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L_2-characteristic classes of Maslov–Trofimov of hamiltonian systems on the Lie algebra of the upper-triangular matrices

by

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Abstract. We generalize the construction of Maslov–Trofimov characteristic classes to the case of some G-manifolds and use it to study certain hamiltonian systems.

1. At present, many examples are known of complete commutative sets of functions on symplectic manifolds. Hence there appears a natural problem of their classification. With this purpose many topological invariants of hamiltonian systems were constructed, integrable in the class of Bott integrals; there exists a classification of isoenergy surfaces of those systems; also bifurcation of Liouville tori with critical value momentum mapping has been studied. This allowed A. T. Fomenko to give a new topological invariant of one-dimensional graphs, in the case of four-dimensional symplectic manifolds (see [6]). V. V. Trofimov proposed another approach to constructing some invariants (see [16, 17]). Every Lagrangian submanifold in a symplectic space has a natural topological invariant, the so-called Maslov index, and more generally, the characteristic classes of Maslov–Arnold (see [2, 10]). In the articles [16, 17] a generalization of this construction has been given for any symplectic manifold. For some applications concerning the generalized index see [8, 12]. In this work we define and investigate a certain generalized index for some class of integrable hamiltonian systems on four-dimensional manifolds.

2. Let $N^n \subset M^{2n}$ denote a Lagrangian submanifold in the symplectic manifold M^{2n}. We consider on M^{2n} a connection Γ^i_{jk} compatible with the symplectic structure. By $C(x_0)$ we denote the set of paths in M^{2n} beginning and ending at $x_0 \in M^{2n}$. Parallel transport along a path $\gamma \in C(x_0)$ gen-

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erates a group of linear transformations of $T_{x_0}M^{2n}$, the so-called holonomy group $H_{x_0}(M^{2n})$ of the given connection. The holonomy group acts naturally on the Grassmann space $\Lambda(T_{x_0}M^{2n})$ of Lagrangian planes. Consider the reduced Grassmann space of Lagrangian planes

$$H\Lambda(T_{x_0}M^{2n}) = \Lambda(T_{x_0}M^{2n})/H_{x_0}(M^{2n}).$$

We have a natural mapping

$$f : N^n \to H\Lambda(T_{x_0}M^{2n})$$

which is generated by parallel transport along paths joining a point $x \in N^n$ to $x_0 \in M^{2n}$. This mapping induces a map in L_2-cohomology

$$f^* : L_2H^*(H\Lambda(T_{x_0}M^{2n})) \to L_2H^*(N^n).$$

We fix $k \in \mathbb{N}$. If $[f^*\omega] \in L_2H^k(N^n)$ for every $[\omega] \in L_2H^k(H\Lambda(T_{x_0}M^{2n}))$, and the value $[f^*(\omega + d\eta)]$ does not depend on η for $\eta \in L_2H^{k-1}(H\Lambda(T_{x_0}M^{2n}))$ then the map f^* is well defined. In the noncompact case it happens rather rarely. Therefore for noncompact manifolds there are strong restrictions on the spaces involved even if (M, g) is flat, where g is a metric structure.

We give appropriate examples. To control the enormous L_2-cohomology group of a metric space we use harmonic forms (see Definition 2 in Section 3).

The above construction has been done for any cohomology theory by V. V. Trofimov (see [16–19]). Next, let J be an almost complex structure on M^{2n} which agrees with the symplectic structure ω (such a J always exists; see [15]). Then for the metric occurring in the definition of L_2-cohomology, we can take $\omega(\xi, J\eta)$, and for any $a \in L_2H^*(H\Lambda(T_{x_0}M^{2n}))$ we can define the characteristic class $a(N^n) \in L_2H^*(N^n)$ of the Lagrangian submanifold $N^n \subset M^{2n}$.

