FUNDAMENTA
MATHEMATICAE
156 (1998)

Gaps in analytic quotients

by

Stevo Todorčević (Toronto, Ont., Beograd and Bellaterra)

Abstract. We prove that the quotient algebra $\mathcal{P}(\mathbb{N})/\mathcal{I}$ over any analytic ideal $\mathcal{I}$ on $\mathbb{N}$ contains a Hausdorff gap.

Gaps of the quotient algebra $\mathcal{P}(\mathbb{N})/\text{fin}$ is a phenomenon discovered long ago by F. Hausdorff ([4], [5]). They are objects of considerable interest in a wide variety of problems. For example, a result of Kunen [14], saying that gaps of $\mathcal{P}(\mathbb{N})/\text{fin}$ discovered by Hausdorff [4] are essentially the only kind of gaps that can be built using ordinary methods, forms the crucial part of Woodin's independence proof of Kaplansky's conjecture about automatic continuity in Banach algebras (see [1]). The results of Kunen have been subsequently synthesized in [21; §8] using a principle of open colouring, OCA ([1]), which (together with a bit of MA) gives a complete description of gaps in the quotient $\mathcal{P}(\mathbb{N})/\text{fin}$ or any of the reduced powers like $\mathbb{N}^{\mathbb{N}}/\text{fin}$ or $\mathbb{R}^{\mathbb{N}}/\text{fin}$. This synthesis was the inspiration for the following result of [22], the terminology of which is given in §1 below.

Theorem 1. *Suppose A and B are two orthogonal families of subsets of $\mathbb{N}$ such that A is analytic and B is downwards closed. Then A is countably generated in $B^{\perp}$ if and only if every countable subset of B can be separated from A if and only if for every sequence of infinite members of B there is a member of $A^{\perp}$ having infinite intersection with every member of the sequence.*

It turns out that this result gives a considerable amount of information about gaps in other quotients $\mathcal{P}(\mathbb{N})/\mathcal{I}$ as well, and the purpose of this note

---

1991 *Mathematics Subject Classification*: Primary 03E15.

Partially supported by NSERC of Canada, the Science Foundation of Serbia, and the Ministry of Education and Science of Spain.

([1]) If X is a set of reals, and if K is an open subset of $[X]^2$, then either there is an uncountable $Y \subseteq X$ such that $[Y]^2 \subseteq K$, or X can be covered by a sequence $\{X_n\}$ of subsets such that $[X_n]^2 \cap K = \emptyset$ for all n.

is to give an account of this. For example, we shall show that if $\mathcal{I}$ is an analytic ideal on $\mathbb{N}$ then the quotient $\mathcal{P}(\mathbb{N})/\mathcal{I}$ contains an (ω_1, ω_1^*)-gap. (Compare this with Problem 4 of [17] originally posed by W. Just.) In fact, we shall essentially show that the gap spectrum of any analytic quotient contains that of the quotient $\mathcal{P}(\mathbb{N})/\mathrm{fin}$. This will be deduced from some quite general facts about the preservation of gaps under certain embeddings of $\mathcal{P}(\mathbb{N})/\mathrm{fin}$ into $\mathcal{P}(\mathbb{N})/\mathcal{I}$. We think that studying the embeddings between different analytic quotients is of independent interest. In fact, some such work has already been done. For example, a synthesis quite similar to that concerning the gaps mentioned above was subsequently applied to the work of S. Shelah [18] about the automorphisms of $\mathcal{P}(\mathbb{N})/\mathrm{fin}$ by B. Veličković [24], when he showed that under OCA (and a bit of MA) all automorphisms of this quotient are induced by bijections between cofinite subsets of $\mathbb{N}$. These ideas are further applied by him and W. Just (see [10]) to other problems solved by Shelah's chain condition method or its extensions (see [7], [8], [9], [11]). It is interesting that all these results are indeed results about embeddings between different analytic quotients.

1. Preliminaries. The power-set $\mathcal{P}(\mathbb{N})$ is equipped with the standard compact metric topology obtained from its identification with the Cantor cube $\{0,1\}^{\mathbb{N}}$. Thus, the topological notions like *open*, *closed*, F_σ, *analytic*, etc. can be applied to families of subsets of $\mathbb{N}$. Similarly, we shall talk about *continuous*, *Borel*, or *Baire* maps defined on $\mathcal{P}(\mathbb{N})$.

For subsets a and b of $\mathbb{N}$, let $a \perp b$ denote the fact that their intersection is finite. For $A, B \subseteq \mathcal{P}(\mathbb{N})$, let $A \perp B$ denote the fact that $a \perp b$ whenever $a \in A$ and $b \in B$. Let $A^\perp$ denote the set of all $b \subseteq \mathbb{N}$ such that $b \perp a$ for all a in A. A subset c of $\mathbb{N}$ *separates* A and B if c is orthogonal to A while its complement is orthogonal to B. We shall say that A is countably generated in some set $D \subseteq \mathcal{P}(\mathbb{N})$ if there is a countable set $C \subseteq D$ such that every a from A is almost included in some c from C.

An ideal $\mathcal{I}$ on $\mathbb{N}$ will always mean *proper* and *nonprincipal*, i.e., containing the ideal Fin of finite subsets of $\mathbb{N}$ and not equal to the power-set of $\mathbb{N}$. We shall say that an ideal $\mathcal{I}$ is *reducible* to an ideal $\mathcal{J}$ if there is $h : \mathbb{N} \to \mathbb{N}$ such that a subset y of $\mathbb{N}$ is in $\mathcal{J}$ if and only if its preimage $h^{-1}[y]$ belongs to $\mathcal{I}$. The orthogonal $\mathcal{I}^\perp$ of an ideal $\mathcal{I}$ is also an ideal which usually contains some information about $\mathcal{I}$ itself. Note, however, that $\mathcal{I}^\perp$ is seldom an analytic ideal so the following result is unexpected.

