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Dominating analytic families

by

Anastasis Kamburelis (Wrocław)

Abstract. Let $\mathcal{A}$ be an analytic family of sequences of sets of integers. We show that either $\mathcal{A}$ is dominated or it contains a continuum of almost disjoint sequences. From this we obtain a theorem by Shelah that a Suslin c.c.c. forcing adds a Cohen real if it adds an unbounded real.

1. Introduction. Let Bor be the σ-field of Borel subsets of the real line. Define the *Random algebra* $\mathcal{R}$ as the factor algebra of Bor modulo the ideal of Lebesgue measure zero sets. Define also the *Cohen algebra* $\mathcal{C}$ as the factor algebra of Bor modulo the ideal of meagre (first category) sets. Both $\mathcal{R}$ and $\mathcal{C}$ satisfy the countable chain condition (c.c.c.).

The Cohen algebra has a simple combinatorial description: it is the unique atomless complete Boolean algebra with a countable dense subset.

A natural problem is to characterize similarly the Random algebra. This problem is not yet solved in a satisfactory way (cf. [F]). In addition to satisfying c.c.c., the Random algebra is *weakly distributive*. This means, in forcing terms, that every sequence of integers from the generic extension is *bounded* (eventually dominated) by a sequence from the ground model. On the other hand, the Cohen algebra is not weakly distributive; so it adds an *unbounded* sequence in the generic extension.

As considered by Shelah ([Sh]), instead of a characterization one may ask whether a given complete Boolean algebra $\mathcal{B}$ contains $\mathcal{R}$ or $\mathcal{C}$ as a *regular* subalgebra. Here regularity means that all maximal antichains in the subalgebra remain maximal in $\mathcal{B}$.

It would be nice to have the following dichotomy for atomless c.c.c. complete Boolean algebras $\mathcal{B}$ adding reals:

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(A) if $\mathcal{B}$ is weakly distributive then $\mathcal{R} < \mathcal{B}$;

(B) if $\mathcal{B}$ is not weakly distributive then $\mathcal{C} < \mathcal{B}$.

Unfortunately, there are *models* where (A) and (B) are false. For example, using $\diamond$ Jensen (see [J], p. 570) constructed an algebra which turns out to be a counterexample to (A). On the other hand, the Mathias forcing with the Ramsey ultrafilter is a counterexample to (B).

However, it is still unknown whether such counterexamples can be found in ZFC alone; or else: is the above dichotomy *consistent* with ZFC?

Shelah ([Sh]) asked what happens if $\mathcal{B}$ has a "simple" description. He proved that (B) is true when the Boolean algebra $\mathcal{B}$ is additionally analytic (Suslin).

The purpose of this paper is to present a proposition about analytic subsets of the space $\mathrm{P}(\omega)^\omega$ (sequences of subsets of integers). Such a subset either is dominated (in a certain sense) or contains a range of a continuum of almost disjoint sequences. From this proposition we obtain an alternative proof of Shelah's theorem.

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2. Notation and some definitions. We will use the standard terminology and notation (see e.g. [J]). Let us recall some of the less common abbreviations.

Sets. ω is the set of natural numbers and ω_1 is the first uncountable cardinal. $[\omega]^{<\omega}$ and $[\omega]^\omega$ are the sets of all finite and resp. infinite subsets of ω.

Topology. A subset of a topological space is *nowhere dense* if its closure has empty interior. It is *meagre* (or *first category*) if it can be written as a countable union of nowhere dense sets. Finally, X has the *Baire property* if the symmetric difference $X \triangle G$ is meagre for some open set G.

$\{0,1\}^\omega$ is the *Cantor space* and ω^ω is the *Baire space*. If s is a finite sequence of integers then $|s|$ is the length of s. If $n < \omega$ then $s^\frown n$ is the sequence of length $|s|+1$ extending s, with last term n. A typical basic open set in the Cantor (Baire) space is defined by $[s] = \{x : s \subset x\}$. We also consider the space $\mathrm{P}(\omega)^\omega$ of all sequences of subsets of ω with the product topology, where $\mathrm{P}(\omega)$ is identified with $\{0,1\}^\omega$. Recall that a set (a subset of some Polish space) is *analytic* ($\mathbf{\Sigma}^1_1$) if it is a continuous image of ω^ω. For more on the projective hierarchy, see e.g. [K].

