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On nonstructure of elementary submodels of a stable homogeneous structure

by

Tapani Hyttinen (Helsinki)

Abstract. We assume that $\mathbf{M}$ is a stable homogeneous model of large cardinality. We prove a nonstructure theorem for (slightly saturated) elementary submodels of $\mathbf{M}$, assuming $\mathbf{M}$ has dop. We do not assume that $\mathrm{th}(\mathbf{M})$ is stable.

In this paper we study elementary submodels of a stable homogeneous L-structure $\mathbf{M}$. We use $\mathbf{M}$ as a monster model used in "classical" stability theory and so as in [HS], we assume that $|\mathbf{M}|$ is strongly inaccessible (= regular and strong limit) and $> |L|$, where $|L|$ is the number of L-formulas. We recall that by [Sh1], if D is a stable finite diagram, then it has a monster model like $\mathbf{M}$ (assuming, of course, the existence of a strongly inaccessible cardinal). As in [HS], we can drop the assumption of $|\mathbf{M}|$ being strongly inaccessible if instead of all elementary submodels of $\mathbf{M}$, we study only suitably small ones.

We assume that the reader is familiar with [HS] and we use conventions, notions and results of [HS] freely. The machinery in [HS] is an improved version of that in [Hy1]. Now we modify it so that it comes closer to the lines of [Hy1].

As mentioned in the abstract, we prove a nonstructure theorem for ($F^{\mathbf{M}}_{\lambda_{\mathrm{r}}(\mathbf{M})}$-saturated) elementary submodels of $\mathbf{M}$, assuming $\mathbf{M}$ has dop (= $\lambda_{\mathrm{r}}(\mathbf{M})$-dop). By a nonstructure theorem we mean a theorem which implies, at least, that for most κ, the number of models of power κ is the maximal one. Often nonstructure theorems imply also that a "Shelah-style" structure theorem does not hold for a class of models. See [HT] for further discussion about nonstructure theorems. In [HS] a structure theorem was proved for $F^{\mathbf{M}}_{\lambda_{\mathrm{r}}(\mathbf{M})}$-saturated models assuming $\mathbf{M}$ is superstable and does not have $\lambda_{\mathrm{r}}(\mathbf{M})$-dop. So in case $\mathbf{M}$ is superstable, we have a dichotomy for

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$F^{\mathbf{M}}_{\lambda_{\mathrm{r}}(\mathbf{M})}$-saturated models, i.e. we have a simple property which determines when $F^{\mathbf{M}}_{\lambda_{\mathrm{r}}(\mathbf{M})}$-saturated models have a structure theorem.

We concentrate on the model theory side of the nonstructure theorem. There is well-developed combinatorics that gives nonstructure theorems as soon as certain model-theoretic results are proved.

1. Basic notions and their properties. By a *model* we mean an elementary submodel of $\mathbf{M}$ of cardinality $< |\mathbf{M}|$; we write $\mathcal{A}$, $\mathcal{B}$ and so on for these. Similarly by a *set* we mean a subset of $\mathbf{M}$ of cardinality $< |\mathbf{M}|$ and we write A, B and so on for these. We will not distinguish elements from finite sequences, so by a, b, c etc. we mean finite sequences (of elements of $\mathbf{M}$) and we write $a \cup A$ and $a \in A$ instead of $\mathrm{rng}(a) \cup A$ and $\mathrm{rng}(a) \subseteq A$. For infinite sequences, we will write $\overline{a}$, $\overline{b}$ and so on. By $\lambda(\mathbf{M})$ we mean the least cardinal in which $\mathbf{M}$ is stable. $\lambda_{\mathrm{r}}(\mathbf{M})$ is the least regular cardinal $\geq \lambda(\mathbf{M})$. By an *automorphism* we mean an automorphism of $\mathbf{M}$.

Next we repeat some definitions and results from [Hy1] and [HS].

We say that a type $p(x)$ over A is *complete* if for all $\phi(x, a)$, $a \in A$, either $\phi(x, a) \in p$ or $\neg\phi(x, a) \in p$. We say that a type p over $\mathbf{M}$ is $\mathbf{M}$-*consistent* if it is realized in $\mathbf{M}$. We write $S(A)$ for the set of all complete $\mathbf{M}$-consistent types over A.

We say that a set $A \subseteq \mathbf{M}$ is $F^{\mathbf{M}}_{\kappa}$-*saturated* if any $\mathbf{M}$-consistent type over a subset of A of power $< \kappa$ is realized in A. Notice that if $\mathcal{A}$ is $F^{\mathbf{M}}_{\omega}$-saturated, then it is an elementary submodel of $\mathbf{M}$.

1.1. Lemma ([Hy1]). *Assume that p is a complete type over A. If p is not $\mathbf{M}$-consistent, then there is a finite set $B \subseteq A$ such that $p \restriction B$ is not $\mathbf{M}$-consistent.*

Proof. Immediate by homogeneity of $\mathbf{M}$. ■

1.2. Definition. (i) ([Sh2]) We say that $p \in S(A)$ *splits over* $B \subseteq A$ if for some $\phi(x, y)$ and $b, c \in A$ with $t(b, B) = t(c, B)$ we have $\phi(x, b) \in p$ and $\neg\phi(x, c) \in p$.

(ii) ([Hy1]) We say that A is *free from C over B*, $A \downarrow_B C$, if for all (finite sequences) $a \in A$ and $c \in C$ there is $D \subseteq B$ of power $< \lambda(\mathbf{M})$ such that $t(a, B \cup c)$ does not split over D.

Notice that this independence notion differs from the one defined in [HS], but as proved there, over $F^{\mathbf{M}}_{\lambda(\mathbf{M})}$-saturated models they are equivalent. Below we list some of the basic properties of this independence notion.

1.3. Lemma ([Hy1]). (i) *There are no A_i, $i < \lambda(\mathbf{M})$, and a such that for all $i < j < \lambda(\mathbf{M})$, $A_i \subseteq A_j$ and $a \not\downarrow_{A_i} A_j$.*

(ii) *For all a and A, there is $B \subseteq A$ of power $< \lambda(\mathbf{M})$ such that $t(a, A)$ does not split over B. In particular, $a \downarrow_A A$.*

(iii) *If* $a \downarrow_B A$, $B \subseteq B' \subseteq A' \subseteq A$ *and* $B' - B$ *is finite*, *then* $a \downarrow_{B'} A'$.

Proof. (i) As [Sh2], I, Lemma 2.7. (ii) Immediate by (i). (iii) Easy. ∎

1.4. Definition. We say that $t(a, A)$ is *stationary* if for all $C \supseteq A$ and b, c the following holds: if $t(b, A) = t(c, A) = t(a, A)$, $b \downarrow_A C$ and $c \downarrow_A C$ then $t(b, C) = t(c, C)$.

1.5. Lemma ([Hy1]). *If* $\mathcal{A}$ *is* $F^{\mathbf{M}}_{\lambda(\mathbf{M})}$-*saturated then* $t(a, \mathcal{A})$ *is stationary.*

Proof. Assume not. Let b, c and C exemplify this. Choose $d \in C$ and $A \subseteq \mathcal{A}$ of power $< \lambda(\mathbf{M})$ such that $t(b, A \cup d) \neq t(c, A \cup d)$ and neither $t(b, \mathcal{A} \cup d)$ nor $t(b, \mathcal{A} \cup d)$ splits over A. Let $d' \in \mathcal{A}$ be such that $t(d', A) = t(d, A)$. Then $t(b, A \cup d') \neq t(c, A \cup d')$, a contradiction. ∎

1.6. Theorem ([Hy1]). *Assume* $\mathcal{A}$ *is* $F^{\mathbf{M}}_{\lambda(\mathbf{M})}$-*saturated and* $t(a, \mathcal{A})$ *does not split over* $A \subseteq \mathcal{A}$ *of power* $< \lambda(\mathbf{M})$. *Then for all* $B \supseteq \mathcal{A}$ *there is* b *such that* $t(b, \mathcal{A}) = t(a, \mathcal{A})$ *and* $t(b, B)$ *does not split over* A.

