FUNDAMENTA
MATHEMATICAE
156 (1998)

On Pettis integral and Radon measures

by

Grzegorz Plebanek (Wrocław)

Abstract. Assuming the continuum hypothesis, we construct a universally weakly measurable function from [0,1] into a dual of some weakly compactly generated Banach space, which is not Pettis integrable. This (partially) solves a problem posed by Riddle, Saab and Uhl [13]. We prove two results related to Pettis integration in dual Banach spaces. We also contribute to the problem whether it is consistent that every bounded function which is weakly measurable with respect to some Radon measure is Pettis integrable.

1. Introduction. Let us start by recalling the following interesting result on Pettis integration.

THEOREM 1.1 (Riddle, Saab & Uhl [13]). *Let (T,Σ,μ) be a finite Radon measure space, E a Banach space, and $\varphi : T \to E^*$ a bounded universally weakly measurable function.*

(a) *If E is separable then φ is Pettis integrable.*

(b) *If E is a weakly compactly generated then φ is Pettis integrable provided φ takes values in some weak* separable subspace of E^*.*

The above theorem has been proved in ZFC; under some additional axioms one can get more general results (see for instance [15], 6-1-3). The authors of [13] asked if part (a) holds for weakly compactly generated (WCG) Banach spaces E, that is, if in (b) the assumption on the range of φ is superfluous. Assuming the absence of measurable cardinals, Andrews [1] showed that part (b) of the theorem holds for every φ which is weak* Borel measurable (see also [12] and [14]).

In this note we show that under the continuum hypothesis (CH) there is a bounded function φ from the unit interval into the dual E^* of some WCG space E such that $\langle x^*,\varphi\rangle$ is a Borel function for every $x^*\in E^*$ but φ is not

1991 *Mathematics Subject Classification*: Primary 46G10; Secondary 28C15.
Partially supported by KBN grant 2P03A 018 13.

Pettis integrable. Thus, at least assuming CH, the problem of Riddle, Saab and Uhl is answered negatively.

Next, using Fremlin's results on additive coverings from [4], we prove that a weak* Borel function $\varphi : T \to E^*$, where E is WCG and (T, Σ, μ) is Radon, has an almost weak* separable range. It follows that Andrew's theorem mentioned above requires no extra set-theoretic assumptions. We also note that putting together resuts from Fremlin [4] and Edgar [2] one can show in ZFC that for every compact space K, the Banach space $M(K)$ (of signed Radon measures on K) is Radon measure-compact (this covers another result from [1]).

The last section is mainly devoted to Pettis integrability of bounded universally weakly measurable functions defined on the Cantor cube 2^κ where $\kappa < \omega_\omega$. We show that such functions are Pettis integrable with respect to the usual product measure provided the measure $\lambda_\mathfrak{c}$ on $2^\mathfrak{c}$ has the Pettis integral property.

We now fix some terminology used in the sequel. Every ordinal number α is identified with $\{\beta : \beta < \alpha\}$. As usual, ω is the first infinite cardinal and ω_1 is the first uncountable cardinal. Given a set X and a cardinal κ, $[X]^\kappa$ stands for the family of all subsets of X of cardinality κ.

We discuss only finite measures. A triple (T, Σ, μ) is called a *Radon measure space* if T is a Hausdorff topological space, the measure μ is complete on the σ-algebra Σ (containing all Borel sets) and inner regular with respect to compact sets.

Given any measure space (T, Σ, μ) and a Banach space E, consider a function $\varphi : T \to E$. Then φ is called *weakly measurable* if for every $x^* \in E^*$ the function $\langle x^*, \varphi\rangle$ is measurable. In case T is a topological space, such a function φ is *universally weakly measurable* if it is weakly measurable with respect to every Radon measure on T. Further, φ is *Pettis integrable* (with respect to μ) if for every $A \in \Sigma$ there is $x_A \in E$ such that

$$\langle x^*, x_A\rangle = \int_A \langle x^*, \varphi(t)\rangle \, d\mu(t)$$

for every $x^* \in E^*$. Talagrand [15] and Musiał [9] survey Pettis integration and related topics.

A Banach space E is said to have the *Pettis Integral Property* (PIP) if every bounded E-valued function defined on some measure space is Pettis integrable. The property Radon-PIP is defined accordingly, as the restriction of PIP to the class of finite Radon measure spaces.

A space E is called *Radon measure-compact* if every E-valued weakly measurable function defined on some Radon measure space (T, Σ, μ) is weakly equivalent to some strongly measurable function ψ (i.e. $\langle x^*, \varphi\rangle = \langle x^*, \psi\rangle$ almost everywhere for every $x^* \in E^*$). It is not difficult to check that Radon measure-compactness implies Radon-PIP (see [2], also for the explanation of the terminology).

Finally, recall that a Banach space E is said to be *weakly compactly generated* (WCG) if there is a weakly compact set $K \subseteq E$ such that E is the closed linear span of K. Basic facts and further references for WCG may be found in [10]. Given a finite measure λ, the space $L_1(\lambda)$ is WCG, since the unit ball of $L^\infty(\lambda)$, seen as a subset of $L_1(\lambda)$, is weakly compact.

2. An example. Let 2^{ω_1} be the Cantor cube $\{0,1\}^{\omega_1}$. Throughout this section we denote by λ the usual product measure on 2^{ω_1}, and consider the Banach space $L^\infty(\lambda)$ which is the dual of the nonseparable WCG space $L_1(\lambda)$. We shall use some well-known properties of λ (the reader is invited to consult the beginning of Section 4 if necessary).