In [1] A. A. Arkhangel'skiĭ has constructed completely integrable hamiltonian systems on the orbits (in general position) for the coadjoint representation of the Lie group of upper-triangular matrices using translation of argument. In the following we shall use the notation and conventions from [1]. In particular, let O^4 denote an orbit in general position for the coadjoint representation of the Lie group Υ_3 of nonsingular 3×3 upper-triangular matrices. Let $\dot{x} = \operatorname{sgrad} F(x + \lambda a)$ be the completely integrable hamiltonian system on O^4 constructed by using the translation of argument by a covector $a = (a_{ij})_{i\geq j}$ to the semi-invariant $F(x) = x_{21}x_{32} - x_{22}x_{31}$. (Here $x = (x_{ij})_{i\geq j}$ is a matrix from $O \subset G^*$, where G is a Lie algebra.) The above system has two first integrals: $F(x)$ and $F_1(x) = a_{21}x_{32} + a_{32}x_{21} - a_{22}x_{31}$ (see [1]). Assume that the covector a is in general position. Let N^2 denote a Lagrangian submanifold obtained as the intersection of level surfaces of those two first integrals. We have the following result for the two-dimensional characteristic classes.

THEOREM 1. (i) *The two-dimensional L_2-Maslov–Trofimov classes for N^2 are trivial.*

(ii) *The oriented two-dimensional L_2-Maslov–Trofimov classes for N^2 are trivial.*

In both cases, oriented and unoriented, one-dimensional classes are not well definable. The reason is that the form $f^*(d\phi)$ (where $d\phi$ is the standard 1-form on $\Lambda(\mathbb{R}^2) \simeq S^1$) cannot represent an L_2-cohomology class on N^2. For more details see Section 4 (or Section 5).

REMARK 1. (a) Note that $H = H_{x_0}(T_{x_0}O^4) = 0$, so (using the orientability of N^2) the image of the natural map

$$f : N^2 \to \Lambda(T_{x_0}O^4)$$

in fact belongs to $\Lambda_2^+ = \Lambda^+(T_{x_0}O^4)$ (the Grassmannian of oriented Lagrangian planes).

(b) It turns out that in many cases the classes defined as above do not reflect all the topological and geometrical properties of Lagrangian manifolds. So we have to modify the definition of the L_2-Maslov–Trofimov classes. In the next section we propose another possible definition generalizing the above one.

3. Let (M^{2n}, ω) denote a symplectic manifold and let μ denote the associated metric defined in terms of the form ω and an almost complex structure J on M^{2n} (see [15]). Let $G = \{\phi_s\}$ denote a group of diffeomorphisms of M^{2n}, preserving both the symplectic and metric structures. If we have a fixed Liouville foliation on M^{2n} we require that every $\phi_s \in G$ preserves it. Notice that, choosing f as in Section 2, we have the following result:

LEMMA. *Let (M^{2n}, ω) be a symplectic manifold with symplectic connection Γ_{ij}^k; suppose that Γ_{ij}^k is the Levi-Civita connection associated with some metric g_{ij}. Consider the Lagrangian submanifold $N^n \subset M^{2n}$ which is a leaf of a Liouville foliation. For any family $\{\phi_s\}$ of diffeomorphisms preserving the form ω, the metric g_{ij} and the given foliation we can define new characteristic classes of the submanifold $N^n/\{\phi_s\} \subset M^{2n}/\{\phi_s\}$. In other words, for each cohomology class $a \in H^*(H\Lambda(T_{x_0}M^{2n}))$ there is a natural characteristic class $a(N^n/\{\phi_s\}) = f^*(a) \in H^*(N^n)$.*

P r o o f. It is plain that the value of the tangent representation $N^n \ni x \to f(x) \in \Lambda(T_{x_0}M^{2n})/H(x_0)$ does not depend on a point $y \in \phi_s^{-1}(x)$ for each ϕ_s. Parallel transport, defined by the Levi-Civita connection on N^n induced from the connection Γ_{jk}^i on M^{2n}, maps the tangent space T_yN^n onto the whole tangent space T_xN^n. However, the same effect is obtained when we transport vectors of the tangent plane T_y using the metric connection of the ambient space. The assertion now follows.

Definition 1. We assume that $G = \{\phi_s\}$ is a maximal family of diffeomorphisms, preserving μ and ω. *The modified L_2-Maslov–Trofimov G-characteristic classes* are defined to be the ones (in the sense of papers [18, 19]) of the subspace N^n/G in the riemannian space M^{2n}/G (we have eliminated possible singularities which can appear in the factorization process).