THEOREM 2. *If $\mathcal{I}$ is an analytic ideal then the ideal generated by $\mathcal{I}$ and its orthogonal $\mathcal{I}^\perp$ is reducible to* Fin *by a finite-to-one map $h : \mathbb{N} \to \mathbb{N}$.*

Proof. Note that if $\mathcal{I}$ and $\mathcal{I}^\perp$ can be separated then the ideal they generate is analytic so the conclusion follows from a theorem of Mathias

[15]. Thus, we may assume that $\mathcal{I}$ and $\mathcal{I}^\perp$ form a gap. By Theorem 1 for the pair $\mathcal{I}$ and $\mathcal{I}^\perp$ we conclude that either there is a sequence $\{c_n\} \subseteq \mathcal{I}^\perp$ of disjoint infinite sets such that no element of $\mathcal{I}^\perp$ has infinite intersection with all c_n's, or there is a sequence $\{d_n\} \subseteq \mathcal{I}^{\perp\perp}$ of disjoint infinite sets such that every element of $\mathcal{I}$ is almost covered by the union of finitely many of the d_n's. Choose a finite-to-one reduction $h : \mathbb{N} \to \mathbb{N}$ of $\mathcal{I}$ to Fin such that for all n, $h^{-1}(n)$ intersects c_i or d_i for all $i \le n$ depending on whether the above application of Theorem 1 gives us the sequence $\{c_n\} \subseteq \mathcal{I}^\perp$ or the sequence $\{d_n\} \subseteq \mathcal{I}^{\perp\perp}$. It is easily checked that for every infinite $x \subseteq \mathbb{N}$ the preimage $h^{-1}(x)$ can't be covered by two sets one from $\mathcal{I}$ and one from $\mathcal{I}^\perp$. This completes the proof.

A mapping $\Phi : \mathcal{P}(\mathbb{N})/\mathcal{I} \to \mathcal{P}(\mathbb{N})/\mathcal{J}$ will usually be given by its lifting $\Phi_* : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ related to it via the following diagram:

$$\begin{array}{ccc} \mathcal{P}(\mathbb{N}) & \xrightarrow{\Phi_*} & \mathcal{P}(\mathbb{N}) \\ \downarrow{\scriptstyle \pi_0} & & \downarrow{\scriptstyle \pi_1} \\ \mathcal{P}(\mathbb{N})/\mathcal{I} & \xrightarrow{\Phi} & \mathcal{P}(\mathbb{N})/\mathcal{J} \end{array}$$

where π_0 and π_1 are the corresponding quotient maps. Thus, when we say that $\Phi : \mathcal{P}(\mathbb{N})/\mathcal{I} \to \mathcal{P}(\mathbb{N})/\mathcal{J}$ is a *continuous*, *Borel*, or *Baire* embedding, we really mean that it has such a lifting $\Phi_* : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$.

The following general fact about Baire liftings is quite useful (see [23; p. 132]).

THEOREM 3. *If a homomorphism Φ from a quotient $\mathcal{P}(\mathbb{N})/\mathcal{I}$ into a quotient $\mathcal{P}(\mathbb{N})/\mathcal{J}$ has a Baire lifting then it also has a continuous lifting.*

Proof. For the convenience of the reader we reproduce the argument from [23; p. 132]. So let $\Phi_* : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ be a fixed Baire lifting of Φ. Then there is a sequence $\{G_i\}$ of dense open subsets of $\mathcal{P}(\mathbb{N})$ such that Φ_* is continuous on $G = \bigcap_{i=1}^\infty G_i$. Choose inductively an increasing sequence $\{n_i\}$ of integers ($n_1 = 1$) and for each i a subset $t_i \subseteq [n_i, n_{i+1})$ such that every $x \subseteq \mathbb{N}$ whose intersection with the interval $[n_i, n_{i+1})$ is equal to t_i belongs to G_j for all $j \le i$. For $\varepsilon = 0, 1$ set

$$x_\varepsilon = \bigcup_{i \equiv \varepsilon \bmod 2} t_i \quad \text{and} \quad M_\varepsilon = \bigcup_{i \equiv \varepsilon \bmod 2} [n_i, n_{i+1}).$$

Then for $\varepsilon = 0, 1$ we have $x_\varepsilon \subseteq M_\varepsilon$ and $x_\varepsilon \cup x \in G$ for every $x \subseteq M_{1-\varepsilon}$. This means that if for $\varepsilon = 0, 1$ we define $\Phi_\varepsilon : \mathcal{P}(M_\varepsilon) \to \mathcal{P}(\mathbb{N})$ by

$$\Phi_\varepsilon(x) = \Phi_*(x \cup x_{1-\varepsilon}) \setminus \Phi_*(x_{1-\varepsilon})$$

we get a continuous lifting of Φ restricted to $\mathcal{P}(M_\varepsilon)/\mathcal{I}$. Hence if we define

$\Phi_{**} : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ by

$$\Phi_{**}(x) = \Phi_0(x \cap M_0) \cup \Phi_1(x \cap M_1)$$

we get a continuous lifting of Φ. This completes the proof.

