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On the insertion of Darboux functions

by

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Abstract. The main goal of this paper is to characterize the family of all functions f which satisfy the following condition: whenever g is a Darboux function and $f < g$ on $\mathbb{R}$ there is a Darboux function h such that $f < h < g$ on $\mathbb{R}$.

1. Preliminaries. We use mostly standard terminology and notation. The letters $\mathbb{R}$ and $\mathbb{N}$ denote the real line and the set of positive integers, respectively. We consider cardinals as ordinals not in one-to-one correspondence with smaller ordinals. The word *interval* means a nondegenerate bounded interval. The word *function* denotes a mapping from $\mathbb{R}$ into $\mathbb{R}$ unless otherwise explicitly stated.

Let $A \subset \mathbb{R}$. We use the symbols $\operatorname{int} A$, $\operatorname{cl} A$, $\operatorname{fr} A$, χ_A, and $|A|$ to denote the interior, the closure, the boundary, the characteristic function, and the cardinality of A, respectively. We write $\mathfrak{c} = |\mathbb{R}|$ and $\aleph_0 = |\mathbb{N}|$. We say that A is *bilaterally* $\mathfrak{c}$*-dense-in-itself* if $|A \cap J| = \mathfrak{c}$ for every interval J with $A \cap J \neq \emptyset$. The shortcut "A is nb$\mathfrak{c}$d" means "A is nonempty and bilaterally $\mathfrak{c}$-dense-in-itself."

Let f be a function. For every $y \in \mathbb{R}$ let $[f < y] = \{x \in \mathbb{R} : f(x) < y\}$. The symbols $[f \le y]$, $[f > y]$, etc., are defined analogously. For every set $A \subset \mathbb{R}$ with $|A| = \mathfrak{c}$ we define $\mathfrak{c}\text{-inf}(f, A) = \inf\{y \in \mathbb{R} : |[f < y] \cap A| = \mathfrak{c}\}$. If $A \subset \mathbb{R}$ and x is a left $\mathfrak{c}$-limit point of A (i.e., $|A \cap (x - \delta, x)| = \mathfrak{c}$ for every $\delta > 0$), then let

$$\mathfrak{c}\text{-}\underline{\lim}(f\restriction A, x^-) = \lim_{\delta\to 0^+} \mathfrak{c}\text{-inf}(f, A \cap (x - \delta, x))$$

and $\mathfrak{c}\text{-}\overline{\lim}(f\restriction A, x^-) = -\,\mathfrak{c}\text{-}\underline{\lim}(-f\restriction A, x^-)$. Similarly we define $\mathfrak{c}\text{-}\underline{\lim}(f\restriction A, x^+)$ and $\mathfrak{c}\text{-}\overline{\lim}(f\restriction A, x^+)$ if x is a right $\mathfrak{c}$-limit point of A. The symbols $\mathcal{C}_f$ and $\mathcal{D}_f$ denote the sets of points of continuity and of discontinuity of f, respectively.

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The following classes of functions are considered.

• $\mathbb{R}^{\mathbb{R}}$ consists of all functions.

• $\mathbf{B}$ consists of all Borel measurable functions.

• $\mathbf{B}_\alpha$ denotes the Baire class α ($\alpha < \omega_1$). Thus $\mathbf{B} = \bigcup_{\alpha<\omega_1} \mathbf{B}_\alpha$.

• $\mathbf{D}$ consists of all *Darboux functions*, i.e., $f \in \mathbf{D}$ iff $f[J]$ is connected for every interval J.

• $\mathbf{U}$ consists of all functions f with the following property: for all $a < \overline{a}$ and each set $A \subset (a, \overline{a})$ with $|A| < \mathfrak{c}$ the set $f[(a, \overline{a}) \setminus A]$ is dense in the interval $[\min\{f(a), f(\overline{a})\}, \max\{f(a), f(\overline{a})\}]$. Recall that $\mathbf{U}$ is the uniform closure of $\mathbf{D}$ [6, Theorem 4.3].

• $\mathbf{C}$ consists of all functions f with the following property: for every open interval P the set $f^{-1}(P)$ is either empty or nb$\mathfrak{c}$d. Equivalently, $f \in \mathbf{C}$ iff for every $x \in \mathbb{R}$ we have $\mathfrak{c}\text{-}\underline{\lim}(|f - f(x)|, x^-) = \mathfrak{c}\text{-}\underline{\lim}(|f - f(x)|, x^+) = 0$.

• $\mathbf{C}_*$ consists of all functions f with the following property: for every $y \in \mathbb{R}$ the set $[f < y]$ is either empty or nb$\mathfrak{c}$d. Equivalently, $f \in \mathbf{C}_*$ iff for every $x \in \mathbb{R}$ we have $\max\{\mathfrak{c}\text{-}\underline{\lim}(f, x^-), \mathfrak{c}\text{-}\underline{\lim}(f, x^+)\} \le f(x)$.

• $\mathbf{C}^*$ consists of all functions f with the following property: for every $y \in \mathbb{R}$ the set $[f > y]$ is either empty or nb$\mathfrak{c}$d. Equivalently, $f \in \mathbf{C}^*$ iff for every $x \in \mathbb{R}$ we have $\min\{\mathfrak{c}\text{-}\overline{\lim}(f, x^-), \mathfrak{c}\text{-}\overline{\lim}(f, x^+)\} \ge f(x)$.

Recall that we have the following proper inclusions:

$$\mathbf{D} \subset \mathbf{U} \subset \mathbf{C} \subset \mathbf{C}_* \cap \mathbf{C}^* \subset \mathbf{C}_*. \tag{1}$$

For the proof of the inequality $\mathbf{D} \ne \mathbf{U}$ see, e.g., [6, p. 72]. The other relations are evident.

2. Introduction. Let f and g be arbitrary functions. The notation "$f < g$" means "$f(x) < g(x)$ for each $x \in \mathbb{R}$." We write $(f, g) \in \mathcal{P}$ (see [7]) if $f < g$ and $|[f < y < g] \cap (a, \overline{a})| = \mathfrak{c}$ whenever $a < \overline{a}$ and $y \in (\min\{f(a), f(\overline{a})\}, \max\{g(a), g(\overline{a})\})$. If $\mathfrak{A}$ and $\mathfrak{B}$ are families of functions, then define

$$\mathcal{P}(\mathfrak{A}) = \{f \in \mathbb{R}^{\mathbb{R}} : (\forall g \in \mathfrak{A})(f < g \Rightarrow (f, g) \in \mathcal{P})\},$$
$$\mathcal{M}(\mathfrak{B}) = \{(f, g) \in \mathbb{R}^{\mathbb{R}} \times \mathbb{R}^{\mathbb{R}} : (\exists h \in \mathfrak{B})(f < h < g)\}$$

and

$$\mathfrak{M}(\mathfrak{A}, \mathfrak{B}) = \{f \in \mathbb{R}^{\mathbb{R}} : (\forall g \in \mathfrak{A})(f < g \Rightarrow (f, g) \in \mathcal{M}(\mathfrak{B}))\}.$$

One can easily verify that if $\mathfrak{A}_1 \subset \mathfrak{A}_2$ and $\mathfrak{B}_1 \supset \mathfrak{B}_2$, then $\mathcal{P}(\mathfrak{A}_1) \supset \mathcal{P}(\mathfrak{A}_2)$ and $\mathfrak{M}(\mathfrak{A}_1, \mathfrak{B}_1) \supset \mathfrak{M}(\mathfrak{A}_2, \mathfrak{B}_2)$.