Consider again the map

$$f^* : L_2H^*(H\Lambda(T_{x_0}M^{2n})) \to L_2H^*(N^n).$$

Denote by $\mathbf{H}^i$ the subspace of L_2-harmonic i-forms in $H^i(H\Lambda(T_{x_0}M^{2n}))$ for some metric in the reduced Grassmann space. Suppose that $f^*(a) \in L_2H^*(N^n)$ for every a in $\mathbf{H}^i$. We can give the following

Definition 2. The *modified L_2-harmonic Maslov–Trofimov G-characteristic classes* of the Lagrangian space N^n are those among the above defined which are images of harmonic forms.

Now let (as above) O^4 be an orbit in general position for the coadjoint representation of the group Υ_3 and let N^2 be a two-dimensional Lagrangian submanifold. Let $\{\phi_s : O^4 \to O^4\}$ denote a maximal family of diffeomorphisms. Then we have the following

Theorem 2. *The one-dimensional modified L_2-harmonic Maslov–Trofimov G-characteristic classes of N^2 are nontrivial.*

Remark 2. (a) In what follows we shall consider only some "natural" symplectic structures on our orbits; their construction will be described below. Note that there exists a deep relation between such a symplectic form and some "sectional operators" in the sense of A. T. Fomenko (see [7]). Such operators have many applications to Hamiltonian mechanics and symmetric spaces (see [7, 9]).

(b) We can also consider a mapping of the Lagrangian submanifold N^n into the Grassmann manifold $\Lambda_k(T_{x_0}M^{2n})$ of isotropic planes for $k = 1, \ldots, n$. In the case $k = n$ we obtain the Grassmann space $\Lambda(T_{x_0}M^{2n})$ of isotropic planes considered above. Vorob'ev and Karasev (see [12]) have proved:

Theorem. $H_1(\Lambda_k(T_{x_0}M^{2n})) = 0$ *for* $k = 1, \ldots, n-1$.

This theorem, among other things, is a reason why the author has introdused the modified Maslov–Trofimov classes.

In order to give the proof of Theorems 1 and 2 we first prove the following

Proposition. *For O^4 as above, there exist global symplectic coordinates in which*

$$\omega = dp_1 \wedge dq_1 + dp_2 \wedge dq_2.$$

P r o o f. The tangent space $T_fO(f)$ of the orbit $O(f)$ containing f at the point $f \in \Re^*$ has the following description:

$$T_fO(f) = \{\mathrm{ad}_g^* f : g \in \Re\},$$

where $\Re$ denotes the Lie algebra $\mathrm{Lie}(G)$ of the Lie group G (see [3]). It is plain that, in the standard basis

$$e_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ , & & \end{pmatrix}$$

$$e_4 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_5 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

of the Lie algebra $T_3 = \mathrm{Lie}(\Upsilon_3)$, the operator ad_g^* acts in the following way:

$$\mathrm{ad}_g^* f = \begin{pmatrix} -g_{12}f_{21} - g_{13}f_{31} & 0 & 0 \\ (g_{11} - g_{22})f_{21} - g_{23}f_{31} & g_{12}f_{21} - g_{23}f_{32} & 0 \\ (g_{11} - g_{33})f_{31} & g_{12}f_{31} + (g_{22} - g_{33})f_{32} & g_{13}f_{31} + g_{23}f_{32} \end{pmatrix}.$$

If we choose coordinates of the vector g in an appropriate way, we obtain the following basis of the tangent space $T_fO(f)$:

$$e_1^* = (1,0,0,0,0,0), \qquad e_2^* = (0,1,0,f_{23}/f_{31},0,-f_{23}/f_{31}),$$
$$e_3^* = (0,f_{21}/f_{31},1,0,0,0), \qquad e_4^* = (-f_{21}/f_{31},0,0,f_{21},1,0).$$

It is known (see [3]) that the Kirillov form is

$$\omega_X(\xi_1,\xi_2) = \langle X, [g_1,g_2]\rangle,$$

where $\xi_1 = \mathrm{ad}_{g_1}^* X$, $\xi_2 = \mathrm{ad}_{g_2}^* X$.

