

## Endomorphism algebras over large domains

by

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**Abstract.** The paper deals with realizations of  $R$ -algebras  $A$  as endomorphism algebras  $\text{End } G \cong A$  of suitable  $R$ -modules  $G$  over a commutative ring  $R$ . We are mainly interested in the case of  $R$  having “many prime ideals”, such as  $R = \mathbb{R}[x]$ , the ring of real polynomials, or  $R$  a non-discrete valuation domain.

**0. Introduction.** This work is based on a previous paper [3] on realization theorems. In [3] an  $R$ -module  $G$  over a commutative ring  $R$  with  $1 \neq 0$  is constructed such that the endomorphism algebra of  $G$  coincides with a given  $R$ -algebra  $A$  (in general modulo an ideal). There is a given countable multiplicatively closed subset  $S$  of  $R$  such that  $A$  and therefore  $G$  is  $S$ -torsion-free and  $S$ -reduced; recall that an  $R$ -module  $G$  is  $S$ -torsion-free if  $gs = 0$  implies  $g = 0$  for any  $s \in S$ ,  $g \in G$ , and it is  $S$ -reduced if  $\bigcap_{s \in S} Gs = 0$ . Such constructions are by now standard, they are discussed in [3] and in some of the references given there. The desired module  $G$  can be constructed between a free  $A$ -module  $B$  and its  $S$ -adic completion  $\widehat{B}$ .

However, it is clear that in many cases  $S$  must be uncountable in order to have  $\bigcap_{s \in S} As = 0$ ; for example, if  $R$  is a valuation domain with a lattice of ideals not coinitial to  $\omega$  and  $A = R$ , then  $\bigcap_{s \in S} As \neq 0$  for all countable  $S \subseteq R \setminus \{0\}$  (see [6]). In this case a different technique is needed to realize a given algebra  $A$  as endomorphism algebra of some module  $G$ . The topological methods may not work any longer for  $|S| > \aleph_0$  since the natural  $S$ -topology (generated by  $Gs$  ( $s \in S$ )) may not be metrizable; see Example 3.8 in [7]. However, if  $S$  is uncountable, which may be necessary as we have seen, a construction of the desired module  $G$  is given in [8]. This construction [8] is difficult and awaits simplification. A first simplification is given in [9]; but here  $R$  is restricted to be a Prüfer ring.

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1991 *Mathematics Subject Classification*: 08A35, 13Cxx, 13L05, 20K20, 20K30.

This work is supported by GIF project No. G-0294-081.06/93 of the German–Israeli Foundation for Scientific Research & Development.

Our present purpose is to link the construction for  $|S| > \aleph_0$  with the much easier topological methods in the countable case as given for example in [3]; by doing so, we do not achieve realization theorems for the same general class of  $R$ -algebras  $A$  as in [8]. It is our intention to present an easier proof. Note that the “local approach” used in this paper might be of interest for considering other problems; it seems likely that results related to a countable multiplicatively closed subset could be generalized to the uncountable case by using this technique.

The following easy observation (Lemma 2.1 of [1]) is the key to replacing the uncountability of  $S$  by a family of pleasant topologies, one for each  $s \in S$ ; in this case  $s$ -reduced and  $s$ -torsion-free refer to the new set  $\{s^n \mid n < \omega\}$ .

**OBSERVATION 0.1.** *Let  $S \subseteq R$ ,  $s \in S$  and let  $G$  be  $S$ -torsion-free. Moreover, let  $s^\omega G = \bigcap_{n < \omega} Gs^n$  and  $G^s = G/s^\omega G$ . Then  $G^s$  is  $s$ -torsion-free and  $s$ -reduced, and every endomorphism  $\sigma$  of  $G$  induces a canonical endomorphism  $\sigma^s$  of  $G^s$  continuous in the  $s$ -topology on  $G^s$ .*

In [3] and [8] unwanted endomorphisms are killed by considering their action on a fixed free module  $B = \bigoplus A$  (and adding new elements to  $B$ ); in this paper we investigate their induced action on  $B^s$ . We are able to control the endomorphisms on  $B$ , getting rid of their induced counterparts for all possible (uncountably many)  $s \in S$ . This way we obtain realization theorems for  $R$ -algebras  $A$  as discussed at the end of this section.

First we want to describe the required tools which may be interesting in their own right. In order to find the desired elements needed to enlarge the base module  $B$  and to kill unwanted homomorphisms, we must be able to embed such a  $B$  into a suitable (larger) algebraically compact module  $\tilde{B}$ . This, of course, is closely related to the work of R. Warfield [11]. However, we are interested in an explicit construction in order to say more about  $\tilde{B}$  (see 1.3). An approach using reduced powers is given in §1. The link to the aforementioned  $s$ -topologies is given by:

**THEOREM 1.6.** *Let  $\tilde{B}$  be an algebraically compact  $R$ -module,  $G$  an  $S$ -RD-submodule of  $\tilde{B}$ ,  $s \in S$  and let  $\widehat{G^s}$  denote the  $s$ -adic completion of  $G^s$ . Then there exists a monomorphism  $\phi : \widehat{G^s} \rightarrow \tilde{B}^s$  where  $\phi|_{G^s}$  is the canonical embedding of  $G^s$  into  $\tilde{B}^s$ .*

In order to apply Shelah’s combinatorial machinery, the Black Box Lemma 3.2, the source of elements waiting for the construction of the final module has to be selected carefully. Such *potential elements* are provided in §2. In §3 we adopt Shelah’s combinatorial idea and present it in a form suitable for application in §4 and §5; as a result we obtain a module  $G$  depending on a given  $R$ -algebra  $A$  and  $G$  will be  $S$ -reduced and  $S$ -torsion-free. In the classical case [3] we also find constructions for  $G$  to be ( $S$ -)

torsion or ( $S$ -) mixed. It is not clear at present how to construct an  $S$ -torsion module  $G$  for  $|S| > \aleph_0$ . However, it is easy to show that it is impossible to construct an  $S$ -mixed module  $G$  in the usual way for  $S$  not coinital to  $\omega$ , as stated in the following observation (see [9], p. 69).

**OBSERVATION 0.2.** *Let  $S, A, R$  be as above with  $S$  not coinital to  $\omega$ ,  $G$  an  $A$ -module and  $T$  a subset of  $G$  consisting of  $S$ -torsion elements. Moreover, assume that a support function  $[\ ]$  is defined for  $G$  with  $|[g]| \leq \aleph_0$  for all  $g \in G$ ,  $[g] = \emptyset$  iff  $g = 0$ , and  $\tau a = 0$  implies  $\tau \notin [ga]$  for all  $\tau \in T$ ,  $g \in G$ ,  $a \in A$ . Then  $G$  is an  $S$ -torsion module.*

However, the module  $G$  constructed here is an  $S$ -torsion-free  $A$ -module; therefore we can identify  $A \subseteq \text{End}_R G$  with scalar multiplication. Traditionally we want to find a two-sided ideal  $\text{Ines } G \triangleleft \text{End } G$  such that  $\text{End}_R G = A \oplus \text{Ines } G$  (the general realization theorem). If the algebra is more special we are able to determine  $\text{Ines } G$  directly.

Let us summarize some of our main results. Since it is convenient to present first a realization in a simple case, we consider in §5 the cotorsion-free case separately, before we have a look at the general case (§6) and at an application (§7). Note that the notion of cotorsion-freeness used here differs slightly from the classical definitions in [3] and [8]. As mentioned before, we will get our realization theorems using local arguments, i.e. we get “local realizations” given by the following theorems.

**THEOREM 5.7.** *If  $A$  is a cotorsion-free  $R$ -algebra, then there exists an  $R$ -module  $G$  with  $\text{End } G^s = A^s$  for each  $s \in S$ .*

**THEOREM 6.4.** *If  $A$  is an  $S$ -reduced and  $S$ -torsion-free  $R$ -algebra, then there exists an  $R$ -module  $G$  with  $\text{End } G^s = A^s \oplus \text{Ines } G^s$  for each  $s \in S$ .*

Note that  $\text{Ines } G^s$  consists of all endomorphisms of  $G^s$  mapping  $\widehat{G^s}$  into  $G^s$ . To lift these local realizations to a global realization  $\text{End } G = A(\oplus \text{Ines } G)$  we need additional assumptions. In the cotorsion-free case (§5) we shall assume that  $A$  is  $F$ -complete with respect to the filtration  $F = \{s^\omega A \mid s \in S\}$  (see [6]). Moreover,  $\text{Ines } G$  must be “well related” (see Definition 6.5) in the general case. Note that an endomorphism  $\phi$  of  $G$  is inessential if all induced endomorphisms  $\phi^s$  of  $G^s$  ( $s \in S$ ) are inessential.

Also, we shall introduce the notion of an  $\aleph_0$ -cotorsion-free module; in a sense the definition given in this paper (§7) generalizes the one used in [3]. If  $A$  is  $\aleph_0$ -cotorsion-free then  $\text{Ines } G$  contains exactly the locally “sub-finitely generated” endomorphisms  $\phi$ , i.e.  $\text{Im } \phi^s$  is contained in a finitely generated submodule of  $G^s$  for each  $s \in S$  (see Definition 7.1). It is easy to show that under this assumption  $\text{Ines } G$  is well related. The main results are now given as follows:

**THEOREM 5.9.** *If  $A$  is  $F$ -complete and cotorsion-free, then there exists an  $R$ -module  $G$  with  $\text{End } G = A$ .*

**THEOREM 7.9.** *If  $A$  is  $F$ -complete and  $\aleph_0$ -cotorsion-free, then there exists an  $R$ -module  $G$  with  $\text{End } G = A \oplus \text{Fin}_l G$ .*

Here  $\text{Fin}_l G$  denotes the ideal of all locally sub-finitely generated endomorphisms.

We finish the introduction with an example of an algebra  $A$  satisfying the hypothesis of 7.9. A non-trivial example of an  $F$ -complete, cotorsion-free algebra  $A$  is given in §5 (see Example 5.8).

Let  $R$  be a maximal valuation ring,  $S = R \setminus \{0\}$  and  $A = R$ . It is easy to see that  $R$  is  $\aleph_0$ -cotorsion-free; moreover,  $R$  is linearly compact in the discrete topology (see [6], p. 20) and hence  $F$ -complete. Note that for a non-discrete  $R$  this really is an example with an uncountable  $S$  not cointegral to  $\omega$ .

**1. Algebraically compact modules.** Let  $R$  be a non-zero commutative ring with  $1 \neq 0$ . Modules will be considered as right  $R$ -modules.

First recall that an  $R$ -module  $M$  is *algebraically compact* if every finitely solvable system of linear equations over  $M$  has a global solution in  $M$ . It is well known that every  $R$ -module is pure embeddable in an algebraically compact module. Note that we call a submodule  $G$  of  $M$  a *pure submodule* if for every finite system of linear equations over  $G$  having a solution in  $M$ , there also exists a solution in  $G$  (notation:  $G \subseteq_* M$ ). A related concept is that of relative divisibility; recall that a submodule  $G$  of the  $R$ -module  $M$  is *relatively divisible* or an *RD-submodule* of  $M$  if  $G \cap Mr = Gr$  for all  $r \in R$  (notation:  $G \subseteq_{\text{rd}} M$ ). It is well known that purity implies relative divisibility and that the concepts coincide for modules over Prüfer rings (see [11]).

Algebraically compact modules can be characterized in different ways, e.g. a module is algebraically compact if and only if it is pure injective (see [11], [12]). Moreover, it is sufficient to consider systems of  $|R| \cdot \aleph_0$  equations, i.e. it is enough to show that an  $R$ -module  $M$  is  $(|R| \cdot \aleph_0)^+$ -algebraically compact, to prove algebraic compactness (see [5], Ch. V). We will use the last mentioned fact to construct an algebraically compact module.

Note that there are many different ways to embed a given module in an algebraically compact module (e.g. see [11], [10], [5]). For the convenience of the reader unfamiliar with the concept we construct an algebraically compact extension  $\tilde{B}$  of a given module  $B$  using an approach via reduced powers which will make the needed properties easy to prove; recall that the *reduced power*  $B^I/F$  of a module  $B$  with respect to a filter  $F$  is given by identifying two elements  $(m_i)_{i \in I}$ ,  $(n_i)_{i \in I}$  of  $B^I$  whenever the set  $\{i \in I \mid m_i = n_i\}$  is

an element of  $F$ . Using the definitions of a filter and the reduced power it is easy to verify that  $S$ -torsion-freeness and decompositions of  $B$  are inherited by reduced powers.

PROPOSITION 1.1. *Let  $B$  be an  $R$ -module.*

- (a) *If  $B$  is  $S$ -torsion-free for some subset  $S$  of  $R$ , then  $B^I/F$  is  $S$ -torsion-free.*
- (b) *If  $B = C \oplus D$ , then  $B^I/F = C^I/F \oplus D^I/F$ . ■*

The first lemma (see Ex. 8 in [5], Ch. V) gives a useful relation between an  $R$ -module  $B$  and the reduced power  $B^I/F$  with respect to a certain filter. It will be our main tool for constructing an algebraically compact extension.

LEMMA 1.2. *Let  $B$  be an  $R$ -module,  $\kappa$  an infinite cardinal,  $I$  the set of all finite subsets of  $\kappa$  and  $F$  the filter generated by the collection  $X_\alpha = \{m \in I \mid \alpha \in m\}$  ( $\alpha < \kappa$ ). Then the diagonal map  $\delta : B \rightarrow B^I/F$  ( $m \mapsto (m)_{i \in I}/F$ ) is a pure embedding and every finitely solvable system of  $\kappa$  equations over  $B\delta$  with coefficients in  $R$  has a solution in  $B^I/F$ .*

PROOF. It is easy to check that  $\delta$  is a pure embedding; the proof is left to the reader. Thus we may identify  $B$  and  $B\delta$ .

We consider a finitely solvable system of  $\kappa$  equations  $\sum_{x \in X} xr_{x,\alpha} = m_\alpha$  ( $m_\alpha \in B$ ,  $r_{x,\alpha} \in R$ ,  $\alpha < \kappa$ ). For every  $i \in I$  there exists a solution  $m_{x,i}$  ( $x \in X$ ) in  $B$  of the corresponding subsystem  $\sum_{x \in X} xr_{x,\alpha} = m_\alpha$  ( $\alpha \in i$ ). We define  $m_x = (m_{x,i})_{i \in I}/F$  for each  $x \in X$ . Since  $X_\alpha = \{i \in I \mid \alpha \in i\} \in F$  is a subset of  $Y_\alpha = \{i \in I \mid \sum_{x \in X} m_{x,i}r_{x,\alpha} = m_\alpha\}$  we get  $Y_\alpha \in F$  for each  $\alpha < \kappa$ . Therefore  $(m_x)_{x \in X}$  is a global solution in  $B^I/F$  of the considered system. ■

We thank the referee for pointing out that we can achieve the same result using any  $\kappa$ -regular ultrafilter  $F$  (see [2], Cor. 4.3.14).

We are now ready to construct an algebraically compact extension.