Proof. We define a type p over B so that $\phi(x, c) \in p$ if there is $d \in \mathcal{A}$ such that $t(d, A) = t(c, A)$ and $\mathbf{M} \models \phi(a, d)$. By Lemma 1.1, it is easy to see that $p \in S(B)$. ∎

1.7. Corollary ([Hy1]). (i) *If* $a \downarrow_{\mathcal{A}} B$ *and* $\mathcal{A}$ *is* $F^{\mathbf{M}}_{\lambda(\mathbf{M})}$-*saturated*, *then there is* $C \subseteq \mathcal{A}$ *of power* $< \lambda(\mathbf{M})$ *such that* $t(a, \mathcal{A} \cup B)$ *does not split over* C.

(ii) *Assume* $\mathcal{A}$ *is* $F^{\mathbf{M}}_{\lambda(\mathbf{M})}$-*saturated*, $A \supseteq \mathcal{A}$ *and* $C \downarrow_{\mathcal{A}} A$. *Then for all* $B \supseteq A$, *there is* D *such that* $t(D, A) = t(C, A)$ *and* $D \downarrow_{\mathcal{A}} B$.

(iii) *Assume* $\mathcal{B} \subseteq B' \subseteq A' \subseteq A$ *and* $\mathcal{B}$ *is* $F^{\mathbf{M}}_{\lambda(\mathbf{M})}$-*saturated. If* $a \downarrow_{\mathcal{B}} A$ *then* $a \downarrow_{B'} A'$.

(iv) *Assume* $\mathcal{A} \subseteq \mathcal{B}$ *are* $F^{\mathbf{M}}_{\lambda(\mathbf{M})}$-*saturated. Then* $a \downarrow_{\mathcal{A}} \mathcal{B}$ *and* $a \downarrow_{\mathcal{B}} C$ *iff* $a \downarrow_{\mathcal{A}} \mathcal{B} \cup C$.

Proof. (i) By Lemma 1.3(ii) and Theorem 1.6, choose b so that $t(b, \mathcal{A}) = t(a, \mathcal{A})$ and $t(b, \mathcal{A} \cup B)$ does not split over some $C \subseteq \mathcal{A}$ of power $< \lambda(\mathbf{M})$. By stationarity, $t(a, \mathcal{A} \cup B) = t(b, \mathcal{A} \cup B)$.

(ii) Immediate by (i), Theorem 1.6 and stationarity.

(iii) Immediate by (i).

(iv) From right to left, this follows from (iii). For the other direction, choose b so that $t(b, \mathcal{B}) = t(a, \mathcal{B})$ and $b \downarrow_{\mathcal{A}} \mathcal{B} \cup C$. By (iii), $b \downarrow_{\mathcal{B}} C$ and so by stationarity, $t(a, \mathcal{B} \cup C) = t(b, \mathcal{B} \cup C)$. ∎

1.8. Lemma ([Hy1]). $a \downarrow_A B$ *and* $b \downarrow_{A \cup a} B$ *iff* $a \cup b \downarrow_A B$.

Proof. From right to left this is easy. We prove the other direction: Assume not. Choose $c \in B$ so that $a \cup b \not\downarrow_A c$. Choose $A' \subseteq A$ of power $< \lambda(\mathbf{M})$ such that

(i) $t(a, A \cup c)$ does not split over A',

(ii) $t(b, A \cup a \cup c)$ does not split over $A' \cup a$.

Then we can find $e, f \in A \cup c$ and $\phi(x, y, z)$ such that $t(e, A') = t(f, A')$ and

$$(*) \qquad \mathbf{M} \models \phi(a, b, e) \wedge \neg\phi(a, b, f).$$

By (i), $t(e, A' \cup a) = t(f, A' \cup a)$. But this and $(*)$ contradict (ii). ∎

1.9. Lemma. *If $I = (I, <)$ is a linear order and $\{a_i \mid i \in I\}$ is infinite and order indiscernible over A, then it is indiscernible over A.*

Proof. Assume not. By Lemma 1.1, we can assume $(I, <)$ is a dense linear order of power $> \lambda(\mathbf{M})$ (without endpoints), with a dense subset of power $\lambda(\mathbf{M})$. As in the classical proof we get a contradiction with $\lambda(\mathbf{M})$-stability of $\mathbf{M}$. ∎

1.10. Theorem ([Hy1]). *Assume $\mathcal{A}$ is $F^{\mathbf{M}}_{\lambda}$-saturated. Then $a \downarrow_{\mathcal{A}} b$ iff $b \downarrow_{\mathcal{A}} a$.*

Proof. Assume not. Choose a and b so that $a \downarrow_{\mathcal{A}} b$ and $b \not\downarrow_{\mathcal{A}} a$. For all $i < \omega$, choose a_i and b_i so that $t(a_i \cup b_i, \mathcal{A}) = t(a \cup b, \mathcal{A})$ and $a_i \cup b_i \downarrow_{\mathcal{A}} \bigcup_{j<i} a_j \cup b_j$. Clearly $\{a_i \cup b_i \mid i < \omega\}$ is infinite. By stationarity, it is easy to see that $\{a_i \cup b_i \mid i < \omega\}$ is order indiscernible over $\mathcal{A}$. By Lemma 1.9, it is indiscernible over $\mathcal{A}$. But by stationarity, $b_i \downarrow_{\mathcal{A}} a_j$ iff $j < i$, a contradiction. ∎

1.11. Definition. We say that $(a_i)_{i\in I}$ is *A-independent* if for all $i \in I$, $a_i \downarrow_A \bigcup\{a_j \mid j \in I,\ j \neq i\}$.

1.12. Lemma ([Hy1]). (i) *If $\mathcal{A}$ is $F^{\mathbf{M}}_{\lambda(\mathbf{M})}$-saturated and for all $i < \alpha$, $a_i \downarrow_{\mathcal{A}} \bigcup\{a_j \mid j < i\}$, then $(a_i)_{i<\alpha}$ is $\mathcal{A}$-independent.*

(ii) *Assume that $(a_i)_{i\in I}$ is A-independent. Then for all $J \subseteq I$,*

$$\bigcup_{i\in J} a_i \downarrow_A \bigcup_{i\in I-J} a_i.$$

Proof. (i) By Lemma 1.8 and Theorem 1.10, this is an easy induction on α.

(ii) By Lemmas 1.3(iii) and 1.8, this is an easy induction on $|J|$. ∎

We define a set $F^{\mathbf{M}}_{\kappa}$ as follows (this concept is essentially the same as (1)-isolation in [Sh1]): $(p, A) \in F^{\mathbf{M}}_{\kappa}$ iff p is $\mathbf{M}$-consistent, $A \subseteq \mathrm{dom}(p)$ has cardinality $< \kappa$ and for all b, $t(b, A) = p\restriction A$ implies $t(b, \mathrm{dom}(p)) = p$. If $(p, A) \in F^{\mathbf{M}}_{\kappa}$ then we also write $p \in F^{\mathbf{M}}_{\kappa}(A)$ and say that $p\restriction A$ *$F^{\mathbf{M}}_{\kappa}$-isolates* p.

We define an $F^{\mathbf{M}}_{\kappa}$-construction as in [Sh2] the general F-construction is defined. Also the other concepts are defined as in [Sh2]: $\mathcal{A}$ is *$F^{\mathbf{M}}_{\kappa}$-primary* over A if it is $F^{\mathbf{M}}_{\kappa}$-constructible over A and $F^{\mathbf{M}}_{\kappa}$-saturated. We say that an $F^{\mathbf{M}}_{\kappa}$-saturated $\mathcal{B}$ is *$F^{\mathbf{M}}_{\kappa}$-prime* over A if for all $F^{\mathbf{M}}_{\kappa}$-saturated $\mathcal{C} \supseteq A$ there is an elementary embedding from $\mathcal{B}$ to $\mathcal{C}$ which fixes A pointwise. $B \supseteq A$ is *$F^{\mathbf{M}}_{\kappa}$-atomic* over A if $t(b, A)$ is $F^{\mathbf{M}}_{\kappa}$-isolated for all $b \in B$.

In [Sh1] the basic facts about these concepts are proved. In [Hy1] there is an alternative way to see these results.