Let $\mathbb{A}$ be the measure algebra of λ; see [5] for terminology and notation concerning measure algebras. In particular, if $B \subseteq 2^{\omega_1}$ is a measurable set then $B^\bullet$ denotes the corresponding element of $\mathbb{A}$.

The following idea is crucial: Say that an ultrafilter $\mathcal{F} \subseteq \mathbb{A}$ has *Kunen's property* if for every double sequence $(a_{nk})_{n,k\in\omega} \subseteq \mathbb{A}$ having for every n the properties:

(i) $-a_{n0} \in \mathcal{F}$,
(ii) $a_{n0} \geq a_{n1} \geq a_{n2} \geq \ldots$,
(iii) $\lim_{k\to\infty} \lambda(a_{nk}) = 0$,

there is $d \in \mathcal{F}$ such that for every n there is k with $d \cdot a_{nk} = \mathbf{0}$.

The following result is due to Kunen [7] (actually, it was proved assuming Martin's axiom).

THEOREM 2.1 (Kunen). *Under CH, in every nonatomic measure algebra of cardinality continuum there exists an ultrafilter with Kunen's property.*

THEOREM 2.2. *Assume the continuum hypothesis. There is a bounded function φ from $[0,1]$ into $L^\infty(\lambda)$ which is not Pettis integrable and such that $\langle x^*, \varphi\rangle$ is a Borel function for every $x^* \in L^\infty(\lambda)^*$.*

Proof. Denoting, as above, the measure algebra of λ by $\mathbb{A}$, we have $|\mathbb{A}| = \mathfrak{c}$ so we may find and fix an ultrafilter $\mathcal{F} \subseteq \mathbb{A}$ with Kunen's property. Enumerate $[0,1]$ as $(t_\alpha)_{\alpha<\omega_1}$. Further, let $(s_\alpha)_{\alpha<\omega_1}$ be an enumeration of all decreasing sequences $s = (s(k))_{k\in\omega} \in \mathbb{A}^\omega$ with the properties $-s(0) \in \mathcal{F}$ and $\lim_{k\to\infty} \lambda(s(k)) = 0$.

(1) Since $\mathcal{F}$ has Kunen's property, for every $\alpha < \omega_1$ there is $b_\alpha \in \mathcal{F}$ with the property that for every $\beta < \alpha$ there is $k \in \omega$ such that $b_\alpha \cdot s_\beta(k) = \mathbf{0}$. Find a set $B_\alpha \subseteq 2^{\omega_1}$, depending on a countable set $I_\alpha \subseteq \omega_1$, such that $b_\alpha = B_\alpha^\bullet$.

(2) Given $\xi < \omega_1$ and $i \in \{0,1\}$, put $C^i_\xi = \{x \in 2^{\omega_1} : x(\xi) = i\}$. Choose any $\xi(\alpha) \in \omega_1 \setminus (I_\alpha \cup \alpha)$ and let $V_\alpha \in \{C^0_{\xi(\alpha)}, C^1_{\xi(\alpha)}\}$ be chosen so that $V^\bullet_\alpha \in \mathcal{F}$.

Now we define a function $\varphi : [0,1] \to L^\infty(\lambda)$ by the formula

$$\varphi(t_\alpha) = \chi_{B_\alpha \cap V_\alpha} - \chi_{B_\alpha \setminus V_\alpha},$$

where χ_B denotes the characteristic function of a set B. We shall check that φ has the required properties.

(3) We claim that if $g \in L_1(\lambda)$ then $\langle g, \varphi(t)\rangle = 0$ for all but countably many $t \in [0,1]$.

Indeed, let g be (represented by) a function depending on coordinates in a countable set $I \subseteq \omega_1$. There is $\beta < \omega_1$ with $I \subseteq \beta$; for $\alpha > \beta$ we have

$$\langle g, \varphi(t)\rangle = \int g(\chi_{B_\alpha \cap V_\alpha} - \chi_{B_\alpha \setminus V_\alpha})\, d\lambda = \int g\chi_{B_\alpha}(\chi_{V_\alpha} - \chi_{V^c_\alpha})\, d\lambda = 0,$$

since the functions $g\chi_{B_\alpha}$ and χ_{V_α} are independent by (2).

Recall that every functional from $L^\infty(\lambda)^*$ is represented by a finitely additive measure m on $\mathbb{A}$. We now consider those nonnegative measures m which are singular with respect to λ, that is, for every $\varepsilon > 0$ there is $a \in \mathbb{A}$ such that $\lambda(a) < \varepsilon$ and $m(-a) < \varepsilon$.

(4) If $m_{\mathcal{F}}$ is a 0-1 measure associated with $\mathcal{F}$ then $m_{\mathcal{F}}(B^\bullet_\alpha \cap V^\bullet_\alpha) = 1$ and $m_{\mathcal{F}}(B^\bullet_\alpha - V^\bullet_\alpha) = 0$ for every α. Consequently, $\langle m_{\mathcal{F}}, \varphi(t)\rangle = 1$ for every $t \in [0,1]$.

(5) Suppose that m is a finitely additive probability measure on $\mathbb{A}$ which is singular with respect to λ, and such that $\inf\{m(d) : d \in \mathcal{F}\} = 0$. We claim that $\langle m, \varphi(t)\rangle = 0$ for all but countably many $t \in [0,1]$.