LEMMA 1.3. *Let  $R$  be of cardinality  $\kappa \geq \aleph_0$  and let  $B$  be a non-zero  $R$ -module. Then there exists an  $R$ -module  $\tilde{B}$  of cardinality less than or equal to  $|B|^\kappa$  such that  $\tilde{B}$  is an algebraically compact  $R$ -module containing  $B$  as a pure submodule.*

PROOF.  $\tilde{B}$  is constructed in such a way that it is  $\kappa^+$ -algebraically compact, which coincides with being algebraically compact. To get solutions for all finitely solvable systems of  $\kappa$  equations we apply Lemma 1.2  $\kappa^+$  times. Therefore let  $I, F$  be as in Lemma 1.2. We get  $\tilde{B}$  as the union of a smooth ascending chain  $\{B_\alpha \mid \alpha < \kappa^+\}$  satisfying the following conditions:

- (1)  $B_\alpha$  is an  $R$ -module of cardinality at most  $|B|^\kappa$ ,
- (2)  $B_\alpha$  is a pure  $R$ -submodule of  $B_{\alpha+1}$ , and

(3) every finitely solvable system of  $\kappa$  equations over  $B_\alpha$  with coefficients in  $R$  has a solution in  $B_{\alpha+1}$ .

For  $\alpha = 0$  let  $B_0 = B$  and if  $\alpha$  is a limit then take  $B_\alpha = \bigcup_{\beta < \alpha} B_\beta$ .

Now let  $\alpha + 1$  be a successor. Assume that  $B_\alpha$  has been constructed satisfying the conditions above. Defining  $B_{\alpha+1} = B_\alpha^I/F$  we immediately see that  $B_{\alpha+1}$  satisfies (2) and (3) by Lemma 1.2 and (1) is given by  $|B_{\alpha+1}| \leq |B_\alpha^I| = |B_\alpha|^{|I|} \leq (|B|^\kappa)^\kappa = |B|^\kappa$ .

Define  $\tilde{B} = \bigcup_{\alpha < \kappa^+} B_\alpha$ ; obviously,  $\tilde{B}$  is an  $R$ -module of cardinality at most  $\kappa^+ \cdot |B|^\kappa = |\tilde{B}|^\kappa$  containing  $B$  as a pure  $R$ -submodule. Moreover, every finitely solvable system of  $\kappa$  equations over  $\tilde{B}$  turns out to be a system over  $B_\alpha$  for some  $\alpha < \kappa^+$  since  $\kappa^+$  is a regular cardinal. Hence, there is a solution in  $B_{\alpha+1} \subseteq \tilde{B}$ . So,  $\tilde{B}$  is  $\kappa^+$ -algebraically compact and thus it is algebraically compact. ■

We would like to point out an interesting fact: the above construction is also suitable for extending the ring structure of  $R$  to an algebraically compact module  $\tilde{R}$  such that  $\tilde{B}$  becomes an  $\tilde{R}$ -module (see e.g. [9]). Note that  $\tilde{B}$  need not be algebraically compact as an  $\tilde{R}$ -module.

We reserve the notation “ $\tilde{\phantom{x}}$ ” for an algebraically compact module constructed as in Lemma 1.3.

The next corollary is an immediate consequence of Proposition 1.1 and the previous lemma; since  $S$ -torsion-freeness and decompositions are inherited by reduced powers, the above construction guarantees that they are also inherited by algebraically compact extensions.

COROLLARY 1.4. (a) *If  $B$  is  $S$ -torsion-free for some subset  $S \subseteq R$ , then  $\tilde{B}$  is also  $S$ -torsion-free.*

(b) *If  $B = C \oplus D$  is any decomposition of an  $R$ -module  $B$ , then  $\tilde{B} = \tilde{C} \oplus \tilde{D}$ .* ■

In the main part of this section we investigate the relation between an algebraically compact module and canonical topological completions of its  $S$ -RD-submodules, where  $S \subseteq R$  is a given multiplicatively closed subset. Note that we call a submodule  $G$  of  $M$  an  $S$ -RD-submodule if  $Ms \cap G = Gs$  for all  $s \in S$ .

For an  $R$ -module  $M$  and  $s \in S$  we define  $s^\omega M = \bigcap_{n < \omega} Ms^n$  and  $M^s = M/s^\omega M$ . Obviously,  $M^s$  is always  $s$ -reduced, i.e. reduced with respect to  $\{s^n \mid n < \omega\}$ , and if  $M$  is  $S$ -torsion-free then  $M^s$  is  $s$ -torsion-free for each  $s \in S$  (see Observation 0.1). Moreover, for each  $s \in S$ , the following holds:

PROPOSITION 1.5. *If  $M$  is algebraically compact, then so is  $M^s$ .*

PROOF. Suppose  $M$  is an algebraically compact  $R$ -module and let  $\sum_{x \in X} xr_{x,\alpha} = \bar{m}_\alpha$ ,  $\bar{m}_\alpha = m_\alpha + s^\omega M$  ( $\alpha < \kappa$ ) be a finitely solvable system of

equations over  $M^s$ . We define a corresponding system of equations over  $M$  by  $\sum_{x \in X} xr_{x,\alpha} + y_\alpha = m_\alpha$ ,  $y_\alpha = y_{\alpha,n}s^n$  ( $\alpha < \kappa$ ,  $n < \omega$ ) where  $y_\alpha$ ,  $y_{\alpha,n}$  are also unknowns. It is easy to check that this system is also finitely solvable. Since  $M$  is algebraically compact by assumption, there is a global solution  $h_x \in M$  ( $x \in X$ ),  $h_\alpha \in s^\omega M$  ( $\alpha < \kappa$ ). Hence  $(h_x + s^\omega M)_{x \in X}$  is a solution of the considered system of equations. Therefore  $M^s$  is algebraically compact. ■

The final theorem gives the basic idea for the “local” approach used to realize certain  $R$ -algebras.

**THEOREM 1.6.** *Let  $M$  be an algebraically compact  $R$ -module,  $G$  an  $S$ - $RD$ -submodule of  $M$ ,  $s \in S$  and let  $\widehat{G}^s$  denote the  $s$ -adic completion of  $G^s$ . Then there exists a monomorphism  $\phi : \widehat{G}^s \hookrightarrow M^s$  where  $\phi|_{G^s}$  is the canonical embedding  $\pi : G^s \hookrightarrow M^s$  ( $g + s^\omega G \mapsto g + s^\omega M$ ).*

**Proof.** Proposition 1.5 allows us to define  $\phi : \widehat{G}^s \rightarrow M^s$  in the following manner: let  $g \in \widehat{G}^s$ . We may express  $g$  as  $g = \sum_{n < \omega} a_n s^n$  where  $a_n \in G^s$  and  $a_n s^n \notin G^s s^{n+1}$  whenever  $a_n s^n$  is non-zero. Since  $x_n - x_{n+1}s = a_n$  ( $n < \omega$ ) is a system of equations over  $G^s \subseteq M^s$  which is finitely solvable, there is a solution  $x_n = h_n$  ( $n < \omega$ ) in  $M^s$ . Now let  $\phi$  be defined by  $g\phi = h_0$ .

To verify that  $\phi$  is well defined we consider an element  $g \in \widehat{G}^s$  with  $g = \sum_{n < \omega} a_n s^n$  and  $g = \sum_{n < \omega} b_n s^n$  and let  $(h_n)_{n < \omega}$ ,  $(k_n)_{n < \omega}$  be the solutions of the corresponding systems of equations. Therefore, for each  $n < \omega$ , we have  $h_n - h_{n+1}s = a_n$  and  $k_n - k_{n+1}s = b_n$ . Hence  $h_0 - k_0 = \sum_{i=0}^{n-1} (a_i - b_i)s^i + (h_n - k_n)s^n$  for each  $n < \omega$ . By our assumption we have  $\sum_{i=0}^{n-1} (a_i - b_i)s^i \in \widehat{G}^s s^n \cap G^s = G^s s^n \subseteq M^s s^n$  and therefore  $h_0 - k_0$  is an element of  $M^s s^n$  for every  $n < \omega$ . Since  $M^s$  is  $s$ -reduced,  $h_0$  and  $k_0$  coincide and thus  $\phi$  is well defined.

As an immediate consequence we find that  $\phi|_{G^s}$  is the canonical embedding  $\pi$ .

It is easy to check that  $\phi$  is an  $R$ -homomorphism, considering the corresponding systems of equations in the definition and using the fact that  $M^s$  is  $s$ -reduced.

Finally, we show that  $\phi$  is injective. Let  $g = \sum_{n < \omega} a_n s^n \in \widehat{G}^s$  with  $a_n s^n \notin G^s s^{n+1}$  for  $a_n s^n \neq 0$ . Suppose  $g\phi = 0$ . Let  $(h_n)_{n < \omega}$  be a solution of the corresponding system of equations. Therefore we get  $0 = g\phi = h_0 = a_0 + h_1s$ . Hence  $a_0 \in G^s s$ , which implies  $a_0 = 0$ . So it follows that  $0 = h_1s = a_1s + h_2s^2$  and by the same argument as before  $a_1s = 0$ . Continuing this procedure implies  $a_n s^n = 0$  for each  $n < \omega$ , i.e.  $g = 0$ . Therefore  $\phi$  is injective and this completes the proof. ■

Note that the previous result is also true with respect to an arbitrary countable multiplicatively closed subset  $C$  of  $S$ . In particular, it follows that

for an algebraically module  $M$  the quotient  $M^C$  is complete in its  $C$ -adic topology, as is well known.

We now proceed to define a suitable “hull” within which we construct the desired module  $G$ .

**2. Potential elements.** In this section we specify the elements which will be suitable for the construction of the desired module  $G$ .

Let  $S \subseteq R$  be a fixed multiplicatively closed subset without zero divisors and let  $A$  be an  $S$ -torsion-free and  $S$ -reduced  $R$ -algebra. Moreover, we choose cardinals  $\kappa, \lambda$  such that  $\lambda = \lambda^\kappa$  and  $\kappa \geq |A| \cdot |S|$ .

As in [3] and [8] we first define a free module  $B$  generated by a basis  $T$  which is equipped with a certain partial order. Let  $T$  be the tree given by  $T = {}^\omega > \lambda = \{\tau : n \rightarrow \lambda \mid n < \omega\}$ . For each element  $\tau$  of  $T$  the *length* of  $\tau$  is the finite set  $l(\tau) = \text{dom } \tau = n = \{0, \dots, n - 1\}$ . The elements of  $T$  are ordered by  $\tau \leq \sigma$  if  $l(\tau) \leq l(\sigma)$  and  $\sigma \upharpoonright l(\tau) = \tau$ . An (infinite) *branch* of  $T$  is a map  $v : \omega \rightarrow \lambda$ ; we can identify  $v$  with a linearly ordered subset of  $T$  by  $v = \{v_n = v \upharpoonright n \mid n < \omega\} \subseteq T$ . Let  $\text{Br } T$  denote the set of all branches of  $T$ . Now we define  $B$  to be the free  $A$ -module generated by the tree  $T$ :  $B = \bigoplus_{\tau \in T} \tau A$ .

Let  $\tilde{B}$  denote the algebraically compact  $R$ -module as obtained by Lemma 1.3, i.e.  $B \subseteq_* \tilde{B}$  and  $|\tilde{B}| \leq |B|^{|R|}$ . Also, by Corollary 1.4,  $\tilde{B}$  is  $S$ -torsion-free and the decompositions of  $B$  are inherited by  $\tilde{B}$ . In particular, for each  $\tau \in T$ , we have  $\tilde{B} = \tau \tilde{A} \oplus (\bigoplus_{\sigma \neq \tau} \sigma A)^\sim$ ; thus there is a unique  $\tau$ -component  $b \upharpoonright \tau \in \tau \tilde{A}$  for any element  $b$  of  $\tilde{B}$ . Therefore we may define a support function in the usual way (see also [3], [8]): for  $g \in \tilde{B}$ ,  $X \subseteq \tilde{B}$  let  $[g] = \{\tau \in T \mid g \upharpoonright \tau \neq 0\}$  and  $[X] = \bigcup_{g \in X} [g]$  be the *support* of  $g$  and of  $X$ , respectively. Since we shall argue mostly “modulo  $s$ ” we also define the  $s$ -support of  $g$  and of  $X$  by  $[g]_s = \{\tau \in T \mid g \upharpoonright \tau \notin \tau s^\omega \tilde{A}\}$  and  $[X]_s = \bigcup_{g \in X} [g]_s$ , for each  $s \in S$ . Obviously, for any  $s \in S$  and for all  $g, h \in \tilde{B}$  with  $g \equiv h \pmod{s^\omega \tilde{B}}$ , the  $s$ -supports  $[g]_s, [h]_s$  coincide. Hence we may define the support of an element  $\bar{g} = g + s^\omega \tilde{B} \in \tilde{B}^s$  ( $s \in S$ ) by  $[\bar{g}] = [g]_s$ . Moreover,  $\bar{g} \upharpoonright \tau = g \upharpoonright \tau + \tau s^\omega \tilde{A}$  defines a (unique)  $\tau$ -component of  $\bar{g}$  for any  $\tau \in T$ ; we get as an immediate consequence  $[\bar{g}] = \{\tau \in T \mid \bar{g} \upharpoonright \tau \neq 0\}$  for any  $\bar{g} \in \tilde{B}^s$  ( $s \in S$ ).

Next we define a norm  $\|\cdot\|$  for the elements and subsets of  $T$  which canonically extends to the elements and subsets of  $\tilde{B}$  and  $\tilde{B}^s = \tilde{B}/s^\omega \tilde{B}$  ( $s \in S$ ) using the supports (see also [3], [8]). We fix a continuous strictly increasing function  $\varrho : \text{cf}(\lambda) + 1 \rightarrow \lambda + 1$  such that  $0\varrho = 0$  and  $\text{cf}(\lambda)\varrho = \lambda$ . The *norm*  $\|\tau\|$  of an element  $\tau \in T$  is defined by  $\|\tau\| = \min\{\nu < \text{cf}(\lambda) \mid \tau \in {}^\omega > (\nu\varrho)\}$  and the norm of a subset  $T'$  of  $T$  is given by  $\|T'\| = \sup_{\tau \in T'} \|\tau\|$ . Note that



$\|\tau\| = \alpha$  means that  $\alpha\varrho$  is the smallest ordinal in  $\text{Im } \varrho$  satisfying  $\tau(i) < \alpha\varrho$  for all  $i < l(\tau)$ . Hence  $\|\tau\|$  is always a successor ordinal. Also note that  $\lambda \subseteq T$  as elements of length 1, hence  $\|\cdot\|$  is also defined for each subset of  $\lambda$ . Using the norm, for a subset  $T'$  of  $T$  and for any ordinal  $\nu < \lambda$ , we define the part of  $T'$  to the right of  $\nu$  by  ${}_\nu T' = \{\tau \in T' \mid \|\tau\| > \nu\}$ .