1.13. Theorem ([Sh1]). *Assume $\kappa \geq \lambda(\mathbf{M})$.*

(i) *For all A there is an $F^{\mathbf{M}}_{\kappa}$-primary model over A.*

(ii) *If $\mathcal{A}$ is $F^{\mathbf{M}}_{\kappa}$-primary over A then it is $F^{\mathbf{M}}_{\kappa}$-prime over A.*

(iii) *Assume κ is regular. If $\mathcal{A}$ is $F^{\mathbf{M}}_{\kappa}$-primary over A then it is $F^{\mathbf{M}}_{\kappa}$-atomic over A.*

(iv) *Assume κ is regular. The $F^{\mathbf{M}}_{\kappa}$-primary model over A is unique up to isomorphism over A.*

1.14. Lemma. *Assume $\kappa \geq \lambda(\mathbf{M})$ is regular and $\mathcal{A}$ is $F^{\mathbf{M}}_{\kappa}$-primary over A. If $B \subseteq \mathcal{A}$ is of power $< \kappa$ then $\mathcal{A}$ is $F^{\mathbf{M}}_{\kappa}$-primary over $A \cup B$.*

Proof. As in [Hy1] we can see that $F^{\mathbf{M}}_{\kappa}$-isolation satisfies the axioms of general isolation notion in [Sh2]. So this lemma can be proved as [Sh2], IV, Lemma 3.6. ∎

We write $t(a, A) \dashv B$ if $B \subseteq A$, and for all b and $F^{\mathbf{M}}_{\lambda(\mathbf{M})}$-saturated $\mathcal{C} \supseteq A$ if $a \downarrow_A \mathcal{C}$ and $b \downarrow_B \mathcal{C}$ then $a \downarrow_{\mathcal{C}} b$. We write $t(a, A) \dashv^a B$ if $B \subseteq A$, and for all b, if $b \downarrow_B A$ then $a \downarrow_A b$.

1.15. Lemma ([HS]). *If $\mathcal{B} \subseteq \mathcal{A}$ are $F^{\mathbf{M}}_{\lambda(\mathbf{M})}$-saturated and $t(a, \mathcal{A}) \dashv^a \mathcal{B}$ then $t(a, \mathcal{A}) \dashv \mathcal{B}$.*

Proof. This follows easily from [HS], Lemmas 4.6 and 1.9(iv). ∎

When we say that $(a_i)_{i \in I}$ is indiscernible, we mean that it is also nontrivial, i.e. for $i \neq j$, $a_i \neq a_j$.

1.16. Lemma. *Let $I = (a_i)_{i<\alpha}$, $\alpha \geq \omega$, be an indiscernible sequence over A. Then there is an indiscernible J over A such that $I \subseteq J$ and $|J| = |\mathbf{M}|$.*

Proof. Immediate by Lemma 1.1. ∎

1.17. Lemma ([Sh1]). *Let $I = (a_i)_{i<\alpha}$ be an indiscernible sequence. Then for all a and $\phi(x, y)$, either $X = \{i < \alpha \mid \models \phi(a_i, a)\}$ or $Y = \{i < \alpha \mid \models \neg\phi(a_i, a)\}$ is of power $< \lambda(\mathbf{M})$.*

Proof. By Lemma 1.3(ii), choose $Z \subseteq \alpha$ of power $< \lambda(\mathbf{M})$ such that $a \downarrow_{\bigcup_{i \in Z} a_i} \cup I$. Then clearly either $X \subseteq Z$ or $Y \subseteq Z$. ∎

1.18. Definition ([Sh1]). Let $I = (a_i)_{i<\alpha}$ be an indiscernible sequence, $\alpha \geq \lambda(\mathbf{M})$ and A a set. We define a type $\mathrm{Av}(I, A)$ to be the set of formulas $\phi(x, a)$, $a \in A$, such that $\{i < \alpha \mid \models \neg\phi(a_i, a)\}$ is of power $< \lambda(\mathbf{M})$.

1.19. Lemma. *Let $I = (a_i)_{i<\alpha}$ be an indiscernible sequence, $\alpha \geq \lambda(\mathbf{M})$ and A a set.*

(i) *If $(|A| + \lambda(\mathbf{M}))^+ \leq \alpha$, then there is $i < \alpha$ such that $t(a_i, A) = \mathrm{Av}(I, A)$.*

(ii) $\mathrm{Av}(I, A)$ *is M-consistent.*

(iii) *If I is indiscernible over A, then a realizes* $\mathrm{Av}(I, I\cup A)$ *iff* $I\cup\{a\}$ *is indiscernible over A.*

(iv) *If A is $F^{\mathbf{M}}_{\lambda(\mathbf{M})}$-saturated, $I\subseteq A$ and a realizes* $\mathrm{Av}(I,A)$, *then for all b, $a\downarrow_A b$ iff a realizes* $\mathrm{Av}(I, A\cup b)$.

Proof. (i) Follows from Lemma 1.17 by the pigeonhole principle.

(ii) Immediate by (i) and Lemma 1.16.

(iii) "$\Rightarrow$" Immediate by (i) and Lemma 1.16.

"$\Leftarrow$" Clearly if both $I\cup\{a\}$ and $I\cup\{b\}$ are indiscernible over A then $t(a, A\cup I) = t(b, A\cup I)$. So "$\Leftarrow$" follows from "$\Rightarrow$".

(iv) By Theorem 1.7(ii) it is enough to prove the claim from left to right. Assume that this does not hold. Let b exemplify this. For $\alpha\le i<\beta=\alpha+(\lambda(\mathbf{M})+|L|)^+$, choose a_i so that $t(a_i, A\cup a\cup\bigcup_{j<i}a_j)=\mathrm{Av}(I, A\cup a\cup\bigcup_{j<i}a_j)$. Let $J=(a_i)_{i<\beta}$. Then by (iii), J is indiscernible. Let b' be such that $t(b',A)=t(b,A)$ and $b'\downarrow_A a\cup J$. By stationarity, we may assume that $b=b'$. By Corollary 1.7(i), choose $A'\subseteq A$ of power $<\lambda(\mathbf{M})$ such that $t(b, A\cup J\cup a)$ does not split over A' and $t(a, A'\cup b)\ne \mathrm{Av}(I, A'\cup b)$.

By Lemma 1.17, the choice of β and the pigeonhole principle, we can find $X\subseteq\beta$ of power $\ge(\lambda(\mathbf{M})+|L|)^+$ such that for all $i\in X$, a_i realizes $t(a,A')$. Similarly we can see that there is $i\in X$ such that $t(a_i, A'\cup b)=\mathrm{Av}(I, A'\cup b)$. But then clearly $t(b, A'\cup a\cup a_i)$ splits over A', a contradiction. ∎

1.20. Lemma. *Assume $I\subseteq J\cap K$, $|I|=\lambda(\mathbf{M})$, $|K|\ge|J|^+$ and J and K are indiscernible sequences over A. Then some $c\in K$ realizes* $\mathrm{Av}(J, A\cup J)$.

Proof. By Lemma 1.3(ii), for some $K'\subseteq K$ with $|K'|<|K|$, for all finite sequences b of elements of J, there are sets $X\subseteq K'$ and $B\subseteq A$ of power $<\lambda(\mathbf{M})$ such that $t(b, A\cup K)$ does not split over $B\cup X$. We claim that any $c\in K-K'$ is as desired.

Assume not. Then we can find a finite sequence b of elements of $J-I$ and sets $X\subseteq K'$ and $B\subseteq A$ of power $<\lambda(\mathbf{M})$ such that $t(b, A\cup K)$ does not split over $B\cup X$ and

$$t(c, B\cup b\cup Y)\ne \mathrm{Av}(J, B\cup b\cup Y), \tag{1}$$

where $Y=X\cap I$. Let $a\in I-X$. Then

$$t(a, B\cup X)=t(c, B\cup X) \tag{2}$$

and by Lemma 1.19(iii),

$$t(a, B\cup b\cup Y)=\mathrm{Av}(J, B\cup b\cup Y). \tag{3}$$

Clearly (1)–(3) imply that $t(b, A\cup K)$ splits over $B\cup X$, a contradiction. ∎

The following property was introduced in [Sh2].

1.21. Definition. Assume $\kappa\ge\lambda(\mathbf{M})$. We say that $\mathbf{M}$ has *κ-dop* if there are $F^{\mathbf{M}}_{\kappa}$-saturated $\mathcal{A}_i$, $i<4$, and $a\notin\mathcal{A}_3$ such that

(a) $\mathcal{A}_0 \subseteq \mathcal{A}_1 \cap \mathcal{A}_2$, $\mathcal{A}_3$ is $F^{\mathbf{M}}_\kappa$-primary over $\mathcal{A}_1 \cup \mathcal{A}_2$,
(b) $\mathcal{A}_1 \downarrow_{\mathcal{A}_0} \mathcal{A}_2$,
(c) $t(a, \mathcal{A}_3) \dashv \mathcal{A}_1$ and $t(a, \mathcal{A}_3) \dashv \mathcal{A}_2$.