The quotient $\mathcal{P}(\mathbb{N})/\mathcal{I}$ of course has its own notion of orthogonality, naturally defined: $a \perp_{\mathcal{I}} b$ iff $a \cap b \in \mathcal{I}$. This leads to the notions analogous to the ones defined above for $\mathcal{P}(\mathbb{N})/\mathrm{fin}$. We shall say that an embedding $\Phi : \mathcal{P}(\mathbb{N})/\mathcal{I} \to \mathcal{P}(\mathbb{N})/\mathcal{J}$ *preserves gaps* (of a certain kind) if the image

$$\langle \Phi''_* A, \Phi''_* B \rangle$$

of every gap $\langle A, B\rangle$ (of that kind) in the quotient $\mathcal{P}(\mathbb{N})/\mathcal{I}$ forms a gap in the other quotient $\mathcal{P}(\mathbb{N})/\mathcal{J}$. We shall also need to distinguish two kinds of gaps in such quotients $\mathcal{P}(\mathbb{N})/\mathcal{I}$. Thus, we say that a gap (A, B) of $\mathcal{P}(\mathbb{N})/\mathcal{I}$ is a *Hausdorff gap* if A and B are both σ-directed under the inclusion modulo $\mathcal{I}$, i.e., for every countable $C \subseteq A$ (or $C \subseteq B$) there is an $a \in A$ (resp., $b \in B$) such that $x \setminus a \in \mathcal{I}$ (resp., $x \setminus b \in \mathcal{I}$) for all $x \in C$. A gap (A, B) of $\mathcal{P}(\mathbb{N})/\mathcal{I}$ is a *Rothberger gap* if one of the sides A or B is countably generated under the inclusion modulo $\mathcal{I}$.

2. Embeddings between analytic quotients. Many of the quotients which we consider here can be (or can contain) saturated algebras under certain assumptions. For example, all F_σ-quotients are saturated if CH holds (see [11; Thm. 1]). When this happens then the quotient in question contains a copy of any other such quotient. The embeddings obtained this way use in one form or another a well-ordering of the continuum and therefore are seldom topologically simple. Hence it is natural to study topological embeddings between analytic quotients or to use an alternative to CH. In this setup the quotient $\mathcal{P}(\mathbb{N})/\mathrm{fin}$ has a special place among other analytic quotients. It is in some sense minimal as it embeds in any other analytic quotient via a continuous embedding. One such embedding can be obtained using a result of Mathias [15] (see also [6] and [20]) which says that every analytic ideal $\mathcal{I}$ is reducible to Fin by a finite-to-one and onto map, i.e., there is a finite-to-one onto map $h : \mathbb{N} \to \mathbb{N}$ such that the preimage of any infinite subset of $\mathbb{N}$ does not belong to $\mathcal{I}$. Given such an h define $\Phi_h : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ by

$$\Phi_h(x) = h^{-1}[x] = \{n : h(n) \in x\}.$$

Clearly, Φ_h is a continuous map which induces an embedding of $\mathcal{P}(\mathbb{N})/\mathrm{fin}$ in $\mathcal{P}(\mathbb{N})/\mathcal{I}$. In fact, Φ_h is a continuous embedding of $\mathcal{P}(\mathbb{N})$ into $\mathcal{P}(\mathbb{N})$ which maps finite sets into finite sets and infinite sets into sets not in $\mathcal{I}$. This argument establishing the minimality of $\mathcal{P}(\mathbb{N})/\,\mathrm{Fin}$ among all analytic quotients suggests also another theme, the search for so simple liftings. For suppose we are given two ideals $\mathcal{I}$ and $\mathcal{J}$ and a Rudin–Keisler reduction $h : \mathbb{N} \to \mathbb{N}$

of $\mathcal{I}$ to $\mathcal{J}$, i.e. a (not necessarily onto) map h such that a subset x of $\mathbb{N}$ belongs to $\mathcal{J}$ if and only if its preimage $h^{-1}[x]$ belongs to $\mathcal{I}$. Consider the map $\Phi_h(x) = h^{-1}[x]$ from $\mathcal{P}(\mathbb{N})$ into $\mathcal{P}(\mathbb{N})$ defined above. Then Φ_h is a continuous lifting for a homomorphism

$$\Phi_h : \mathcal{P}(\mathbb{N})/\mathcal{I} \to \mathcal{P}(\mathbb{N})/\mathcal{J}.$$

Note that the kernel of Φ_h is the ideal generated by $\mathcal{I} \cup \{\mathbb{N} \setminus \mathrm{range}(h)\}$. Thus Φ_h is an embedding of $\mathcal{P}(\mathbb{N})/\mathcal{I}$ into $\mathcal{P}(\mathbb{N})/\mathcal{J}$ if and only if $\mathbb{N}\setminus\mathrm{range}(h)$ belongs to $\mathcal{I}$, which basically means that the mapping h is onto (or it can be replaced by an onto map without any effect). The search for liftings of the form Φ_h has been the most prominent theme in this area ever since Shelah's groundbreaking work ([18; IV]). As an example of a result of this sort we mention the following fact made explicit in Farah [3] and appearing implicitly in the previous work of Just [9], Shelah [18] and Veličković [23].

THEOREM 4. *If a homomorphism* $\Phi : \mathcal{P}(\mathbb{N})/\mathrm{fin} \to \mathcal{P}(\mathbb{N})/\mathrm{fin}$ *has a Baire lifting*, *then it has a lifting of the form* Φ_h *for some finite-to-one mapping* $h : \mathbb{N} \to \mathbb{N}$.

From this, one immediately gets the following result made explicit by Just [9] (see also [18], [23]).

THEOREM 5. *If a homomorphism* $\Phi : \mathcal{P}(\mathbb{N})/\mathrm{fin} \to \mathcal{P}(\mathbb{N})/\mathrm{fin}$ *has a Baire lifting then its kernel is an ideal generated over* Fin *by a single subset of* $\mathbb{N}$.