In [3] elements for constructing the desired module are found within the  $S$ -adic completion of a free module  $B$ . As mentioned in §0 this is no longer possible for uncountable  $S$ . The required elements will be chosen from the algebraically compact module  $\tilde{B}$  but not all are suitable. Following an idea in [8] we define potential elements needed in the construction; they are taken from  $\tilde{B}^{\aleph_0}$ , the set of all elements of  $\tilde{B}$  with countable support. For certain submodules  $U$  of  $\tilde{B}$  we shall need “preimages” of elements of the  $s$ -adic completion of  $(U + s^\omega \tilde{B})/s^\omega \tilde{B}$  (“ $s$ ” refers to the set  $\{s^n \mid n < \omega\}$ ). To be more precise we define a series  $(g^k)_{k < \omega}$  of elements of  $\tilde{B}$  to be an  $(s, U)$ -chain ( $s \in S, U \subseteq \tilde{B}$ ) if  $g^k - g^{k+1}s \in U$  and, for some  $\nu < \|g^0\|$ ,  ${}_\nu[g^k] \subseteq [g^0]$  for each  $k < \omega$ . We are now ready for

DEFINITION 2.1. We define the set  $\text{POT} = \text{POT}(B)$  of potential elements in  $\tilde{B}^{\aleph_0}$  inductively as follows:

- (i)  $B \subseteq \text{POT}$ ;
- (ii) if  $(g^k)_{k < \omega}$  is an  $(s, U)$ -chain of elements of  $\tilde{B}^{\aleph_0}$  and  $s \in S, U \subseteq \text{POT}$ , then  $g^k$  is potential for all  $k < \omega$ ;
- (iii) if  $bs \in \text{POT}$  and  $s \in S$  then  $b$  is potential;
- (iv) elements of an  $A$ -module generated by potential elements are potential.

An  $A$ -module  $U \subseteq \text{POT}$  is called a potential module.

If a module  $U$  is an  $S$ -RD-submodule of  $\tilde{B}$  we may consider  $\widehat{U^s}$ , the  $s$ -adic completion of  $U^s = U/s^\omega U$  ( $s \in S$ ), as a submodule of  $\tilde{B}^s$  (see Theorem 1.6). Note that in this case we may identify  $U/s^\omega U$  with  $(U + s^\omega \tilde{B})/s^\omega \tilde{B}$ . Since this is not possible in general, let us agree on using  $U^s$  for  $(U + s^\omega \tilde{B})/s^\omega \tilde{B}$  whenever  $U$  is not an  $S$ -RD-submodule of  $\tilde{B}$ .

The notion of a canonical module has been proven useful (see e.g. [3], [8]); an  $S$ -RD-submodule of  $\tilde{B}$  generated by at most  $\kappa$  potential elements containing its support  $[P]$  is called a canonical module. Let  $\mathcal{C}$  denote the set of all canonical modules; as an immediate consequence of the definition we get

LEMMA 2.2. For  $P \in \mathcal{C}, X \subseteq \text{POT}$  with  $|X| \leq \kappa$  there exists  $P' \in \mathcal{C}$  with  $P \cup X \subseteq P'$ . Moreover,  $\mathcal{C}$  is non-empty and closed under unions of countable ascending chains. ■

In [3] and [8] the branches of  $T$  have been used to define elements playing a crucial role in the construction of the desired module. Instead of going into the rather unusual notion of branches with leaves (see [8]) we use the idea from [3] replacing the  $S$ -adic limits by a family of corresponding  $(s, B)$ -chains.

**DEFINITION 2.3.** For any branch  $v$  of  $T$  and  $s \in S$ , we define potential elements  $v^{k,s}$  ( $k < \omega$ ) as a solution in  $(\bigoplus_{\tau \in v} \tau A)^\sim$  of the following, finitely solvable system of equations over  $B$ :  $x_k - x_{k+1}s = v_k$  ( $= v|k$ ) ( $k < \omega$ ).

The above-defined elements have some nice properties:

**LEMMA 2.4.** Let  $v \in \text{Br } T$ ,  $s, q \in S$ ,  $a \in A$ ,  $\bar{a} = a + q^\omega A \in A^q$ , and  $\overline{v^{k,s}} = v^{k,s} + q^\omega \tilde{B}$ . Then:

- (i)  $v^{k,s} \upharpoonright v_n = v_n s^{n-k}$  for any  $k \leq n < \omega$ ,
- (ii)  $[v^{k,s}] = \{v_n \mid n \geq k\}$  for each  $k < \omega$ ,
- (iii) the sequence  $(v^{k,s})_{k < \omega}$  is an  $(s, B)$ -chain,
- (iv)  $[v^{k,s}a] \subseteq [v^{k,s}] \subseteq v$ ,
- (v)  $a = 0 \Leftrightarrow v^{k,s}a = 0 \Leftrightarrow [v^{k,s}a]$  is finite  $\Leftrightarrow v \setminus [v^{k,s}a]$  is infinite,
- (vi)  $(\bar{a}s^n = \bar{0} \text{ for some } n < \omega) \Leftrightarrow \overline{v^{k,s}a} \in B^q \Leftrightarrow [v^{k,s}a]$  is finite  $\Leftrightarrow v \setminus [v^{k,s}a]$  is infinite.

**Proof.** By 2.3 we immediately get  $[v^{k,s}] \subseteq \{v_n \mid n \geq k\}$  and  $v^{0,s} = \sum_{i=0}^n v_i s^i + v^{k+1,s} s^{k+1}$  for each  $n, k < \omega$ . Therefore  $v^{0,s} \upharpoonright v_n = (\sum_{i=0}^n v_i s^i) \upharpoonright v_n + v^{k+1,s} s^{k+1} \upharpoonright v_n = v_n s^n$ , which implies  $[v^{0,s}] = \{v_n \mid n < \omega\}$ . Moreover,  $v^{k,s} \upharpoonright v_n = (v^{0,s} - \sum_{i=0}^{k-1} v_i s^i) s^{-k} \upharpoonright v_n = v_n s^{n-k}$  for all  $k \leq n < \omega$ . Now parts (i) and (ii) are obvious and (iii) follows by the definition of an  $(s, B)$ -chain, 2.3 and part (ii). Moreover, (iv) is immediate from (ii) and the definition of the support.

Next we show (v). Clearly,  $a = 0 \Rightarrow v^{k,s}a = 0 \Rightarrow [v^{k,s}a]$  is finite  $\Rightarrow v \setminus [v^{k,s}a]$  is infinite.

Conversely, assume  $v \setminus [v^{k,s}a]$  is infinite. Therefore there are infinitely many  $n \geq k$  such that  $v^{k,s}a \upharpoonright v_n = 0$ . On the other hand, we have  $v^{k,s}a \upharpoonright v_n = v_n a s^{n-k}$  for each  $n \geq k$  by (i). Hence  $a s^{n-k} = 0$  for infinitely many  $n \geq k$ . Since  $A$  is  $S$ -torsion-free that implies  $a = 0$ .

Finally, we consider (vi). Assuming that there is an  $n < \omega$  with  $\bar{a}s^n = 0$  (in  $A^q$ ) we get either  $\overline{v^{k,s}a} = \bar{0} \in B^q$  for  $n = 0$  or, for  $n > 0$ ,

$$\begin{aligned} v^{k,s}a &\equiv \sum_{i=k}^{n+k-1} v_i a s^{i-k} + v^{k+n,s} a s^n \\ &\equiv \sum_{i=k}^{n+k-1} v_i a s^{i-k} + v^{k+n,s} a s^n \equiv \sum_{i=0}^{n+k-1} v_i a s^{i-k} \pmod{q^\omega \tilde{B}}, \end{aligned}$$

which induces  $\overline{v^{k,s}a} \in B^q$ . Hence  $\overline{[v^{k,s}a]}$  is finite and  $v \setminus \overline{[v^{k,s}a]}$  is infinite.

Now, if  $v \setminus \overline{[v^{k,s}a]}$  is infinite, we get  $\bar{0} = \overline{v^{k,s}a} \upharpoonright v_i = v^{k,s}a \upharpoonright v_i + q^\omega v_i \tilde{A}$  for infinitely many  $i < \omega$ . Therefore  $v^{k,s}a \upharpoonright v_i = v_i a s^{i-k} \in q^\omega v_i \tilde{A}$  for infinitely many  $k \leq i < \omega$ . Hence  $as^n \in q^\omega \tilde{A} \cap A = q^\omega A$  for some  $n < \omega$  and this completes the proof. ■

**3. Construction.** Now we are going to construct the required  $R$ -module  $G$ . As in [3] and [8] we shall use Black Box arguments to prove realization theorems. Different versions of the Black Box are known; the one presented here is very similar to that given in [3].

First we need to say what we mean by a “trap”; since we are concerned only with discrete realizations we can omit one of the parameters used in the definition of “trap” in [3], but we shall need to use the elements  $s$  of  $S$  as an additional parameter. As we want to catch and “kill” homomorphisms via their induced actions on the corresponding quotients, it seems natural to consider endomorphisms on quotients from the outset. Our definition of a trap then becomes:

DEFINITION 3.1. A quadruple  $(f, P, s, \phi)$  is called a *trap* if  $f : {}^\omega > \kappa \rightarrow T$  is a tree embedding,  $P$  is a canonical module,  $s \in S$ , and  $\phi \in \text{End } P^s$  satisfying the following conditions:

- (a)  $\text{Im } f \subseteq P$ ,
- (b)  $[P]$  is a subtree of  $T$ ,
- (c)  $\|P\|$  is a limit ordinal of cofinality  $\omega$ , and
- (d)  $\|v\| = \|P\|$  for each  $v \in \text{Br}(\text{Im } f)$ .

We are now ready to present the Black Box in a suitable form.

THE BLACK BOX LEMMA 3.2. *For an ordinal  $\lambda^* \geq \lambda$  there exists a transfinite sequence  $(f_\alpha, P_\alpha, s_\alpha, \phi_\alpha)_{\alpha < \lambda^*}$  of traps such that, for  $\alpha, \beta < \lambda^*$ ,*

- (i)  $\beta < \alpha \Rightarrow \|P_\beta\| \leq \|P_\alpha\|$ ,
- (ii)  $\beta \neq \alpha \Rightarrow \text{Br}(\text{Im } f_\beta) \cap \text{Br}(\text{Im } f_\alpha) = \emptyset$ ,
- (iii)  $\beta + \kappa^{\aleph_0} \leq \alpha \Rightarrow \text{Br}(\text{Im } f_\alpha) \cap \text{Br}([P_\beta]) = \emptyset$ ,
- (iv) *if  $K$  is a potential module,  $X$  a subset of  $K$  with  $|X| \leq \kappa$ ,  $s \in S$  and  $\phi \in \text{End } K^s$ , then there exists an  $\alpha < \lambda^*$  such that*

$$X \leq P_\alpha, \quad \|X\| < \|P_\alpha\|, \quad s = s_\alpha, \quad \text{and} \quad \phi \upharpoonright P_\alpha^s = \phi_\alpha. \quad \blacksquare$$

A detailed proof of the existence of a slightly different Black Box is given in [3]. Besides the aforementioned differences it is necessary to replace the “fixed”  $S$ -adic completion, as used in the version of the prediction principle (iv) in [3], by arbitrary potential modules (see also [8]); this is due to the fact that in general there is no “universal” module to which suitable homomorphisms can be (uniquely) extended.

Note that the Black Box is very robust under changes of its setting; the only real concern is the cardinality of the objects in question. Lemma 1.3 and the choice of  $\kappa$  and  $\lambda$  guarantee that all cardinalities of interest are bounded by  $\lambda$ .

CONSTRUCTION 3.3. Choose a transfinite sequence  $(f_\alpha, P_\alpha, s_\alpha, \phi_\alpha)_{\alpha < \lambda^*}$  as in Lemma 3.2. Moreover, let  $\infty$  be a fixed element which does not belong to  $\tilde{B}$ .

We will construct inductively a sequence  $(b_\beta)_{\beta < \lambda^*}$  in  $\text{POT} \cup \{\infty\}$  and an ascending smooth chain  $(G_\mu)_{\mu \leq \lambda^*}$  of potential modules such that, for all  $\mu \leq \lambda^*$ ,

$$(I_\mu) \quad b_\beta + s_\beta^\omega \tilde{B} \notin G_\mu^{s_\beta} \quad \text{for each } \beta < \mu.$$

If  $\mu = 0$  we put  $G_0 = B = \bigoplus_{\tau \in T} \tau A$ . Therefore  $G_0$  is a potential module by definition.

If  $\mu$  is a limit we assume that the potential modules  $G_\alpha$  and the elements  $b_\beta$  are given for all  $\alpha, \beta < \mu$  such that  $(I_\alpha)$  is satisfied for each  $\alpha < \mu$ . We take  $G_\mu = \bigcup_{\alpha < \mu} G_\alpha$ , which is obviously also potential and it satisfies  $(I_\mu)$  since  $b_\beta + s_\beta^\omega \tilde{B} \notin G_\alpha^{s_\beta}$  for all  $\beta < \alpha < \mu$ .

Now let  $\mu = \alpha + 1$  be a successor,  $G_\alpha$  a potential module and let the elements  $b_\beta$  ( $\beta < \alpha$ ) be given satisfying  $(I_\alpha)$ . Suppose it is possible to choose a branch  $v_\alpha \in \text{Br}(\text{Im } f_\alpha)$ , an  $(s_\alpha, G_\alpha)$ -chain  $(g_\alpha^k)_{k < \omega}$ ,  $b_\alpha \in \text{POT} \cup \{\infty\}$ , and  $G_{\alpha+1}$  in such a way that  $(I_{\alpha+1})$  and the following conditions are satisfied:

- (II $_{\alpha+1}$ )  $G_{\alpha+1} = G_\alpha + \sum_{k < \omega} g_\alpha^k A$ ,
- (III $_\alpha$ )  $\sup_{k < \omega} \|g_\alpha^k - v_\alpha^{k, s_\alpha}\| < \|v_\alpha\|$  ( $= \|P_\alpha\|$ ),
- (IV $_\alpha$ )  $g_\alpha^k + s_\alpha^\omega \tilde{B} \in \widehat{P_\alpha^{s_\alpha}}$  for each  $k < \omega$ , and
- (V $_{\alpha+1}$ ) either

$$b_\alpha + s_\alpha^\omega \tilde{B} = (g_\alpha^0 + s_\alpha^\omega \tilde{B})\phi_\alpha \quad (1)$$

or

$$b_\alpha = \infty. \quad (2)$$

We then make such a choice and depending on the outcome of  $(V_\alpha)$ , we call  $\alpha$  a *strong ordinal* in case (1) and a *weak ordinal* in case (2). Note that whenever it is possible to get  $\alpha$  to be strong we do so. If such a choice is not possible, then we call  $\alpha$  *useless* and we put  $G_{\alpha+1} = G_\alpha$ ,  $g_\alpha^k = 0$  ( $k < \omega$ ), and  $b_\alpha = \infty$ . (We shall show that in fact this case never arises.) However, in every case  $G_{\alpha+1}$  consists of potential elements.

Finally, let  $G$  be given by

$$G = G_{\lambda^*} = B + \sum_{\alpha < \lambda^*} \sum_{k < \omega} g_\alpha^k A$$

with  $b_\beta + s_\beta^\omega \tilde{B} \notin G^{s_\beta}$  for each  $\beta < \lambda^*$ . ■

Note that the above construction is also very similar to the one given in [3]; the obvious changes are due to the “local” approach used in this paper.