In the case of stable theories, the following property is equivalent to dop ([Sh2]).

1.22. Definition. Assume $\kappa \geq \lambda(\mathbf{M})$. We say that $\mathbf{M}$ has *κ-sdop* if the following holds: there are $F^{\mathbf{M}}_\kappa$-saturated $\mathcal{A}_i$, $i < 4$, of power κ and $I = (a_i)_{i<\lambda(\mathbf{M})} \subseteq \mathcal{A}_3$ such that

(a) $\mathcal{A}_0 \subseteq \mathcal{A}_1 \cap \mathcal{A}_2$, $\mathcal{A}_3$ is $F^{\mathbf{M}}_\kappa$-primary over $\mathcal{A}_1 \cup \mathcal{A}_2$,
(b) $\mathcal{A}_1 \downarrow_{\mathcal{A}_0} \mathcal{A}_2$,
(c) I is an indiscernible sequence over $\mathcal{A}_1 \cup \mathcal{A}_2$.

The following theorem is proved in [HS]. There we used a different independence notion, but since over $F^{\mathbf{M}}_{\lambda(\mathbf{M})}$-saturated models they are equivalent, it is easy to see that dop and sdop do not depend on which independence notion is used. The claim (i) in the theorem can be proved as in the case of stable theories ([Sh2]) by using the properties of $\downarrow$ listed above. (ii) is harder.

1.23. Theorem ([HS]). *Assume $\kappa = \xi^+ > \lambda_{\mathrm{r}}(\mathbf{M})$ and $\mathbf{M}$ is ξ-stable.*

(i) *If $\mathbf{M}$ has κ-sdop then it has κ-dop.*
(ii) *If $\mathbf{M}$ has $\lambda_{\mathrm{r}}(\mathbf{M})$-dop then it has κ-sdop.*

1.24. Lemma. *Assume κ is a regular cardinal $> \lambda(\mathbf{M})$. Let $\mathcal{A}$ be $F^{\mathbf{M}}_\kappa$-primary over A and $(a_i)_{i<\alpha} \subseteq \mathcal{A}$ be an indiscernible sequence over A. Then $\alpha < \kappa^+$.*

Proof. For a contradiction assume $\alpha = \kappa^+$. Let $\mathcal{A}_0$ be $F^{\mathbf{M}}_\kappa$-primary over $A \cup \bigcup_{i<\lambda(\mathbf{M})} a_i$ and $\mathcal{A}_1$ be $F^{\mathbf{M}}_\kappa$-primary over $A \cup \bigcup_{i<\kappa} a_i$. By Lemma 1.14, we may assume that $\mathcal{A} = \mathcal{A}_0$, and by Theorem 1.13(ii) that there is an automorphism f that takes $\mathcal{A}$ into $\mathcal{A}_1$ and fixes $A \cup \bigcup_{i<\lambda(\mathbf{M})} a_i$ pointwise.

Let $I = (a_i)_{i<\kappa}$. By Lemma 1.20, there is $i^* < \kappa^+$ such that $t(f(a_{i^*}), A \cup I) = \mathrm{Av}(I, A \cup I)$. Because $\mathcal{A}_1$ is $F^{\mathbf{M}}_\kappa$-primary over $A \cup \bigcup_{i<\kappa} a_i$ there is $B \subseteq A \cup \bigcup_{i<\kappa} a_i$ of cardinality $< \kappa$ such that $t(f(a_{i^*}), B)$ $F^{\mathbf{M}}_\kappa$-isolates $t(f(a_{i^*}), A \cup \bigcup_{i<\kappa} a_i)$, which is impossible, because clearly there is $i < \kappa$ for which $t(a_i, B) = \mathrm{Av}(I, A \cup I){\restriction}B$. ■

In the next chapter we will need the following theorem. It is proved in [HS]. In the case ξ is regular it is trivial, but for singular ξ all the machinery developed in [HS] is needed.

1.25. Theorem ([HS]). *Assume* $\mathbf{M}$ *is* ξ*-stable.*

(i) *If* $A \subseteq \mathcal{A}$, $|A| \le \xi$ *and* $\mathcal{A}$ *is* $F^{\mathbf{M}}_{\xi}$*-saturated, then there is* $A \subseteq B \subseteq \mathcal{A}$ *such that* $|B| = \xi$ *and* B *is* $F^{\mathbf{M}}_{\xi}$*-saturated.*

(ii) *Let* A_i, $i < \xi \cdot \xi$, *be an increasing continuous sequence of sets such that*

(a) *for all* $i < \xi \cdot \xi$, A_{i+1} *is an* $F^{\mathbf{M}}_{\xi}$*-saturated model of power* ξ,

(b) *for all* $i < \xi \cdot \xi$ *and* a *there is* $b \in A_{i+1}$ *such that* $t(b, A_i) = t(a, A_i)$.

Then $\bigcup_{i<\xi\cdot\xi} A_i$ *is* $F^{\mathbf{M}}_{\xi}$*-saturated.*

Proof. (i) Choose $B \subseteq \mathcal{A}$ so that it is $F^{\mathbf{M}}_{\xi}$-primary over A. By [HS], Theorem 3.14, B is as desired. (ii) follows from the proof of [HS], Theorem 3.14. ■

2. Nonstructure. From now on we assume that $\lambda = \lambda_{\mathrm{r}}(\mathbf{M})^{+}$, $\mathbf{M}$ has λ-sdop, $\kappa > \lambda$ is regular $\xi > \kappa^{+}$ is regular and $\xi^{(\kappa^{+})} = \xi$. We will prove a nonstructure theorem for $F^{\mathbf{M}}_{\kappa}$-saturated elementary submodels of $\mathbf{M}$.

Remark. A nonstructure thererem for $F^{\mathbf{M}}_{\kappa}$-saturated models implies a nonstructure theorem for $F^{\mathbf{M}}_{\kappa'}$-saturated models for all $\kappa' < \kappa$ and also for elementary submodels of $\mathbf{M}$.

We follow the proofs of the related results in [Sh2]. The main differences are that we have not created a theory of strong types for $\mathbf{M}$ (in [HS] this is done) and that some types may have no free extensions over some sets. In fact, the author does not see any reason why $t(I, \mathcal{A}_1 \cup \mathcal{A}_2)$ in the definition of λ-sdop should not be of such type. We overcome these problems by "replacing $t(A, B) \perp t(C, B)$ by $t(A, B \cup C) \in F^{\mathbf{M}}_{\lambda}(B)$".

Notice also that $t(a, A) \in F^{\mathbf{M}}_{\lambda}(B)$, $B \subseteq A$, does not imply that $a \downarrow_B A$. Of course, if there is b such that $t(b, B) = t(a, B)$ and $b \downarrow_B A$, then $t(a, A) \in F^{\mathbf{M}}_{\lambda}(B)$ does imply that $a \downarrow_B A$.

2.1. Definition. Assume $C \subseteq B$. Then we write $t(A, B) \in F^{\mathbf{M}}_{\lambda}(C)$ if for all finite $a \in A$, there is $D \subseteq C$ of power $< \lambda$ such that $t(a, B) \in F^{\mathbf{M}}_{\lambda}(D)$.