Thus the only analytic quotients $\mathcal{P}(\mathbb{N})/\mathcal{I}$ that can be embedded into $\mathcal{P}(\mathbb{N})/\mathrm{fin}$ by a reasonable embedding are the trivial ones, corresponding to *atomic* ideals, the ideals generated by Fin and a single subset of $\mathbb{N}$. If we do not put any restriction on the embedding, the principle of open colouring, OCA, can be used to obtain the same result:

THEOREM 6 (OCA). *If* $\mathcal{I}$ *is a nonatomic analytic ideal on* $\mathbb{N}$ *then* $\mathcal{P}(\mathbb{N})/\mathcal{I}$ *is not isomorphic to a subalgebra of* $\mathcal{P}(\mathbb{N})/\mathrm{fin}$.

Proof. This result essentially appears as Corollary 19 in Just's paper [10] which surveys the work about OCA and homomorphisms of $\mathcal{P}(\mathbb{N})/\mathrm{fin}$. Corollary 19 of [10] appears with an additional assumption (that all PCA sets of reals have the property of Baire) which turns out to be unnecessary as we shall now see.

Let $\mathcal{J}$ be an ideal generated by $\mathcal{I}$ and $\mathcal{I}^{\perp}$. By Theorem 2 we can choose a finite-to-one and onto map $h : \mathbb{N} \to \mathbb{N}$ which reduces $\mathcal{J}$ to Fin. Clearly, we may assume h is monotonic. Suppose OCA holds but there is an embedding

$$\Phi : \mathcal{P}(\mathbb{N})/\mathcal{I} \to \mathcal{P}(\mathbb{N})/\mathrm{Fin},$$

and let $\Phi_* : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ be one of its liftings. Let σ_n $(n \in \mathbb{N})$ be the enumeration of all finite 0-1-sequences, first by length and then lexicographically. For $x \in 2^{\mathbb{N}}$, set

$$a_x = h^{-1}[\{n \in \mathbb{N} : \sigma_n \subset x\}].$$

Then a_x ($x \in 2^{\mathbb{N}}$) is an uncountable "neat" (see [10], [23]) almost disjoint family of sets all positive with respect to the ideal $\mathcal{J}$. An application of OCA shows that there must be an $x \in 2^{\mathbb{N}}$ and a $\mathcal{J}$-positive subset M of a_x such that the restriction of Φ to $\mathcal{P}(M)/\mathcal{J}$ is induced by a continuous lifting, the key step in the whole argument (see [10; Theorem 11] and [23; Lemma 2.2]). Since Φ is also a homomorphism from $\mathcal{P}(\mathbb{N})/\mathrm{Fin}$ into $\mathcal{P}(\mathbb{N})/\mathrm{Fin}$, by Theorem 5, we conclude that $\mathcal{I} = \ker(\Phi)$ when restricted to M is atomic. But this means that M belongs to $\mathcal{J}$, a contradiction. This finishes the proof.

Recall that an ideal $\mathcal{I}$ on $\mathbb{N}$ is a *P-ideal* iff it is σ-directed under the inclusion modulo the Fréchet ideal, i.e., for every sequence $\{a_n\}$ of elements of $\mathcal{I}$ there is an element b of $\mathcal{I}$ such that $a_n \setminus b$ is finite for all n. Note that any Rudin–Keisler reduction $h : \mathbb{N} \to \mathbb{N}$ between two P-ideals can be replaced by a reduction which is moreover finite-to-one. So the following special case of the general problem about liftings of the form Φ_h described above is particularly interesting.

PROBLEM 1. Suppose that $\mathcal{I}$ is an analytic P-ideal on $\mathbb{N}$ and that Φ is a homomorphism from $\mathcal{P}(\mathbb{N})/\mathrm{fin}$ into $\mathcal{P}(\mathbb{N})/\mathcal{I}$ with a Baire lifting. Does Φ have a lifting of the form Φ_h for some finite-to-one function $h : \mathbb{N} \to \mathbb{N}$?

The following result related to this problem was proved recently by Farah [3].

THEOREM 7. *Suppose that $\mathcal{J}$ is an analytic P-ideal and that*

$$\Phi : \mathcal{P}(\mathbb{N})/\mathrm{Fin} \to \mathcal{P}(\mathbb{N})/\mathcal{J}$$

is a homomorphism with a Baire lifting. Then there exist a strictly increasing sequence $\{n_k\}$ of integers, a sequence $\{v_k\}$ of disjoint finite subsets of $\mathbb{N}$, and a sequence of mappings $H_k : \mathcal{P}([n_k, n_{k+1})) \to \mathcal{P}(v_k)$ such that

$$\Psi_H(x) = \bigcup_{k=1}^{\infty} H_k(x \cap [n_k, n_{k+1}))$$

defines a lifting $\Psi_H : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ of Φ.

Note that the lifting Φ_h is a special case of Ψ_H by putting $n_k = k$, $v_k = h^{-1}(k)$, $H_k(\{k\}) = v_k$ and $H_k(\emptyset) = \emptyset$. The following application of this result will be needed in §3.

THEOREM 8. *Suppose a quotient $\mathcal{P}(\mathbb{N})/\mathcal{I}$ is embeddable into a quotient $\mathcal{P}(\mathbb{N})/\mathcal{J}$ by a Baire embedding Φ. If $\mathcal{J}$ is an analytic P-ideal then so is $\mathcal{I}$.*