Indeed, for every $n \ge 1$, choose $d_n \in \mathcal{F}$ such that $m(d_n) < 1/n$. Put $a_{n0} = -d_n$ and, employing singularity of m, find a sequence $(a_{nk})_k$, where $a_{n0} \ge a_{n1} \ge \ldots$, $m(a_{kn}) > 1-2/n$ while $\lambda(a_{nk}) < 1/k$. Now there is $\beta < \omega_1$ such that every sequence $(a_{nk})_k$ appears in $\{s_\eta : \eta < \beta\}$. It follows that for every $\alpha \ge \beta$ and every n there is k with $b_\alpha \cdot a_{nk} = \mathbf{0}$. Hence $m(b_\alpha) < 2/n$ for every n, so we have $m(b_\alpha) = 0$. Therefore $\langle m, \varphi(t_\alpha)\rangle = 0$ for all $\alpha \ge \beta$, since $\varphi(t_\alpha)$ vanishes outside B_α.

It follows from (3)–(5) that $\langle x^*, \varphi(t)\rangle$ is a Borel function for every $x^* \in L^\infty(\lambda)^*$ (since we can write $x^* = g+m$, where $g \in L_1(\lambda)$, m being singular). Moreover, φ is weak* equivalent to zero by (3), and φ is not weakly zero by (4). Thus φ is not Pettis integrable and the proof is complete.

It is not difficult to check that $L^\infty(\lambda)$ has Lebesgue-PIP under ¬CH and Martin's axiom. Thus, contrary to a hope expressed in [13], set theory does play an important role in recognizing Pettis integrable functions valued in duals of WCG spaces.

The idea of using ultrafilters with Kunen's property comes from [3], where it is proved, imposing no extra axioms, that $L^\infty(\lambda)$ is not realcompact in its weak topology, which is equivalent to saying that the space does not have PIP with respect to 0-1 measures. As the existence of ultrafilters with Kunen's property cannot be proved in ZFC, in fact a filter with an analogous property containing "enough" elements was used in [3]. Such a filter might as well be used in the above proof and therefore CH is not essential at this point. CH is really needed for a suitable enumeration of sequences in the measure algebra, which enables us to worry only about countably many sequences at each step. Let us note, however, that relaxing CH to the axiom

there is a family $\mathcal{J}$ of cardinality ω_1 of Lebesgue-null subsets of the unit interval such that every null set is covered by some $N \in \mathcal{J}$,

one can modify the above argument in order to get a bounded weakly measurable not integrable φ from $[0,1]$ into $L^\infty(\lambda)$.

3. Two results on Pettis integration. In this section we show how one can eliminate the assumption on measurable cardinals from two results related to Pettis integration against Radon measures. The basic idea is to use Fremlin's theory of measure additive coverings (see [4]).

Following [4], say that a measure space (T,Σ,μ) has *property* AF_1 if for every family $\mathcal{D} \subseteq \Sigma$ of pairwise disjoint null sets we have $\mu(\bigcup \mathcal{D}) = 0$, provided $\mathcal{D}$ is Σ-additive, that is, $\bigcup \mathcal{D}' \in \Sigma$ for every subfamily $\mathcal{D}' \subseteq \mathcal{D}$. Property $\mathrm{AF}_{<\omega}$ is defined in a similar manner, with "pairwise disjoint" replaced by "point-finite". A family $\mathcal{D}$ of subsets of T is *point-finite* if $|\{D \in \mathcal{D} : t \in D\}| < \omega$ for every $t \in T$. It is worth recalling here that if a point-finite family consists of sets of positive measure then it is necessarily countable.

It is fairly obvious that every finite measure space has AF_1 if and only if there are no measurable cardinals. Much more subtle is the case of Radon measures; for the following results due to Fremlin, see [4], 6M and 11D.

THEOREM 3.1 (Fremlin). *Let (T,Σ,μ) be a finite Radon measure space.*

(a) *(T,Σ,μ) has property $AF_{<\omega}$.*

(b) *Let $(f_a)_{a\in\Delta}$ be a family of measurable functions defined on T which are zero almost everywhere. If $\sum_{a\in\Delta'} f_a$ is a measurable finite function for every $\Delta' \subseteq \Delta$ then $\sum_{a\in\Delta} f_a = 0$ almost everywhere.*

THEOREM 3.2. *Let E be a WCG Banach space and let (T,Σ,μ) be a finite Radon measure space. If $\varphi : T \to E^*$ is a weak* Borel measurable function then there is a weak* separable subspace F of E^* and a μ-null set N such that $\varphi(T \setminus N) \subseteq F$.*

Proof. Being a WCG Banach space, E is generated by a weakly compact set of the form $K = \{0\} \cup \{e_\alpha : \alpha < \kappa\}$, where κ is some cardinal and $K \setminus \{0\}$ is weakly discrete (see [10], 6.36).

For every $\alpha < \kappa$ and $n \geq 1$ put $V_{\alpha,n} = \{x^* \in E^* : |\langle x^*, e_\alpha\rangle| > 1/n\}$. Note that the family $(V_{\alpha,n})_\alpha$ is point-finite for every n. Indeed, for every $x^* \in E^*$ the set $\{x \in K : \langle x^*, x\rangle \geq 1/n\}$ is a compact subset of $K \setminus \{0\}$, hence it is finite.