Let $\xi_1 = \sum_{i=1}^4 \lambda_i e_i^*$ and $\xi_2 = \sum_{i=1}^4 \mu_i e_i^*$. Then

$$\omega_X(\xi_1,\xi_2) = \omega_X\Big(\sum_i \lambda_i e_i^*, \sum_j \mu_j e_j^*\Big),$$
$$\sum_{i,j} \omega_X(e_i^*,e_j^*) = \sum_{i,j} \langle X, [g_i,g_j]\rangle,$$

where $e_i^* = \mathrm{ad}_{g_i}^* X$.

We have

$$g_1 = \begin{pmatrix} 0 & 0 & -1/x_{31} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad g_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1/x_{31} \\ 0 & 0 & 0 \end{pmatrix},$$

$$g_3 = \begin{pmatrix} 1/x_{31} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad g_4 = \begin{pmatrix} 0 & 1/x_{31} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

As a result we obtain

$$\begin{aligned}\omega_X(\xi_1, \xi_2) = (\lambda_1\mu_3 - \lambda_3\mu_1)x_{31} + (\lambda_3\mu_4 - \lambda_4\mu_3)x_{21} \\ + (1/x_{31})(\lambda_2\mu_4 - \lambda_4\mu_2).\end{aligned}$$

Then $\omega_X(e_1^*, e_3^*) = x_{31}$, $\omega_X(e_3^*, e_4^*) = x_{21}$, $\omega_X(e_2^*, e_4^*) = 1/x_{31}$ and $\omega_X(e_i^*, e_j^*) = 0$ for other pairs (i,j). In the canonical basis of the dual space of the algebra T_3 we have

$$\begin{aligned}
e_1^* &= \frac{\partial}{\partial f_{11}}, \\
e_2^* &= \frac{\partial}{\partial f_{21}} + \frac{f_{32}}{f_{31}}\frac{\partial}{\partial f_{22}}, \\
e_3^* &= \frac{f_{21}}{f_{31}}\frac{\partial}{\partial f_{21}} + \frac{\partial}{\partial f_{31}}, \\
e_4^* &= -\frac{f_{21}}{f_{31}}\frac{\partial}{\partial f_{11}} + \frac{f_{21}}{f_{31}}\frac{\partial}{\partial f_{22}} + \frac{\partial}{\partial f_{32}}.
\end{aligned}$$

Now we project the fields $e_1^*, \ldots, e_4^*$ (and denote the projections by the same symbols) to the plane generated by the fields $\partial/\partial u^1, \ldots, \partial/\partial u^4$ parallel to the vectors $\partial/\partial f_{22}, \partial/\partial f_{33}$, where $u^1 = f_{11}$, $u^2 = f_{21}$, $u^3 = f_{31}$, $u^4 = f_{32}$. Then

$$\begin{aligned}
e_1^* &= \frac{\partial}{\partial u^1}, & e_2^* &= \frac{\partial}{\partial u^2}, \\
e_3^* &= \frac{u^2}{u^3}\frac{\partial}{\partial u^2} + \frac{\partial}{\partial u^3}, & e_4^* &= -\frac{u^2}{u^3}\frac{\partial}{\partial u^1} + \frac{\partial}{\partial u^4}.
\end{aligned}$$

Put $\omega_{ij} = \omega(\partial/\partial u^i, \partial/\partial u^j)$. Then

$$\omega = \sum_{i<j} du^i \wedge du^j = u^3 du^1 \wedge du^3 - \frac{u^2}{(u^3)^2} du^3 \wedge du^4 + \frac{1}{u^3} du^2 \wedge du^4.$$

It is plain that

$$\begin{aligned}
\omega &= du^1 \wedge d\left(\frac{(u^3)^2}{2}\right) + u^2 d\left(\frac{1}{u^3}\right) \wedge du^4 + \frac{1}{u^3} du^2 \wedge du^4 \\
&= du^1 \wedge d\left(\frac{(u^3)^2}{2}\right) + d\left(\frac{u^2}{u^3}\right) \wedge du^4 \\
&= du^1 \wedge d\left(\frac{(u^3)^2}{2}\right) + d\left(\frac{u^2}{u^3}\right) \wedge du^4.
\end{aligned}$$

The change of variables

$$p_1 = u^1, \quad q^1 = (u^3)^2/2, \quad p_2 = u^2/u^3, \quad q^2 = u^4$$

yields the desired canonical expression for ω. It is plain that the above change of variables gives the canonical coordinates in the domains $u^3 > 0$ and $u^3 < 0$.