**4. Properties of the constructed module.** In this section we are going to assemble some properties of  $G$ , e.g. we shall describe the essential part of the support of the elements of  $G$  and  $G^s$  ( $s \in S$ ). Again, the results and methods used are very similar to those in [3]; indeed, we get the same results with respect to the quotients  $G^s$  for each  $s \in S$ . Moreover, we also need to include corresponding results for the elements of  $G$ . However, detailed proofs are given for the convenience of the reader.

First we summarize a few properties satisfied by the elements which we use to extend the submodules  $G_\alpha$  in the strong or weak case.

LEMMA 4.1. *Let  $\alpha < \lambda^*$  be any weak or strong ordinal and  $s = s_\alpha$ . Then there exists  $\nu < \|v_\alpha\|$  such that, for all  $a \in A$ ,  $q \in S$  and  $k < \omega$ ,*

- (i)  $g_\alpha^k a \upharpoonright \tau = v_\alpha^{k,s} a \upharpoonright \tau$  for each  $\tau$  with  $\|\tau\| > \nu$ ,
- (ii)  $\nu[\overline{g_\alpha^k a}] \subseteq \nu[g_\alpha^k a] = \nu[v_\alpha^{k,s} a] \subseteq v_\alpha$ ,
- (iii)  $a = 0 \Leftrightarrow g_\alpha^k a = 0 \Leftrightarrow \nu[g_\alpha^k a]$  is finite, and
- (iv)  $(\overline{a}s^n = \overline{0} \text{ for some } n < \omega) \Leftrightarrow \overline{g_\alpha^k a} \in G_\alpha^q \Leftrightarrow \nu[\overline{g_\alpha^k a}]$  is finite,

where  $\overline{g_\alpha^k} = g_\alpha^k + q^\omega \widetilde{B}$ ,  $\overline{a} = a + q^\omega A$ .

PROOF. By (III $_\alpha$ ) in Construction 3.3 we have  $\sup_{k < \omega} \|g_\alpha^k - v_\alpha^{k,s}\| < \|v_\alpha\|$ . Since  $\|v_\alpha\|$  is a limit ordinal by Definition 3.1 we may choose  $\nu < \|v_\alpha\|$  such that  $\nu > \sup_{k < \omega} \|g_\alpha^k - v_\alpha^{k,s}\|$ . Therefore we get  $(g_\alpha^k - v_\alpha^{k,s}) \upharpoonright \tau = 0$  for each  $\tau$  with  $\|\tau\| > \nu$ . Hence, for any  $a \in A$ , we have  $g_\alpha^k a \upharpoonright \tau = v_\alpha^{k,s} a \upharpoonright \tau$  whenever  $\|\tau\| > \nu$ . So  $\nu[\overline{g_\alpha^k a}] \subseteq \nu[g_\alpha^k a] = \nu[v_\alpha^{k,s} a] \subseteq v_\alpha$  according to Lemma 2.4, which proves (i) and (ii).

Clearly  $a = 0 \Rightarrow g_\alpha^k a = 0 \Rightarrow \nu[g_\alpha^k a]$  is finite.

To complete (iii) assume that  $\nu[g_\alpha^k a] = \nu[v_\alpha^{k,s} a]$  is finite. By Lemma 2.4 we have  $[v_\alpha^{k,s} a] \subseteq [v_\alpha^{k,s}] = \{v_{\alpha,i} = v_\alpha \upharpoonright i \mid i \geq k\} = M$ . It follows that  $\nu[v_\alpha^{k,s} a]$  is a finite subset of  $M$  and therefore there is  $n \geq k$  such that  $\nu[v_\alpha^{k,s} a] \subseteq \{v_{\alpha,i} \mid k \leq i \leq n\}$ . Now, since  $\|v_{\alpha,i}\| \leq \|v_{\alpha,j}\|$  for all  $i \leq j < \omega$ , it follows that  $[v_\alpha^{k,s} a]$  is finite. Hence  $a = 0$  by Lemma 2.4 and therefore (iii) is proved.

For (iv) we assume  $\overline{a}s^n = 0$  (in  $A^q$ ) for some  $n < \omega$ . Using  $g_\alpha^{k+n} a s^n \equiv 0 \pmod{q^\omega \widetilde{B}}$  we get  $g_\alpha^k a \equiv g_\alpha^k a - g_\alpha^{k+n} a s^n \pmod{q^\omega \widetilde{B}}$ . Since  $g' = g_\alpha^k - g_\alpha^{k+n} s^n$  is an element of  $G_\alpha$  by the definition of a chain,  $\overline{g_\alpha^k a} = \overline{g'} a$  is an element of  $G_\alpha^q$ .

Therefore  $\nu[\overline{g_\alpha^k a}] \subseteq [\overline{g_\alpha^k a}] \subseteq v_\alpha \cap [g']$  is finite because  $v_\alpha$  does not appear before the  $(\alpha + 1)$ th step by 3.3 and Lemma 3.2.

Finally, assume that  ${}_\nu \overline{[g_\alpha^k a]}$  is finite. Therefore  ${}_\nu \overline{[v_\alpha^{k,s} a]}$  is finite according to (ii). Now, in the same way as before, we find that  $[v_\alpha^{k,s} a]$  is finite. Hence  $as^n = 0$  (in  $A^q$ ) for some  $n < \omega$  by Lemma 2.4, which completes the proof. ■

In the next lemma we describe the supports of the elements of  $G$  and  $G^s$  ( $s \in S$ ). As also in [3] and [8], this will be the main tool for testing if a given potential element belongs to  $G$  or not.

**THE RECOGNITION LEMMA 4.2.** *Let  $g \in G \setminus B$ ,  $s \in S$ , and  $\bar{g} = g + s^\omega \tilde{B} \in G^s$ .*

- (a) (i) *There exists a unique  $\alpha < \lambda^*$  such that  $g \in G_{\alpha+1} \setminus G_\alpha$ .*
- (ii) *Moreover, either  $\bar{g} \in B^s$  or there is a unique  $\beta \leq \alpha$  such that  $\bar{g} \in G_{\beta+1}^s \setminus G_\beta^s$ .*
- (b) (i) *With  $\alpha$  as in (a)(i) there is a strictly decreasing sequence of ordinals  $\alpha = \alpha(0) > \dots > \alpha(r)$  in  $\lambda^*$  ( $r < \omega$ ) with  $\|P_{\alpha(i)}\| = \|P_\alpha\|$  for  $i \leq r$  and an ordinal  $\nu < \|P_\alpha\|$  such that*

$${}_\nu [g] = F \cup \bigcup_{i \leq r} {}_\nu [v_{\alpha(i)}] \quad (\text{disjoint union})$$

where  $F$  is a finite set of elements of  $T$  each of norm greater than  $\|P_\alpha\|$ .

- (ii) *With  $\beta$  as in (a)(ii) there is a strictly decreasing sequence of ordinals  $\beta = \beta(0) > \dots > \beta(k)$  in  $\lambda^*$  ( $k < \omega$ ) with  $\|P_{\beta(i)}\| = \|P_\beta\|$  for  $i \leq k$  and an ordinal  $\mu < \|P_\beta\|$  such that*

$${}_\mu [\bar{g}] = F' \cup \bigcup_{i \leq k} {}_\mu [v_{\beta(i)}] \quad (\text{disjoint union})$$

where  $F'$  is a finite set of elements of  $T$  each of norm greater than  $\|P_\beta\|$ .

- (c) (i) *For any  $\gamma < \lambda^*$  with  $\|P_\gamma\| = \|P_\alpha\|$  there exist  $a \in A$  and  $l < \omega$  such that, for almost all  $\tau \in v_\gamma$ , we have  $g \upharpoonright \tau = \tau a s_\gamma^{l(\tau)-l}$ .*
- (ii) *For any  $\delta < \lambda^*$  with  $\|P_\delta\| = \|P_\beta\|$  there exist  $a' \in A^s$  and  $l' < \omega$  such that, for almost all  $\tau \in v_\delta$ , we have  $\bar{g} \upharpoonright \tau = \tau a' s_\delta^{l(\tau)-l'}$ .*

**Proof.** Since the modules  $G_\alpha$  ( $\alpha \leq \lambda^*$ ) and therefore the modules  $G_\alpha^s$  ( $\alpha \leq \lambda^*$ ) form an ascending smooth chain, (a) is obviously satisfied.

Now  $g \in G_{\alpha+1} = B + \sum_{\gamma \leq \alpha} \sum_{k < \omega} g_\alpha^k A$ . By 3.3 we see that  $(g_\gamma^k)_{k < \omega}$  is an  $(s_\gamma, G_\gamma)$ -chain for each  $\gamma \leq \alpha$ . Moreover, every strictly decreasing chain of ordinals is finite. Therefore we can split the sums in finitely many steps in such a way that we may consider  $g$  as an element of  $B + \sum_{i \leq n} g_{\alpha(i)}^m A$  for ordinals  $\alpha = \alpha(0) > \dots > \alpha(n)$  and for some  $n, m < \omega$ , i.e.  $g = b + \sum_{i \leq n} g_{\alpha(i)}^m a_i$  ( $b \in B$ ,  $a_i \in A$ ).

Now, for every weak or strong ordinal  $\gamma$  we have  $\|g_\gamma^m\| = \|P_\gamma\|$  ( $m < \omega$ ) by Lemmas 2.4 and 4.1. Moreover, the Black Box induces  $\|P_\gamma\| \leq \|P_{\gamma'}\| \leq \|P_\alpha\|$  whenever  $\gamma \leq \gamma' \leq \alpha$ . Therefore there is  $r \leq n$  such that  $\|P_{\alpha(i)}\| = \|P_\alpha\|$  for  $i \leq r$  and  $\|P_{\alpha(i)}\| < \|P_\alpha\|$  otherwise. Hence  $g = b + x + \sum_{i \leq r} g_{\alpha(i)}^m a_i$  where  $\|x\| \leq \max_{r < i \leq n} \|g_{\alpha(i)}\| < \|P_\alpha\|$ .

Since  $\|P_\alpha\|$  is a limit,  $[b]$  does not contain any element of norm  $\|P_\alpha\|$ . The branches  $v_{\alpha(i)}$  ( $i \leq r$ ) are different and therefore the pairwise intersections are finite. Thus we may choose  $\nu < \|P_\alpha\|$  (large enough) such that

- $\nu[g_{\alpha(i)}^m a] = \nu[v_{\alpha(i)} a]$  for  $i \leq r$ ,
- $\nu[v_{\alpha(i)}] \cap \nu[v_{\alpha(j)}] = \emptyset$  for  $i \neq j \leq r$ ,
- $\|x\| < \nu$ , and
- either  $\|\tau\| \leq \nu$  or  $\|\tau\| > \|P_\alpha\|$  for any  $\tau \in [b]$ .

Defining  $F = \{\tau \in [b] \mid \|\tau\| > \|P_\alpha\|\}$  we get  $\nu[g] = F \cup \bigcup_{i \leq r} \nu[v_{\alpha(i)}]$ , which is a disjoint union by our choice of  $\nu$ . This proves part (i) of (b).

To get the second part of (b) we can use similar arguments.

Note that  $\|\overline{g_\gamma^m}\| = \|P_\gamma\|$  whenever  $s_\gamma^j \notin s^\omega A$  for all  $j < \omega$ . Since  $\overline{g_{\beta(i)}^j} a'_i \in G_{\beta(i)}^s$  whenever  $a'_i s_{\beta(i)}^j = 0$  (in  $A^s$ ) by Lemma 4.1, we may assume that  $s_{\beta(i)}^j \notin s^\omega A$  for all  $i \leq k$ , i.e. we get  $\bar{g} = \bar{b} + \sum_{i \leq n} g_{\beta(i)}^m a'_i$  ( $\bar{b} \in B^s$ ,  $a'_i \in A^s$ ).

Finally, we get (c) choosing  $a = a_i$  ( $a' = a'_i$ ) for  $\gamma = \alpha(i)$  ( $\delta = \beta(i)$ ) for some  $i \leq r$  ( $i \leq k$ ) and  $a = 0$  ( $a' = 0$ ) for  $\gamma \notin \{\alpha(0), \dots, \alpha(r)\}$  ( $\delta \notin \{\beta(0), \dots, \beta(k)\}$ ). ■

As an immediate consequence of the above Recognition Lemma we have:

**COROLLARY 4.3.** *An element  $g \in G$  is contained in  $B$  iff  $[g]$  is finite, and an element  $\bar{g} \in G^s$  ( $s \in S$ ) is contained in  $B^s$  iff  $[\bar{g}]$  is finite. ■*

Now we are ready to prove further properties of  $G$ .

**LEMMA 4.4.**  *$G$  is an RD-submodule of  $\tilde{B}$  and  $G$  is  $S$ -reduced and  $S$ -torsion-free.*

**PROOF.** We immediately see that  $G \subseteq \tilde{B}$  is  $S$ -torsion-free since  $\tilde{B}$  is  $S$ -torsion-free by Corollary 1.4. We prove inductively that  $G$  is an RD-submodule of  $\tilde{B}$ .

For  $\nu = 0$  we have  $G_0 = B \subseteq_* \tilde{B}$  and therefore  $G_0 \subseteq_{\text{rd}} \tilde{B}$ . If  $\nu$  is a limit and if  $G_\mu \subseteq_{\text{rd}} \tilde{B}$  for all  $\mu < \nu$ , then  $G_\nu = \bigcup_{\mu < \nu} G_\mu \subseteq_{\text{rd}} \tilde{B}$ , since RD-purity is of finite character.

Now we investigate  $\nu = \alpha + 1$  assuming  $G_\alpha \subseteq_{\text{rd}} \tilde{B}$ . If  $\alpha$  is a useless ordinal, then there is nothing to show since  $G_\alpha = G_{\alpha+1}$ . Otherwise we consider  $\tilde{b}r \in G_{\alpha+1} \setminus G_\alpha$  with  $r \in R$  and  $\tilde{b} \in \tilde{B}$ . There are  $k < \omega$ ,  $0 \neq a \in A$ , and  $g \in G_\alpha$  such that

$$(1) \quad \tilde{b}r = g + g_\alpha^k a.$$

By Lemma 4.2 we see that  $[g] \cap v_\alpha$  is finite. On the other hand, since  $a \neq 0$ , there is  $\nu < \|v_\alpha\|$  such that

$$(2) \quad \nu[g_\alpha^k a] = \nu[v_\alpha^{k,s_\alpha}] \subseteq v_\alpha \quad \text{is infinite}$$

by Lemma 4.1. Therefore, for all  $\tau \in \nu[g_\alpha^k a] \setminus [g]$ ,

$$(3) \quad \tilde{b}r \upharpoonright \tau = (g_\alpha^k a) \upharpoonright \tau.$$

By (2), (3) and Lemma 2.4 it follows that  $\tilde{b}r \upharpoonright \tau = (g_\alpha^k a) \upharpoonright \tau = (v_\alpha^{k,s_\alpha}) \upharpoonright \tau = \tau a s_\alpha^{l(\tau)-k}$  for each  $\tau \in \nu[g_\alpha^k a] \setminus [g]$ . Hence  $a s_\alpha^{l(\tau)-k}$  is an element of  $Ar$ , i.e.  $a s_\alpha^{l(\tau)-k} = a'r$  for some  $a' \in A$ . According to the definition of an  $(s_\alpha, G_\alpha)$ -chain,  $g' = g_\alpha^k a - g_\alpha^{l(\tau)} a s_\alpha^{l(\tau)-k}$  is an element of  $G_\alpha$ . Thus, using (1), we have  $g + g' = \tilde{b}r - g_\alpha^{l(\tau)} a s_\alpha^{l(\tau)-k} = (\tilde{b} - g_\alpha^{l(\tau)} a')r \in G_\alpha$ . Since  $G_\alpha \subseteq_{\text{rd}} \tilde{B}$  by assumption, there exists  $h \in G_\alpha$  such that  $g + g' = hr = (\tilde{b} - g_\alpha^{l(\tau)} a')r$ , which implies  $\tilde{b}r = (h + g_\alpha^{l(\tau)} a')r$  where  $h + g_\alpha^{l(\tau)} a' \in G_{\alpha+1}$ . Therefore,  $G_{\alpha+1} \subseteq_{\text{rd}} \tilde{B}$ . Finally,  $G = \bigcup_{\alpha \leq \lambda^*} G_\alpha \subseteq_{\text{rd}} \tilde{B}$ .

