset G$ such that $|A| < |H|$.*

(2) *If U is an open symmetric neighbourhood of the identity with $U^4 \subset V$, then the family $\mathcal{U} = \{U \cdot g : g \in H\}$ is discrete in G.*

Proof. Indeed, if there were an A as in (1), then for some $a \in A$ the set $a \cdot U$ would contain at least two elements of H; so let $h, g \in H \cap (a \cdot U)$, $h \neq g$. Then there exist $u, v \in U$ such that $h = au$ and $g = av$. Therefore $g^{-1}h = v^{-1}u \in U^2 \subset V$ and $e \neq hg^{-1} \in H$, which is a contradiction.

To prove (2), take any $g \in G$. We prove that $U \cdot g$ can intersect at most one element of $\mathcal{U}$. If not, then there are $u, u_1, v, v_1 \in U$ and distinct $f, h \in H$ such that $uf = u_1 g$ and $vh = v_1 g$. Therefore $e \neq fh^{-1} = u^{-1}u_1 v_1^{-1} v \in U^4 \subset V$ and this contradiction proves the second part of the lemma. ∎

To continue the proof of the theorem, suppose that $\kappa < \lambda$ and take a subset $P \subset N$ such that $\kappa < |P| = \gamma < \lambda$. Then $|\langle P \rangle| = \gamma$ and hence $H = \langle P \rangle$ is closed and discrete in G. By Lemma 3.11(1) the group G is not κ-bounded, which is a contradiction. Hence our theorem is proved for all limit cardinals λ.

Now if λ is a successor cardinal, then $\lambda = \mathrm{cf}(\lambda)$. The group N is submaximal and the cardinality and dispersion character of N coincide, so we can take a canonical decomposition $\{H_\alpha : \alpha < \lambda\}$ for N. It follows from Proposition 3.2 that there is a cofinal set $A \subset \lambda$ such that $H_A = \bigcup \{H_\alpha : \alpha \in A\}$ is closed and discrete in G and does not contain the identity of G.

Now pick a point $x_\alpha \in H_\alpha$ for every $\alpha \in A$. The set $Y = \{x_\alpha : \alpha \in A\}$ is closed and discrete in G, being a subset of H_A. Since A is cofinal in λ, the cardinality of Y is λ. There is an open neighbourhood U of the identity such that $U \cap H_A = \emptyset$. Choose an open symmetric neighbourhood V of the identity with $V^2 \subset U$.

Let us verify that

$$(*) \qquad \text{if } P \subset G \text{ and } |P| < \lambda, \text{ then } P \cdot V \neq G.$$

Assume, on the contrary, that $P \cdot V = G$ for some $P \subset G$ with $|P| < \lambda$. The set Y has cardinality λ, while $|P| < \lambda$. Therefore, there is a $p \in P$ such that $p \cdot V$ contains at least two different points of Y, say x_α and x_β, where $\alpha < \beta$. Thus, there are $u, v \in V$ such that $pu = x_\alpha$ and $pv = x_\beta$. It follows that $p = x_\alpha u^{-1} = x_\beta v^{-1}$ so that

$$(**) \qquad x_\beta^{-1} \cdot x_\alpha = v^{-1} \cdot u.$$

It follows from 3.1(7) that $x_\beta^{-1} \in H_\beta$, and hence $x_\beta^{-1} \cdot x_\alpha \in H_\beta \subset H_A$ by 3.1(6). But it is immediate from $(**)$ that $x_\beta^{-1} \cdot x_\alpha \in V \cdot V \subset U$; this contradiction proves that $P \cdot V \neq G$. Now, $(*)$ implies $\kappa \geq \lambda$. ∎

The following corollary strengthens Theorem 8.4 of [ArCo] which states that the density and cardinality of any submaximal group are equal.

3.12. COROLLARY. *$c(G) = |G|$ for every submaximal group G. In particular, a submaximal group with the Suslin property is countable.*

Proof. It is known that if $c(G) \le \kappa$, then G is κ-bounded [Gu]. Now apply Theorem 3.11 to conclude that $|G| \le \kappa$. ∎

The following lemma will be of use for proving hereditary paracompactness of some submaximal topological groups.

3.13. LEMMA. *Let X be a regular space. Suppose that $X = \bigcup\{H_n : n \in \omega\}$, where the subsets H_n have the following properties*:

(1) *H_i is closed and discrete in X for all $i \in \omega$;*

(2) *$H_i \cap H_j = \emptyset$ if $i \ne j$;*

(3) *for every $x \in X$ there is an open neighbourhood V_x of x such that for any $i \in \omega$ the family $\{V_x : x \in H_i\}$ is discrete in X.*

Then X is weakly collectionwise Hausdorff.

Proof. Let D be a closed discrete subset of X. Then $D = \bigcup\{D_n : n \in \omega\}$, where $D_n = D \cap H_n$ for all $n \in \omega$.

For every point $x \in D_0$, let W_x be an open neighbourhood of x such that $W_x \subset V_x$ and $\overline{W}_x \cap (D \setminus \{x\}) = \emptyset$. Then the family $\delta_0 = \{W_x : x \in D_0\}$ is discrete and $\overline{\bigcup \delta_0} \cap (D \setminus D_0) = \emptyset$.

Suppose that $n \ge 1$ and that for every $k < n$ and for all $x \in D_k$ we have defined an open neighbourhood W_x of x with $W_x \subset V_x$ and such that the families $\delta_k = \{W_x : x \in D_k\}$, $k = 0, \ldots, n-1$, have the following properties:

(i) δ_k is discrete for every $k = 0, 1, \ldots, n-1$;

(ii) $\overline{\bigcup \delta_k} \cap (D \setminus D_k) = \emptyset$;

(iii) $(\bigcup \delta_i) \cap (\bigcup \delta_j) = \emptyset$ if $i \ne j$.