REMARK 3. The symplectic form considered in the Proposition is not invariant; therefore it is not a Kirillov form.

4. Proof of the theorems from Sections 2 and 3. First we study the preimage of the momentum mapping. Integrating the formula for $\mathrm{ad}_g^* f$ from the previous section we obtain the orbit $O = O(X)$ of the coadjoint action of Υ_3. It turns out that this orbit can be described by two equations in $\Re^*$:

$$x_{11} + x_{22} + x_{33} = c_1, \qquad \frac{x_{21}x_{32} - x_{22}x_{31}}{x_{31}} = c_2.$$

Now because the momentum mapping is

$$F = x_{21}x_{32} - x_{22}x_{31}, \qquad F_1 = a_{21}x_{32} + a_{32}x_{21} - a_{22}x_{31} - a_{31}x_{22},$$

its preimage $N^2 = \{F = k_1, F_1 = k_2\}$ is described as follows:

$$(2q^1)^{1/2} = k_1/c_2,$$
$$a_{21}q^2 + a_{32}p_2(2q_1)^{1/2} - \frac{c_2a_{31}}{k_1}p_2q^2 = -a_{31}c_2 + k_2 + \frac{a_{22}k_1}{c_2}.$$

Put

$$a = a_{21}, \qquad b = \frac{a_{32}k_1}{c_2}, \qquad c = -\frac{c_2a_{31}}{k_1},$$
$$C = \frac{{k_1}^2}{2{c_2}^2}, \qquad D = -a_{31}c_2 + k_2 + \frac{a_{22}k_1}{c_2}.$$

Then the surface N^2 can be given by the equations

$$q^1 = C, \qquad aq^2 + bp_2 + cp_2q^2 = D.$$

We make an additional change of variables $p_2 \to p_2 + a/c$, $q^2 \to q^2 + b/c$. In these coordinates the equations defining N^2 take the very simple form $q^1 = C$, $p_2q^2 = A$. Notice that p_1 is arbitrary.

Since the holonomy group is trivial we have the tangent mapping $f : N^2 \to \Lambda(T_{x_0}O^4)$, sending a point $x \in N^n$ to the tangent plane at $x_0 \in O$. It is easy to see that the mapping f can be factorized in the following sense. Let us represent N^2 as the product $\mathbb{R}^1 \times M^1 = \{(p_1; (p_2, q^2)) : p_1 \in \mathbb{R},\ p_2q^2 = A\}$. Each plane tangent to N^2 is the product of the fixed line $l_0 : dq^1 = 0$ and the line $l : y = -\frac{A}{p_2^2}x$ tangent to the hyperbola M^1.

Therefore we have the following commutative diagram:

$$\begin{tikzcd} f : N^2 \arrow[rr] \arrow[dr, "\widetilde{f}"'] & & \Lambda(T_{x_0}O) \\ & \Lambda(T_{x_0}\mathbb{R}^2) \arrow[ur, "i"'] & \end{tikzcd}$$

where $\widetilde{f}$ sends a point to the line l and $i(l) = l_0 \times l$. Now we have $\Lambda(T_{x_0}\mathbb{R}^2) \simeq \mathbb{R}P^1 \simeq S^1$. Therefore the two-dimensional Maslov–Trofimov classes of N^2 are trivial. This concludes the proof of Theorem 1.