It is quite evident that the relation $f < g$ does not imply $(f, g) \in \mathcal{M}(\mathbf{D})$. (See also Lemma 3.6.) So we can ask two questions:

1. Which assumptions on f and g (in addition to $f < g$) imply $(f, g) \in \mathcal{M}(\mathbf{D})$?

2. If $f < g$ and $(f,g) \notin \mathcal{M}(\mathbf{D})$, how "regular" can the functions f and g be?

We now discuss briefly these questions.

1. In 1966 J. G. Ceder and M. L. Weiss proved the following theorem [8, Theorem 1]. (See also [7, Theorem 1].)

THEOREM 2.1. $\mathcal{P} \subset \mathcal{M}(\mathbf{D})$.

They also showed that $\mathbf{D} \cap \mathbf{B}_1 \subset \mathfrak{M}(\mathbf{D} \cap \mathbf{B}_1, \mathbf{D} \cap \mathbf{B}_2)$ [8, Theorem 4], and asked whether $\mathbf{D} \cap \mathbf{B}_1 \subset \mathfrak{M}(\mathbf{D} \cap \mathbf{B}_1, \mathbf{D} \cap \mathbf{B}_1)$. This question has been answered in the affirmative by A. M. Bruckner, J. G. Ceder, and T. L. Pearson [4, Theorem 1]. The latter authors also proved the next theorem, which contains the answer to the first question in case $f, g \in \mathbf{D}$ [5, Theorem 1].

THEOREM 2.2. *Let $f, g \in \mathbf{D}$. Then $(f,g) \in \mathcal{M}(\mathbf{D})$ if and only if $f < g$ and for all $a < \overline{a}$ and $y \in (\min\{f(a), f(\overline{a})\}, \max\{g(a), g(\overline{a})\})$ the set $[f < y < g] \cap (a, \overline{a})$ is nonempty and bilaterally dense-in-itself.*

In 1968 J. G. Ceder and T. L. Pearson proved the following theorem [7, Theorem 5].

THEOREM 2.3. *Every continuous function belongs to $\mathcal{P}(\mathbf{C})$.*

By Theorem 2.1, it follows that each continuous function belongs to $\mathfrak{M}(\mathbf{C}, \mathbf{D})$. In Section 4 we characterize the class $\mathfrak{M}(\mathfrak{A}, \mathbf{D})$ for $\mathfrak{A} \in \{\mathbf{D}, \mathbf{U}, \mathbf{C}, \mathbf{C}_* \cap \mathbf{C}^*, \mathbf{C}_*, \mathbf{C}^*, \mathbb{R}^{\mathbb{R}}\}$.

2. In 1966 J. G. Ceder and M. L. Weiss constructed functions $f, g \in \mathbf{D} \cap \mathbf{B}_2$ such that $f < g$ and $(f,g) \notin \mathcal{M}(\mathbf{D})$ [8, Example 1]. A. M. Bruckner, J. G. Ceder, and T. L. Pearson showed in 1973 that there exist $f \in \mathbf{D} \cap \mathbf{B}_1$ and $g \in \mathbf{D} \cap \mathbf{B}_2$ such that $f < g$ and $(f,g) \notin \mathcal{M}(\mathbf{D})$ [4, Example, p. 165]. They also claimed that if $f \in \mathbf{D}$ and the set $f[\mathcal{C}_f \cap J]$ is dense in $f[J]$ for each interval J, then $f \in \mathfrak{M}(\mathbf{D}, \mathbf{D})$ [4, Theorem 2]. We will see that this assertion is false. In fact, this result does not hold even if we moreover assume that f is continuous except on a countable set and f satisfies Banach's condition T_2 (Example 5.4). So [4, Corollary, p. 166] is also incorrect.

3. Auxiliary results. The next lemma follows by [7, Lemma 4, p. 285]. (See also [12, Lemma I.3.2].)

LEMMA 3.1. *Let $A \subset \mathbb{R}$ be nb$\mathfrak{c}$d and $f : A \to \mathbb{R}$. Then*

$$|\{x \in A : \max\{\mathfrak{c}\text{-}\underline{\lim}(|f - f(x)|, x^-), \mathfrak{c}\text{-}\underline{\lim}(|f - f(x)|, x^+)\} > 0\}| < \mathfrak{c}. \ \blacksquare$$

LEMMA 3.2. *Assume that $A \subset \mathbb{R}$ is nb$\mathfrak{c}$d, and f is a function such that for each $x \in A$ we have $\max\{\mathfrak{c}\text{-}\underline{\lim}(f{\restriction}A, x^-), \mathfrak{c}\text{-}\underline{\lim}(f{\restriction}A, x^+)\} < \infty$. There is a function $g : A \to \mathbb{R}$ such that*

$$f(x) < g(x) \quad \textit{for each } x \in A \tag{2}$$

and

(3) *for each interval* J, *if* $A \cap J \neq \emptyset$, *then* $g[A \cap J] = (\mathfrak{c}\text{-}\inf(f, A \cap J), \infty)$.

Proof. Set $B = \{x \in \mathbb{R} : \max\{\mathfrak{c}\text{-}\underline{\lim}(f, x^-), \mathfrak{c}\text{-}\underline{\lim}(f, x^+)\} > f(x)\}$. Then $|B| < \mathfrak{c}$. (See Lemma 3.1.) Arrange all intervals intersecting A in a transfinite sequence, $\{J_\alpha : \alpha < \mathfrak{c}\}$. For each $\alpha < \mathfrak{c}$ and $n \in \mathbb{N}$ put $y_{\alpha,n} = \max\{\mathfrak{c}\text{-}\inf(f, A \cap J_\alpha) + n^{-1}, -n\}$, and define $K_{\alpha,n} = [f < y_{\alpha,n}] \cap A \cap J_\alpha \setminus B$. Then $|K_{\alpha,n}| = \mathfrak{c}$ for each α and n. Use [10, Lemma 5] to construct a family, $\{Q_{\alpha,n} : \alpha < \mathfrak{c}, n \in \mathbb{N}\}$, consisting of pairwise disjoint sets of cardinality $\mathfrak{c}$, such that each $Q_{\alpha,n}$ is a subset of $K_{\alpha,n}$. For each α and n let $g_{\alpha,n} : Q_{\alpha,n} \to (y_{\alpha,n}, \infty)$ be a surjection. Define $g(x) = g_{\alpha,n}(x)$ if $x \in Q_{\alpha,n}$ for some $\alpha < \mathfrak{c}$ and $n \in \mathbb{N}$, and $g(x) = \max\{\mathfrak{c}\text{-}\underline{\lim}(f\restriction A, x^-), \mathfrak{c}\text{-}\underline{\lim}(f\restriction A, x^+), f(x)\} + 1$ if $x \in A \setminus \bigcup_{\alpha<\mathfrak{c}} \bigcup_{n\in\mathbb{N}} Q_{\alpha,n}$.

Clearly (2) holds. To prove (3) fix an interval J with $A \cap J \neq \emptyset$. Then $J = J_\alpha$ for some $\alpha < \mathfrak{c}$. Hence

$$g[A \cap J] \supset \bigcup_{n\in\mathbb{N}} g_{\alpha,n}[Q_{\alpha,n}] = (\mathfrak{c}\text{-}\inf(f, A \cap J), \infty).$$

On the other hand, by assumption, for each $x \in A \cap J$ we have

$$g(x) > \max\{\mathfrak{c}\text{-}\underline{\lim}(f\restriction A, x^-), \mathfrak{c}\text{-}\underline{\lim}(f\restriction A, x^+)\} \geq \mathfrak{c}\text{-}\inf(f, A \cap J). \ \blacksquare$$

LEMMA 3.3. *Let* $f \in \mathbb{R}^{\mathbb{R}}$. *There is a function* $g \in \mathbf{C}^*$ *with* $g > f$.