Every point $x \in D_n$ has an open neighbourhood W_x in X such that $W_x \subset V_x$, $\overline{W}_x \cap (D \setminus \{x\}) = \emptyset$, and

$$\overline{W}_x \cap \Big(\overline{\bigcup \delta_0} \cup \ldots \cup \overline{\bigcup \delta_{n-1}}\Big) = \emptyset.$$

It is clear that the family $\delta_n = \{W_x : x \in D_n\}$ satisfies (i)–(iii) as well. Thus,when we construct δ_n for all natural n, the family $\{W_x : x \in D\}$ will be disjoint and $x \in W_x$ for every $x \in D$. This proves that X is weakly collectionwise Hausdorff. ∎

3.14. THEOREM. *Every submaximal group G of non-measurable cardinality is hereditarily paracompact.*

Proof. By Theorem 2.2 it suffices to prove that G is weakly collectionwise Hausdorff. It is clear that G has an open subgroup whose cardinality and dispersion character are equal. If we prove hereditary paracompactness of this open subgroup of G, then G will be hereditarily paracompact. Therefore, without loss of generality we may assume that $|G| = \kappa$ is an uncountable cardinal which coincides with the dispersion character of G.

Take a canonical decomposition $\{H_\alpha : \alpha < \kappa\}$ of G. Applying Theorem 3.3 find a family $\{A_n : n \in \omega\}$ of subsets of κ such that $\bigcup\{A_n : n \in \omega\} = \kappa$ and $H_n = H_{A_n} = \bigcup\{H_\alpha : \alpha \in A_n\}$ is closed and discrete in G. For each $n \in \omega$ find an open neighbourhood U_n of the identity e with $U_n \cap H_n \subset \{e\}$. There exists a symmetric open neighbourhood V_n of e such that $V_n^4 \subset U_n$.

For every $\alpha \in A_n$, let $W_\alpha^n = V_n \cdot H_\alpha$. We will show that the family $\gamma_n = \{W_\alpha^n : \alpha \in A_n\}$ is discrete in G for all $n \in \omega$.

Let $g \in G$. It suffices to prove that $O = V_n \cdot g$ can intersect at most one element of γ_n.

Suppose, on the contrary, that there exist $\alpha, \beta \in A_n$, $\alpha < \beta$, such that $W_\alpha^n \cap O \neq \emptyset \neq W_\beta^n \cap O$. Then there are $p \in H_\alpha$ and $q \in H_\beta$ with the property that $(V_n \cdot p) \cap O \neq \emptyset \neq (V_n \cdot q) \cap O$. Pick $u, v, u_1, v_1 \in V_n$ such that $ug = vp$ and $u_1 g = v_1 q$. Note that these equalities imply $e \neq pq^{-1} = v^{-1} u u_1^{-1} v_1 \in V^4 \subset U$. But $q^{-1} \in H_\beta$ by 3.1(7) and $pq^{-1} \in H_\beta$ by 3.1(6). The contradiction with $H_n \cap U_n \subset \{e\}$ shows that each γ_n is discrete in G.

For every $\alpha \in A_n$, the subgroup $G_\alpha = \bigcup\{H_\nu : \nu \leq \alpha\}$ is closed and discrete in G because $|G_\alpha| < \kappa$. Let U be an open neighbourhood of e such that $U \cap G_\alpha = \{e\}$. Choose a symmetric open neighbourhood V of e with $V^4 \subset U$. Then the family $\mu = \{V \cdot x : x \in G_\alpha\}$ is discrete in G—this follows from Lemma 3.11(2).

Take any $n \in \omega$. If $\alpha \in A_n$ and $p \in H_\alpha$, let $V_p = (V \cdot p) \cap W_\alpha^n$. It is straightforward to check that the family $\mu_n = \{V_p : p \in H_n\}$ is discrete in G and $p \in V_p$ for every $p \in H_n$.

Thus, we have obtained a family $\{H_n : n \in \omega\}$ for the group G which satisfies the conditions (1)–(3) of Lemma 3.13. Thus G is hereditarily paracompact by Theorem 2.2. ∎

3.15. COROLLARY. *If a submaximal topological group G has non-measurable cardinality, then* $\dim(G) = 0$. *In particular, G cannot be connected.*

Proof. If a normal space is a countable union of its strongly zero-dimensional (in this case even discrete) subspaces, then it is strongly zero-dimensional [En, Theorem 7.2.1]. ∎

3.16. COROLLARY. *If there are no measurable cardinals, then every submaximal topological group is hereditarily paracompact and zero-dimensional in the sense of the dimension* dim. *In particular, no submaximal infinite topological group is connected.*

This corollary gives a partial answer to Problem 8.18 of [ArCo]. It turns out that for Abelian submaximal groups no set-theoretic assumptions are needed to prove that they are paracompact and strongly σ-discrete.

3.17. PROPOSITION. *Any infinite Abelian submaximal topological group is hereditarily paracompact and strongly σ-discrete.*

Proof. Let G be an Abelian submaximal group. It is proved in [Pr] that every irresolvable (and hence every submaximal) Abelian group contains a countable open subgroup. Therefore G is a discrete union of its countable subspaces and the result follows. ∎

3.18. COROLLARY. *Any Abelian submaximal topological group is strongly zero-dimensional.*

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