Now let us recall the basic notation and definitions concerning L_2-cohomology. For any p-forms ω_1, ω_2 on an arbitrary (not necessarily symplectic) riemannian manifold (M^n, g_{ij}) we put

$$\{\omega_1, \omega_2\} = \sum \frac{1}{p!} g^{i_1 j_1} \dots g^{i_p j_p} T_{i_1 \dots i_p} S_{j_1 \dots j_p}$$

where

$$\omega_1 = \sum_{i_1 < \dots < i_p} T_{i_1 \dots i_p} dx^{i_1} \wedge \dots \wedge dx^{i_p},$$

$$\omega_2 = \sum_{j_1 < \dots < j_p} S_{j_1 \dots j_p} dx^{j_1} \wedge \dots \wedge dx^{j_p}.$$

It is plain that $\omega_1 \wedge *\omega_2 = |g|^{1/2}\{\omega_1, \omega_2\} dx^1 \wedge \dots \wedge dx^n$, where $*$ denotes the Hodge $*$ operator. We define the L_2-norm of a p-form ω as follows: $\|\omega\|^2 = \int_M \omega \wedge *\omega$. We define the L_2-cohomology of M as follows:

$$L_2 H^p(M) = \frac{\{C^\infty \cap L_2 \ p\text{-forms } \omega : d\omega = 0\}}{\{C^\infty \cap L_2 \ (p-1)\text{-forms } \eta : d\eta \in L_2\}}.$$

Now consider the (above described) tangent mapping f of the Lagrangian submanifold N^2 to the space $\mathbb{R}P^1 \simeq S^1$. Let $d\varphi$ be the standard form on S^1; it is well known that $d\varphi = d(\arctan(y/x))$. We have $\widetilde{f}^*(d\varphi) = \widetilde{\omega}$. On the other hand,

$$\widetilde{\omega} = \widetilde{f}^*(d\varphi) = \widetilde{f}^*\left\{d\left(\arctan\frac{y}{x}\right)\right\} = d\left\{\arctan\left(\frac{-A}{p_2^2}\right)\right\} = \frac{2Ap_2 dp_2}{A^2 + p_2^4}.$$

Now observe that the group of diffeomorphisms preserving the symplectic structure ω, the metric

$$\mu = d(p_1)^2 + d(p_2)^2 + d(p_3)^2 + d(p_4)^2,$$

and the Liouville foliation described above consists of the family of transformations $\varphi_s = (p_1 + s, p_2, q^1, q^2)$ together with the single transformation $\kappa(p_1, p_2, q^1, q^2) = (p_1, -p_2, q^1, -q^2)$. The 1-form $\widetilde{\omega}$ defined as above can be considered on N^2 and on its quotient $P = N^2/G$ as well.

Notice that P can be regarded as a component of the hyperbola M^1.

We now calculate the norms involved in Theorems 1 and 2, and conclude the proof of Theorem 2. It is plain that the metrics induced on P and N^2 are of the form

$$dS_P^2 = \left(1 + \frac{A^2}{p_2^4}\right) dp_2^2 \quad \text{and} \quad dS_N^2 = \left(1 + \frac{A^2}{p_2^4}\right) dp_2^2 + dp_1^2$$

respectively.

For the L_2-norm of the 1-form $\widetilde{\omega}$ we have

$$(*) \qquad \begin{aligned} \|\widetilde{\omega}\|_P^2 &= \int_P \frac{4A^2 p_2^4\, dp_2}{(p_2^4 + A^2)^{5/2}} = \int_0^\infty \frac{4A^2 p_2^4\, dp_2}{(p_2^4 + A^2)^{5/2}}, \\ \|\widetilde{\omega}\|_N^2 &= \int_{N^2} \left(1 + \frac{A^2}{p_2^4}\right)^{-1/2} \left(\frac{2Ap_2}{A^2 + p_2^4}\right)^2 dp_1 \wedge dp_2. \end{aligned}$$

Since the last integrand does not depend on p_1, the last norm is ∞. Compare the remarks after the formulation of Theorem 1.

Now we transform the first of the expressions $(*)$. Put $p_2 = x$. The integral

$$\int_0^\infty \frac{4A^2 x^4\, dx}{(x^4 + A^2)^{5/2}} = \int_0^\infty \frac{4A^2 x^4\, dx}{(A^2(x^4 + 1))^{5/2}}$$

after the change of variables $x/|A|^{1/2} = y$ takes the form

$$\begin{aligned} \int_0^\infty \frac{4A^2 y^4 A^2 |A|^{1/2}\, dy}{|A|^5 (y^4 + 1)^{5/2}} &= \int_0^\infty \frac{4y^4 dy}{|A|^{1/2} (y^4 + 1)^{5/2}} \\ &= \frac{4}{|A|^{1/2}} \int_0^\infty \frac{x^4\, dx}{(x^4 + 1)^{5/2}}. \end{aligned}$$