Proof. Define $A = \{x \in \mathbb{R} : \max\{\mathfrak{c}\text{-}\underline{\lim}(f, x^-), \mathfrak{c}\text{-}\underline{\lim}(f, x^+)\} < \infty\}$. Then by Lemma 3.1, we have $|\mathbb{R} \setminus A| < \mathfrak{c}$. So we can use Lemma 3.2 to construct a function $g : A \to \mathbb{R}$ such that conditions (2) and (3) hold. Extend g to the whole real line setting $g(x) = f(x) + 1$ for $x \notin A$. Clearly $g > f$. Moreover, by (3), for each $x \in \mathbb{R}$ we have $\mathfrak{c}\text{-}\overline{\lim}(g, x^-) = \mathfrak{c}\text{-}\overline{\lim}(g, x^+) = \infty$. Thus $g \in \mathbf{C}^*$. $\blacksquare$

The proof of the next proposition is similar to that of [5, Theorem 2]. (See also [12, Corollary VI.1.4].)

PROPOSITION 3.4. *For every function* f *the following are equivalent*:

(i) *there is a function* $g \in \mathbf{D}$ *with* $g > f$;
(ii) *there is a function* $g \in \mathbf{U}$ *with* $g > f$;
(iii) *there is a function* $g \in \mathbf{C}$ *with* $g > f$;
(iv) *there is a function* $g \in \mathbf{C}_* \cap \mathbf{C}^*$ *with* $g > f$;
(v) *there is a function* $g \in \mathbf{C}_*$ *with* $g > f$;
(vi) *for each* $x \in \mathbb{R}$ *we have* $\max\{\mathfrak{c}\text{-}\underline{\lim}(f, x^-), \mathfrak{c}\text{-}\underline{\lim}(f, x^+)\} < \infty$.

Proof. The implications (i)⇒(ii)⇒(iii)⇒(iv)⇒(v) are evident. To prove (v)⇒(vi) recall that, by definition, for each $x \in \mathbb{R}$ we have

$$\max\{\mathfrak{c}\text{-}\underline{\lim}(f, x^-), \mathfrak{c}\text{-}\underline{\lim}(f, x^+)\} \leq \max\{\mathfrak{c}\text{-}\underline{\lim}(g, x^-), \mathfrak{c}\text{-}\underline{\lim}(g, x^+)\} \leq g(x) < \infty.$$

(vi)⇒(i). Use Lemma 3.2 with $A = \mathbb{R}$ to construct a function g satisfying (2) and (3). Clearly $g \in \mathbf{D}$ and $g > f$. ∎

We denote the class of functions which satisfy condition (i) of Proposition 3.4 by $\mathbf{A}$. Clearly $\mathbf{C}_* \subset \mathbf{A}$. The next lemma shows that $\mathbf{A} \cap \mathfrak{M}(\mathbf{D}, \mathbf{C}_*) \subset \mathbf{C}_*$.

LEMMA 3.5. *Let $f \in \mathbf{A} \setminus \mathbf{C}_*$. There is a function $g \in \mathbf{D}$ such that $g > f$ and $(f, g) \notin \mathcal{M}(\mathbf{C}_*)$.*

Proof. By assumption, there is a $y \in \mathbb{R}$ and an interval I such that $0 < |B| < \mathfrak{c}$, where $B = [f < y] \cap I$. Set $A = \mathbb{R} \setminus B$. Use Lemma 3.2 to construct a function $g : A \to \mathbb{R}$ such that (2) and (3) hold. Extend g to the whole real line setting $g(x) = \max\{\mathfrak{c}\text{-}\underline{\lim}(f, x^-), \mathfrak{c}\text{-}\underline{\lim}(f, x^+)\}$ for $x \in B$. One can easily verify that $g > f$ and $g \in \mathbf{D}$. Let h be an arbitrary function with $f < h < g$. Then for each $x \in B$ we have

$$\begin{aligned} h(x) < g(x) &= \max\{\mathfrak{c}\text{-}\underline{\lim}(f, x^-), \mathfrak{c}\text{-}\underline{\lim}(f, x^+)\} \\ &\le \max\{\mathfrak{c}\text{-}\underline{\lim}(h, x^-), \mathfrak{c}\text{-}\underline{\lim}(h, x^+)\}. \end{aligned}$$

Thus $h \notin \mathbf{C}_*$ and $(f, g) \notin \mathcal{M}(\mathbf{C}_*)$. ∎

LEMMA 3.6. *Let $f \in \mathbb{R}^{\mathbb{R}}$. There is a function $g > f$ with $(f, g) \notin \mathcal{M}(\mathbf{D})$. If moreover $f \in \mathbf{A}$, then we can choose $g \in \mathbf{C}_*$.*

Proof. If f is constant, then define $g(x) = f(x) + |x| + \chi_{\{0\}}(x)$. It is evident that $g > f$ and $g \in \mathbf{C}_*$. If $f < h < g$, then

$$\mathfrak{c}\text{-}\overline{\lim}(h, 0^-) \le \mathfrak{c}\text{-}\overline{\lim}(g, 0^-) = f(0) < h(0).$$

Thus $h \notin \mathbf{D}$ and $(f, g) \notin \mathcal{M}(\mathbf{D})$.

If f is not constant, then let $y \in \mathbb{R}$ be such that $[f < y] \neq \emptyset \neq [f \ge y]$. If $f \notin \mathbf{A}$, then define $g(x) = y$ if $f(x) < y$, and $g(x) = f(x) + 1$ otherwise. It is clear that $g > f$. Let h be an arbitrary function with $f < h < g$. Observe that if $f(x) < y$, then $h(x) < g(x) = y$, and $f(x) \ge y$ implies $h(x) > f(x) \ge y$. Hence $[h = y] = \emptyset$. Furthermore, $[h < y] \neq \emptyset \neq [h > y]$. Thus $h \notin \mathbf{D}$ and $(f, g) \notin \mathcal{M}(\mathbf{D})$.

Finally, let $f \in \mathbf{A}$. If $f \notin \mathfrak{M}(\mathbf{D}, \mathbf{D})$, then by definition, there exists a function $g \in \mathbf{D} \subset \mathbf{C}_*$ such that $g > f$ and $(f, g) \notin \mathcal{M}(\mathbf{D})$. Otherwise define $g(x) = y$ if $f(x) < y$, and $g(x) = \max\{\mathfrak{c}\text{-}\underline{\lim}(f, x^-), \mathfrak{c}\text{-}\underline{\lim}(f, x^+), f(x)\} + 1$ if $f(x) \ge y$. Then clearly $g > f$, and the relation $(f, g) \notin \mathcal{M}(\mathbf{D})$ can be proved as in the previous case. To complete the proof we will verify that $g \in \mathbf{C}_*$.

Let $x \in \mathbb{R}$, $\overline{y} > g(x)$, and let $J \ni x$ be an interval. If $[f < y] \cap J \neq \emptyset$, then

$$|[g < \overline{y}] \cap J| \ge |[f < y] \cap J| = \mathfrak{c}.$$

(Notice that $\overline{y} > g(x) \ge y$, and by Lemma 3.5, $f \in \mathbf{C}_*$.)