Forcing. A *forcing* is a partially ordered set $(P, \le)$. Elements $p, q \in P$ are *compatible* if there is $r \in P$ such that $r \le p$ and $r \le q$; otherwise p, q are *incompatible* and we write $p \perp q$ in this case. $\mathrm{RO}(P)$ is the canonical complete Boolean algebra associated with P. The Boolean value of a formula φ is denoted by $[\![\varphi]\!]$. We write $p \Vdash \varphi$ (*p forces* φ) iff $p \leq [\![\varphi]\!]$; and $\Vdash \varphi$ means that $[\![\varphi]\!] = \mathbf{1}$.

3. Ellentuck topology. For $s, A \subseteq \omega$ write $s < A$ if $\forall m \in s\ \forall n \in A\ m < n$. Put

$$(s, A) = \{u \in [\omega]^\omega : s \subseteq u \subseteq s \cup A\}.$$

The *Ellentuck topology* (ET for short) on $[\omega]^\omega$ is the topology with neighbourhood system consisting of sets of the form (s, A), where $s \in [\omega]^{<\omega}$, $A \in [\omega]^\omega$ and $s < A$.

It is easy to check that $(s, A) \subseteq (t, B)$ iff $s \supseteq t$, $A \subseteq B$ and $s - t \subseteq B$. In forcing, this is the *definition* of the Mathias ordering (cf. [B]). It is known that ET is richer than the usual topology on $[\omega]^\omega$ (inherited from the Cantor space). A set $X \subseteq [\omega]^\omega$ is called *completely Ramsey* if for every (s, A) there is $B \in [A]^\omega$ such that either $(s, B) \subseteq X$ or $(s, B) \cap X = \emptyset$ (we then say that (s, B) *decides* X). The main result about ET ([GP], [E]) is that completely Ramsey sets are precisely sets with the Baire property in ET. Another interesting property is that in ET every meagre set is nowhere dense. The following consequence of this will be used in the proof of Lemma 4.2: if $\{D_n\}$ is a sequence of sets with the Baire property in ET and $(s, A) \subseteq \bigcup_n D_n$ then there is $(t, B) \subseteq (s, A)$ such that $(t, B) \subseteq D_n$ for some n.

Notice also the following: if $\{(t_n, B_n)\}$ is a *decreasing* sequence of neighbourhoods and $|t_n| \to \infty$ then $\bigcap_n (t_n, B_n) = \{u\}$ where $u = \bigcup_n t_n$. Finally, we shall use Silver's Theorem ([S]):

Every analytic ($\mathbf{\Sigma}^1_1$) *subset of* $[\omega]^\omega$ *is completely Ramsey.*

4. Main lemma. Let us agree that $\max \emptyset = 0$.

LEMMA 4.1. *Let* $\{D_n\}$ *be a sequence of subsets of* $[\omega]^\omega$ *with the Baire property in ET. For every* (s, A) *there is* $B \subseteq A$ *such that for every finite* $t \subseteq B$, *if* $m = \max(s \cup t)$ *then* $(s \cup t, B - (m + 1))$ *decides* D_m.