2.2. Lemma. *There are* $F^{\mathbf{M}}_{\lambda}$*-saturated models* $\mathcal{A}_i$ *of cardinality* λ, $i < 3$, *and an indiscernible sequence* I *over* $\mathcal{A}_1 \cup \mathcal{A}_2$ *of power* $\lambda(\mathbf{M})$ *such that*

(i) $\mathcal{A}_0 \subseteq \mathcal{A}_1 \cap \mathcal{A}_2$, $\mathcal{A}_1 \downarrow_{\mathcal{A}_0} \mathcal{A}_2$,

(ii) *there is* $D \subseteq \mathcal{A}_1 \cup \mathcal{A}_2$ *of power* $< \lambda$ *with the following property*: *if* C_i, $i < 3$, *are such that* $C_0 \downarrow_{\mathcal{A}_0} \mathcal{A}_1 \cup \mathcal{A}_2$ *and for* $i \in \{1, 2\}$ *and all* $c_i \in C_i$, *there is* $D_i \subseteq \mathcal{A}_i \cup C_0$ *of power* $< \kappa$ *such that* $t(c_i, \mathcal{A}_1 \cup \mathcal{A}_2 \cup C_0 \cup C_{3-i}) \in F^{\mathbf{M}}_{\lambda}(D_i)$, *then*

$$t(I, \mathcal{A}_1 \cup \mathcal{A}_2 \cup C_0 \cup C_1 \cup C_2) \in F^{\mathbf{M}}_{\lambda}(D).$$

Proof. Let $\mathcal{A}_i$, $i < 3$, and I be as in the definition of λ-sdop, $|I| = \lambda(\mathbf{M})$. Clearly these satisfy (i). We show (ii). By Theorem 1.13(iii), let $D \subseteq \mathcal{A}_1 \cup \mathcal{A}_2$ of power $< \lambda$ be such that $t(I, \mathcal{A}_1 \cup \mathcal{A}_2) \in F^{\mathbf{M}}_{\lambda}(D)$. Clearly we may assume that $|C_0| < \kappa$ and that $C_1 = c_1$ and $C_2 = c_2$ are finite. Let $a \in \mathcal{A}_1 \cup \mathcal{A}_2$ be arbitrary. Then it is enough to show that

$$(*) \qquad t(I, D \cup a \cup C_0 \cup c_1 \cup c_2) \in F^{\mathbf{M}}_{\lambda}(D).$$

Choose $F^{\mathbf{M}}_{\lambda(\mathbf{M})}$-saturated models $\mathcal{D}_i$, $i < 3$, such that

(a) $\mathcal{D}_i \subseteq \mathcal{A}_i$ has cardinality $< \lambda$ and $\mathcal{D}_0 = \mathcal{D}_1 \cap \mathcal{D}_2$,
(b) $\mathcal{D}_1 \cup \mathcal{D}_2 \downarrow_{\mathcal{D}_0} \mathcal{A}_0$, $\mathcal{D}_1 \downarrow_{\mathcal{D}_2} \mathcal{A}_2$ and $\mathcal{D}_2 \downarrow_{\mathcal{D}_1} \mathcal{A}_1$,
(c) $D \cup a \subseteq \mathcal{D}_1 \cup \mathcal{D}_2$,
(d) $C_0 \downarrow_{\mathcal{D}_0} \mathcal{D}_1 \cup \mathcal{D}_2$,
(e) for $i \in \{1, 2\}$, $t(c_i, \mathcal{A}_1 \cup \mathcal{A}_2 \cup C_0 \cup c_{3-i}) \in F^{\mathbf{M}}_{\lambda}(\mathcal{D}_i \cup C_0)$.

The only difficulty in seeing the existence of such sets is how to pick them so that they are $F^{\mathbf{M}}_{\lambda(\mathbf{M})}$-saturated. This can be solved by using Theorem 1.25. ($\mathbf{M}$ is $\lambda_{\mathrm{r}}(\mathbf{M})$-stable by [HS].)

Choose $C'_0 \subseteq \mathcal{A}_0$ so that $t(C'_0, \mathcal{D}_0) = t(C_0, \mathcal{D}_0)$. Then by (b) and (d) above, because $\mathcal{D}_0$ is $F^{\mathbf{M}}_{\lambda(\mathbf{M})}$-saturated, $t(C'_0, \mathcal{D}_1 \cup \mathcal{D}_2) = t(C_0, \mathcal{D}_1 \cup \mathcal{D}_2)$. For $i \in \{1, 2\}$, choose $c'_i \in \mathcal{A}_i$ so that $t(c'_i \cup C'_0, \mathcal{D}_i) = t(c_i \cup C_0, \mathcal{D}_i)$. By (e) above, there is an automorphism f such that $f{\restriction}(\mathcal{D}_1 \cup \mathcal{D}_2) = \mathrm{id}_{\mathcal{D}_1 \cup \mathcal{D}_2}$, $f(C_0) = C'_0$ and for $i \in \{1, 2\}$, $f(c_i) = c'_i$. By the choice of D and (c) above, $(*)$ follows. ■

2.3. Corollary. *In Lemma* 2.2 *we may require that* $|I| = \kappa^+$.

Proof. This follows immediately from Lemma 1.19. ■

Let $\mathcal{A}_i$, $i < 3$, and I be as in Corollary 2.3. Let U be a set and R a binary relation on U. We define an $F^{\mathbf{M}}_{\kappa}$-saturated model $\mathcal{A}_{(U,R)}$ as follows. For all $i \in U$ we choose B_i and C_i so that $t(B_i, \mathcal{A}_0) = t(\mathcal{A}_1, \mathcal{A}_0)$, $t(C_i, \mathcal{A}_0) = t(\mathcal{A}_2, \mathcal{A}_0)$ and $\{B_i \mid i \in U\} \cup \{C_i \mid i \in U\}$ is independent over $\mathcal{A}_0$. Since $\mathcal{A}_0$ is $F^{\mathbf{M}}_{\lambda(\mathbf{M})}$-saturated, we can find these by Lemmas 1.12(i) and 1.7(ii).

For all $i, j \in U$ we choose I_{ij} so that $t(I_{ij} \cup \mathcal{B}_i \cup \mathcal{C}_j, \mathcal{A}_0) = t(I \cup \mathcal{A}_1 \cup \mathcal{A}_2, \mathcal{A}_0)$. Then we let $\mathcal{A}_{(U,R)}$ be $F^{\mathbf{M}}_{\kappa}$-primary over $\bigcup\{\mathcal{B}_i \mid i \in U\} \cup \bigcup\{\mathcal{C}_i \mid i \in U\} \cup \bigcup\{I_{ij} \mid (i, j) \in R\}$.

2.4. Lemma. *Let* $i,j\in U$, $i\neq j$, $|U|\geq 3$ *and*

$$\begin{aligned}D_0^R(i,j)&=\bigcup\{\mathcal{B}_u\mid u\neq i\}\cup\bigcup\{\mathcal{C}_u\mid u\neq j\}\\&\quad\cup\bigcup\{I_{uv}\mid u\neq i,\ v\neq j,\ (u,v)\in R\},\\D_1^R(i,j)&=\bigcup\{I_{uv}\mid u=i,\ v\neq j,\ (u,v)\in R\},\\D_2^R(i,j)&=\bigcup\{I_{uv}\mid u\neq i,\ v=j,\ (u,v)\in R\}.\end{aligned}$$

Then

(i) $D_0^R(i,j)\downarrow_{\mathcal{A}_0}\mathcal{B}_i\cup\mathcal{C}_j$,
(ii) $t(I_{ij},\mathcal{B}_i\cup\mathcal{C}_j\cup\bigcup_{n<3}D_n^R(i,j))\in F_\lambda^{\mathbf{M}}(\mathcal{B}_i\cup\mathcal{C}_j)$.

Proof. Clearly it is enough to prove this for all finite U. In particular, it is enough to assume $U=\omega$ and R is finite. We prove this by induction on $|R|$.

Notice that if $(i,j)\in R$, then

$$\bigcup_{n<3}D_n^R(i,j)\cup I_{ij}\cup\mathcal{B}_i\cup\mathcal{C}_j=\bigcup\{\mathcal{B}_i\cup\mathcal{C}_i\mid i\in U\}\cup\bigcup\{I_{uv}\mid (u,v)\in R\},$$

otherwise

$$\bigcup_{n<3}D_n^R(i,j)\cup\mathcal{B}_i\cup\mathcal{C}_j=\bigcup\{\mathcal{B}_i\cup\mathcal{C}_i\mid i\in U\}\cup\bigcup\{I_{uv}\mid (u,v)\in R\}.$$

For $R=\emptyset$ the claim is clear by Lemmas 1.12(ii) and 2.2.

So assume that $R=R'\cup\{(u,v)\}$ ($R\neq R'$) and that we have proved the claim for all proper subsets of R. There are four cases:

1. $(i,j)\in R$: Let $R^*=R-\{(i,j)\}$. Then $D_n^R(i,j)=D_n^{R^*}(i,j)$ for $n<3$, and so (i) and (ii) follow immediately from the induction assumption.

2. $(i,j)\notin R$, $u\neq i$ and $v\neq j$: By the induction assumption,

$$D_0^{R'}(i,j)\downarrow_{\mathcal{A}_0}\mathcal{B}_i\cup\mathcal{C}_j$$

and

$$t(I_{uv},D_0^{R'}(i,j)\cup\mathcal{B}_i\cup\mathcal{C}_j)\in F_\lambda^{\mathbf{M}}(D_0^{R'}(i,j)).$$

Since $\mathcal{A}_0$ is $F_{\lambda(\mathbf{M})}^{\mathbf{M}}$-saturated, (i) follows.