Proof. Since $\mathcal{I} = \ker(\Phi)$ and since Φ has a continuous lifting (Theorem 3), we conclude that $\mathcal{I}$ is analytic. Suppose $\mathcal{I}$ is not a P-ideal and fix a sequence $\{a_n\}$ of disjoint infinite elements of $\mathcal{I}$ such that no element of $\mathcal{I}$ almost includes a_n for all n. For $n \in \mathbb{N}$ let $\pi_n : a_n \to \mathbb{N}$ be the increasing onto map. Then for every $n \in \mathbb{N}$ and $x \subseteq \mathbb{N}$ we can define $a_n(x)$ to be the π_n-preimage of x, i.e. the set $\{i \in a_n : \pi_n(i) \in x\}$. Let

$$\mathcal{K} = \Big\{x \subseteq \mathbb{N} : \bigcup_{n=1}^{\infty} a_n(x) \in \langle \mathcal{I} \cup \{a_n\}^{\perp}\rangle\Big\},$$

where $\langle \mathcal{I} \cup \{a_n\}^{\perp}\rangle$ is the ideal generated by $\mathcal{I}$ and the orthogonal of $\{a_n : n \in \mathbb{N}\}$. Then $\mathcal{K}$ is an analytic ideal so by Mathias [15] there is a finite-to-one onto map $h : \mathbb{N} \to \mathbb{N}$ which reduces $\mathcal{K}$ to Fin. Let $\phi : \mathbb{N}^2 \to \mathbb{N}$ be a bijection. For $(i,j) \in \mathbb{N}^2$ set

$$b_{ij} = \bigcup_{n=1}^{i} a_n(h^{-1}(\phi(i,j))).$$

Then by the definition of $\mathcal{K}$ and the property of h one easily checks that for every infinite sequence $\{(i_l, j_l)\}$ of elements of $\mathbb{N}^2$, with $\{i_l\}$ strictly increasing, the union of $b_{i_l j_l}$ $(l \in \mathbb{N})$ is not in $\mathcal{I}$. By Theorem 7 there are a strictly increasing sequence $\{n_k\}$ of integers, a sequence $\{v_k\}$ of disjoint finite sets of integers, and a sequence $H_k : \mathcal{P}([n_k, n_{k+1})) \to \mathcal{P}(v_k)$ of mappings such that the corresponding

$$\Psi_H : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$$

is a lifting of Φ. For each $i \in \mathbb{N}$ pick an infinite set x_i such that different sets of the form b_{ij} $(i \in \mathbb{N},\ j \in x_i)$ intersect different intervals $[n_k, n_{k+1})$. For $i \in \mathbb{N}$, set

$$b_i = \bigcup_{j \in x_i} b_{ij}.$$

Clearly, $b_i \in \mathcal{I}$ for all i. Since $\mathcal{J}$ is a P-ideal there is $c \in \mathcal{J}$ such that $\Psi_H(b_i)\setminus c$ is finite for all i. By the definition of Ψ_H and the choice of x_i's, this means that for each i we can find $k_i \in x_i$ such that $\Psi_H(b_{ik_i}) \subseteq c$. It follows that $\Psi_H(\bigcup_{i=1}^{\infty} b_{ik_i}) \subseteq c$, contradicting the fact that the union $\bigcup_{i=1}^{\infty} b_{ik_i}$ is not a member of $\mathcal{I}$. This finishes the proof.

Remark. Theorem 8 is true even if we assume that there is a Borel reduction of the equivalence relation $E_{\mathcal{I}}$ on $\mathcal{P}(\mathbb{N})$ associated with $\mathcal{I}$ in the equivalence relation $E_{\mathcal{J}}$ associated with $\mathcal{J}$ (see [12]). This can easily be deduced from some results of Kechris and Louveau [13] and Solecki [19].

Following the analogy described above one can prove the following result related to Theorem 8 in the same way Theorem 6 relates to Theorem 5.

Theorem 9 (OCA). *Suppose that $\mathcal{I}$ and $\mathcal{J}$ are analytic ideals and that the quotient $\mathcal{P}(\mathbb{N})/\mathcal{I}$ is embeddable into the quotient $\mathcal{P}(\mathbb{N})/\mathcal{J}$. If $\mathcal{J}$ is a P-ideal then so is $\mathcal{I}$.*

3. Preservation of gaps. In this section we consider topological embeddings of $\mathcal{P}(\mathbb{N})/\mathrm{fin}$ into analytic quotients and investigate the preservation of the gap structure.

THEOREM 10. *If $\mathcal{I}$ is an analytic ideal on $\mathbb{N}$ then every Baire embedding Φ of $\mathcal{P}(\mathbb{N})/\mathrm{fin}$ into $\mathcal{P}(\mathbb{N})/\mathcal{I}$ preserves Hausdorff gaps.*

Proof. By Theorem 3, we may assume that Φ has a continuous lifting $\Phi_* : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$. Suppose (A, B) is a Hausdorff gap in $\mathcal{P}(\mathbb{N})/\mathrm{fin}$ but its image $(\Phi_*'' A, \Phi_*'' B)$ is not a gap in $\mathcal{P}(\mathbb{N})/\mathcal{I}$. Choose c such that

$$\Phi_*(a) \setminus c \in \mathcal{I} \quad \text{for all } a \in A, \tag{1}$$

$$\Phi_*(b) \cap c \in \mathcal{I} \quad \text{for all } b \in B. \tag{2}$$

Let

$$A_* = \{a \subseteq \mathbb{N} : \Phi_*(a) \setminus c \in \mathcal{I}\}.$$

Then A_* is an analytic set which includes A.

CLAIM. *$A_* \perp B$.*

Proof. Take $a \in A_*$ and $b \in B$. Then $\Phi_*(a \cap b)$ is equal, modulo $\mathcal{I}$, to the intersection of $\Phi_*(a)$ and $\Phi_*(b)$. By (2) and the definition of A_* we have

$$\Phi_*(a) \cap \Phi_*(b) \subseteq (\Phi_*(a) \setminus c) \cup (\Phi_*(b) \cap c) \in \mathcal{I}. \tag{3}$$

It follows that $\Phi_*(a \cap b) \in \mathcal{I}$. Since Φ_* is a lifting of an embedding of $\mathcal{P}(\mathbb{N})/\mathrm{fin}$ into $\mathcal{P}(\mathbb{N})/\mathcal{I}$, we conclude that $a \cap b$ must be finite.