In the opposite case put $B = \{t \in \mathbb{R} : \max\{\mathfrak{c}\text{-}\underline{\lim}(f, t^-), \mathfrak{c}\text{-}\underline{\lim}(f, t^+)\} > f(t)\}$. By Lemma 3.1, we have $|B| < \mathfrak{c}$. Observe that $g(t) = f(t)+1$ whenever $t \in J \setminus B$, and

$$\mathfrak{c}\text{-}\inf(f, J) \leq \max\{\mathfrak{c}\text{-}\underline{\lim}(f, x^-), \mathfrak{c}\text{-}\underline{\lim}(f, x^+)\} \leq g(x) - 1 < \overline{y} - 1.$$

Thus $|[f < \overline{y} - 1] \cap J| = \mathfrak{c}$ and $|[g < \overline{y}] \cap J| \geq |[f < \overline{y} - 1] \cap J \setminus B| = \mathfrak{c}$. ∎

LEMMA 3.7. *Let I' be a closed interval and $y \in \mathbb{R}$. Suppose that a function $f \in \mathbf{A}$ is such that the sets $B = [f < y] \cap I'$ and $B' = \mathbb{R} \setminus B$ are nb$\mathfrak{c}$d. There exists a function $g \in \mathbf{C}_* \cap \mathbf{C}^*$ such that $g > f$ and $(f, g) \notin \mathcal{M}(\mathbf{D})$. If moreover* $\max\{\mathfrak{c}\text{-}\underline{\lim}(f\restriction B', x^-), \mathfrak{c}\text{-}\underline{\lim}(f\restriction B', x^+)\} < \infty$ *for each $x \in B'$ (resp.* $\mathfrak{c}\text{-}\inf(f, B' \cap J) = y$ *for every interval $J \subset I'$ with $B \cap J \neq \emptyset \neq B' \cap J$), then we can choose $g \in \mathbf{C}$ (resp. $g \in \mathbf{D}$).*

Proof. Put $A = \{x \in B' : \max\{\mathfrak{c}\text{-}\underline{\lim}(f\restriction B', x^-), \mathfrak{c}\text{-}\underline{\lim}(f\restriction B', x^+)\} < \infty\}$. Then by Lemma 3.1, we have $|B' \setminus A| < \mathfrak{c}$. So we can use Lemma 3.2 to construct a function $g : A \to \mathbb{R}$ such that (2) and (3) hold. Extend g to the whole real line setting $g(x) = \max\{\mathfrak{c}\text{-}\underline{\lim}(f, x^-), \mathfrak{c}\text{-}\underline{\lim}(f, x^+), f(x)\} + 1$ for $x \in B' \setminus A$ and $g(x) = y$ for $x \in B$. Then clearly $g > f$.

Let $f < h < g$. Observe that $x \in B$ implies $h(x) < g(x) = y$. On the other hand, if $x \in B' \cap I'$, then $h(x) > f(x) \geq y$. Hence $[h = y] \cap I' = \emptyset$. Since $B \neq \emptyset \neq B' \cap I'$, we obtain $h \notin \mathbf{D}$. Thus $(f, g) \notin \mathcal{M}(\mathbf{D})$.

Fix an $x \in \mathbb{R}$. We consider three cases.

First let $x \in B$. Then $\mathfrak{c}\text{-}\underline{\lim}(|g - g(x)|, x^-) = \mathfrak{c}\text{-}\underline{\lim}(|g - g(x)|\restriction B, x^-) = 0$. Similarly $\mathfrak{c}\text{-}\underline{\lim}(|g - g(x)|, x^+) = 0$.

If $x \in A$, then by (3), $\mathfrak{c}\text{-}\underline{\lim}(|g - g(x)|, x^-) = \mathfrak{c}\text{-}\underline{\lim}(|g - g(x)|, x^+) = 0$.

Finally, let $x \in B' \setminus A$. Then $\mathfrak{c}\text{-}\overline{\lim}(g, x^-) = \mathfrak{c}\text{-}\overline{\lim}(g, x^+) = \infty > g(x)$. (Recall that B' is nb$\mathfrak{c}$d, so $A \cap J \neq \emptyset$.) On the other hand,

- if x is a left $\mathfrak{c}$-limit point of B, then $\mathfrak{c}\text{-}\underline{\lim}(g, x^-) \leq y \leq f(x) < g(x)$;
- otherwise $\mathfrak{c}\text{-}\underline{\lim}(|g - g(x)|, x^-) = 0$. (We have used (3) and the fact that $f \in \mathbf{A}$.)

Similarly we can show that $\mathfrak{c}\text{-}\underline{\lim}(g, x^+) \leq g(x)$.

Consequently, $g \in \mathbf{C}_* \cap \mathbf{C}^*$. Moreover, the first additional assumption implies $A = B'$, whence $g \in \mathbf{C}$.

Now suppose that the second additional assumption holds. Then the first additional assumption holds as well, so $A = B'$. Let J be an interval. If $A \cap J = \emptyset$, then $g[J] = \{y\}$. If $B \cap J = \emptyset$, then by (3), the set $g[J] = g[A \cap J]$ is an interval. Finally, $B \cap J \neq \emptyset \neq A \cap J$ yields $\mathfrak{c}\text{-}\inf(f, A \cap J) \leq y$. Hence and by (3), $g[J]$ is an interval with end points $\mathfrak{c}\text{-}\inf(f, A \cap J)$ and ∞. Thus $g \in \mathbf{D}$. ∎

LEMMA 3.8. *Let f be an arbitrary function and $g \in \mathbf{C}^*$. Assume that $a < \overline{a}$ and $y \in (\min\{f(a), f(\overline{a})\}, \max\{g(a), g(\overline{a})\})$ are such that the set $A' = [f \geq y] \cap (a, \overline{a})$ is not nb$\mathfrak{c}$d. Then $|[f < y < g] \cap (a, \overline{a})| = \mathfrak{c}$.*

Proof. Choose a closed interval J such that $\operatorname{int} J \subset (a, \overline{a})$, $[g > y] \cap J \neq \emptyset$, and $|[f \geq y] \cap J| < \mathfrak{c}$. (If $A' = \emptyset$, then we can set $J = [a, \overline{a}]$.) Using the fact that $g \in \mathbf{C}^*$ we obtain

$$|[f < y < g] \cap (a, \overline{a})| \geq |[f < y < g] \cap J| = |[g > y] \cap J| = \mathfrak{c}. \ \blacksquare$$

4. Main theorems. The next theorem follows directly from Lemma 3.6. (Notice that if $f \notin \mathbf{A}$, then by Proposition 3.4, $f \in \mathfrak{M}(\mathbf{C}_*, \mathbf{D})$ vacuously.)

THEOREM 4.1. (a) *$\mathcal{P}(\mathbb{R}^{\mathbb{R}}) = \mathfrak{M}(\mathbb{R}^{\mathbb{R}}, \mathbf{D}) = \emptyset$.*

(b) *$\mathcal{P}(\mathbf{C}_*) = \mathfrak{M}(\mathbf{C}_*, \mathbf{D}) = \mathbb{R}^{\mathbb{R}} \setminus \mathbf{A}$.* $\blacksquare$

THEOREM 4.2. *For every function $f \in \mathbf{A}$ the following are equivalent*:

(i) $f \in \mathfrak{M}(\mathbf{C}_* \cap \mathbf{C}^*, \mathbf{D})$;

(ii) *for every open interval I and $y \in \mathbb{R}$, if the set $[f \geq y] \cap I$ is nb$\mathfrak{c}$d, then* $\operatorname{cl} I \subset [f \geq y]$;

(iii) $f \in \mathcal{P}(\mathbf{C}^*)$;

(iv) $f \in \mathcal{P}(\mathbf{C}_* \cap \mathbf{C}^*)$;

(v) $f \in \mathfrak{M}(\mathbf{C}^*, \mathbf{D})$.