Putting $A_{\alpha,n} = \varphi^{-1}(V_{\alpha,n})$ we have, for every n, a point-finite family $(A_{\alpha,n})_{\alpha<\kappa}$ of measurable subsets of T. Since every point-finite family of sets of positive measure is necessarily countable, the set $I_n = \{\alpha < \kappa : \mu(A_{\alpha,n}) > 0\}$ is countable. We let $I = \bigcup_n I_n$.

Now, for a fixed n, $(A_{\alpha,n})_{\alpha\in\kappa\setminus I}$ is a point-finite family of null sets. Since φ is weak* Borel and every $V_{\alpha,n}$ is weak* open, such a family is Σ-additive. Thus part (a) of Theorem 3.1 implies that the set $N_n = \bigcup_{\alpha\in\kappa\setminus I} A_{\alpha,n}$ has measure zero.

Put $N = \bigcup_n N_n$ and let

$$F = \{x^* \in E^* : \langle x^*, e_\alpha\rangle = 0 \text{ for all } \alpha \in \kappa \setminus I\}.$$

Note that the space F is weak* separable. Indeed, a countable family $(e_\alpha)_{\alpha\in I}$ separates elements of F, for if $x^* \in F$ and $\langle x^*, e_\alpha\rangle = 0$ for every $\alpha \in I$ then $x^*_{|K} = 0$ and, since K generates E, $x^* = 0$. We have $\varphi(t) \in F$ for every $t \in T \setminus N$ and the proof is complete.

The following result was proved by Andrews [1] under the absence of measurable cardinals (however, for a slightly more general class of Banach spaces).

COROLLARY 3.3. *Let E be a WCG Banach space, let (T, Σ, μ) be a finite Radon measure space and let $\varphi : T \to E^*$ be a weak* Borel measurable function. If φ is bounded and universally weakly measurable then it is Pettis integrable.*

Proof. Follows immediately from Theorem 3.2 and Theorem 1.1(b).

Using his Theorem 3.1 Fremlin ([4], 11E) proved that the Banach space $l_1(\kappa)$ is Radon measure compact for every κ. Combining Fremlin's idea from the proof of that result with Edgar's proof of Theorem 3.4 from [2] one can write the following.

THEOREM 3.4 (Edgar + Fremlin). *Let $(E_\alpha)_{\alpha<\kappa}$ be a family of Radon measure-compact Banach spaces and denote by E the l_1-direct sum $(\sum_{\alpha<\kappa} E_\alpha)_1$. Then E is Radon measure-compact.*

Proof. Let (T, Σ, μ) be a finite Radon space and take any weakly measurable function $\varphi : T \to E$. We can write $\varphi = (\varphi_\alpha)_{\alpha<\kappa}$, where every function $\varphi_\alpha : T \to E_\alpha$ is weakly measurable.

Repeating Edgar's argument, we first check that the set I of those α for which φ_α is not weakly equivalent to 0 is countable. Indeed, otherwise we find for every $\alpha \in I$ a functional $x^*_\alpha \in E^*_\alpha$ of norm one such that

$$\mu(\{t : \langle x_\alpha^*, \varphi_\alpha(t)\rangle \neq 0\}) > 0.$$

It follows easily that for every $\alpha \in I$ there is $r_\alpha > 0$ such that

$$\mu(\{t : |\langle x_\alpha^*, \varphi_\alpha(t)\rangle| \geq r_\alpha\}) > r_\alpha.$$

Since I is uncountable, there is $r > 0$ such that $r_\alpha \geq r$ for infinitely many α. Hence there is $t \in T$ for which $|\langle x_\alpha^*, \varphi_\alpha(t)\rangle| \geq r$ for infinitely many α, but this contradicts $\varphi(t) \in E$, E being an l_1-sum.

Denoting by π the natural projection from E onto $E_0 = (\sum_{\alpha\in I} E_\alpha)_1$, we now prove that φ is weakly equivalent to $\pi \circ \varphi$. Every $x^* \in E^*$ can be written as $x^* = (x_\alpha^*)_{\alpha<\kappa}$, where $x_\alpha^* \in E_\alpha^*$.

Putting $g_\alpha(t) = \langle x_\alpha^*, \varphi_\alpha(t)\rangle$, we note that $(g_\alpha)_{\alpha\in\kappa\setminus I}$ is a Σ-additive family of measurable functions which are zero almost everywhere. Indeed, for any $\Delta \subseteq \kappa \setminus I$ we have

$$\sum_{\alpha\in\Delta} g_\alpha(t) = \langle x_\Delta^*, \varphi(t)\rangle,$$

where $x_\Delta^* = (y_\alpha^*)_\alpha$ is defined by $y_\alpha^* = x_\alpha^*$ for $\alpha \in \Delta$ and $y_\alpha^* = 0$ otherwise.

Now Theorem 3.1 gives $\sum_{\alpha\in\kappa\setminus I} g_\alpha = 0$ almost everywhere, and hence $\langle x^*, \varphi(t)\rangle = \langle x^*, \pi \circ \varphi\rangle(t)$ for almost all $t \in T$. Using Radon measure-compactess of every E_α and $|I| \leq \omega$, it is routine to check that in turn $\pi \circ \varphi$ is weakly equivalent to a strongly measurable function. This finishes the proof.

Recall that for a finite measure λ, the Banach space $L_1(\lambda)$ is measure-compact (as $L_1(\lambda)$ is WCG, see [2]). Adapting 3.6 of [2], we get the following.