Thus, A_* and B are orthogonal and A_* is analytic, which leads us to the hypothesis of Theorem 1. Since B is σ-directed every countable subset of B can be separated from A_*, so by Theorem 1 there is a countable subset C of $B^\perp$ such that every element of A_* is almost included in some element of C. Since $A \subseteq A_*$ is σ-directed under almost-inclusion there must be a single element d in C such that $a \subseteq^* d$ for all $a \in A$. This means that d separates A and B, contradicting our assumption that (A, B) is a gap in $\mathcal{P}(\mathbb{N})/\mathrm{fin}$. This completes the proof of Theorem 10.

The preservation of Rothberger gaps seems less frequent as the following example shows.

EXAMPLE 1. For $i \in \mathbb{N}$, set

$$\mathbb{N}_i = \{2^{i-1}(2j-1) : j \in \mathbb{N}\}$$

and let A be the family $\{\mathbb{N}_i\}$ of these sets. Let B be its orthogonal, i.e.,

$$B = \{b \subseteq \mathbb{N} : b \cap \mathbb{N}_i \text{ is finite for all } i\}.$$

Thus (A, B) is a typical example of a Rothberger gap. Let $\mathcal{I}_1$ be the ideal of subsets of $\mathbb{N} \times 2$ generated (over Fin) by the families

$$a \times \{0\} \quad (a \in A) \qquad \text{and} \qquad b \times \{1\} \quad (b \in B).$$

Note that the projection map $\pi : \mathbb{N} \times 2 \to \mathbb{N}$ is a finite-to-one reduction of $\mathcal{I}_1$ to the Fréchet ideal, i.e., that no π-preimage of an infinite subset of $\mathbb{N}$ is covered by finitely many sets taken from our two generating families. The corresponding map Φ_π from $\mathcal{P}(\mathbb{N})/\text{fin}$ into $\mathcal{P}(\mathbb{N} \times 2)/\mathcal{I}_1$ does not preserve the Rothberger gap (A, B) since it maps it to

$$\langle \{a \times 2 : a \in A\}, \{b \times 2 : b \in B\} \rangle,$$

which is split in $\mathcal{P}(\mathbb{N} \times 2)/\mathcal{I}_1$ by the set $\mathbb{N} \times \{1\}$.

While this example shows that Rothberger gaps may not necessarily be preserved even under the embeddings of the form Φ_h, this still leaves us the possibility of a reasonably general preservation result for certain kind of ideals on $\mathbb{N}$. We shall illustrate this by considering two classes of ideals on $\mathbb{N}$. Note that $\mathcal{I}_1$ is not a P-ideal, i.e., that $\mathcal{I}_1$ is not σ-directed under the inclusion modulo Fin. It is interesting that this class of analytic ideals preserves the gaps of $\mathcal{P}(\mathbb{N})/\text{fin}$ under all topological embeddings.

THEOREM 11. *If $\mathcal{I}$ is an analytic P-ideal then every Baire embedding Φ of $\mathcal{P}(\mathbb{N})/\text{fin}$ into $\mathcal{P}(\mathbb{N})/\mathcal{I}$ preserves all gaps.*

Proof. Let Φ_* be a continuous lifting of Φ and let (A, B) be a given gap of $\mathcal{P}(\mathbb{N})/\text{fin}$. Suppose $(\Phi_*'' A, \Phi_*'' B)$ is not a gap modulo $\mathcal{I}$ and fix a subset c on $\mathbb{N}$ which splits $\Phi_*'' A$ from $\Phi_*'' B$ modulo $\mathcal{I}$, i.e., such that

$$\Phi_*(a) \setminus c \in \mathcal{I} \quad \text{for all } a \in A, \tag{4}$$

$$\Phi_*(b) \cap c \in \mathcal{I} \quad \text{for all } b \in B. \tag{5}$$

Set

$$A_* = \{a \subseteq \mathbb{N} : \Phi_*(a) \setminus c \in \mathcal{I}\} \quad \text{and} \quad B_* = \{b \subseteq \mathbb{N} : \Phi_*(b) \cap c \in \mathcal{I}\}.$$

Then A_* and B_* are two orthogonal analytic ideals on $\mathbb{N}$. To see this fix $a \in A_*$ and $b \in B_*$. Then $\Phi_*(a \cap b)$ is equal modulo $\mathcal{I}$ to the intersection of $\Phi_*(a)$ and $\Phi_*(b)$, which belongs to $\mathcal{I}$ since it is covered by the union of $\Phi_*(a) \setminus c$ and $\Phi_*(b) \cap c$, which are members of $\mathcal{I}$ by the definitions of A_* and B_*. Note that $A \subseteq A_*$ and $B \subseteq B_*$, so (A_*, B_*) is also a gap in $\mathcal{P}(\mathbb{N})/\text{fin}$. It follows that the Claim below, together with Theorem 1, leads to a contradiction caused by our assumption that the gap (A, B) is not preserved.

CLAIM. *Both A_* and B_* are σ-directed under the inclusion modulo* Fin.

Proof. By symmetry it suffices to show only that A_* is σ-directed. Let $\mathcal{J}$ be the ideal on $\mathbb{N}$ generated by $\mathcal{I}$ and the set c. Clearly, $\mathcal{J}$ is also an analytic P-ideal. Note that by the definition of A_*, the continuous map Φ_* is also a lifting of an embedding of the quotient $\mathcal{P}(\mathbb{N})/A_*$ into the quotient $\mathcal{P}(\mathbb{N})/\mathcal{J}$. By Theorem 8 we conclude that A_* must also be a P-ideal, which was to be shown.