Let $\{I_m\mid m<n\}$ be an enumeration of $D_1^R(i,j)$. Then by the induction assumption, for all $m<n$,

$$t\Big(I_m,\mathcal{B}_i\cup\mathcal{C}_j\cup\bigcup_{k<m}I_k\cup\bigcup_{n\in\{0,2\}}D_n^R(i,j)\Big)\in F_\lambda^{\mathbf{M}}(\mathcal{B}_i\cup D_0^R(i,j)).$$

So by the basic properties of $F_\lambda^{\mathbf{M}}$-isolation ($F_\lambda^{\mathbf{M}}$-isolation satisfies Axiom VII for a general isolation notion in [Sh2]),

$$t\Big(D_1^R(i,j), \mathcal{B}_i \cup \mathcal{C}_j \cup \bigcup_{n\in\{0,2\}} D_n^R(i,j)\Big) \in F_\lambda^{\mathbf{M}}(\mathcal{B}_i \cup D_0^R(i,j)).$$

Similarly we can see that

$$t\Big(D_2^R(i,j), \mathcal{B}_i \cup \mathcal{C}_j \cup \bigcup_{n\in\{0,1\}} D_n^R(i,j)\Big) \in F_\lambda^{\mathbf{M}}(\mathcal{C}_j \cup D_0^R(i,j)).$$

So by Lemma 2.2, (ii) follows.

3. $(i,j) \notin R$, $u = i$ and $v \neq j$: Because $D_0^R(i,j) = D_0^{R'}(i,j)$, (i) is immediate by the induction assumption.

As in case 2 above, by the induction assumption,

$$t\Big(D_1^R(i,j), \mathcal{B}_i \cup \mathcal{C}_j \cup \bigcup_{n\in\{0,2\}} D_n^R(i,j)\Big) \in F_\lambda^{\mathbf{M}}(\mathcal{B}_i \cup D_0^R(i,j))$$

and

$$t\Big(D_2^R(i,j), \mathcal{B}_i \cup \mathcal{C}_j \cup \bigcup_{n\in\{0,1\}} D_n^R(i,j)\Big) \in F_\lambda^{\mathbf{M}}(\mathcal{C}_j \cup D_0^R(i,j)).$$

So by Lemma 2.2, (ii) follows.

4. $(i,j) \notin R$, $u \neq i$ and $v = j$: As in case 3. ∎

2.5. Lemma. *For all $i,j \in U$, $(i,j) \in R$ iff there is $J \subseteq \mathcal{A}_{(U,R)}$ (of cardinality κ^+) such that $t(J \cup \mathcal{B}_i \cup \mathcal{C}_j, \mathcal{A}_0) = t(I \cup \mathcal{A}_1 \cup \mathcal{A}_2, \mathcal{A}_0)$.*

Proof. From left to right the claim is trivial. We prove the other direction: For a contradiction assume $(i,j) \notin R$. By Lemma 2.4(ii), J is indiscernible over $\bigcup\{\mathcal{B}_u \mid u \in U\} \cup \bigcup\{\mathcal{C}_u \mid u \in U\} \cup \bigcup\{I_{uv} \mid (u,v) \in R\}$. By Lemma 1.24, $|J| < \kappa^+$, a contradiction. ∎

We let $\psi(x,y)$, $x = x_1{}^\frown x_2$, $y = y_1{}^\frown y_2$, $\mathrm{length}(x_1) = \mathrm{length}(x_2) = \mathrm{length}(y_1) = \mathrm{length}(y_2) = \lambda$, be a formula which says that there is J such that $t(J \cup x_1 \cup y_2, \emptyset) = t(I \cup \mathcal{A}_1 \cup \mathcal{A}_2, \emptyset)$. We do not care in which language ψ is, as long as isomorphism preserves truth in that language. We let $\phi(x,y) = \psi(x,y) \wedge \neg\psi(y,x)$, i.e. we make ψ asymmetric.

2.6. Definition ([Sh3]). Let $\phi'(x,y) = \phi(y,x)$ and let $\eta = (\eta,<)$ be a linear ordering. Assume A is a model and for all $i \in \eta$ there is $\overline{a}_i \in A^{\mathrm{length}(x)}$. Then we say that $(\overline{a}_i)_{i\in\eta}$ is weakly (κ,ϕ)-*skeleton-like* in A if

(i) for all $i,j \in \eta$, $\models \phi(\overline{a}_i, \overline{a}_j)$ iff $i < j$,

(ii) for all $\overline{a} \in A^{\mathrm{length}(x)}$ there is $K \subseteq \eta$ of power $< \kappa$ such that if $i,j \in \eta$ and $t(i,K) = t(j,K)$ (in η), then

$$\models \phi(\overline{a}_i, \overline{a}) \leftrightarrow \phi(\overline{a}_j, \overline{a})$$

and similarly for ϕ'.

We say that a linear ordering η is κ^+-*dense* if for all $A, B \subseteq \eta$ of power $< \kappa^+$, the following holds: if $a < b$ for all $a \in A$ and $b \in B$, then there is $c \in \eta$ such that $a < c < b$ for all $a \in A$ and $b \in B$.

2.7. LEMMA. *Assume η is a κ^+-dense linear ordering. Then $(\mathcal{B}_i \cup \mathcal{C}_i)_{i\in\eta}$ is weakly (κ, ϕ)-skeleton-like in $\mathcal{A}_\eta$.*

Proof. By Lemma 2.5, (i) of Definition 2.6 holds. So we need to prove (ii).

Let $A \subseteq \mathcal{A}_\eta$ be of power λ. Because A_η is $F^{\mathbf{M}}_\kappa$-atomic over $\bigcup\{\mathcal{B}_i \cup \mathcal{C}_i \cup I_{ij} \mid i, j \in \eta,\ i < j\}$, there is $K \subseteq \eta$ of power $< \kappa$ such that for all finite sequences $a \in A$, there is $D \subseteq \bigcup\{\mathcal{B}_i \cup \mathcal{C}_i \cup \mathcal{B}_j \cup \mathcal{C}_j \cup I_{ij} \mid i < j,\ i, j \in K\}$ of power $< \kappa$ such that

$$t\Big(a, \bigcup\{\mathcal{B}_i \cup \mathcal{C}_i \cup \mathcal{B}_j \cup \mathcal{C}_j \cup I_{ij} \mid i < j,\ i, j \in \eta\}\Big) \in F^{\mathbf{M}}_\kappa(D).$$

Then K is as required:

It is enough to show that if $x, y \in \eta$ and $t(x, K) = t(y, K)$ (in η), then

$$\models \psi(A, \mathcal{B}_x \cup \mathcal{C}_x) \leftrightarrow \psi(A, \mathcal{B}_y \cup \mathcal{C}_y)$$

and

$$\models \psi(\mathcal{B}_x \cup \mathcal{C}_x, A) \leftrightarrow \psi(\mathcal{B}_y \cup \mathcal{C}_y, A).$$

Since these are similar, we only prove the first. By symmetry, it is enough to prove "$\rightarrow$".

Assume $\models \psi(A, \mathcal{B}_x \cup \mathcal{C}_x)$. Let $\mathcal{B} \subseteq A$ and $I' \subseteq \mathcal{A}_\eta$ be such that $t(I' \cup \mathcal{B} \cup \mathcal{C}_x, \emptyset) = t(I \cup \mathcal{A}_1 \cup \mathcal{A}_2, \emptyset)$. Then it is easy to see that there is $K' \subseteq \eta$ such that

(i) $K \cup \{x\} \subseteq K'$, $|K'| = \kappa^+$,

(ii) $\mathcal{B} \cup I' \subseteq \mathcal{A}_{K'}$, where $\mathcal{A}_{K'}$ is $F^{\mathbf{M}}_\kappa$-primary over $\bigcup\{\mathcal{B}_i \cup \mathcal{C}_i \mid i \in K'\} \cup \bigcup\{I_{i,j} \mid i < j,\ i, j \in K'\}$.