Similarly, using the fact that  $A$  is  $S$ -reduced, it is easy to show by transfinite induction that  $G = \bigcup_{\alpha \leq \lambda^*} G_\alpha$  is an  $S$ -reduced  $R$ -module as well. ■

Note that we now may identify  $(G + s^\omega \tilde{B})/s^\omega \tilde{B}$  with  $G/s^\omega G$  since  $G$  is an RD- and hence an  $S$ -RD-submodule of  $\tilde{B}$ . Moreover, we may consider  $\widehat{G^s}$ , the  $s$ -adic completion of  $G^s = G/s^\omega G$ , as a submodule of  $\tilde{B}^s$  by Theorem 1.6.

The key lemma for the non-existence of useless ordinals and for “killing” unwanted endomorphisms is given next.

LEMMA 4.5. *Let  $\alpha < \lambda^*$ ,  $\nu < \|P_\alpha\|$ , and for each  $v \in \text{Br}(\text{Im } f_\alpha)$  let  $(g_v^k)_{k < \omega}$  be an  $(s_\alpha, G_\alpha)$ -chain such that, for all  $k < \omega$ ,  $\nu[g_v^k - v^{k,s_\alpha}] = \emptyset$ . Then there exists  $v \in \text{Br}(\text{Im } f_\alpha)$  such that*

$$(1) \quad b_\beta + s_\beta^\omega \tilde{B} \notin (G_{\alpha+1}(v))^{s_\beta} \quad \text{for all } \beta < \alpha$$

where  $G_{\alpha+1}(v) = G_\alpha + \sum_{k < \omega} g_v^k A$ .

PROOF. Let  $s = s_\alpha$  and suppose that the conclusion is false. Then there exists, for each  $v \in \text{Br}(\text{Im } f_\alpha)$ , an ordinal  $\beta = \beta(v)$  such that  $b_\beta + q^\omega \tilde{B} \in (G_{\alpha+1}(v))^q$  where  $q = s_\beta$ . By our Construction 3.3 that means  $b_\beta \neq \infty$  and  $b_\beta + q^\omega \tilde{B} = (g_\beta^0 + q^\omega \tilde{B})\phi_\beta \in \widehat{P}_\beta^q$ . Moreover, there are  $a = a(v) \in A^q$  and  $k = k(v) < \omega$  such that

$$(2) \quad (b_\beta + q^\omega \tilde{B}) - g_v^k a \in G_\alpha^q.$$

Since  $(g_v^k)_{k < \omega}$  is an  $(s, G_\alpha)$ -chain and  $b_\beta + q^\omega \tilde{B} \notin G_\alpha^q$  by assumption, we have  $a s^n \neq 0$  (in  $A^q$ ) for each  $n < \omega$ . Clearly we also have  $\nu[g_v^k a] = \nu[v^{k,s_\alpha} a]$ .

Now, using Lemma 2.4, we find that  $\nu[g_v^k a]$  is infinite. For all  $\gamma < \alpha$  it is certainly true that  $\|v_\gamma\| \leq \|P_\alpha\| = \|v\|$  and  $v \neq v_\gamma$ . Therefore there is an



infinite subset  $X$  of  $v$  such that  $X \subseteq [b_\beta + q^\omega \tilde{B}] \subseteq [P_\beta]$  by our Recognition Lemma. Since  $[P_\beta]$  is a subtree of  $T$  this implies  $v \subseteq [P_\beta]$ . Hence  $v$  is an element of  $\text{Br}(\text{Im } f_\alpha) \cap \text{Br}([P_\beta])$ . The Black Box tells us that this is only possible for  $\beta < \alpha < \beta + \kappa^{\aleph_0}$ .

We have shown by now that for each  $v \in \text{Br}(\text{Im } f_\alpha)$ , there exist  $\beta(v) < \alpha$ ,  $k(v) < \omega$ , and  $a(v) \in A^{s_{\beta(v)}}$  such that

$$(3) \quad \beta(v) < \alpha < \beta(v) + \kappa^{\aleph_0} \quad \text{and} \quad (b_{\beta(v)} + q^\omega \tilde{B}) - g_v^{k(v)} a(v) \in G_\alpha^{s_{\beta(v)}}.$$

Now let  $\beta_0$  be the smallest ordinal satisfying  $\beta_0 < \alpha < \beta_0 + \kappa^{\aleph_0}$ . This implies

$$\beta_0 \leq \beta(v) < \alpha < \beta_0 + \kappa^{\aleph_0} \quad \text{for all } v \in \text{Br}(\text{Im } f_\alpha).$$

Therefore  $|\{\beta(v) \mid v \in \text{Br}(\text{Im } f_\alpha)\}| < \kappa^{\aleph_0} = |\text{Br}(\text{Im } f_\alpha)|$ . Hence there are different branches  $v, u \in \text{Br}(\text{Im } f_\alpha)$  with  $\beta(v) = \beta(u) = \beta$ . Subtracting the corresponding equations in (3) gives  $g_v^{k(v)} a(v) - g_u^{k(u)} a(u) \in G_\alpha^{s_\beta}$ . Arguing as before we show that an infinite subset of  $v$  is contained in  ${}_\nu[g_u^{k(u)} a(u)] \subseteq u$ , which contradicts the assumption that  $v, u$  are different branches. ■

**COROLLARY 4.6.** *There are no useless ordinals. An ordinal  $\alpha < \lambda^*$  is strong or weak according as  $(g_\alpha^0 + s_\alpha^\omega \tilde{B})\phi_\alpha$  lies outside or in  $G^{s_\alpha}$ .*

**PROOF.** Take  $g_v^k = v^{k, s_\alpha}$  for each  $v \in \text{Br}(\text{Im } f_\alpha)$  and apply Lemma 4.5. ■

After we have considered the  $R$ -module  $G$  from a more general point of view, we are now going to investigate special cases.

**5. The cotorsion-free case.** Considering the classical definition of a cotorsion-free module in the countable case (see e.g. [3], [4]) there seem to be different ways to generalize it to the uncountable case. In [8] an  $(\omega)$ -cotorsion-free module is defined by having 0 as the only  $(\omega)$ -complete submodule where  $(\omega)$ -completeness is an obvious replacement for completeness in the  $S$ -adic completion for countable  $S$ . Our notion of cotorsion-freeness is adapted to the modules under consideration.

**DEFINITION 5.1.** An  $R$ -module  $M$  is defined to be *cotorsion-free* if it is  $S$ -torsion-free,  $S$ -reduced, and  $\text{Hom}(\widehat{R^s}, M^s) = 0$  for each  $s \in S$ .

Note that it is easy to check that a module which is cotorsion-free in the above sense is also  $(\omega)$ -cotorsion-free with respect to the definition given in [8] and both definitions coincide if  $S$  is countable.

The above version of cotorsion-freeness allows us to follow “locally” the arguments in [3], which are sketched below for the convenience of the reader.

Using the special form of  $G$  we are able to show that  $G$  is cotorsion-free if  $A$  is. We assume that  $A$  is cotorsion-free throughout this section.

**LEMMA 5.2.** *If  $A$  is cotorsion-free then  $G$ , constructed as in 3.3, is also cotorsion-free.*

Proof. We already know that  $G$  is  $S$ -torsion-free and  $S$ -reduced by Lemma 4.4. Now it remains to show that  $\text{Hom}(\widehat{R}^s, G^s) = 0$  for each  $s \in S$ . Clearly  $B$  is cotorsion-free as a direct sum of copies of  $A$ .

Suppose that there is a non-zero homomorphism  $\phi : \widehat{R}^s \rightarrow G^s$  for some  $s \in S$ , hence  $0 \neq \bar{g} = 1\phi \in G^s \subseteq \widehat{G}^s$ . We have  $r\phi = 1\phi r = \bar{g}r \in G^s$  for all  $r \in \widehat{R}^s$  by continuity. Hence the supports are defined and

$$(1) \quad [\bar{g}r] \subseteq [\bar{g}] \quad \text{for each } r \in \widehat{R}^s.$$

(i) First we consider the case  $\bar{g} \in B^s$ . Then  $[\bar{g}]$  is finite and therefore  $[\bar{g}r]$  is finite for each  $r \in \widehat{R}^s$  as well by (1). By Corollary 4.3 it follows that  $\bar{g}r \in B^s$  for each  $r \in \widehat{R}^s$ , hence  $\phi \in \text{Hom}(\widehat{R}^s, B^s) = 0$  contrary to the assumption.

(ii) Now suppose  $\bar{g} \in G^s \setminus B^s$ . Thus, by Lemma 4.2, there are  $\beta < \lambda^*$  with  $\bar{g} \in G_{\beta+1} \setminus G_\beta$ ,  $\mu < \|\bar{g}\|$ , and ordinals  $\beta = \beta(0) > \beta(1) \dots > \beta(k)$  ( $k < \omega$ ) such that

$$(2) \quad \mu[\bar{g}] = F \cup \bigcup_{i \leq k} \mu[v_{\beta(i)}]$$

where  $F$  is finite. Moreover, we may assume that there is an  $a \in A^s$  such that

$$(3) \quad \bar{g} - \bar{g}_\beta^0 a \in G_\beta^s$$

(otherwise multiply  $\phi$  by  $s_\beta^n$  for some suitable  $n$ ).

To get a contradiction we define a homomorphism  $\psi : \widehat{R}^s \rightarrow A^s$ . Let  $r\phi = \bar{g}r \in G^s$  for any  $r \in \widehat{R}^s$ . By (1) and (2),  $[\bar{g}r]$  cannot contain infinitely many elements from a branch  $v_\gamma$  with  $\gamma > \beta$ . Therefore, by Lemma 4.2 again,  $\bar{g}r$  is an element of  $G_{\beta+1}^s$ . Thus we find  $q \in \{s_\beta^n \mid n < \omega\}$  and  $a' \in A^s$  with

$$(4) \quad \bar{g}qr - \bar{g}_\beta^0 a' \in G_\beta^s.$$

Multiplying (3) by  $qr$  we get  $(\bar{g} - \bar{g}_\beta^0 a)qr \in G_\beta^s \widehat{R}^s$ , hence  $\bar{g}_\beta^0(a' - aqr) \in G_\beta^s \widehat{R}^s$  by (4). Moreover,  $[\bar{g}_\beta^0(a' - aqr)] \subseteq [\bar{g}_\beta^0]$ . The intersection  $[\bar{g}_\beta^0(a' - aqr)] \cap v_\beta$  must be finite by the Recognition Lemma. Therefore  $\bar{g}_\beta^0(a' - aqr) \upharpoonright (v_\beta \upharpoonright k) = 0$  for almost all  $k < \omega$ , i.e.  $a' \in \widetilde{A}^s q \cap A^s = A^s q$ . Hence  $a' = a_* q$  for some  $a_* \in A^s$  and (4) reduces to  $\bar{g}r - \bar{g}_\beta^0 a_* \in G_\beta^s$ . Clearly  $a_* \neq 0$  (in  $A^s$ ) by our choice of  $\beta$ . Finally, we define  $\psi : \widehat{R}^s \rightarrow A^s$  by  $r\psi = a_*$  and get a non-zero homomorphism, contradicting the hypothesis on  $A$ . ■

Note that, if  $G$  is cotorsion-free, there are no non-zero homomorphisms from any  $s$ -complete module  $\widehat{K}^s$  into  $G^s$  ( $s \in S$ ).

The desired result  $\text{End } G = A$  will follow from its local form  $\text{End } G^s = A^s$  for each  $s \in S$ , and this needs

PROPOSITION 5.3. For  $s \in S$  let  $K$  be a potential module containing  $G$ , and  $\phi \in K^s$  with  $\phi|G^s \notin A^s$ . Then there exists a canonical module  $P \subseteq G$  such that

$$(1) \quad \widehat{P}^s(\phi - a) \not\subseteq G^s \quad \text{for each } a \in A^s.$$

PROOF. Let  $P_0$  be an arbitrary canonical module contained in  $G$  and assume  $\widehat{P}_0^s(\phi - a_*) \subseteq G^s$  for some  $a_* \in A^s$ . Therefore we have  $\phi|P_0^s = a_*$  since  $\text{Hom}(\widehat{P}_0^s, G^s) = 0$ . We know that  $\phi|G^s \notin A^s$ , i.e. there is a  $\bar{g} = g + s^\omega \tilde{B} \in G^s$  with  $\bar{g}\phi \neq \bar{g}a_*$ . Now let  $P$  be a canonical module containing  $P_0$  and  $g$ ; also assume that  $P$  does not satisfy (1). Then we get  $\widehat{P}^s(\phi - a') \subseteq G^s$  for some  $a' \in A^s$ . Therefore it follows that  $\widehat{P}_0^s(a' - a_*) \subseteq G^s$ , which implies  $a' = a_*$ ; but then  $\bar{g}\phi = \bar{g}a' = \bar{g}a_* \in G^s$ , contradicting our choice of  $\bar{g}$ . ■

Now we extend 5.3(1) to homomorphisms of the form  $\phi s^n - a$  where either  $n$  is zero or  $s$  does not divide  $a$ ; let

$$\Delta_s = \{(a, n) \in A^s \times \omega \mid n = 0 \vee (n \geq 1 \wedge a \notin A^s s)\} \quad \text{for any } s \in S.$$

COROLLARY 5.4. For  $s \in S$  let  $K$  be a potential module containing  $G$ , and  $\phi \in \text{End } K^s$  with  $\phi|G^s \notin A^s$ . Then there exists a canonical module  $P \subseteq G$  such that

$$\widehat{P}^s(\phi s^n - a) \not\subseteq G^s \quad \text{for all } (a, n) \in \Delta_s.$$

PROOF. Choose  $P$  as in Proposition 5.3, i.e.  $\widehat{P}^s(\phi - a) \not\subseteq G^s$  for each  $a \in A^s$ . If  $n \geq 1$  then  $\widehat{P}^s(\phi s^n - a) \subseteq G^s$  is only possible for  $\phi s^n - a = 0$ , hence  $s$  divides  $a$  and  $(a, n) \notin \Delta_s$ . ■