Proof. The implications (iii)⇒(v) and (iv)⇒(i) follow from Theorem 2.1, and (iii)⇒(iv) and (v)⇒(i) are evident.

(i)⇒(ii). Assume that (ii) fails. There exist an open interval I and $y \in \mathbb{R}$ such that $[f \geq y] \cap I$ is nb$\mathfrak{c}$d and $[f < y] \cap \operatorname{cl} I \neq \emptyset$. By Lemma 3.5, if $f \notin \mathbf{C}_*$, then $f \notin \mathfrak{M}(\mathbf{C}_* \cap \mathbf{C}^*, \mathbf{D})$. So suppose $f \in \mathbf{C}_*$. Then $[f < y] \cap I$ is nb$\mathfrak{c}$d. Consequently, there is a closed interval $I' \subset I$ such that $\operatorname{fr} I' \subset [f \geq y]$ and $[f < y] \cap I' \neq \emptyset$. By Lemma 3.7, we obtain $f \notin \mathfrak{M}(\mathbf{C}_* \cap \mathbf{C}^*, \mathbf{D})$.

(ii)⇒(iii). Take a $g \in \mathbf{C}^*$ with $g > f$, $a < \overline{a}$, and $y \in (\min\{f(a), f(\overline{a})\}, \max\{g(a), g(\overline{a})\})$. Put $A' = [f \geq y] \cap (a, \overline{a})$. We have $[f < y] \cap [a, \overline{a}] \neq \emptyset$, so by (ii), the set A' is not nb$\mathfrak{c}$d. Thus by Lemma 3.8, we get $|[f < y < g] \cap (a, \overline{a})| = \mathfrak{c}$. Consequently, $(f, g) \in \mathcal{P}$. $\blacksquare$

THEOREM 4.3. *For every function $f \in \mathbf{A}$ the following are equivalent*:

(i) $f \in \mathfrak{M}(\mathbf{C}, \mathbf{D})$;

(ii) *for every open interval I and $y \in \mathbb{R}$, if the set $A' = [f \geq y] \cap I$ is nb$\mathfrak{c}$d, then either* $\operatorname{cl} I \subset [f \geq y]$ *or* $\max\{\mathfrak{c}\text{-}\underline{\lim}(f{\restriction}A', x^-), \mathfrak{c}\text{-}\underline{\lim}(f{\restriction}A', x^+)\} = \infty$ *for some* $x \in A'$;

(iii) $f \in \mathcal{P}(\mathbf{C})$.

Proof. The implication (iii)⇒(i) follows from Theorem 2.1.

(i)⇒(ii). Assume that (ii) fails. There exist an open interval I and $y \in \mathbb{R}$ such that the set $A' = [f \geq y] \cap I$ is nb$\mathfrak{c}$d, $[f < y] \cap \operatorname{cl} I \neq \emptyset$, and

$\max\{\mathfrak{c}\text{-}\underline{\lim}(f,x^-),\mathfrak{c}\text{-}\underline{\lim}(f,x^+)\} < \infty$ for each $x \in A'$. By Lemma 3.5, if $f \notin \mathbf{C}_*$, then $f \notin \mathfrak{M}(\mathbf{C},\mathbf{D})$. So suppose $f \in \mathbf{C}_*$. Then $[f < y] \cap I$ is nb$\mathfrak{c}$d. Consequently, there is a closed interval $I' \subset I$ such that $\operatorname{fr} I' \subset [f \geq y]$ and $[f < y] \cap I' \neq \emptyset$. By Lemma 3.7, we obtain $f \notin \mathfrak{M}(\mathbf{C},\mathbf{D})$.

(ii)⇒(i). Take a $g \in \mathbf{C}$ with $g > f$, $a < \overline{a}$, and $y \in (\min\{f(a), f(\overline{a})\}, \max\{g(a), g(\overline{a})\})$. If $A' = [f \geq y] \cap (a,\overline{a})$ is not nb$\mathfrak{c}$d, then $|[f < y < g] \cap (a,\overline{a})| = \mathfrak{c}$. (We use Lemma 3.8.) In the opposite case notice that $[f < y] \cap [a,\overline{a}] \neq \emptyset$. So by (ii), there exists an $x \in A'$ such that $\max\{\mathfrak{c}\text{-}\underline{\lim}(f\restriction A', x^-), \mathfrak{c}\text{-}\underline{\lim}(f\restriction A', x^+)\} = \infty$. Let $\overline{y} > g(x)$. Choose a closed interval $J \subset (a,\overline{a})$ such that $x \in J$ and $|[y \leq f < \overline{y}] \cap J| < \mathfrak{c}$. Since $y \leq f(x) < g(x) < \overline{y}$ and $g \in \mathbf{C}$, we have

$$|[f < y < g] \cap (a,\overline{a})| \geq |[f < y < g] \cap J| \geq |[y < g < \overline{y}] \cap J \setminus [y \leq f < \overline{y}]| = \mathfrak{c}.$$

Consequently, $(f,g) \in \mathcal{P}$. ∎

Theorem 4.4. *For every function $f \in \mathbf{A}$ the following are equivalent*:

(i) $f \in \mathfrak{M}(\mathbf{D},\mathbf{D})$;

(ii) *for every open interval I and $y \in \mathbb{R}$, if the set $A' = [f \geq y] \cap I$ is nb$\mathfrak{c}$d, then either* $\operatorname{cl} I \subset [f \geq y]$ *or there is an interval $J \subset I$ such that $A' \cap J \neq \emptyset$, $|J \setminus A'| = \mathfrak{c}$, and $\mathfrak{c}\text{-}\inf(f, A' \cap J) > y$*;

(iii) $f \in \mathcal{P}(\mathbf{U})$;

(iv) $f \in \mathcal{P}(\mathbf{D})$;

(v) $f \in \mathfrak{M}(\mathbf{U},\mathbf{D})$.

P r o o f. The implications (iii)⇒(v) and (iv)⇒(i) follow from Theorem 2.1, and (iii)⇒(iv) and (v)⇒(i) are evident.

(i)⇒(ii). Assume that (ii) fails. There are an open interval I and $y \in \mathbb{R}$ such that the set $A' = [f \geq y] \cap I$ is nb$\mathfrak{c}$d, $[f < y] \cap \operatorname{cl} I \neq \emptyset$, and for each interval $J \subset I$ if $A' \cap J \neq \emptyset$ and $|J \setminus A'| = \mathfrak{c}$, then $\mathfrak{c}\text{-}\inf(f, A' \cap J) = y$. By Lemma 3.5, if $f \notin \mathbf{C}_*$, then $f \notin \mathfrak{M}(\mathbf{D},\mathbf{D})$. So suppose $f \in \mathbf{C}_*$. Then $[f < y] \cap I$ is nb$\mathfrak{c}$d. Consequently, there is a closed interval $I' \subset I$ such that $\operatorname{fr} I' \subset [f \geq y]$ and $[f < y] \cap I' \neq \emptyset$. By Lemma 3.7, we obtain $f \notin \mathfrak{M}(\mathbf{D},\mathbf{D})$.