CLAIM. *$\mathcal{A}_{K'}$ is $F^{\mathbf{M}}_\kappa$-primary over $\mathcal{B} \cup \bigcup\{\mathcal{B}_i \cup \mathcal{C}_i \mid i \in K'\} \cup \bigcup\{I_{i,j} \mid i < j,\ i, j \in K'\}$.*

Proof. Since $|\mathcal{B}| < \kappa$, this follows from Lemma 1.14. $\blacksquare_{\text{Claim}}$

Since η is κ^+-dense, there is $K'' \subseteq \eta$ and an order preserving onto function $f : K' \to K''$ such that $f\restriction K = \mathrm{id}_K$ and $f(x) = y$. Let $\mathcal{A}_{K''} \subseteq \mathcal{A}_\eta$ be $F^{\mathbf{M}}_\kappa$-primary over $\mathcal{B} \cup \bigcup\{\mathcal{B}_i \cup \mathcal{C}_i \mid i \in K''\} \cup \bigcup\{I_{i,j} \mid i < j,\ i, j \in K''\}$. By the choice of K and Lemma 2.4(ii), there is an isomorphism $g : \mathcal{A}_{K'} \to \mathcal{A}_{K''}$ such that $g\restriction\mathcal{B} = \mathrm{id}_{\mathcal{B}}$ and for $i, j \in K'$, $i < j$, $g\restriction(\mathcal{B}_i \cup \mathcal{C}_i \cup \mathcal{B}_j \cup \mathcal{C}_j \cup I_{ij})$ is the natural isomorphism onto $\mathcal{B}_{f(i)} \cup \mathcal{C}_{f(i)} \cup \mathcal{B}_{f(j)} \cup \mathcal{C}_{f(j)} \cup I_{f(i)f(j)}$. Then $g(I')$ is the required indiscernible sequence. $\blacksquare$

We have now proved all the model theory needed for nonstructure theorems, i.e. in [Sh3] it is shown that from Lemma 2.7 a nonstructure theorem follows. Notice that so far we have used only a small fraction of the cardinal assumptions that we made at the beginning of this section.

In the rest of this section, we show that from Lemma 2.7 and [Sh3] we can also get a stronger nonstructure theorem than what is explicitly proved in [Sh3]. The price we pay is that we need all the cardinal assumptions that we made at the beginning of this section. These assumptions are used in the construction of the linear orderings $\Phi(A)$. Notice that our (stronger) nonstructure theorem implies that if $\mathbf{M}$ has $\lambda_{\mathrm{r}}(\mathbf{M})$-dop, then we cannot prove a "Shelah-style" structure theorem for $F^{\mathbf{M}}_{\kappa}$-saturated models. This does not follow from the result that for most $\xi > \kappa$, the number of $F^{\mathbf{M}}_{\kappa}$-saturated models of power ξ is the maximal one.

Notice also that the construction of the linear orderings $\Phi(A)$ is essentially the one used by J. Conway to construct ω_1-like dense linear orderings. Also the proof of the fact that our models are L_{∞,ξ^+}-equivalent is essentially the same as the proof of the fact that ω_1-like dense linear orderings are L_{∞,ω_1}-equivalent (see [NS]).

We let D_{ξ^+} be the filter on ξ^+ generated by the closed unbounded subsets of ξ^+ and the set $\{\delta < \xi^+ \mid \mathrm{cf}(\delta) \geq \kappa\}$. If $f : \xi^+ \to \xi^+$, then by f/D_{ξ^+} we mean the $\sim$-equivalence class of f where $f \sim g$ iff $\{\delta < \xi^+ \mid f(\delta) = g(\delta)\} \in D_{\xi^+}$.

2.8. Definition ([Sh3]). Assume I is a linear ordering and $\mathrm{cf}(I) = \xi^+$. Let $(I_i)_{i<\xi^+}$ be an increasing continuous sequence of proper initial segments of I such that $I = \bigcup_{i<\xi^+} I_i$. Define $f : \xi^+ \to \xi^+$ by $f(\alpha) = \mathrm{cf}((I - I_\alpha)^*)$, where $(I - I_\alpha)^*$ is the inverse of $I - I_\alpha$.

We define $\mathrm{inv}^1_\kappa(I)$ as follows: If there is a closed unbounded set C in ξ^+ such that $f(\alpha) \geq \kappa$ for all $\alpha \in C$ with $\mathrm{cf}(\alpha) \geq \kappa$, then $\mathrm{inv}^1_\kappa(I) = f/D_{\xi^+}$, otherwise we say that $\mathrm{inv}^1_\kappa(I)$ is undefined.

Let τ and τ' be linear orderings. We define $\tau + \tau'$ as follows: The universe of $\tau + \tau'$ is $(\{0\} \times \tau) \cup (\{1\} \times \tau')$, and $(m, i) < (n, j)$ if $m < n$ or $m = n$ and $i < j$. We define $\tau \times \tau'$ as follows: The universe of $\tau \times \tau'$ is $\tau \times \tau'$, and $(i, j) < (i', j')$ if $j < j'$ or $j = j'$ and $i < i'$.

We define a linear ordering $\tau = (\tau, <)$ as follows: We let the universe of τ be the set of all functions $f : \kappa^+ \to \xi$ such that for all $i < \kappa^+$ there is $j > i$ such that $f(j) \neq 0$. We order τ so that $f < g$ if $f(i) < g(i)$, where $i < \kappa^+$ is the least ordinal such that $f(i) \neq g(i)$. For all $A \subseteq \xi^+$ we define a linear ordering $\Phi(A)$ as follows: We let $\tau_\alpha = \tau$ if $\alpha \notin A$, and otherwise $\tau_\alpha = \tau \times \kappa^*$, where κ^* is the inverse of κ. The universe of $\Phi(A)$ is $\bigcup_{\alpha<\xi^+} \{\alpha\} \times \tau_\alpha$ and $\Phi(A)$ is ordered so that $(\alpha, f) < (\beta, g)$ iff $\alpha < \beta$ or $\alpha = \beta$ and $f < g$.

2.9. Lemma. (i) *τ is a κ^+-dense linear ordering of power ξ, $\tau \times \xi \cong \tau$ and for all $\alpha < \xi$, $\tau \times (\alpha + 1) \cong \tau$.*

(ii) $\tau \times (\xi + \kappa^*) \cong \tau$.

(iii) *If* $A \subseteq \{\alpha < \xi^+ \mid \mathrm{cf}(\alpha) \geq \kappa^+\}$, *then* $\Phi(A)$ *is* κ^+*-dense.*

(iv) *Let*

$$\Phi(A,\alpha) = \{\alpha\} \times \tau_\alpha, \quad \Phi(A,<\alpha) = \bigcup_{\beta<\alpha} \Phi(A,\beta), \quad \Phi(A,\beta,\alpha) = \bigcup_{\beta\leq\gamma<\alpha} \Phi(A,\gamma).$$

Then $|\Phi(A,<\alpha)| \leq \xi$ *for all* $\alpha < \xi^+$ *and if* $E \subseteq \{\alpha < \xi^+ \mid \mathrm{cf}(\alpha) = \xi\}$ *then the following is true*: *if* $\beta \leq \alpha < \xi^+$ *and* $\beta \notin E$, *then* $\Phi(E,\beta,\alpha+1) \cong \Phi(\emptyset,\beta,\alpha+1)$.

(v) $\mathrm{inv}^1_\kappa(\Phi(A)) \neq \mathrm{inv}^1_\kappa(\Phi(A'))$ *if* $(A \triangle A') \cap \{\alpha < \xi^+ \mid \mathrm{cf}(\alpha) \geq \kappa\}$ *is stationary*, *where* $A \triangle A'$ *means the symmetric difference of* A *and* A'.

Proof. (i) Immediate by the definitions.

(ii) For all $i < \kappa$, we let τ^i be the set of those $f \in \tau$ such that for all $j < i$, $f(j) = 0$ and $f(i) \neq 0$. We let τ^κ be the set of those $f \in \tau$ such that for all $j < \kappa$, $f(j) = 0$. We order these by the induced order. Then clearly for all $i \leq \kappa$, $\tau^i \cong \tau$ and so $\tau + \tau \times \kappa^* \cong \tau$. By this and the first part of (i), the claim follows.

(iii) Immediate.

(iv) Using (i) and (ii), by an easy induction on α we can see that $\Phi(E,\beta,\alpha+1) \cong \tau$, from which the claim follows.

(v) Immediate by the definitions. ■

Let E_i, $i < \xi^+$, be such that

(i) for all $i < \xi^+$, $E_i \subseteq \{\alpha < \xi^+ \mid \mathrm{cf}(\alpha) = \xi\}$ is stationary,

(ii) for all $i < j < \xi^+$, $E_i \cap E_j = \emptyset$.