COROLLARY 3.5. *If K is a compact space then the Banach space $M(K)$ of finite signed Radon measures on K is Radon measure-compact.*

Proof. Let $(\lambda_\alpha)_{\alpha<\kappa}$ be a maximal family of mutually singular Radon probability measures on K. Then $M(K)$ is isometric to the l_1-direct sum $(\sum_{\alpha<\kappa} L_1(\lambda_\alpha))_1$. Since every $L_1(\lambda_\alpha)$ is Radon measure-compact, the assertion follows from Theorem 3.4.

Assuming the absence of measurable cardinals, Andrews [1] showed that if K is a Talagrand compact space then $M(K)$ has the property: every bounded universally weakly measurable function $\varphi : T \to M(K)$ is Pettis integrable. This might suggest that Pettis integrability in $M(K)$ relies upon some special features of a space K. Of course, Corollary 3.5 explains that a fairly general and stronger result can be derived from Edgar's and Fremlin's ideas.

4. On PIP and Radon measures. The first part of the present section collects some (essentially) known results related to Pettis integrability and Radon measures in general setting. Next we prove a positive result on Pettis integrability of universally weakly measurable functions defined on Cantor cubes 2^κ, where $\kappa < \omega_\omega$.

For every cardinal κ we have a standard Radon measure space of Maharam type κ, namely $(2^\kappa, \Sigma_\kappa, \lambda_\kappa)$, where 2^κ denotes the Cantor cube $\{0,1\}^\kappa$ and λ_κ is the usual product measure (see [5] and [6], A2G). Recall that the σ-algebra Σ_κ of λ_κ-measurable sets is in fact a completion of the Baire σ-algebra of 2^κ with respect to λ_κ. This implies the following useful fact: For every $A \in \Sigma_\kappa$ there are sets B_1, B_2 depending on a countable number of coordinates and such that $B_1 \subseteq A \subseteq B_2$, $\lambda_\kappa(B_2 \setminus B_1) = 0$. A set $B \subseteq 2^\kappa$ is said to *depend on a set* $I \subseteq \kappa$ (of coordinates) if $B = \pi_I^{-1}(\pi_I(B))$, where $\pi_I : 2^\kappa \to 2^I$ is the natural projection. Accordingly, a function $g : 2^\kappa \to \mathrm{R}$ depends on coordinates in a set I if g can be written as $g = g' \circ \pi_I$. Every Σ_κ-measurable function f equals almost everywhere to a function depending on countably many coordinates. Moreover, there is a countable set $I \subseteq \kappa$ such that f is Σ_I-measurable, where Σ_I is the completion of the σ-algebra of Baire sets depending on coordinates in I.

If (T, Σ, μ) is any nonatomic measure space with $\mu(T) > 0$, there are several cardinal numbers related to $\mathcal{N}_\mu$, the ideal of μ-null sets (see [5]). In particular, $\mathrm{non}(\mathcal{N}_\mu)$ is the minimal cardinality of a set $X \subseteq T$ which is not in $\mathcal{N}_\mu$ while $\mathrm{cov}(\mathcal{N}_\mu)$ is the minimal cardinality of a subfamily of $\mathcal{N}_\mu$ covering T. In the sequel, we denote by $\mathbb{L}_\kappa$ the ideal of λ_κ-null subsets of 2^κ.

It will be convenient to say that a Radon measure space (T, Σ, μ) (or a measure μ) has PIP if every Banach space has μ-PIP. Accordingly, we shall say that μ has PIP(u) if every bounded universally weakly measurable function from T into some Banach space is Pettis integrable with respect to μ (recall that UPIP considered in the literature is a property of a Banach space and not of a measure space, see [1, 12–14].

Clearly, PIP is a stronger property than PIP(u). On the other hand, we do not know any example showing that the properties are really distinct. Note that if for some κ the measure λ_κ has PIP then every Radon measure space (T, Σ, μ) of Maharam type $\le \kappa$ has PIP. Indeed, there is a function $g : 2^\kappa \to T$ which is inverse-measure-preserving, that is, $g^{-1}(A) \in \Sigma_\kappa$ and $\mu(A) = \lambda_\kappa(g^{-1}(A))$ for every $A \in \Sigma$ (see A2K of [6]). Now if $\varphi : T \to E$ is bounded and weakly measurable then so is $\varphi \circ g : 2^\kappa \to E$. Thus $\varphi \circ g$ is Pettis integrable, which implies that so is φ (this is a special case of 4-1-7 of [15]). It does not seem so obvious whether an analogous result is true for PIP(u).

Property PIP for measures has a convenient characterization in the language of the topology τ_{p} of pointwise convergence considered on sets of measurable functions; this subject is surveyed by Vera [16]. Recall that a measure μ has PIP if and only if the mapping $g \in C \to \int g \, d\mu$ is τ_{p}-continuous on every convex and τ_{p}-compact set C of bounded measurable functions (see

Edgar [2], Theorem 4.2). An analogous characterization of PIP(u), involving sets of universally measurable functions, can be checked in the same way.

There is a classical example due to Phillips, showing that under CH the Lebesgue measure on $[0,1]$ does not have PIP (see e.g. [9], 7.1). We note that it may be modified as follows.

Suppose that (T,Σ,μ) is a measure space with the property:

$(*)$ For some κ there is a family $(N_\xi)_{\xi<\kappa}$ such that:

(i) $\mu(N_\xi)=0$ for every $\xi<\kappa$;
(ii) $N_\eta\subseteq N_\xi$ whenever $\eta<\xi<\kappa$;
(iii) $A=\bigcup_{\xi<\kappa}N_\xi\in\Sigma$ and $\mu(A)>0$.