As explained in §2, the only embeddings of $\mathcal{P}(\mathbb{N})/\mathrm{fin}$ into $\mathcal{P}(\mathbb{N})/\mathcal{I}$ that we presently know have a lifting of the form Φ_h for some reduction $h : \mathbb{N} \to \mathbb{N}$ of $\mathcal{I}$ to the Fréchet ideal, so some readers might appreciate a direct proof of the Claim under the assumption that Φ has a lifting of the form Φ_h. To see for example that A_* is σ-directed in this case, choose a countable subset A_0 of A_*. Since $\mathcal{I}$ is a P-ideal there is an $x \in \mathcal{I}$ such that

$$(\Phi_h(a) \setminus c) \setminus x) \text{ is finite for all } a \in A_0. \tag{6}$$

This means that for each $a \in A_0$ we can find $k_a \in \mathbb{N}$ such that

$$\Phi_h(a \setminus \{1, \ldots, k_a\}) \setminus c \subseteq x \quad \text{ for all } a \in A_0. \tag{7}$$

Set

$$d = \bigcup_{a \in A_0} (a \setminus \{1, \ldots, k_a\}).$$

Clearly, every element of A_0 is almost included in d, so we are left to show that d is an element of A_*, i.e., that $\Phi_h(d) \setminus c$ is a member of $\mathcal{I}$. To see this note that

$$\Phi_h(d) \setminus c = \bigcup_{a \in A_0} \Phi_h(a \setminus \{1, \ldots, k_a\}) \setminus c \subseteq x. \tag{8}$$

This completes the proof of the Claim and of Theorem 11.

The second class of ideals which allows a general preservation result is introduced by K. Mazur in [16]. Call an ideal $\mathcal{I}$ on $\mathbb{N}$ a *Mazur ideal* if there is $h : \mathbb{N} \to \mathbb{N}$, and for each $n \in \mathbb{N}$ a monotone family $\mathcal{I}_n$ of subsets of $h^{-1}(n)$, such that

(9) $h^{-1}(n) \in \mathcal{I}$ but $h^{-1}(n) \neq x \cup y$ for any two elements $x, y \in \mathcal{I}_n$,
(10) for every $x \in \mathcal{I}$ there is $m \in \mathbb{N}$ such that $x \cap h^{-1}(n) \in \mathcal{I}_n$ for all $n \geq m$.

As pointed out in [16], this class of ideals is rather large. For example, every F_σ-ideal belongs to this class, as well as the union of a strictly increasing sequence of arbitrary ideals. Its relationship with the general class of analytic ideals is, however, still unclear. The reason for their introduction in [16] was that an adaptation of the original Hausdorff argument shows that every quotient $\mathcal{P}(\mathbb{N})/\mathcal{I}$ over such an ideal contains an (ω_1, ω_1^*)-gap. We shall now see that, in fact, much more can be said about the gap spectrum of such quotients. To state this, note that (9) and (10) in particular mean that h is a reduction of $\mathcal{I}$ to the Fréchet ideal, so the corresponding map

$$\Phi_h(x) = \bigcup_{n \in x} h^{-1}(n)$$

defines an embedding of $\mathcal{P}(\mathbb{N})/\mathrm{fin}$ into $\mathcal{P}(\mathbb{N})/\mathcal{I}$.

Theorem 12. *If $\mathcal{I}$ is a Mazur ideal and if $h : \mathbb{N} \to \mathbb{N}$ is a reduction witnessing this fact, then the corresponding embedding $\Phi_h : \mathcal{P}(\mathbb{N})/\mathrm{fin} \to \mathcal{P}(\mathbb{N})/\mathcal{I}$ preserves all gaps.*

Proof. Define $F_h : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ by

$$F_h(a) = \{n \in \mathbb{N} : h^{-1}(n) \setminus a \in \mathcal{I}_n\}.$$

Then we have the following relationships between Φ_h and F_h, which are easily checked using (9) and (10):

$$\Phi_h(a) \setminus c \in \mathcal{I} \quad \text{implies} \quad a \setminus F_h(c) \in \mathrm{Fin}, \tag{11}$$

$$\Phi_h(a) \cap c \in \mathcal{I} \quad \text{implies} \quad a \cap F_h(c) \in \mathrm{Fin}. \tag{12}$$

It follows that if A and B are two orthogonal families such that their images $\Phi_h'' A$ and $\Phi_h'' B$ are split in $\mathcal{P}(\mathbb{N})/\mathcal{I}$ by some c then $F_h(c)$ splits A and B in $\mathcal{P}(\mathbb{N})/\mathrm{fin}$. This completes the proof.

4. Gap spectra of analytic quotients. In this section we summarize the results about gap spectra of algebras $\mathcal{P}(\mathbb{N})/\mathcal{I}$ that follow from the preservation theorems proved in §2. We also mention some possible ramifications of these results.

Theorem 13. *A quotient algebra $\mathcal{P}(\mathbb{N})/\mathcal{I}$ over an analytic ideal contains any type of a Hausdorff gap which appears in $\mathcal{P}(\mathbb{N})/\mathrm{fin}$. If $\mathcal{I}$ is an analytic P-ideal or a Mazur ideal then the gap spectrum of $\mathcal{P}(\mathbb{N})/\mathcal{I}$ includes that of $\mathcal{P}(\mathbb{N})/\mathrm{fin}$.*

Corollary 14. *If $\mathcal{I}$ is an analytic ideal on $\mathbb{N}$ then $\mathcal{P}(\mathbb{N})/\mathcal{I}$ contains an (ω_1, ω_1^*)-gap.*

Corollary 15. *The quotient algebra $\mathcal{P}(\mathbb{N})/\mathcal{I}$ over any F_σ-ideal or any analytic P-ideal contains an (ω_1, ω_1^*)-gap as well as a $(\mathfrak{b}, \omega^*)$-gap.* [Here $\mathfrak{b}$ is the minimal cardinality of an unbounded subset of the reduced power $\mathbb{N}^{\mathbb{N}}/\mathrm{fin}$.]