Now, using integration by parts, we obtain

$$\begin{aligned} \frac{4}{|A|^{1/2}} \int_0^\infty \frac{x^4 dx}{(x^4 + 1)^{5/2}} &= -\frac{1}{6} \frac{4}{|A|^{1/2}} \int_0^\infty d(x(x^4 + 1)^{-3/2}) \\ &= \frac{2}{3|A|^{1/2}} \int_0^\infty (x^4 + 1)^{-3/2}\, dx. \end{aligned}$$

In the last integral we change the variables: $x \to 1/t$, to obtain

$$\frac{2}{3|A|^{1/2}} \int_0^\infty (x^4 + 1)^{-3/2}\, dx = \frac{2}{3|A|^{1/2}} \int_0^\infty t^4 (1 + t^4)^{-3/2}\, dt.$$

By integration by parts, the last integral takes the form

$$-\frac{1}{3|A|^{1/2}} \int_0^\infty t\, d(t^4 + 1)^{-1/2} = \frac{2}{3|A|^{1/2}} \int_0^\infty t^4 (1 + t^4)^{-1/2}\, dt.$$

The last integral is of elliptic type. We now rewrite it in the standard form

$$\int \frac{R(u)\,du}{((1+u^2)(1+k^2u^2))^{1/2}}.$$

In order to do it, we note that $t^4+1=(t^2+\sqrt{2}\,t+1)(t^2-\sqrt{2}\,t+1)$ and make the change of variables $t=(\mu z+nu)/(z+1)$. We omit the standard calculations. Our integral takes the form

$$\begin{aligned}\frac{1}{3|A|^{-1/2}}\int_0^\infty t^4(1+t^4)^{-1/2}\,dt \\ &= \frac{2}{3|A|^{-1/2}}\int_{-1}^{1}\frac{(z+1)\,dz}{((z^2c+d)(z^2+c))^{1/2}} \\ &= \frac{2}{3|A|^{-1/2}}\int_{-1}^{1}\frac{(z+1)^{-1}\,dz}{\sqrt{cd}((z^2c/d+1)(z^2d/c+1))^{1/2}}.\end{aligned}$$

Put $z(c/d)^{1/2}=u$. Then $z^2c/d=u^2$, and $k=(2-\sqrt{2})/(2+\sqrt{2})$, and our integral takes the form

$$\frac{\sqrt{2}-1}{3|A|^{1/2}}\int\frac{u^{-1}\,du}{((u^2+1)(u^2k^2+1))^{1/2}}.$$

We must show that $\widetilde{\omega}$ is not exact. This follows from the equality

$$Ad\left(\arctan\frac{x^2}{A^2}\right)=\frac{2Ax\,dx}{A^2+x^2}$$

and $\int_0^\infty \arctan t\,dt=\infty$.

Our assertions now follow.

Notice that in our example the class $[f^*(\eta)]\in H^1(P)$ depends on the choice of $\eta\in[a(\phi)]\in H^1(S^1)$. Generally f^* is not well defined.

5. It turns out that using "the methods of chains of subalgebras" (see [20]) one can construct another completely integrable hamiltonian system on the Lie algebra $\mathrm{Lie}(\Upsilon_3)$. In order to do this we can consider the following chain of subalgebras (see [20]):

$$\left\{\begin{pmatrix} a_{11} & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\right\}\subset\left\{\begin{pmatrix} a_{11} & a_{12} & 0\\ 0 & a_{22} & 0\\ 0 & 0 & 0\end{pmatrix}\right\}\subset\left\{\begin{pmatrix} a_{11} & a_{12} & a_{13}\\ 0 & a_{22} & a_{23}\\ 0 & 0 & a_{33}\end{pmatrix}\right\}.$$

For the system $\dot{x}=\operatorname{sgrad}H$ one can prove theorems analogous to Theorems 1, 2 where H depends functionally on $g(x)=a_{11}x_{11}$ and $h(x)=x_{11}+x_{22}$.

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