Then there is a bounded weakly measurable function $\varphi:T\to l^\infty(\kappa)$ that is not Pettis integrable.

Let us briefly recall how it works. We may assume that κ is regular. For $t\in A$ set $i(t)=\inf\{\xi: t\in N_\xi\}$. Define $\varphi(t)=\chi_{\{\eta: i(t)\le\eta\}}$ for $t\in A$ and $\varphi=0$ outside A. To see that φ is as required, recall that every $m\in l^\infty(\kappa)^*$ is a finitely additive measure defined for all subsets of κ. Now if there is $\xi<\kappa$ such that m is concentrated on ξ then $\langle m,\varphi(t)\rangle=0$ for $t\in A\setminus N_\xi$. If $m(\xi)=0$ for all $\xi<\kappa$ we have $\langle m,\varphi(t)\rangle=m(\kappa)$ for every $t\in A$.

Note that if (T,Σ,μ) is a Radon measure space for which $(*)$ holds with $\kappa=\omega_1$ then the function φ above can be made universally weakly measurable since one can assume that N_ξ's are Borel sets.

The above mentioned property $(*)$ is precisely the negation of the axiom AF_∞ used in [4]: A measure space (T,Σ,μ) has property AF_∞ if for every Σ-additive family $\mathcal{D}$ of μ-null sets one has $\mu(\bigcup\mathcal{D})=0$ (compare with AF_1 mentioned in Section 3). As the example above shows that property PIP implies AF_∞, we point out that the following question seems to be open.

Problem. Is it true that AF_∞ implies PIP or PIP(u) for every Radon measure space?

Fremlin and Talagrand proved that if $\mathrm{non}(\mathbb{L}_\omega)<\mathrm{cov}(\mathbb{L}_\omega)$ then the Lebesgue measure has PIP (see [15], 5-5-2). In fact the same proof gives a more general result.

Theorem 4.1 (Fremlin, Talagrand). *If* $\mathrm{non}(\mathbb{L}_\kappa)<\mathrm{cov}(\mathbb{L}_\kappa)$ *then the measure* λ_κ *has PIP. In particular, it is relatively consistent that every Radon measure of Maharam type* $\le\mathfrak{c}$ *has PIP.*

Theorem 4.1 and the example above show that if $\mathrm{non}(\mathbb{L}_\mathfrak{c})=\omega_1$ then $\lambda_\mathfrak{c}$ has PIP if and only if $\mathrm{cov}(\mathbb{L}_\mathfrak{c})>\omega_1$, so it is undecidable in ZFC whether the measure $\lambda_\mathfrak{c}$ has PIP. We can ask if there is *any* κ for which the measure λ_κ honestly fails to have PIP (so that we could check it without any set-theoretic assumptions). Note that Theorem 4.1 is not applicable if κ is too large. For instance, if $\kappa > 2^{\mathfrak{c}}$ then $\mathrm{non}(\mathbb{L}_\kappa) > \mathfrak{c}$, while always $\mathrm{cov}(\mathbb{L}_\kappa) \le \mathfrak{c}$ (see 6.17e(v) of [5]).

The main result of this section is based on two lemmata we now prove.

LEMMA 4.2. *Let $\kappa > \mathfrak{c}$ be a regular cardinal such that $\eta^\omega < \kappa$ whenever $\eta < \kappa$. For every family $(N_\xi)_{\xi<\kappa} \subseteq \mathbb{L}_\kappa$ there is a set $X \in [\kappa]^\kappa$ for which the set $\bigcup_{\xi\in X} N_\xi$ is of inner λ_κ-measure zero.*

Proof. We may find for every $\xi < \kappa$ a set $Z_\xi \in \mathbb{L}_\kappa$ such that $N_\xi \subseteq Z_\xi$ and Z_ξ depends on a countable set $I_\xi \subseteq \kappa$. By the Erdős–Rado theorem on quasi-disjoint families, see e.g. [8], Theorem 1.6, there is a set $Y \in [\kappa]^\kappa$ such that $(I_\xi)_{\xi\in Y}$ is a Δ-system with a root R, that is, $I_\xi \cap I_\eta = R$ whenever $\xi, \eta \in Y$, $\xi \ne \eta$.

Denote by π_R the natural projection onto 2^R. In the sequel, we identify 2^κ with the power set of κ, treating every $t \in 2^\kappa$ as a subset of κ (so, for instance, π_R is given by $\pi_R(t) = R \cap t$; if a set Z depends on I then for every $t \in 2^\kappa$, we have $t \in Z$ iff $t \cap I \in Z$). Let λ_R denote the product measure on 2^R.

For every ξ the set $\pi_R(2^\kappa \setminus Z_\xi)$ is of full λ_R-measure. Since R is countable and $\kappa > \mathfrak{c}$ is regular, it follows that there are $X \in [Y]^\kappa$ and a fixed $F \subseteq 2^R$ with $\lambda_R(F) = 1$ such that $F \subseteq \pi_R(2^\kappa \setminus Z_\xi)$ for every $\xi \in X$. We claim that the set $W = \bigcup_{\xi\in X} Z_\xi$ has inner measure zero.