These results can be considered as parts of an answer to the following general

Problem 2. Determine the gap spectrum of $\mathcal{P}(\mathbb{N})/\mathcal{I}$ for every analytic ideal $\mathcal{I}$ on $\mathbb{N}$.

In view of Theorem 13 and the results about the gap spectrum of $\mathcal{P}(\mathbb{N})/\mathrm{fin}$ accumulated so far (see [21; §8]) it is natural to expect that OCA, or the stronger principle PFA, might be relevant to the full solution to this problem. For example, it is natural to expect that under one of these two principles the spectrum of any analytic quotient is equal to $\{(\omega_2, \omega^*), (\omega_1, \omega_1^*), (\omega, \omega_2{}^*)\}$. The phenomenon that a quotient $\mathcal{P}(\mathbb{N})/\mathcal{I}$ is not embeddable in $\mathcal{P}(\mathbb{N})/\mathrm{fin}$ (see Theorem 6) but it has the same gap spectrum as $\mathcal{P}(\mathbb{N})/\mathrm{fin}$ is by no means automatic. For example, in [2] and [3; §22], Farah describes a situation in which $\mathcal{P}(\mathbb{N})/\mathrm{fin}$ has no well-ordered chains of length ω_2 but the analytic quotient $\mathcal{P}(\mathbb{N})/\mathcal{I}_2$, where $\mathcal{I}_2$ is the ideal generated by the families A and B of Example 1, contains an (ω_2, ω_2^*)-gap. Note that $\mathcal{I}_2$ is a Mazur ideal so its gap spectrum (properly) includes that of Fin.

We have seen above that the cardinal invariant $\mathfrak{b}$ shows up in the gap spectrum of essentially every analytic quotient, so it is natural to ask the following

Problem 3. Is there any other standard cardinal invariant of the continuum which occurs in the gap spectrum of some analytic quotient?

References

[1] H. G. Dales and W. H. Woodin, *An Introduction to Independence for Analysts*, Cambridge Univ. Press, 1987.
[2] I. Farah, *Embedding partially ordered sets into* ${}^{\omega}\omega$, Fund. Math. 151 (1996), 53–95.
[3] —, *Analytic ideals and their quotients*, Ph.D. thesis, Univ. of Toronto, 1997.
[4] F. Hausdorff, *Die Graduierung nach dem Endverlauf*, Abh. König. Sächs. Ges. Wiss. Math.-Phys. Kl. 31 (1909), 296–334.
[5] —, *Summen von* $\aleph_1$ *Mengen*, Fund. Math. 26 (1936), 241–255.
[6] S.-A. Jalali-Naini, *The monotone subsets of Cantor space, filters and descriptive set theory*, doctoral dissertation, Oxford, 1976.
[7] W. Just, *The space* $(\omega^*)^{n+1}$ *is not always a continuous image of* $(\omega^*)^n$, Fund. Math. 132 (1989), 59–72.
[8] —, *Nowhere dense P-subsets of* ω^*, Proc. Amer. Math. Soc. 106 (1989), 1145–1146.
[9] —, *A modification of Shelah's oracle chain condition with applications*, Trans. Amer. Math. Soc. 329 (1992), 325–341.
[10] —, *A weak version of AT from OCA*, in: Math. Sci. Res. Inst. Publ. 26, Springer, 1992, 281–291.
[11] W. Just and A. Krawczyk, *On certain Boolean algebras* $\mathcal{P}(\omega)/I$, Trans. Amer. Math. Soc. 285 (1984), 411–429.
[12] A. S. Kechris, *Lectures on definable group actions and equivalence relations*, in preparation.
[13] A. S. Kechris and A. Louveau, *The classification of hypersmooth Borel equivalence relations*, J. Amer. Math. Soc. 10 (1997), 215–242.
[14] K. Kunen, (κ, λ^*) *gaps under MA*, note of August 1976.
[15] A. R. D. Mathias, *A remark on rare filters*, in: Infinite and Finite Sets, A. Hajnal *et al.* (ed.), Colloq. Math. Soc. János Bolyai 10, North-Holland, 1975, 1095–1097.
[16] K. Mazur, F_σ*-ideals and* $\omega_1\omega_1^*$*-gaps in the Boolean algebra* $\mathcal{P}(\omega)/I$, Fund. Math. 138 (1991), 103–111.
[17] M. Scheepers, *Gaps in* ω^ω, in: Israel Math. Conf. Proc. 6, Bar-Ilan Univ., 1993, 439–561.
[18] S. Shelah, *Proper Forcing*, Lecture Notes in Math. 940, Springer, 1982.
[19] S. Solecki, *Analytic ideals*, Bull. Symbolic Logic 2 (1996), 339–348.

[20] M. Talagrand, *Compacts de fonctions mesurables et filtres non mesurables*, Studia Math. 67 (1980), 13–43.
[21] S. Todorčević, *Partition Problems in Topology*, Amer. Math. Soc., Providence, 1989.
[22] —, *Analytic gaps*, Fund. Math. 150 (1996), 55–66.
[23] B. Veličković, *Definable automorphisms of* $\mathcal{P}(\omega)/\mathrm{fin}$, Proc. Amer. Math. Soc. 96 (1986), 130–135.
[24] —, *OCA and automorphisms of* $\mathcal{P}(\omega)/\mathrm{fin}$, Topology Appl. 49 (1992), 1–12.

Department of Mathematics
University of Toronto
Toronto, Ontario
Canada M5S 3G3
E-mail: stevo@math.toronto.edu

Centre de Recerca Matematica
Institut D'Estudias Catalans
Apartat 50
E-08193 Bellaterra, Catalonia, Spain

Matematicki Institut
Kneza Mihaila 35
11000 Beograd, Yugoslavia

Received 21 August 1997