Proof. This is a standard argument (comp. [B]). Inductively construct a sequence $b_0 < b_1 < b_2 < \ldots$ of elements of A and a sequence $B_0 \supseteq B_1 \supseteq B_2 \supseteq \ldots$ of subsets of A such that $b_n < B_{n+1}$. Let $B_0 = A$. Given B_n let $t_1, \ldots, t_k$ enumerate all subsets of $\{b_i : i < n\}$. Now construct $B^n_0 \supseteq B^n_1 \supseteq \ldots \supseteq B^n_k$ as follows. Let $B^n_0 = B_n$. Given B^n_{i-1} find $B^n_i \subseteq B^n_{i-1}$ such that $(s \cup t_i, B^n_i)$ decides D_m where $m = \max(s \cup t_i)$. Finally, let $b_n = \min B^n_k$ and $B_{n+1} = B^n_k - \{b_n\}$. Let $B = \{b_n : n < \omega\}$. If now $t \subseteq B$ is finite and $m = \max(s \cup t)$, let n be minimal such that $t \subseteq \{b_i : i < n\}$. Then $t = t_i$ for some i at induction step n. It follows that $(s \cup t, B^n_k)$ decides D_m. But $(s \cup t, B - (m + 1)) \subseteq (s \cup t, B^n_k)$. ∎

Lemma 4.2. *Let* $\{D_n\}$ *be a sequence of subsets of* $[\omega]^\omega$ *with the Baire property in ET such that* $n \in \bigcap D_n$ *and* $[\omega]^\omega = \bigcap_m \bigcup_{n>m} D_n$. *For every* (s, A) *there is* $(t, B) \subseteq (s, A)$ *such that* $(t, B) \subseteq D_{\max t}$.

Proof. Let (s, A) be arbitrary. First, using Lemma 4.1 find $C \subseteq A$ such that $(s \cup t, C - (m+1))$ decides D_m if $t \subseteq C$ is finite and $m = \max(s \cup t)$. We have $(s, C) \subseteq \bigcup_{n>\max(s)} D_n$ so let $(r, D) \subseteq (s, C) \cap D_n$ for some $n > \max(s)$. Then $r \cup D \in D_n$, hence $n \in r \cup D$. By enlarging r if necessary, we can assume that actually $n \in r$. Now $r = s \cup a \cup \{n\} \cup b$ where $s < a < \{n\} < b$. Put $t = s \cup a \cup \{n\}$ and $B = C - (n + 1)$. Then $(t, B) \subseteq (s, A)$ and (t, B) decides D_n. But also $(t, B) \cap D_n \neq \emptyset$ because $r \cup D \in (t, B)$. It follows that $(t, B) \subseteq D_n$ as required. ∎

Definition 4.1. Let $\mathcal{F}$ be a family consisting of basic open sets in ET. Let us say that

- $\mathcal{F}$ is *dense* if for every (s, A) there is $(t, B) \subseteq (s, A)$ such that $(t, B) \in \mathcal{F}$;
- $\mathcal{F}$ is *semi-open* if for all $(t, B) \in \mathcal{F}$ and $C \subseteq B$ we have $(t, C) \in \mathcal{F}$.

Notice that $\mathcal{F} = \{(t, B) : (t, B) \subseteq D_{\max t}\}$ is dense and semi-open, for $\{D_n\}$ as in Lemma 4.2. The next lemma is the main result of this section. Shelah's original argument uses ramified Mathias forcing over elementary submodel.

Lemma 4.3. *Let* $\{\mathcal{F}_m\}$ *be a sequence of dense and semi-open families. There exists a sequence* $\{(t_s, B_s) : s \in \bigcup_{n<\omega} \{0, 1\}^n\}$ *such that*

1. $|t_{s^\frown\varepsilon}| > |t_s|$ *for every* s *and* $\varepsilon = 0, 1$;
2. $r \subseteq s$ *implies* $(t_s, B_s) \subseteq (t_r, B_r)$;
3. $(t_s, B_s) \in \mathcal{F}_{|s|}$;
4. *let* $S_x = \bigcup_{s \subset x} [\max t_s, \min B_s)$ *for* $x \in \{0, 1\}^\omega$; *then for* $x \neq y$ *we have* $S_x \cup S_y =^* \omega$, *i.e., the family* $\{\omega - S_x : x \in \{0, 1\}^\omega\}$ *is almost disjoint.*