It suffices to check that whenever $\lambda_\kappa(Z) > 0$ and Z depends on a countable set $I \subseteq \kappa$, then $Z \setminus W \ne \emptyset$. Of course, we can assume that $R \subseteq I$. Note that the set $X_0 = \{\xi \in X : (I_\xi \setminus R) \cap I \ne \emptyset\}$ is countable. It follows that $\lambda_\kappa(Z \setminus \bigcup_{\xi\in X_0} Z_\xi) > 0$, and there is $t \in Z \setminus \bigcup_{\xi\in X_0} Z_\xi$ such that $\pi_R(t) \in F$. For every $\xi \in X$ we may find $s_\xi \notin Z_\xi$ such that $\pi_R(s_\xi) = \pi_R(t)$, i.e. $s_\xi \cap R = t \cap R$. Putting $J = I \cup \bigcup_{\xi\in X_0} I_\xi$, we consider $u \in 2^\kappa$ given by

$$u = (t \cap J) \cup \bigcup_{\xi\in X\setminus X_0} (s_\xi \cap I_\xi).$$

Since $I_\eta \cap I = R$ for $\eta \in X \setminus X_0$, we have

$$u \cap I = (t \cap I) \cup \bigcup_{\xi\in X\setminus X_0} (s_\xi \cap I_\xi \cap I) = (t \cap I) \cup (t \cap R) = t \cap I.$$

Now $t \in Z$ and Z depends on I, so we get $u \in Z$.

For $\eta \in X_0$ we have $u \cap I_\eta = (t \cap I_\eta) \cup (t \cap R) = t \cap I_\eta$; hence $u \notin Z_\eta$. For $\eta \in X \setminus X_0$ we have $u \cap I_\eta = (t \cap J \cap I_\eta) \cup (s_\eta \cap I_\eta) = s_\eta \cap I_\eta$; hence, again, $u \notin Z_\eta$. It follows that $u \in Z \setminus W$, and we are done.

LEMMA 4.3. *Suppose that κ is the minimal cardinal such that λ_κ does not have PIP(u). If $\mathrm{non}(\mathbb{L}_\kappa) \le \kappa$ then there is a family $(N_\xi)_{\xi<\kappa} \subseteq \mathbb{L}_\kappa$ for which $\bigcup_{\xi\in X} N_\xi$ is a measurable set of full measure whenever $X \in [\kappa]^\kappa$.*

Proof. (1) A characterization of PIP(u) mentioned above implies that there is a convex and τ_{p}-compact set C of universally measurable functions for which the mapping $g \in C \to \int g\,d\mu$ is not τ_{p}-continuous. By a lemma due to Talagrand [15], 5-1-2 (see also [11]), there are $f, g \in C$ such that $T = \{f \neq g\}$ has positive measure and g is in the τ_{p}-closure of $C(f) = \{h \in C : h = f \text{ a.e.}\}$. We shall find sets $N_\xi \subseteq T$ having the property $T \setminus \bigcup_{\xi \in X} N_\xi \in \mathbb{L}_\kappa$ for every $X \in [\kappa]^\kappa$. This gives the assertion of the lemma in view of homogeneity of λ_κ.

(2) We claim that if $D \subseteq C(f)$ and $|D| < \kappa$ then $\mathrm{cl}_{\tau_{\mathrm{p}}}(D) \subseteq C(f)$.

We shall check that the claim is a consequence of the fact that λ_η has PIP(u) whenever $\eta < \kappa$. Suppose the contrary; let $d \in C$ be a function such that d is in the closure of D but $d \notin C(f)$.

There is a set $X \subseteq \kappa$ with $|X| < \kappa$ such that all functions from D, and also f, d, are measurable with respect to Σ_X, where Σ_X is the completion, with respect to λ_κ, of the σ-algebra of Baire sets that depend on coordinates in X (since $|D| < \kappa$, the existence of such a set X follows from the remark made at the beginning of this section).

Let $K = \{t \in 2^\kappa : t \subseteq X\}$, that is, $t \in K$ if and only if $t(\xi) = 0$ for all $\xi \notin X$, and let λ_X be the natural product measure on 2^X. Identifying K with 2^X we may treat λ_X as a measure on K. Note that if N is a set in Σ_X then $\lambda_\kappa(N) = 0$ if and only if $\lambda_X(N \cap K) = 0$.

Given $h \in C$, let h_K denote the restriction of h to K; put $C_K = \{h_K : h \in C\}$. Now C_K is a convex and τ_{p}-compact set of universally measurable functions and the integral with respect to λ_X is not τ_{p}-continuous on C_K. Indeed, d_K is not equal to f_K λ_X-a.e., while for every $h \in C(f)$ we have $h = f$ λ_κ-a.e. and hence $h_{|K} = f_{|K}$ λ_X-a.e. It follows that λ_X does not have PIP(u), contrary to $|X| < \kappa$.

(3) Let γ be the minimal cardinality of a set $Q \subseteq 2^\kappa$ having the property that for every $h \in C$, if $h_{|Q} = g_{|Q}$ then $h = g$ a.e. (that is, C is determined by Q in the terminology of [16]). We have $\gamma \leq \mathrm{non}(\mathbb{L}_\kappa) \leq \kappa$, since functions from C are measurable. We claim that in fact $\gamma = \kappa$.

Indeed, let Q have the property as above. If $|Q| < \kappa$ then, using $g \in \mathrm{cl}_{\tau_{\mathrm{p}}}(C(f))$, we may easily find a family $D \subseteq C(f)$ with $|D| = |Q|$ such that there is $h' \in \mathrm{cl}_{\tau_{\mathrm{p}}}(D)$ with $h'_{|Q} = g_{|Q}$. But this gives $h' = g$ a.e., which contradicts (2).