Next we show that a homomorphism of  $G$ , unwanted when viewed on  $G^s$  ( $s \in S$ ), extends to a module  $G'$ , and when the extension is viewed on  $(G')^s$  it cannot occur, which “kills” the candidate. This is based on the existence of certain elements of  $\widehat{G}^s$  which are mapped outside of  $(G')^s$ . For this purpose we define a constant branch  $w = w(\eta) : \omega \rightarrow \{\eta\}$  for each  $\eta < \lambda$ . As an immediate consequence of the Recognition Lemma we get the following

PROPOSITION 5.5. For any  $g \in G$  or  $g \in G^s$  ( $s \in S$ ) the support  $[g]$  cannot contain an infinite subset of a constant branch of norm  $\|g\|$ . ■

LEMMA 5.6. For  $s \in S$  let  $K$  be a potential module containing  $G$ , and  $\phi \in \text{End } K^s$  with  $\phi|G^s \notin A^s$ . Then there exists an  $(s, G)$ -chain  $(x^k)_{k < \omega}$  such that

$$(1) \quad (x^0 + s^\omega \tilde{B})\phi = \bar{x}^0\phi \notin \left(G + \sum_{k < \omega} x^k A\right)^s.$$

Proof. Choose  $P$  as in Corollary 5.4 and pick an ordinal  $\mu < \lambda$  such that

$$(2) \quad \max\{\|P\|, \|P^s\phi\|\} < \|\mu\|.$$

Consider the constant branch  $w = w(\mu)$ . Either the  $(s, B)$ -chain  $(w^{k,s})_{k < \omega}$  will satisfy 5.6 or else (1) does not hold for this chain. Then there is a least  $r < \omega$  such that, for some  $a \in A^s$ ,

$$(3) \quad \overline{w^{0,s}}\phi - \overline{w^{r,s}}a \in G^s.$$

Then  $(a, r) \in \Delta_s$  by the minimality of  $r$ . There exists an element  $\bar{y} = (y + s^\omega \tilde{B}) \in \widehat{P^s}$  with

$$(4) \quad \bar{y}(\phi s^r - a) \notin G^s.$$

We may extend  $y$  to an  $(s, P)$ -chain  $(y^k)_{k < \omega}$  with  $y \equiv y^0 \pmod{s^\omega \tilde{B}}$  taking a solution of  $y_k - y_{k+1}s = p_k$  in  $(\bigoplus_{\tau \in [P]} \tau A)^\sim \subseteq \tilde{B}$  where  $\bar{y} = \sum_{i < \omega} \bar{p}_i s^i$  for  $p_i \in P$ ,  $i < \omega$ . Next define  $x^k = y^k + w^k$  ( $k < \omega$ ). The sequence  $(x^k)_{k < \omega}$  is an  $(s, G)$ -chain with  $\nu = \|y\| \leq \|P\| < \|\mu\| = \|w^0\| = \|x^0\|$ . We claim that  $(x^k)_{k < \omega}$  satisfies (1). Otherwise there are  $r' \geq r$  and  $a' \in A^s$  with

$$(5) \quad (\overline{w^{0,s}} + \bar{y})\phi - (\overline{w^{r',s}} + \bar{y}^{r'})a' \in G^s.$$

Now subtracting (5) and (3) gives  $\bar{y}\phi - \bar{y}^{r'}a' - \overline{w^{r',s}}a' + \overline{w^{r,s}} \in G^s$ . Using  $w^{r,s} - w^{r',s}s^{r'-r} \in B$  we get

$$(6) \quad (\bar{y}\phi - \bar{y}^{r'}a') + \overline{w^{r',s}}(as^{r'-r} - a') \in G^s.$$

Note that  $\|(\bar{y}\phi - \bar{y}^{r'}a')\| < \|\mu\|$  and  $w$  is a constant branch of norm  $\|\mu\|$ . Therefore  $[\overline{w^{r',s}}(as^{r'-r} - a')]$  must be finite by Proposition 5.5. Since  $as^{r'-r} = a'$  (in  $A^s$ ) by Lemma 2.4, condition (6) reduces to  $\bar{y}\phi - \bar{y}^{r'}as^{r'-r} \in G^s$ . Multiplying this by  $s^r$  gives  $\bar{y}\phi s^r - \bar{y}a = \bar{y}(\phi s^r - a) \in G^s$ , contradicting (4). ■

Now we show that an unwanted homomorphism killed by an extension 5.6 has already been treated this way while constructing  $G$ , hence

**THEOREM 5.7.** End  $G^s = A^s$  for each  $s \in S$ .

Proof. Let  $s \in S$ ,  $\phi \in \text{End } G^s$  and assume  $\phi \notin A^s$ . Then, by Lemma 5.6, there exists an  $(s, G)$ -chain  $(x^k)_{k < \omega}$  satisfying

$$(1) \quad \overline{x^0}\phi \notin \left(G + \sum_{k < \omega} x^k A\right)^s.$$

Since  $\{x^k - x^{k+1}s \mid k < \omega\}$  is a countable subset of  $G$  there exists  $\beta < \lambda^*$  such that  $(x^k)_{k < \omega}$  is an  $(s, G_\beta)$ -chain. Moreover, it is easy to check that there exist potential elements  $z^k$  ( $k < \omega$ ) such that  $z^k + s^\omega \tilde{B} = (x^k + s^\omega \tilde{B})\phi$ .

Now the Black Box provides an  $\alpha < \lambda^*$  with

$$(2) \quad \begin{aligned} \{x^k, z^k \mid k < \omega\} &\subseteq P_\alpha, & \|\{x^k, z^k \mid k < \omega\}\| &< \|P_\alpha\|, \\ s &= s_\alpha, & \phi_\alpha &= \phi \upharpoonright P_\alpha^s \end{aligned}$$

where we may assume  $\beta \leq \alpha$ . We have  $g_\alpha = g_\alpha^0 \in G$  satisfying  $g_\alpha + s^\omega \tilde{B} \in \widehat{P}_\alpha^s$  by our construction of  $G$ .

If  $\alpha$  is strong then  $(g_\alpha + s^\omega \tilde{B})\phi = (g_\alpha + s^\omega \tilde{B})\phi_\alpha = b_\alpha + s^\omega \tilde{B} \notin G^s$ , which contradicts  $\phi \in \text{End } G^s$ . We shall show that  $\alpha$  is strong indeed.

Consider any branch  $v \in \text{Br}(\text{Im } f_\alpha)$ . We claim that there are  $\varepsilon = \varepsilon(v) \in \{0, 1\}$  such that

$$(3) \quad (v^{0,s} + \varepsilon x^0 + s^\omega \tilde{B})\phi_\alpha \notin \left(G_\alpha + \sum_{k < \omega} (v^{k,s} + \varepsilon x^k)A\right)^s.$$

Otherwise there exist  $k > \omega$  and  $a_0, a_1 \in A$  with

$$(v^{0,s} + \varepsilon x^0 + s^\omega \tilde{B})\phi_\alpha - (v^{k,s} + \varepsilon x^k + s^\omega \tilde{B})a_\varepsilon \in G_\alpha^s \quad (\varepsilon = 0, 1),$$

and the subtraction of both terms in (3) gives

$$(4) \quad (x^0 + s^\omega \tilde{B})\phi_\alpha - x^k a_1 + v^{k,s}(a_0 - a_1) + s^\omega \tilde{B} \in G_\alpha^s$$

where  $\|(x^0 + s^\omega \tilde{B})\phi_\alpha - x^k a_1 + s^\omega \tilde{B}\| < \|P_\alpha\| = \|v\|$  by (2). Therefore, by the Recognition Lemma and Lemma 2.4, we have  $v^{k,s}(a_0 - a_1) \equiv 0 \pmod{s^\omega \tilde{B}}$  and thus (4) reduces to  $(x^0 + s^\omega \tilde{B})\phi_\alpha - x^k a_1 + s^\omega \tilde{B} \in G_\alpha^s$ , contradicting (1).

Hence we have proved, for each  $v \in \text{Br}(\text{Im } f_\alpha)$ , that there exists an  $(s, G_\alpha)$ -chain  $(g_v^k = v^{k,s} + \varepsilon x^k)_{k < \omega}$  which obviously satisfies the hypothesis of Lemma 4.5 and is such that  $(g_v^0 + s^\omega \tilde{B})\phi_\alpha \notin (G_\alpha + \sum_{k < \omega} g_v^k A)^s$ . Applying Lemma 4.5 it is now easy to see that  $\alpha$  is a strong ordinal, which completes the proof. ■

Note that all the results and proofs above follow very closely the arguments given in [3].

For lifting the “local results” in Theorem 5.7 to  $\text{End } G = A$  we need an additional assumption on  $A$ . The notion of  $F$ -completeness with respect to a filtration  $F$  as introduced in [6] turns out to be exactly what we need. Let  $F$  be the filtration of  $A$  given by  $\{s^\omega A \mid s \in S\}$ . We will assume that  $A$  is  $F$ -complete, which means that for every family  $C = (a_s + s^\omega A)_{s \in S}$  of cosets with the finite intersection property there is  $a \in A$  with  $a \in \bigcap_{s \in S} (a_s + s^\omega A)$ . Note that for  $S$  countable this condition is no restriction since  $F$  contains only 0 in this case. Moreover, a module which is complete in the  $R$ -filtration is also  $F$ -complete and thus any linearly compact module is  $F$ -complete (see [6] for details).

Before we prove the realization theorem we give a non-trivial example of a cotorsion-free and  $F$ -complete algebra  $A$ .

EXAMPLE 5.8. Let  $R = \mathbb{R}[x]$  be the polynomial ring over the real numbers and  $S$  be the multiplicative closure of the set  $\{p_a = x - a \mid a \in \mathbb{R}\}$ . Therefore  $S$  is uncountable and  $R$  is  $S$ -torsion-free.

We define  $R_{(a)} = \{f/g \mid f, g \in R; (g, p_a) = 1\}$  for each  $a \in \mathbb{R}$  and  $A = \prod_{a \in \mathbb{R}} R_{(a)}$ . It is easy to verify that  $A$  is  $S$ -torsion-free and  $S$ -reduced. Moreover, we have  $\bigcap_{s \in S'} As \neq 0$  for each countable subset  $S'$  of  $S$ , i.e. we cannot replace  $S$  by a countable subset.

It remains to show that  $\text{Hom}(\widehat{R^s}, A^s) = 0$  for each  $s \in S$  (to get  $A$  cotorsion-free) and  $A$  is  $F$ -complete.

We first determine  $\widehat{R^s}$  and  $A^s$ . For each  $s \in S$  we have  $s^\omega R = 0$ ; thus  $R^s = R$  and  $\widehat{R^s}$  consists of all formal power series of the form  $\sum_{n=0}^\infty f_n s^n$  where  $f_n \in R$  for each  $n < \omega$ . By continuity arguments we conclude

$$(1) \quad \text{Hom}_R(\widehat{R^s}, R) = 0 \quad \text{for each } s \in S.$$

By  $p_a^\omega R_{(a)} = 0$  and  $p_a^\omega R_{(b)} = R_{(b)}$  for  $b \neq a$  it follows that  $p_a^\omega A = \prod_{b \neq a} R_{(b)}$  and thus  $A^{p_a} = A/p_a^\omega A \cong R_{(a)}$ . In general we have  $A^s \cong \prod_{i \leq n} R_{(a_i)} = \bigoplus_{i \leq n} R_{(a_i)}$  for  $s = p_{a_1}^{r_1} \cdot \dots \cdot p_{a_n}^{r_n}$ .

Therefore it is enough to consider homomorphisms  $\phi : \widehat{R^s} \rightarrow R_{(a)}$  to get  $\text{Hom}_R(\widehat{R^s}, A^s) = 0$  for each  $s \in S$ .

Suppose  $\phi$  is such a homomorphism and  $1\phi = f/g \in R_{(a)}$ . Multiplying  $\phi$  by  $g$  gives  $\phi g : \widehat{R^s} \rightarrow R$  and thus  $\phi g = 0$  by (1). Therefore  $\phi = 0$  and  $A$  is cotorsion-free.

To show that  $A$  is  $F$ -complete with respect to  $F = \{s^\omega A \mid s \in S\}$  we consider a family  $(f_s + s^\omega A)_{s \in S}$  of cosets having the finite intersection property. We may assume  $f_s \in \prod_{i \leq n} R_{(a_i)}$  since  $f_s + s^\omega A \in A^s \cong \prod_{i \leq n} R_{(a_i)}$  ( $s = p_{a_1}^{r_1} \cdot \dots \cdot p_{a_n}^{r_n}$ ).

Let  $f_a$  denote  $f_s$  for  $s = p_a$  and define  $g = (f_a)_{a \in \mathbb{R}} \in A$ . We claim that  $g \in \bigcap_{s \in S} (f_s + s^\omega A)$ , i.e.  $g \equiv f_s \pmod{s^\omega A}$  for each  $s \in S$ .

Obviously, for each  $a \in \mathbb{R}$ ,

$$(2) \quad g \equiv f_a \pmod{p_a^\omega A}.$$

For  $s = p_{a_1}^{r_1} \cdot \dots \cdot p_{a_n}^{r_n}$  we have  $f_s \equiv f_{a_i} \pmod{p_{a_i}^\omega A}$  for each  $i \leq n$  by the finite intersection property and  $s^\omega A \subseteq p_{a_i}^\omega A$ . Therefore  $f_s \equiv g \pmod{p_{a_i}^\omega A}$  ( $i \leq n$ ) by (2) and thus  $f_s \equiv g \pmod{s^\omega A}$  as  $\bigcap_{i \leq n} p_{a_i}^\omega A = s^\omega A$ . Hence  $A$  is  $F$ -complete. ■

Finally, we get

THEOREM 5.9. *If  $A$  is cotorsion-free and  $F$ -complete then  $\text{End } G = A$ .*

PROOF. We consider an endomorphism  $\phi$  of  $G$ . For each  $s \in S$  we get a canonical endomorphism  $\phi^s$  of  $G^s = G/s^\omega G$ . By Theorem 5.7 there are  $a_s \in A$  ( $s \in S$ ) such that  $\phi^s = a_s + s^\omega A$ . We claim that  $(a_s + s^\omega A)_{s \in S}$  has the finite intersection property. Let  $N$  be a finite subset of  $S$  and  $q \in S$  the

product of all elements in  $N$ . For all  $g \in G$  and for each  $s \in N$  we have  $g\phi \equiv ga_s \pmod{s^\omega G}$  and  $g\phi \equiv ga_q \pmod{q^\omega G}$ . Therefore  $a_s \equiv a_q \pmod{s^\omega A}$  for each  $s \in N$ , since  $q^\omega G \subseteq s^\omega G$ , i.e.  $a_q \in \bigcap_{s \in N} (a_s + s^\omega A)$ . Since  $A$  is  $F$ -complete it follows that there is an  $a \in A$  with  $a \equiv a_s \pmod{s^\omega A}$  for every  $s \in S$ . Hence  $\phi^s = a + s^\omega A$  for each  $s \in S$ . Therefore  $\phi = a$  because  $G$  is  $S$ -reduced by Lemma 4.4. ■

**6. Inessentials.** In this section we state the most general case (we will give an application in §7). Its proof follows by modifications of the arguments given in §5 combined with obvious changes similar to [3, §§ 4, 5]; see also [9, II, §§3, 4]. The definition of an inessential endomorphism is, as usual, strongly related to the notion of cotorsion-freeness: in the cotorsion-free case there are no non-zero inessential endomorphisms. Moreover, the notion of inessentials depends on the topic of consideration (e.g. see [3]). In our context we shall use Definition 6.1, which differs from the one used in [8]; note that the latter should read as follows:  $\phi$  is inessential if  $\text{Im } \phi$  is contained in (rather than equal to) an  $\omega$ -complete module.