(ii)⇒(iii). Take a $g \in \mathbf{U}$ with $g > f$, $a < \overline{a}$, and $y \in (\min\{f(a), f(\overline{a})\}, \max\{g(a), g(\overline{a})\})$. If the set $A' = [f \geq y] \cap (a,\overline{a})$ is not nb$\mathfrak{c}$d, then $|[f < y < g] \cap (a,\overline{a})| = \mathfrak{c}$. (We use Lemma 3.8.) In the opposite case notice that $[f < y] \cap [a,\overline{a}] \neq \emptyset$. By (ii), there is an interval $J \subset (a,\overline{a})$ such that $A' \cap J \neq \emptyset$, $|J \setminus A'| = \mathfrak{c}$, and $\overline{y} = \mathfrak{c}\text{-}\inf(f, A' \cap J) > y$. If $J \subset [g > y]$, then

$$|[f < y < g] \cap (a,\overline{a})| \geq |[f < y] \cap J| = |J \setminus A'| = \mathfrak{c}.$$

In the opposite case observe that $[g > \overline{y}] \cap J \supset [f \geq \overline{y}] \cap J \neq \emptyset$. So, since $g \in \mathbf{U}$, we obtain $|[y < g < \overline{y}] \cap J| = \mathfrak{c}$. Thus

$$|[f < y < g] \cap (a,\overline{a})| \geq |[f < y < g] \cap J| \geq |[y < g < \overline{y}] \cap J \setminus [y \leq f < \overline{y}]| = \mathfrak{c}.$$

(We have used the fact that $|[y \le f < \overline{y}] \cap J| < \mathfrak{c}$.) Consequently, $(f,g) \in \mathcal{P}$. ■

REMARK 4.1. By Theorem 4.1 and (1), we have

$$\begin{aligned}\mathbb{R}^{\mathbb{R}} \setminus \mathbf{A} &= \mathfrak{M}(\mathbf{C}_*, \mathbf{D}) = \mathfrak{M}(\mathbf{C}_* \cap \mathbf{C}^*, \mathbf{D}) \setminus \mathbf{A} \\ &= \mathfrak{M}(\mathbf{C}, \mathbf{D}) \setminus \mathbf{A} = \mathfrak{M}(\mathbf{U}, \mathbf{D}) \setminus \mathbf{A} = \mathfrak{M}(\mathbf{D}, \mathbf{D}) \setminus \mathbf{A}.\end{aligned}$$

On the other hand, by Lemmas 3.5 and 3.3, and Proposition 3.4, we obtain

$$\begin{aligned}\mathfrak{M}(\mathbf{C}^*, \mathbf{D}) &= \mathfrak{M}(\mathbf{C}_* \cap \mathbf{C}^*, \mathbf{D}) \cap \mathbf{A} \subset \mathfrak{M}(\mathbf{C}, \mathbf{D}) \cap \mathbf{A} \subset \mathfrak{M}(\mathbf{U}, \mathbf{D}) \cap \mathbf{A} \\ &= \mathfrak{M}(\mathbf{D}, \mathbf{D}) \cap \mathbf{A} \subset \mathbf{C}_*.\end{aligned} \tag{4}$$

We will show later that the above inclusions are proper. (See Examples 5.1–5.3.)

THEOREM 4.5. *If* $\mathfrak{A} \in \{\mathbf{D}, \mathbf{U}, \mathbf{C}, \mathbf{C}_* \cap \mathbf{C}^*, \mathbf{C}^*\}$, *then*

$$\mathfrak{A} \cap \mathfrak{M}(\mathfrak{A}, \mathbf{D}) = \mathbf{D} \cap \mathfrak{M}(\mathbf{C}^*, \mathbf{D}).$$

Proof. By (4) and Proposition 3.4, we obtain $\mathfrak{A} \subset \mathbf{A}$. Let $f \in \mathfrak{A} \setminus \mathbf{D}$. There is an open interval I such that $[f > y] \cap \operatorname{cl} I \ne \emptyset \ne [f < y] \cap \operatorname{cl} I$ and $[f = y] \cap I = \emptyset$. Put $A' = [f > y] \cap I = [f \ge y] \cap I$.

• If $f \in \mathbf{C}^*$, then A' is nb$\mathfrak{c}$d. Thus $f \notin \mathfrak{M}(\mathbf{C}^*, \mathbf{D}) = \mathfrak{M}(\mathbf{C}_* \cap \mathbf{C}^*, \mathbf{D}) \cap \mathbf{A}$. (See Theorem 4.2.)

• If $f \in \mathbf{C}$, then moreover $\max\{\mathfrak{c}\text{-}\underline{\lim}(f \restriction A', x^-), \mathfrak{c}\text{-}\underline{\lim}(f \restriction A', x^+)\} \le f(x) < \infty$ for each $x \in A'$. Thus $f \notin \mathfrak{M}(\mathbf{C}, \mathbf{D})$. (See Theorem 4.3.)

It follows that $\mathbf{C}^* \cap \mathfrak{M}(\mathbf{C}^*, \mathbf{D}) \subset \mathbf{D}$ and $\mathfrak{A} \cap \mathfrak{M}(\mathfrak{A}, \mathbf{D}) \subset \mathbf{U}$.

Now let $f \in \mathbf{U} \setminus \mathfrak{M}(\mathbf{C}^*, \mathbf{D})$. By Theorem 4.2, there are an open interval I and $y \in \mathbb{R}$ such that the set $A' = [f \ge y] \cap I$ is nb$\mathfrak{c}$d and $[f < y] \cap \operatorname{cl} I \ne \emptyset$. Since $f \in \mathbf{U}$, for each interval $J \subset I$ if $A' \cap J \ne \emptyset$ and $|[f < y] \cap J| = \mathfrak{c}$, then $\mathfrak{c}\text{-}\inf(f, A' \cap J) = y$. Thus $f \notin \mathfrak{M}(\mathbf{D}, \mathbf{D})$. (See Theorem 4.4.) By (1), (4), and the first part of the proof, we obtain

$$\begin{aligned}\mathfrak{A} \cap \mathfrak{M}(\mathfrak{A}, \mathbf{D}) &\subset \mathbf{U} \cap \mathfrak{M}(\mathfrak{A}, \mathbf{D}) \subset \mathbf{U} \cap \mathfrak{M}(\mathbf{D}, \mathbf{D}) \subset \mathbf{U} \cap \mathfrak{M}(\mathbf{C}^*, \mathbf{D}) \\ &\subset \mathbf{C}^* \cap \mathfrak{M}(\mathbf{C}^*, \mathbf{D}) \subset \mathbf{D} \cap \mathfrak{M}(\mathbf{C}^*, \mathbf{D}) \subset \mathfrak{A} \cap \mathfrak{M}(\mathfrak{A}, \mathbf{D}). \ ■\end{aligned}$$

The next theorem is a generalization of Theorem 2.3.

THEOREM 4.6. *Let* $f \in \mathbf{C}_*$ *be such that for each* $y \in \mathbb{R}$ *the set* $[f < y]$ *is ambiguous*, *i.e.*, *it is both an* F_σ *and a* G_δ *set. Then* $f \in \mathcal{P}(\mathbf{C}^*)$. *In particular*, *every upper semicontinuous function belongs to* $\mathcal{P}(\mathbf{C}^*)$.

Proof. Take an open interval I and $y \in \mathbb{R}$ such that the sets $A' = [f \ge y] \cap I$ and $[f < y] \cap \operatorname{cl} I$ are nonempty. Since $f \in \mathbf{C}_*$, the set $B = [f < y] \cap I$ is nb$\mathfrak{c}$d. Observe that A' and B are disjoint nonempty ambiguous sets, and $A' \cup B = I$. So by [16, Lemma 7], A' is not nb$\mathfrak{c}$d. By Theorem 4.2, $f \in \mathcal{P}(\mathbf{C}^*)$. ■

5. Examples

EXAMPLE 5.1. $\mathbf{C}_* \cap \mathbf{C}^* \cap \mathbf{B}_2 \cap \mathfrak{M}(\mathbf{C}, \mathbf{D}) \setminus \mathfrak{M}(\mathbf{C}_* \cap \mathbf{C}^*, \mathbf{D}) \neq \emptyset$.