(4) Now let $Q = (q_\xi)_{\xi<\kappa}$ be a set as in (3); define $Q_\xi = (q_\beta)_{\beta<\xi}$. Using (2) we may find for every $\xi < \kappa$ a function $g_\xi \in C(f)$ such that $g_{\xi|Q_\xi} = g_{|Q_\xi}$. Now define $N_\xi = \{g_\xi \neq f\} \cap T$; we have $N_\xi \in \mathbb{L}_\kappa$.

If $X \in [\kappa]^\kappa$ then any cluster point h of $(g_\xi)_{\xi \in X}$ satisfies $h_{|Q} = g_{|Q}$ and thus $h = g$ a.e. Consequently, $A(X) = \bigcup_{\xi \in X} N_\xi$ satisfies $\{h = g\} \cap T \subseteq A(X) \subseteq T$, and the proof is complete.

THEOREM 4.4. *If the measure $\lambda_{\mathfrak{c}}$ has PIP(u) then so does λ_κ for every $\kappa < \omega_\omega$.*

Proof. The assertion is clear if $\mathfrak{c} > \omega_\omega$. Otherwise we have $\mathfrak{c} = \omega_n$ for some n. Then $\omega_k^\omega = \omega_k$ for $k \geq n$, which implies $\mathrm{non}(\mathbb{L}_{\omega_k}) \leq \omega_k$ (see [5], 6.17). The rest follows from Lemmas 4.2 and 4.3.

COROLLARY 4.5. *It is relatively consistent that λ_κ has PIP(u) for every $\kappa < \mathfrak{c} + 2^{\mathfrak{c}} + 2^{2^{\mathfrak{c}}} + \ldots$*

Proof. Indeed, if $\omega_1 = \mathrm{non}(\mathbb{L}_{\mathfrak{c}}) < \mathrm{cov}(\mathbb{L}_{\mathfrak{c}}) = \omega_2 = \mathfrak{c}$, while $\omega_{n+1} = 2^{\omega_n}$ for every $n \geq 2$, $\lambda_{\mathfrak{c}}$ has PIP in view of Theorem 4.1, and $\omega_\omega = \mathfrak{c}+2^{\mathfrak{c}}+2^{2^{\mathfrak{c}}}+\ldots$

We do not know if Theorem 4.4 holds true when PIP(u) is replaced by PIP. Note that universal measurability was employed only in step 2 of the proof of Lemma 4.3.

It seems that without further assumptions nothing can be said on Pettis integrability with respect to the measure λ_{ω_ω}. This is partially supported by the following remark due to Fremlin [4], page 106. Namely, if $\kappa = \sup_n \kappa_n$, where $\kappa_{n+1} = 2^{\kappa_n}$, and $\kappa^+ = 2^\kappa$ then the measure λ_κ does not have property AF_∞ and therefore does not have PIP.

Acknowledgements. I am grateful to the referee who found a gap in the previous version of Lemma 4.3 and suggested several improvements.

References

[1] K. T. Andrews, *Universal Pettis integrability*, Canad. J. Math. 37 (1985), 141–159.

[2] G. A. Edgar, *Measurability in a Banach space II*, Indiana Univ. Math. J. 28 (1979), 559–579.

[3] R. Frankiewicz and G. Plebanek, *On nonaccessible filters in measure algebras and functionals on $L^\infty(\lambda)^*$*, Studia Math. 108 (1994), 191–200.

[4] D. H. Fremlin, *Measure-additive coverings and measurable selectors*, Dissertationes Math. 260 (1987).

[5] —, *Measure algebras*, in: Handbook of Boolean Algebras, J. D. Monk (ed.), North-Holland, 1989, Vol. III, Chap. 22.

[6] —, *Real-valued measurable cardinals*, in: Israel Math. Conf. Proc. 6, 1993, 961–1044.

[7] K. Kunen, *Some points in βN*, Math. Proc. Cambridge Philos. Soc. 80 (1975), 385–398.

[8] —, *Set Theory*, Stud. Logic 102, North-Holland, 1980.

[9] K. Musiał, *Topics in the theory of Pettis integration*, Rend. Inst. Mat. Univ. Trieste 23 (1991), 177–262.

[10] S. Negrepontis, *Banach spaces and topology*, in: Handbook of Set-Theoretic Topology, K. Kunen and J. Vaughan (eds.), North-Holland, 1984, 1045–1142.

[11] G. Plebanek, *On Pettis integrals with separable range*, Colloq. Math. 64 (1993), 71–78.

[12] L. H. Riddle and E. Saab, *On functions that are universally Pettis integrable*, Illinois J. Math. 29 (1985), 509–531.

[13] L. H. Riddle, E. Saab and J. J. Uhl, *Sets with the weak Radon–Nikodym property in dual Banach spaces*, Indiana Univ. Math. J. 32 (1983), 527–541.

[14] G. F. Stefansson, *Universal Pettis integrability*, Proc. Amer. Math. Soc. 125 (1993), 1431–1435.

[15] M. Talagrand, *Pettis integral and measure theory*, Mem. Amer. Math. Soc. 51 (1984).

[16] G. Vera, *Pointwise compactness and continuity of the integral*, Rev. Mat. (1996) (numero supl.), 221–245.

Institute of Mathematics
Polish Academy of Sciences
Kopernika 18
51-617 Wrocław, Poland
E-mail: grzes@math.uni.wroc.pl

Received 24 June 1997;
in revised form 22 September and 1 December 1997