We now have

DEFINITION 6.1. The *inessential endomorphisms* of  $G^s$  and  $G$  are defined by

(a)  $\text{Ines } G^s = \{\phi \in \text{End } G^s \mid \widehat{G^s}\phi \subseteq G^s\}$  for all  $s \in S$ .

(b)  $\text{Ines } G = \{\phi \in \text{End } G \mid \phi^s \in \text{Ines } G^s \text{ for each } s \in S\}$  where  $\phi^s$  denotes the induced endomorphism of  $G^s$ .

Note that we are identifying  $\phi$  in (a) with its unique extension to  $\widehat{G^s}$ . It is easy to check that  $\text{Ines } G^s$  and  $\text{Ines } G$  are ideals of  $\text{End } G^s$  and  $\text{End } G$ , respectively. Moreover, we can verify  $A^s \cap \text{Ines } G^s = 0$  by applying Proposition 5.5 to the element  $\overline{w^{0,s}} \in \widehat{G^s}$  for a constant branch  $w$ .

The key arguments are assembled in the following three statements.

PROPOSITION 6.2. For  $s \in S$  let  $K$  be a potential module containing  $G$ , and  $\phi \in \text{End } K^s$  with  $\phi|_{G^s} \notin A^s \oplus \text{Ines } G^s$ . Then there exists a canonical module  $P \subseteq G$  such that  $\widehat{P^s}(\phi s^n - a) \not\subseteq G^s$  for all  $(a, n) \in \Delta_s$  where  $\Delta_s$  is defined as in §5. ■

LEMMA 6.3. For  $s \in S$  let  $K$  be a potential module containing  $G$ , and  $\phi \in \text{End } K^s$  with  $\phi|_{G^s} \notin A^s \oplus \text{Ines } G^s$ . Then there exists an  $(s, G)$ -chain  $(x^k)_{k < \omega}$  such that

$$(x^0 + s^\omega \widetilde{B})\phi = \overline{x^0}\phi \notin \left(G + \sum_{k < \omega} x^k A\right)^s. \blacksquare$$

THEOREM 6.4. If  $A$  is an  $S$ -reduced and  $S$ -torsion-free  $R$ -algebra, then  $\text{End } G^s = A^s \oplus \text{Ines } G^s$  for each  $s \in S$ , where  $G$  is the  $R$ -module constructed in 3.3. ■

Extension of the local result 6.4 to  $\text{End } G = A \oplus \text{Ines } G$  again requires  $A$  and the ideals  $\text{Ines } G^s$  to be of a special form.

**DEFINITION 6.5.** *Ines } G is well related if, for any  $\phi \in \text{End } G$  with  $\phi^q \in \text{Ines } G^q$  ( $q \in S$ ), the endomorphism  $\phi^s$  of  $G^s$  belongs to  $\text{Ines } G^s$  for each  $s$  dividing  $q$ .*

**THEOREM 6.6.** *If  $A$  is  $S$ -reduced,  $S$ -torsion-free,  $F$ -complete and  $\text{Ines } G$  is well related, then  $\text{End } G = A \oplus \text{Ines } G$ .*

**PROOF.** Obviously,  $A \cap \text{Ines } G = 0$  since  $A^s \cap \text{Ines } G^s = 0$  for all  $s \in S$ .

Let  $\phi$  be an endomorphism of  $G$ . For all  $s \in S$ , by Theorem 6.4 there are  $a_s \in A$  and  $\psi_s \in \text{Ines } G^s$  such that  $\phi^s = \bar{a}_s + \psi_s$ . Defining  $\varrho_s = \phi - a_s$  we can identify  $\psi_s$  with  $\varrho_s^s$ . We claim that  $(a_s + s^\omega A)_{s \in S}$  has the finite intersection property. Let  $N \subseteq S$  finite and  $q$  be the product of all elements of  $N$ . Then  $\varrho_q^s = (\phi - a_q)^s \in \text{Ines } G^s$  for each  $s \in N$ , since  $\text{Ines } G$  is well related. Hence  $\phi^s = \varrho_q^s + \bar{a}_q = \varrho_s^s + \bar{a}_s$  and this induces  $a_q \equiv a_s \pmod{s^\omega A}$ . Therefore  $a_q \in \bigcap_{s \in N} (a_s + s^\omega A)$ . Since  $A$  is  $F$ -complete there exists  $a \in A$  with  $a \equiv a_s \pmod{s^\omega A}$  for all  $s \in S$ . Defining  $\varrho$  by  $\varrho = \phi - a$  we get  $\phi = a + \varrho$  with  $\varrho^s = \varrho_s^s = \psi_s \in \text{Ines } G^s$ , i.e.  $\varrho \in \text{Ines } G$ . ■

Note that the above result has been obtained using a different approach for a more general class in [8]; the above less general result is, however, considerably simpler in its approach.

In the last section we give an example for well-related inessentials.

**7. The  $\aleph_0$ -cotorsion-free case.** In the final section we consider the  $\aleph_0$ -cotorsion-free case. For  $S$  uncountable no such notion has been introduced yet. Our notion of  $\aleph_0$ -cotorsion-freeness generalizes the definition for countable  $S$ ; it is also adapted to the local approach. Assuming that  $S$  is linearly ordered we shall show that the ideal  $\text{Ines } G$  is well related when  $G$  is  $\aleph_0$ -cotorsion-free.

Hence we get a realization  $\text{End } G = A \oplus \text{Ines } G$ . Throughout this section let  $S$  be linearly ordered, i.e.  $s$  divides  $q$  or  $q$  divides  $s$  for all  $s, q \in S$ .

First we introduce the notion of a sub-finitely generated homomorphism, or sfg-homomorphism, and a “local” version of the same concept. Note that this differs from the definition of finite rank used in [3] but that for the situations where  $\aleph_0$ -cotorsion-freeness has previously been considered, e.g.  $p$ -adic modules or subgroups of the Baer–Specker group, the two concepts coincide.

**DEFINITION 7.1.** (a) Let  $M, H$  be  $s$ -reduced,  $s$ -torsion-free  $R$ -modules ( $s \in S$ ). A homomorphism  $\phi : M \rightarrow H$  is *sub-finitely generated* if there exists a finitely generated submodule  $H_0 \subseteq H$  such that  $M\phi \subseteq H_0$ .



(b) Let  $M, H$  be  $S$ -reduced and  $S$ -torsion-free  $R$ -modules. A homomorphism  $\phi : M \rightarrow H$  is *locally sub-finitely generated* if the induced homomorphisms  $\phi^s : M^s \rightarrow H^s$  are sub-finitely generated for all  $s \in S$ .

Note that we shall also use the notation “locally sfg-homomorphism” for a locally sub-finitely generated homomorphism. We are now ready for

DEFINITION 7.2. An  $S$ -torsion-free and  $S$ -reduced  $R$ -module  $M$  is  $\aleph_0$ -cotorsion-free if for all  $s \in S$  and for each homomorphism  $\phi : \widehat{F^s} \rightarrow M^s$ , the restriction  $\phi|_{F^s}$  is sub-finitely generated for any free  $R$ -module  $F = \bigoplus_{i \in I} e_i R$ .

In the following we outline the arguments to prove that, in the  $\aleph_0$ -cotorsion-free case, the inessential endomorphisms coincide with the locally sfg-endomorphisms. As before, the proofs are similar to the corresponding ones in [3].

It will be useful to consider the support of an element  $g \in G^s$  with respect to its “top”.

DEFINITION 7.3. Let  $s \in S, g \in G^s$ . The *top of the support* of  $g$  is defined by

$$[g]^* = \begin{cases} \{\sigma \in T \mid \|\sigma\| = \|g\| \wedge \sigma \in [g]\} & \text{for } \|g\| \text{ successor,} \\ \{\alpha < \lambda^* \mid \|v_\alpha\| = \|g\| \wedge [g] \setminus v_\alpha \text{ finite}\} & \text{for } \|g\| \text{ limit.} \end{cases}$$

Note that  $[g]^*$  is always finite and non-empty for  $g \neq 0$ .

The next lemma is the key for the further conclusions in this section.

LEMMA 7.4. Let  $s \in S, G$  be constructed as in 3.3, and  $\phi : \widehat{F^s} \rightarrow G^s$ . Then

- (a)  $\bigcup_{i \in I} [\bar{e}_i \phi]^*$  is finite, and
- (b)  $\{\|\bar{e}_i \phi\| \mid i \in I\}$  is finite.

Proof. Part (b) follows from (a), hence it is enough to prove (a). For  $I$  finite the conclusion is obviously true. Now let  $I$  be infinite and suppose that (a) is false. Then  $\bigcup_{i \in I} [\bar{e}_i \phi]^*$  is infinite and there exists a sequence  $(i_n)_{n < \omega}$  in  $I$  such that, for all  $n < \omega$ ,

$$(1) \quad X_n = [f_n \phi]^* \setminus \bigcup_{k < n} [f_k \phi]^* \neq \emptyset$$

where  $f_n = \bar{e}_{i_n}$ . We may assume that

$$(2) \quad \|f_n \phi\| \leq \|f_{n+1} \phi\| \quad \text{for all } n < \omega.$$

Moreover, we may assume that all  $\|f_n \phi\|$  are either successors or limits (otherwise change to a subsequence). If all  $\|f_n \phi\|$  are successors then, by Definition 7.3, (1) and (2), we can choose  $\sigma_n \in X_n$  satisfying the following

conditions:

$$(3) \quad \sup_{n < \omega} \|\sigma_n\| = \sup_{n < \omega} \|f_n \phi\|, \quad f_n \phi \upharpoonright \sigma_n \neq 0, \\ f_k \phi \upharpoonright \sigma_n = 0 \quad \text{for } k < n.$$

If all  $\|f_n \phi\|$  are limit ordinals, then we may choose ordinals  $\alpha(n) \in X_n$  which are all different by (1). Hence all branches  $v_{\alpha(n)}$  are different and we may choose  $\sigma_n \in v_{\alpha(n)}$  satisfying (3). Additionally, we may assume that

$$(4) \quad \text{whenever } \sigma_n \in v \text{ for infinitely many } n < \omega \text{ and for some } v \in \text{Br } T, \\ \text{then } \sigma_n \in v \text{ for all } n.$$

Finally, choose an increasing sequence  $m(n)$  of natural numbers such that, for all  $n < \omega$ ,

$$(5) \quad n \cdot l(\sigma_n) + n \leq m(n) \quad \text{and} \quad f_n s^{m(n)} \phi \upharpoonright \sigma_n \not\equiv 0 \pmod{\sigma_n \tilde{A}^s s^{m(n+1)}}.$$

Now we consider the element  $f = \sum_{k < \omega} f_k s^{m(k)} \in \widehat{F}^s$ , which we can split into

$$f = \sum_{k \leq n} f_k s^{m(k)} + f^{(n)} s^{m(n+1)} \quad (n < \omega)$$

where  $f^{(n)} \in \widehat{F}^s$  for each  $n < \omega$ . Using (3) we get

$$f \phi \upharpoonright \sigma_n = f_n s^{m(n)} \phi \upharpoonright \sigma_n + f^{(n)} s^{m(n+1)} \phi \upharpoonright \sigma_n \\ \equiv f_n s^{m(n)} \phi \upharpoonright \sigma_n \pmod{\sigma_n \tilde{A}^s s^{m(n+1)}}$$

where  $f \phi, f^{(n)} \phi \in G^s$ . Therefore, using (5), for each  $n < \omega$ ,

$$(6) \quad \sigma_n \in [f \phi].$$

By continuity we get  $f \phi = \sum_{k < \omega} f_k \phi s^{m(k)}$ . Hence  $\|f \phi\| = \sup_{n < \omega} \|\sigma_n\|$  by (3). The Recognition Lemma provides infinitely many  $\sigma_n$  which belong to a finite union of branches. Hence there exists  $\beta < \lambda^*$  such that  $\sigma_n \in v_\beta$  for infinitely many  $n < \omega$ , thus  $\sigma_n \in v_\beta$  for all  $n < \omega$  by (4). Again using the Recognition Lemma we find  $k < \omega, a \in A^s$  with  $f \phi \upharpoonright \sigma_n = \sigma_n a s_\beta^{l(\sigma_n) - k}$  for large  $n$ . On the other hand, we have  $f \phi \upharpoonright \sigma_n \in \sigma_n \tilde{A}^s s^{m(n)}$ , hence

$$(7) \quad a s_\beta^{l(\sigma_n) - k} \in \tilde{A}^s s^{m(n)} \cap A^s = A^s s^{m(n)} \quad \text{for almost all } n.$$

If  $s_\beta \in s^\omega R$  then  $a s_\beta^{l(\sigma_n) - k} \equiv 0$  for almost all  $n$ , and for  $s_\beta \notin s^\omega R$  we find  $t < \omega$  such that  $s_\beta \notin R s^t$ . Therefore  $s^t \in R s_\beta$  since  $S$  is linearly ordered; also,  $a \in A^s s^n$  for almost all  $n$  by (5) and (7). Hence  $a = 0$  because  $A^s$  is  $s$ -reduced. We deduce  $f \phi \upharpoonright \sigma_n = 0$  in any case, contradicting (6). ■

Now we are prepared to say more about a homomorphism  $\phi : \widehat{F}^s \rightarrow G^s (s \in S)$ . The next lemma will be used to prove that  $G$  is  $\aleph_0$ -cotorsion-free if  $A$  is.

LEMMA 7.5. Let  $F = \bigoplus_{i \in I} e_i R$  and  $G$  be given by Construction 3.3. Then any homomorphism  $\phi : \widehat{F}^s \rightarrow G^s$  ( $s \in S$ ) can be written as a sum  $\phi = \phi_1 + \phi_2$  of homomorphisms  $\phi_1, \phi_2 \in \text{Hom}(\widehat{F}^s, G^s)$  such that  $\|\bar{e}_i \phi_1\|$  is a limit for all  $i \in I$ , and  $\bar{e}_i \phi_2 \in \bigoplus_{\tau \in T'} \tau A^s$  for all  $i \in I$  and for some finite subset  $T'$  of  $T$ .

PROOF. For each  $i \in I$  the Recognition Lemma provides  $b_i \in B$ ,  $g_i \in G$  such that

$$(1) \quad \bar{e}_i \phi = g_i + b_i \text{ with } \|g_i\| \text{ a limit and } \|\tau\| > \|g_i\| \text{ for all } \tau \in [b_i].$$

Let  $T' = \bigcup_{i \in I} [b_i]$ , which is finite by Definition 7.3 and Lemma 7.4. Moreover, let  $\nu$  be the maximum of all  $\|\tau\|$  with  $\tau \in T'$ ; hence  $\nu$  is a successor. We prove 7.5 by induction on  $\nu$ . If  $\nu = 0$  then  $T' = \emptyset$ ; hence  $\phi_1 = \phi$  and  $\phi_2 = 0$  by (1).