These exist by [Sh2], Appendix, Theorem 1.3(2). For all $C \subseteq \xi^+$, let $E_C = \bigcup_{i\in C} E_i$. Let $F(C) = \sum_{i<\xi^+} \Phi_i(C)$, where $\Phi_i(C)$, $i < \xi^+$, are disjoint copies of $(\Phi(E_C))^*$ (= inverse of $\Phi(E_C)$). We write $\mathcal{A}_C$ for $\mathcal{A}_{F(C)}$. Notice that $F(C)$ is κ^+-dense.

2.10. Lemma. *There are sets* $C_i \subseteq \xi^+$, $i < 2^{(\xi^+)}$, *such that for* $i \neq j$, $\mathcal{A}_{C_i} \not\cong \mathcal{A}_{C_j}$.

Proof. Since for all $C \neq D$, $\mathrm{inv}^1_\kappa(\Phi(E_C)) \neq \mathrm{inv}^1_\kappa(\Phi(E_D))$, we can prove the claim exactly as Theorem 3.11(a) of [Sh3] (the second method). ■

2.11. Lemma. *For all* $C, D \subseteq \xi^+$, $\mathcal{A}_C \equiv \mathcal{A}_D(L_{\infty,\xi^+})$.

Proof. It is enough to show that $\exists$ has a winning strategy for $G^\omega_{\xi^+}(\mathcal{A}_C, \mathcal{A}_D)$, the Ehrenfeucht–Fraisse game of length ω in which the players choose sets of power $< \xi^+$.

We write $S(C)$ for

$$\bigcup\{\mathcal{B}_i \mid i \in F(C)\} \cup \bigcup\{\mathcal{C}_i \mid i \in F(C)\} \cup \bigcup\{I_{ij} \mid i < j\}$$

and similarly for $S(D)$. Let $(S(C), \{c_i \mid i < \alpha\}, (C_i \mid i < \alpha))$ and $(S(D), \{d_i \mid i < \alpha\}, (D_i \mid i < \beta))$ be $F^{\mathbf{M}}_{\kappa}$-constructions of $\mathcal{A}_C$ and $\mathcal{A}_D$, respectively (see [Sh2]). Because $\xi^{<\kappa} = \xi$, if we choose the constructions carefully we may assume $\alpha = \beta = \xi^+$.

Let f_i be the natural one-one and onto function from $\Phi_i(C)$ to $\Phi(E_C)$. By $F(C, <\gamma)$ we mean $\bigcup_{i<\gamma} f_i^{-1}(\Phi(E_C, <\gamma))$. We write $S(C, \gamma)$ for

$$\bigcup\{\mathcal{B}_i \mid i \in F(C, <\gamma)\} \cup \bigcup\{\mathcal{C}_i \mid i \in F(C, <\gamma)\} \\ \cup \bigcup\{I_{ij} \mid i < j,\ i, j \in F(C, <\gamma)\}$$

and similarly for $S(D, \gamma)$. We write

$$A(C, \gamma) = S(C, \gamma) \cup \{c_i \mid i < \gamma\}$$

and similarly for $A(D, \gamma)$.

Assume now that the players have played n rounds and $\exists$ has played so that she has chosen $\alpha^{n-1} < \xi^+$ of cofinality κ and partial isomorphisms $f^{n-1} : A(C, \alpha^{n-1}) \to A(D, \alpha^{n-1})$ and $g^{n-1} : F(C, <\alpha^{n-1}) \to F(D, <\alpha^{n-1})$ so that

(i) $A(C, \alpha^{n-1})$ and $A(D, \alpha^{n-1})$ are $F^{\mathbf{M}}_{\kappa}$-saturated,

(ii) for all $i < j$, $i, j \in F(C, < \alpha^{n-1})$, $f^{n-1}\restriction(B_i \cup C_i)$ and $f^{n-1}\restriction I_{ij}$ are the natural isomorphisms onto $B_{g^{n-1}(i)} \cup C_{g^{n-1}(i)}$ and $I_{g^{n-1}(i)g^{n-1}(j)}$, respectively,

(iii) if $c_i \in A(C, \alpha^{n-1})$ then $C_i \subseteq S(C, \alpha^{n-1}) \cup \{c_j \mid j < i\}$, if $d_i \in A(D, \alpha^{n-1})$ then $D_i \subseteq S(D, \alpha^{n-1}) \cup \{d_j \mid j < i\}$,

(iv) $\exists$ has chosen her moves according to f^{n-1}.

Notice that by (i) and (iii), $A(C, \alpha^{n-1})$ is $F^{\mathbf{M}}_{\kappa}$-primary over $S(C, \alpha^{n-1})$ and $A(D, \alpha^{n-1})$ is $F^{\mathbf{M}}_{\kappa}$-primary over $S(D, \alpha^{n-1})$.

Let A be the move of $\forall$ in round n. By symmetry we may assume $A \subseteq \mathcal{A}_C$.

It is easy to see that for all γ there is $\delta > \gamma$ of cofinality κ such that $A(C, \delta)$ is $F^{\mathbf{M}}_{\kappa}$-saturated and if $c_i \in A(C, \delta)$ then $C_i \subseteq S(C, \delta) \cup \{c_j \mid j < i\}$. So for all γ there is $\gamma^* > \gamma$ of cofinality κ such that $A(C, \gamma^*)$ and $A(D, \gamma^*)$ are $F^{\mathbf{M}}_{\kappa}$-saturated and if $c_i \in A(C, \gamma^*)$ and $d_i \in A(D, \gamma^*)$ then $C_i \subseteq S(C, \gamma^*) \cup \{c_j \mid j < i\}$ and $D_i \subseteq S(D, \gamma^*) \cup \{d_j \mid j < i\}$, respectively.

Let $\gamma > \alpha^{n-1}$ be such that $A \subseteq A(C, \gamma)$. We let $\alpha^n = \gamma^*$. Since $\alpha^{n-1} \notin E_C \cup E_D$, by the properties of the orderings $\Phi(E')$ we can find a partial isomorphism $g^n : F(C, <\alpha^n) \to F(D, <\alpha^n)$ such that $g^{n-1} \subseteq g^n$ (apply Lemma 2.9(iv) κ times).

As in the proof of Theorem 1.13(iii) (this is the same proof as the proof of [Sh2], Theorem IV, 3.2), we see that for all $e \in A(C, \alpha^{n-1})$ there is $E \subseteq S(C, \alpha^{n-1})$ of cardinality $< \kappa$ such that $t(e, S(C)) \in F^{\mathbf{M}}_{\kappa}(E)$ and similarly for D. This implies that we can find a partial isomorphism f :

$A(C,\alpha^{n-1}) \cup S(C,\alpha^n) \to A(D,\alpha^{n-1}) \cup S(D,\alpha^n)$ such that $f^{n-1} \subseteq f$ and (ii) above holds for f.

By the definition of $A(C,\alpha^{n-1})$, $A(C,\alpha^n)$ is $F^{\mathbf{M}}_\kappa$-primary over $A(C,\alpha^{n-1}) \cup S(C,\alpha^n)$ and similarly for D. By Theorem 1.13(iii) we can find $f^n \supseteq f$ so that f^n, g^n and α^n satisfy (i)–(iii) above. So $\exists$ can continue the game by choosing her answer according to f^n. ∎

Notice that $\exists$ can modify the strategy described above, so that she can play ξ rounds without losing.

In the lemmas above we have proved:

2.12. Corollary. *Assume $\kappa > \lambda_{\mathrm{r}}(\mathbf{M})^+$ is regular and $\mathbf{M}$ has $\lambda_{\mathrm{r}}(\mathbf{M})$-dop. If $\xi > \kappa^+$ is regular and $\xi^{(\kappa^+)} = \xi$, then there are $F^{\mathbf{M}}_\kappa$-saturated models $\mathcal{A}_i$, $i < 2^{(\xi^+)}$, of cardinality ξ^+ such that for $i \neq j$, $\mathcal{A}_i \not\cong \mathcal{A}_j$ and $\mathcal{A}_i \equiv \mathcal{A}_j(L_{\infty,\xi^+})$.*

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Department of Mathematics
P.O. Box 4
00014 University of Helsinki
Finland
E-mail: thyttine@cc.helsinki

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