Construction. Use the Baire Category Theorem to construct a family, $\{F_n : n \in \mathbb{N}\}$, consisting of pairwise disjoint nonempty nowhere dense perfect sets, such that for each interval I there is an $n \in \mathbb{N}$ with $F_n \subset I$. Define $f(x) = n$ if $x \in F_n$ for some $n \in \mathbb{N}$, and $f(x) = 0$ otherwise. Then clearly $f \in \mathbf{B}_2$. Moreover, for each $x \in \mathbb{R}$ we have

$$\mathfrak{c}\text{-}\underline{\lim}(f, x^-) = \mathfrak{c}\text{-}\underline{\lim}(f, x^+) = 0 < f(x) < \infty = \mathfrak{c}\text{-}\overline{\lim}(f, x^-) = \mathfrak{c}\text{-}\overline{\lim}(f, x^+).$$

Thus $f \in \mathbf{C}_* \cap \mathbf{C}^*$.

Take an open interval I and $y \in \mathbb{R}$ such that the set $A' = [f \geq y] \cap I$ is nb$\mathfrak{c}$d and $[f < y] \cap \operatorname{cl} I \neq \emptyset$. There is an $n \geq y$ with $F_n \subset I$. Choose an $x \in F_n$ which is not a left limit point of F_n. Notice that $y > 0$, so for each $\overline{y} > y$ and each sufficiently small $\delta > 0$ we have $[y \leq f < \overline{y}] \cap (x - \delta, x) = \emptyset$. Thus $\mathfrak{c}\text{-}\underline{\lim}(f \restriction A', x^-) = \infty$. By Theorem 4.3, we obtain $f \in \mathfrak{M}(\mathbf{C}, \mathbf{D})$.

Finally, observe that $[f \geq 1] \cap (0, 1)$ is nb$\mathfrak{c}$d, and $[f < 1] \cap [0, 1] \neq \emptyset$. So by Theorem 4.2, $f \notin \mathfrak{M}(\mathbf{C}_* \cap \mathbf{C}^*, \mathbf{D})$. ∎

EXAMPLE 5.2. $\mathbf{C} \cap \mathbf{B}_2 \cap \mathfrak{M}(\mathbf{D}, \mathbf{D}) \setminus \mathfrak{M}(\mathbf{C}, \mathbf{D}) \neq \emptyset$.

Construction. Let $F \subset \mathbb{R} \setminus \{-\pi/4, \pi/4\}$ be an F_σ set such that $|F \cap I| = |I \setminus F| = \mathfrak{c}$ for each interval I. (Cf. Example 5.1.) Define $f(x) = |\arctan x| \cdot \chi_F(x)$. Clearly $f \in \mathbf{C} \cap \mathbf{B}_2$. Using Theorem 4.4, one can easily show that $f \in \mathfrak{M}(\mathbf{D}, \mathbf{D})$. Moreover, $[f \geq 1] \cap (0, 1)$ is nb$\mathfrak{c}$d, and $[f < 1] \cap [0, 1] \neq \emptyset$. Since f is bounded, Theorem 4.3 yields $f \notin \mathfrak{M}(\mathbf{C}, \mathbf{D})$. ∎

EXAMPLE 5.3. $\mathbf{C} \cap \mathcal{P}(\mathbf{C}^* \cap \mathbf{B}) \setminus \mathfrak{M}(\mathbf{D}, \mathbf{D}) \neq \emptyset$.

Construction. Let B be a *Bernstein set* (i.e., a totally imperfect set whose complement is also totally imperfect) and $f = \chi_B$. It is clear that $f \in \mathbf{C}$. Notice that $[f \geq 1] \cap (0, 1)$ is nb$\mathfrak{c}$d, and $[f < 1] \cap [0, 1] \neq \emptyset$. Since $f \leq 1$, Theorem 4.4 shows that $f \notin \mathfrak{M}(\mathbf{D}, \mathbf{D})$.

Take a $g \in \mathbf{C}^* \cap \mathbf{B}$ with $g > f$. First observe that $[g \leq 1]$ is at most countable. Indeed, otherwise there is a nonempty perfect set $K \subset [g \leq 1]$. Then $K \cap B \neq \emptyset$ and $g(x) \leq 1 = f(x)$ for each $x \in K \cap B$, an impossibility.

Let $a < \overline{a}$ and $y \in (\min\{f(a), f(\overline{a})\}, \max\{g(a), g(\overline{a})\})$. Clearly $y > 0$. If $y \leq 1$, then $|[f < y < g] \cap (a, \overline{a})| \geq |[f = 0] \cap (a, \overline{a}) \setminus [g \leq 1]| = \mathfrak{c}$, and in the opposite case $|[f < y < g] \cap (a, \overline{a})| = |[g > y] \cap (a, \overline{a})| = \mathfrak{c}$. Consequently, $f \in \mathcal{P}(\mathbf{C}^* \cap \mathbf{B})$. ∎

The above example suggests the following problem.

PROBLEM 5.1. Let $\mathfrak{A} \in \{\mathbf{D}, \mathbf{U}, \mathbf{C}, \mathbf{C}_* \cap \mathbf{C}^*, \mathbf{C}_*, \mathbf{C}^*, \mathbb{R}^\mathbb{R}\}$. Characterize the classes $\mathcal{P}(\mathfrak{A} \cap \mathbf{B})$ and $\mathfrak{M}(\mathfrak{A} \cap \mathbf{B}, \mathbf{D})$.

In the next example we will need several new notions. Let $h \in \mathbb{R}^{\mathbb{R}}$. We say that h is a *strong Świątkowski function* [11] if whenever $a < \overline{a}$ and y is a number between $h(a)$ and $h(\overline{a})$, there is an $x \in \mathcal{C}_h \cap (a, \overline{a})$ with $h(x) = y$. (Clearly strong Świątkowski functions are both Darboux and quasi-continuous in the sense of Kempisty [9].) We say that h satisfies *Banach's condition* T_2 (see [2]) if the set $\{y \in \mathbb{R} : |[h = y]| > \aleph_0\}$ has Lebesgue measure zero. We say that h is a *honorary Baire class two function* [1] if $|[h \neq \overline{h}]| \leq \aleph_0$ for some $\overline{h} \in \mathbf{B}_1$. Finally, h is *almost continuous* in the sense of Stallings [15] if every open set $V \subset \mathbb{R}^2$ containing the graph of h contains the graph of some continuous function as well. Recall that almost continuous functions have the Darboux property, and that the converse is not true [15]. Moreover, in Baire class one these two notions coincide [3].

T. Natkaniec showed in 1992 that there are almost continuous functions f and g such that $f < g$ and $(f, g) \notin \mathcal{M}(\mathbf{D})$ [14, Example 1.8.1]. (See also [13].) Example 5.4 generalizes this result as well as many results mentioned in Section 2.

Example 5.4. Let C be the Cantor ternary set. There are bounded functions f and g satisfying the following conditions:

- f is nonpositive, $\mathcal{D}_f$ is a countable subset of C (so $f \in \mathbf{B}_1$), f is strong Świątkowski, and it satisfies Banach's condition T_2;
- g is nonnegative, $\mathcal{D}_g = C$, g is a honorary Baire class two function, it is almost continuous, strong Świątkowski, and satisfies Banach's condition T_2;
- $f < g$ and $(f, g) \notin \mathcal{M}(\mathbf{D})$.