Suppose 7.5 is true for all homomorphisms such that  $\max\{\|\tau\| \mid \tau \in T''\} < \nu$  with  $T''$  a finite subset of  $T$  as in 7.5. Consider  $\phi : \widehat{F}^s \rightarrow G^s$  with  $\max\{\|\tau\| \mid \tau \in T'\} = \nu$  and let  $Z = \{\tau \in T' \mid \|\tau\| = \nu\}$ . Define  $\psi : \widehat{F}^s \rightarrow B^s$  by  $f\psi = \sum_{\tau \in Z} f\phi \upharpoonright \tau$ . We need to check that  $f\psi \in B^s$  for each  $f \in \widehat{F}^s$ . Since  $Z \subseteq T'$  is finite, it is enough to show  $f\phi \upharpoonright \tau \in B^s$  for each  $\tau \in Z$ . By our choice of  $\nu$  we certainly have  $\|f\phi\| \leq \nu$  for all  $f \in \widehat{F}^s$ . If  $\|f\phi\| < \nu$ , then  $f\phi \upharpoonright \tau = 0$  for all  $\tau \in Z$ . If  $\|f\phi\| = \nu$  we get  $f\phi = g_f + b_f$  where  $\|g_f\| < \nu$  and  $b_f \in B^s$  with  $\|b_f\| = \|f\phi\|$ . Hence  $f\phi \upharpoonright \tau = b_f \upharpoonright \tau \in B^s$ . Thus  $\psi : \widehat{F}^s \rightarrow \bigoplus_{\tau \in Z} \tau A^s$  and obviously  $\|f(\phi - \psi)\| < \nu$  for all  $f \in \widehat{F}^s$ . So there are homomorphisms  $\phi'_1, \phi'_2$  satisfying 7.5 for  $\phi - \psi = \phi'_1 + \phi'_2$ . Hence  $\phi_1 = \phi'_1$  and  $\phi_2 = \phi'_2 + \psi$  are the desired homomorphisms. ■

LEMMA 7.6. If  $A$  is  $\aleph_0$ -cotorsion-free then so is  $G$ .

PROOF. We want to show that  $G_\alpha$  is  $\aleph_0$ -cotorsion-free for all  $\alpha \leq \lambda^*$ .

(i) Let  $\alpha = 0$ ,  $s \in S$ ,  $\phi : \widehat{F}^s \rightarrow G_0^s = B^s$ . Then, by Lemma 7.5, there exists a finite subset  $T'$  of  $T$  such that  $\phi : \widehat{F}^s \rightarrow \bigoplus_{\tau \in T'} \tau A^s$ . Since  $A$  is  $\aleph_0$ -cotorsion-free,  $\phi \upharpoonright F^s$  is sub-finitely generated.

By Lemma 7.5 it is enough to consider homomorphisms  $\phi : \widehat{F}^s \rightarrow G_\alpha^s$  with  $\|\bar{e}_i \phi\|$  a limit for all  $i \in I$ .

(ii) Assume  $\alpha$  is a limit and  $G_\beta$  is  $\aleph_0$ -cotorsion-free for all  $\beta < \alpha$ . By Lemma 7.4 there exist ordinals  $\alpha(1) < \dots < \alpha(n)$  ( $n < \omega$ ) such that  $\bigcup_{i \in I} [\bar{e}_i \phi]^* = \{\alpha(i) \mid i \leq n\}$ . We get  $\alpha(i) < \alpha$  for all  $i \leq n$ . Hence  $\widehat{F}^s \phi \subseteq G_{\alpha(n)+1}^s$  using  $\|f\phi\| \leq \|v_{\alpha(n)}\|$  and  $[f\phi] \subseteq \bigcup_{i \in I} [\bar{e}_i \phi]$  for all  $f \in \widehat{F}^s$ . Thus  $\phi \upharpoonright F^s$  is a sfg-homomorphism.

(iii) Next assume  $G_{\alpha+1} = G_\alpha + \sum_{k < \omega} g_\alpha^k A$  and  $G_\alpha$  is  $\aleph_0$ -cotorsion-free. Let  $s \in S$  and  $\phi : \widehat{F}^s \rightarrow G_{\alpha+1}^s$  with  $\|\bar{e}_i \phi\|$  limits. If  $s_\alpha \in s^\omega R$  then, for all  $k < \omega$ ,  $g_\alpha^k \equiv g_\alpha^{k+1} s_\alpha + g_k \equiv g_k \pmod{s^\omega G}$  for some  $g_k \in G_\alpha$ , thus  $G_{\alpha+1}^s = G_\alpha^s$ .

Therefore in this case  $F^s\phi$  is contained in a finitely generated submodule of  $G_\alpha^s$ . Hence we consider the case  $s_\alpha \notin s^\omega R$ . There is a (minimal)  $m < \omega$  such that

$$(1) \quad s^m = s_\alpha \cdot q \quad (q \in R).$$

For all  $f \in \widehat{F^s}$  we have

$$(2) \quad f\phi = g_f + g_\alpha^{k_f} a_f$$

where  $g_f \in G_\alpha^s$ ,  $a_f \in A^s$  and  $k_f < \omega$  minimal with  $k_f = 0$  or  $a_f \notin A^s s_\alpha$ . Let  $k_i = k_{\bar{e}_i}$  ( $i \in I$ ).

(a) First we assume  $\sup\{k_i \mid i \in I\}$  is finite. We shall show that in this case  $\sup\{k_f \mid f \in \widehat{F^s}\} < \omega$ . Let  $k = \max\{k_i \mid i \in I\}$ . Then  $k_f \leq k$  for all  $f \in F^s$ . Now we consider  $f \in \widehat{F^s}$  and suppose  $k_f > k$ ; we have  $f = \sum_{i < \omega} f_i s^i$  where  $f_i \in F^s$ . We can decompose  $f$  into  $f = \sum_{i < n} f_i s^i + f^{(n)} s^n$ , with  $f^{(n)} \in \widehat{F^s}$  for each  $n < \omega$ . Using (2) and  $k_f > k$  we get

$$(3) \quad f\phi = g_f + g_\alpha^{k_f} a_f, \quad a_f \notin A^s s_\alpha.$$

On the other hand, we have

$$f\phi = \left( \sum_{i < n} f_i s^i \right) \phi + f^{(n)} s^n \phi = g_n + g_\alpha^{k'} a_n + f^{(n)} s^n \phi \quad \text{for all } n < \omega,$$

where  $g_n \in G_\alpha^s$ ,  $a_n \in A^s$ ,  $k' \leq k$ . Since  $(g_\alpha^k)_{k < \omega}$  is an  $(s_\alpha, G_\alpha)$ -chain, there is a  $g'_n \in G_\alpha^s$ , for each  $n < \omega$ , such that

$$(4) \quad f\phi = g'_n + g_\alpha^{k_f} a_n s_\alpha^{k_f - k'} + f^{(n)} \phi s^n.$$

We get  $f\phi \upharpoonright \sigma = \sigma a_f s_\alpha^{l(\sigma) - k_f} = \sigma a_n s_\alpha^{k_f - k'} s_\alpha^{l(\sigma) - k_f} + f^{(n)} \phi s^n \upharpoonright \sigma$  for almost all  $\sigma \in v_\alpha$  using (3) and (4). By (1) we may choose  $n$  large enough such that  $s^n \in R s_\alpha s_\alpha^{l(\sigma) - k_f}$  and therefore, from  $k_f > k \geq k'$ , we get  $a_f s_\alpha^{l(\sigma) - k_f} \in A^s s_\alpha s_\alpha^{l(\sigma) - k_f}$  and  $a_f \in A^s s_\alpha$ . This contradicts (3) and thus we have  $k_f \leq k$  for all  $f \in \widehat{F^s}$ .

Since  $(g_\alpha^k)_{k < \omega}$  is an  $(s_\alpha, G_\alpha)$ -chain we may write  $f\phi = g'_f + g_\alpha^k a'_f$  for all  $f \in \widehat{F^s}$ . We define  $\psi : \widehat{F^s} \rightarrow G_\alpha^s$  by  $f\psi = g'_f$  and  $\varrho : \widehat{F^s} \rightarrow g_\alpha^k A^s$  by  $f\varrho = g_\alpha^k a'_f$ . Hence  $\phi = \psi + \varrho$  and since  $G_\alpha, A$  are  $\aleph_0$ -cotorsion-free we get a finitely generated submodule  $H$  of  $G_\alpha + g_\alpha^k A^s$  such that  $F^s\phi \subseteq H$ . Therefore  $G_{\alpha+1}$  is  $\aleph_0$ -cotorsion-free if we can exclude  $\sup\{k_i \mid i \in I\} = \omega$ .

(b) Assume for contradiction that there is a homomorphism  $\phi : \widehat{F^s} \rightarrow G_{\alpha+1}^s$  ( $s \in S$ ) such that  $\sup\{k_i \mid i \in I\} = \omega$  where  $\bar{e}_i \phi = g_i + g_\alpha^{k_i} a_i$  ( $k_i$  minimal). Then we can choose  $n_j$  ( $j < \omega$ ) in  $I$  and  $\sigma_j$  in  $v_\alpha$  ( $j < \omega$ ) such that

$$(5) \quad \begin{aligned} k_{n_0} < \dots < k_{n_j} < \dots, \quad \bar{e}_{n_j}\phi|_{\sigma_j} = \sigma_j a_{n_j} s_\alpha^{l(\sigma_j)-k_{n_j}} \neq 0, \\ \bar{e}_{n_i}\phi|_{\sigma_j} = 0 \quad \text{for } i > j. \end{aligned}$$

Moreover, let  $m(j) = k_{n_j} \cdot m$  for all  $j$  ( $m$  is given by (1)). Now we consider the element  $f = \sum_{j < \omega} \bar{e}_{n_j} s^{m(j)}$ ; we have  $f\phi = g_f + g_\alpha^{k_f} a_f$ . For almost all  $j < \omega$ , we get

$$\begin{aligned} f\phi|_{\sigma_j} &= \sigma_j a_f s_\alpha^{l(\sigma_j)-k_f} = \left( \sum_{i \leq j} \bar{e}_{n_i} s^{m(i)} \phi \right) |_{\sigma_j} \\ &= \sum_{i \leq j} \sigma_j a_{n_i} (s_\alpha q)^{k_{n_i}} \cdot s_\alpha^{l(\sigma_j)-k_{n_i}} = \left( \sum_{i \leq j} \sigma_j a_{n_i} q^{k_{n_i}} \right) s_\alpha^{l(\sigma_j)}. \end{aligned}$$

Hence, by our choice of  $k_f$  (minimal), we get  $k_f = 0$  (otherwise  $s_\alpha$  divides  $a_f$ ). Therefore we get  $a_f = \sum_{i \leq j} a_{n_i} q^{k_{n_i}}$  for almost all  $j < \omega$ , which implies  $a_{n_{j+1}} q^{k_{n_{j+1}}} = 0$  (in  $A^s$ ). Since  $s_\alpha \cdot q = s^m$  and  $A^s$  is  $s$ -torsion-free we have  $a_{n_{j+1}} = 0$  for almost all  $j < \omega$ , but this contradicts (5). Hence  $G = \bigcup_{\alpha \leq \lambda^*} G_\alpha$  is  $\aleph_0$ -cotorsion-free. ■

Next we describe the inessential endomorphisms of  $G$  for  $\aleph_0$ -cotorsion-free  $G$ .

**THEOREM 7.7.** *Let  $A$  be  $\aleph_0$ -cotorsion-free and  $\phi \in \text{End } G^s$  for some  $s \in S$ . Then  $\phi$  is inessential iff  $\phi$  is sub-finitely generated.*

**Proof.** (i) Let  $\phi \in \text{Ines } G^s$ , i.e.  $\widehat{G^s}\phi \subseteq G^s$ . As an  $R$ -module  $G$  is an epimorphic image of a free module  $F$ . Hence  $G^s$  is an epimorphic image of  $F^s$ ; let  $\pi : F^s \rightarrow G^s$  be the epimorphism. We can extend  $\pi$  to a homomorphism  $\widehat{\pi} : \widehat{F^s} \rightarrow \widehat{G^s}$ . Considering  $\widehat{\pi}\phi : \widehat{F^s} \rightarrow G^s$  we see that  $\widehat{\pi}\phi|_{F^s} = \pi\phi$  is a sfg-homomorphism, since  $G$  is  $\aleph_0$ -cotorsion-free. Therefore  $\phi : G^s \rightarrow G^s$  is sub-finitely generated.

(ii) Conversely, let  $\phi \in \text{End } G^s$  with  $G^s\phi \subseteq H_0$  for some finitely generated submodule  $H_0 \subseteq G^s$ . By Theorem 6.4 there are  $\psi \in \text{Ines } G^s$  and  $a \in A^s$  such that  $\phi = a - \psi$ . There exists a finitely generated module  $H_1 \subseteq G^s$  such that  $G^s\psi \subseteq H_1$  by (i). Therefore  $G^s a = G^s(\phi + \psi) \subseteq H_0 + H_1$ ; but this is only possible for  $a = 0$  (in  $A^s$ ), since the  $\tau a$  ( $\tau \in T$ ) are all linearly independent for  $a \neq 0$ . Hence  $\phi = -\psi \in \text{Ines } G^s$ . ■

As an immediate consequence we have

**COROLLARY 7.8.** *Let  $A$  be  $\aleph_0$ -cotorsion-free and  $\phi \in \text{End } G$ . Then  $\phi$  is inessential iff  $\phi$  is locally sub-finitely generated.* ■

Now let  $\text{Fin}_l G = \text{Ines } G$  denote the ideal of all locally sub-finitely generated endomorphisms of  $G$ . Thus our realization theorem becomes

**THEOREM 7.9.** *If  $A$  is  $F$ -complete and  $\aleph_0$ -cotorsion-free then  $\text{End } G = A \oplus \text{Fin}_l G$ .*

**Proof.** By Theorem 6.6 it is enough to show that  $\text{Fin}_l G = \text{Ines } G$  is well related in this case. Let  $s, q \in S$  with  $s$  dividing  $q$ , and  $\phi \in \text{End } G$  such that  $\phi^q \in \text{Ines } G^q$ . Moreover, let  $\pi$  be the canonical projection  $\pi : G^q \rightarrow G^s$  ( $q^\omega G \subseteq s^\omega G$ ); obviously,  $\pi\phi^s = \phi^q\pi$ . Since  $\phi^q$  is a sfg-endomorphism there is a finitely generated submodule  $H \subseteq G^q$  such that  $G^q\phi^q \subseteq H$  by assumption. Therefore  $G^q\phi^q\pi = G^q\pi\phi^s = G^s\phi^s \subseteq H\pi$ , which is a finitely generated submodule of  $G^s$ . Hence  $\phi^s$  is sub-finitely generated and inessential. So we have shown that  $\text{Ines } G$  is well related, which completes the proof. ■

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*Received 18 August 1996;  
 in revised form 22 December 1997*