Construction. Let $\mathcal{I} = \{I_n : n \in \mathbb{N}\}$ and $\mathcal{J} = \{J_k : k \in \mathbb{N}\}$ be families of components of $[0, 1] \setminus C$ such that

$$\left(\operatorname{cl}\bigcup \mathcal{I}\right) \cap \left(\operatorname{cl}\bigcup \mathcal{J}\right) = C. \tag{5}$$

Let $\mathcal{I}_0 = \emptyset$. We will construct a sequence, $\{\mathcal{I}_n : n \in \mathbb{N}\}$, such that for each n the following conditions hold:

(a) $\mathcal{I}_{n-1} \subset \mathcal{I}_n \subset \mathcal{I}$;
(b) $\operatorname{cl}\bigcup \mathcal{I}_n = \bigcup_{I \in \mathcal{I}_n} \operatorname{cl} I$;
(c) if $I \in \mathcal{I}_{n-1}$ and $x \in \operatorname{fr} I$, then $x \in \operatorname{cl}(\bigcup \mathcal{I}_n \setminus I)$;
(d) $I_n \in \mathcal{I}_n$.

Let $n \in \mathbb{N}$ and suppose that we have already defined families $\mathcal{I}_0, \ldots, \mathcal{I}_{n-1}$ so that the above conditions hold. Define

$$B = \bigcup_{I \in \mathcal{I}_{n-1} \cup \{I_n\}} \left((\operatorname{fr} I) \setminus \operatorname{cl}\left(\bigcup \mathcal{I}_{n-1} \setminus I\right)\right).$$

Clearly $|B| \leq \aleph_0$. Let $B = \{x_p : p < r\}$, where $r \in \mathbb{N} \cup \{\infty\}$. For each $p < r$ use (5) to choose a monotone sequence of intervals, $\{\widetilde{I}_{p,m} : m \in \mathbb{N}\} \subset \mathcal{I}$, converging to x_p and such that $\bigcup_{m\in\mathbb{N}}\widetilde{I}_{p,m}\subset(x_p-p^{-1},x_p+p^{-1})$. Finally, define $\mathcal{I}_n=\mathcal{I}_{n-1}\cup\{I_n\}\cup\bigcup_{p<r}\{\widetilde{I}_{p,m}:m\in\mathbb{N}\}$. One can easily verify that conditions (a)–(d) are satisfied.

For each $n\in\mathbb{N}$ and each $I\in\mathcal{I}_n$ let $f_{n,I}:\operatorname{cl}I\to[-2^{1-n},-2^{-n}]$ be a continuous surjection such that $f_{n,I}[\operatorname{fr}I]=\{-2^{-n}\}$ and $|f_{n,I}^{-1}(y)|\le 2$ for each $y\in\mathbb{R}$. Similarly, for each $k\in\mathbb{N}$ let $g_k:\operatorname{cl}J_k\to[k^{-1},1]$ be a continuous surjection such that $g_k[\operatorname{fr}J_k]=\{1\}$ and $|g_k^{-1}(y)|\le 2$ for each $y\in\mathbb{R}$. Define functions f and g as follows:

$$f(x)=\begin{cases} f_{n,I}(x) & \text{if } x\in\operatorname{cl}I,\ I\in\mathcal{I}_n,\ n\in\mathbb{N},\\ 0 & \text{otherwise,}\end{cases}$$

$$g(x)=\begin{cases} g_k(x) & \text{if } x\in\operatorname{cl}J_k,\ k\in\mathbb{N},\\ 0 & \text{if } x\in\bigcup_{I\in\mathfrak{I}}\operatorname{cl}I,\\ 1 & \text{otherwise.}\end{cases}$$

It is clear that $f<g$ and $f\le 0\le g$, $\mathcal{D}_f=\bigcup_{I\in\mathfrak{I}}\operatorname{fr}I\subset C$, $\mathcal{D}_g=C$, and both f and g are strong Świątkowski. Moreover, $\{y\in\mathbb{R}:|[f=y]|>\aleph_0\}=\{0\}$ and $\{y\in\mathbb{R}:|[g=y]|>\aleph_0\}=\{0,1\}$. Thus both f and g satisfy Banach's condition T_2.

Define $\overline{g}(x)=g(x)$ if $x\in\mathbb{R}\setminus C$, and $g(x)=1$ if $x\in C$. Then $\overline{g}\in\mathbf{B}_1$ and $|[g\ne\overline{g}]|=\aleph_0$. So g is a honorary Baire class two function.

Let $f<h<g$. Then both $[h<0]$ and $[h>0]$ are nonempty, and $[h=0]=\emptyset$. Thus $h\notin\mathbf{D}$ and $(f,g)\notin\mathcal{M}(\mathbf{D})$.

Finally, we prove that g is almost continuous. Let $V\subset\mathbb{R}^2$ be an open set which contains the graph of g. Let S denote the set of all $x\in\mathbb{R}$ such that for every $t\in(-\infty,x)\setminus C$ there is a continuous function $h:(-\infty,t]\to\mathbb{R}$ with $h(t)=g(t)$ whose graph is contained in V. Evidently $(-\infty,0]\subset S$. We verify that $s=\sup S=\infty$. By way of contradiction suppose $s\in[0,\infty)$. Choose a $\tau>0$ such that

$$(s-\tau,s+\tau)\times(g(s)-\tau,g(s)+\tau)\subset V.$$

We now show $s+\tau\in S$, contradicting the definition of s.

Let $t\in(-\infty,s+\tau)\setminus C$. Without loss we may assume that $t\ge s$. Let $\overline{s}\in C$ be such that $C\cap(\overline{s},t]=\emptyset$. There is a $t_1\in(s-\tau,s)\setminus C$ such that $|g(t_1)-g(s)|<\tau$. Construct a continuous function $h_1:(-\infty,t_1]\to\mathbb{R}$ with $h_1(t_1)=g(t_1)$ whose graph is contained in V. We consider two cases.

Case 1. First suppose that $\overline{s}\le s$. Observe that $g\restriction[a,\overline{a}]$ is continuous whenever $C\cap(a,\overline{a})=\emptyset$. Define $h(x)=h_1(x)$ if $x\le t_1$ and $h(x)=g(x)$ if $x\in[s,t]$, and extend h linearly in the interval $[t_1,s]$. Then $h:(-\infty,t]\to\mathbb{R}$, h is continuous, $h(t)=g(t)$, and the graph of h is contained in V.

Case 2. In the opposite case let $\overline{\tau}\in(0,\overline{s}-s)$ be such that

$$(\overline{s}-\overline{\tau},\overline{s}+\overline{\tau})\times(g(\overline{s})-\overline{\tau},g(\overline{s})+\overline{\tau})\subset V.$$

Let $k > 1/\overline{\tau}$ be such that $J_k \subset (\overline{s} - \overline{\tau}, \overline{s})$. There are $t_2, t_3 \in J_k$ such that $t_2 < t_3$, $|g(t_2) - g(s)| < \tau$, and $|g(t_3) - g(\overline{s})| < \overline{\tau}$. Define $h(x) = h_1(x)$ if $x \le t_1$ and $h(x) = g(x)$ if $x \in [t_2, t_3] \cup [\overline{s}, t]$, and extend h linearly in the intervals $[t_1, t_2]$ and $[t_3, \overline{s}]$. Then $h : (-\infty, t] \to \mathbb{R}$, h is continuous, $h(t) = g(t)$, and the graph of h is contained in V.

We have proved that $s + \tau \in S$, an impossibility. Thus $s = \infty$.

Let $h : (-\infty, 2] \to \mathbb{R}$ be a continuous function whose graph is contained in V such that $h(2) = g(2)$. Extend h to the whole real line setting $h(x) = g(x)$ for $x > 2$. The extended function is continuous and its graph is contained in V. Thus g is almost continuous. ∎

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