Proof. We define t_s, B_s together with $C_s \in [\omega]^\omega$ by induction on $|s|$. Let $(t_\emptyset, C_\emptyset) \in \mathcal{F}_0$ be arbitrary. Assume that $(t_s, C_s) \in \mathcal{F}_m$ have been defined for all $s \in \{0, 1\}^m$. Following the lexicographic ordering of $\{0, 1\}^m$, for each $s \in \{0, 1\}^m$ do the following. First choose $n \in C_s$ greater than $\max t_r$ for all t_r *defined so far*, and let $B_s = C_s - n$. Then $(t_s, B_s) \in \mathcal{F}_m$ because $\mathcal{F}_m$ is semi-open. Next, using density, pick from $\mathcal{F}_{m+1}$ *any* two sets $(t_{s^\frown\varepsilon}, C_{s^\frown\varepsilon}) \subseteq (t_s, B_s)$ such that $|t_{s^\frown\varepsilon}| > |t_s|$ for $\varepsilon = 0, 1$. This completes the inductive definition.

Conditions 1–3 are obviously satisfied. If now $x, y \in \{0, 1\}^\omega$ and $x \neq y$ let N be such that $x|N = y|N$ and (say) $x(N) < y(N)$. Then for all $n > N$ the sets $B_{x|n}$ were defined before $B_{y|n}$. So $\min B_{y|n} > \max t_{x|n^\frown\varepsilon}$ and hence $\min B_{y|n} > \max t_{x|n+1}$. Also, $\max t_{y|n} < \min B_{x|n}$ because $t_{y|n}$ was defined at a previous stage. Finally, $\min B_{x|n} < \max t_{x|n+1}$ because $\emptyset \neq t_{x|n+1} -$

$t_{x|n} \subseteq B_{x|n}$. From those inequalities we see that $[\min B_{x|n}, \max t_{x|n+1}) \subseteq [\max t_{y|n}, \min B_{y|n})$ for all $n > N$. Thus, for $n > N$ the intervals in S_x and S_y overlap and so $S_x \cup S_y =^* \omega$. ∎

5. Dominating infinite sets

DEFINITION 5.1. For an infinite set $B \subseteq \omega$, let $e_B \in \omega^\omega$ be the canonical enumeration of B. Thus $e_B(0) = \min B$ and $n \leq e_B(n)$. For $A \subseteq \omega$ and $B \in [\omega]^\omega$ we shall write $A \preceq B$ (*B dominates A*) if

$$\exists m\ \forall n > m \quad A \cap [e_B(n), e_B(n+1)) \neq \emptyset.$$

Of course $A \preceq B$ implies that A is infinite. Note that $A \preceq B$ and $C \in [B]^\omega$ implies $A \preceq C$. Also, given arbitrary $A, C \in [\omega]^\omega$ there is $B \in [C]^\omega$ dominating A. Notice that $\preceq$ may not be transitive, but if $A \preceq B \preceq C$ and $D = \{e_C(2n) : n < \omega\}$ then $A \preceq D$. We also say that B dominates a *family* $\mathcal{A} \subseteq [\omega]^\omega$ if B dominates every $A \in \mathcal{A}$.

For two functions $f, g \in \omega^\omega$ write $f \preceq g$ (*g dominates f*) if

$$\exists m\ \forall n > m \quad f(n) \leq g(n).$$

We say that g dominates a family $\mathcal{F} \subseteq \omega^\omega$ if g dominates every $f \in \mathcal{F}$. The *bounding number* $\mathbf{b}$ is the cardinal

$$\mathbf{b} = \min\{|\mathcal{F}| : \mathcal{F} \subseteq \omega^\omega \text{ and no } g \text{ dominates } \mathcal{F}\}.$$

It turns out that the similar number for $[\omega]^\omega$ equals $\mathbf{b}$.

LEMMA 5.1. $\min\{|\mathcal{A}| : \mathcal{A} \subseteq [\omega]^\omega$ *and no B dominates* $\mathcal{A}\} = \mathbf{b}$.

Proof. Let $\mathcal{A} \subseteq [\omega]^\omega$ and $|\mathcal{A}| < \mathbf{b}$. We shall find $B \in [\omega]^\omega$ dominating $\mathcal{A}$. Let $g \in \omega^\omega$ be strictly increasing and such that $e_A \preceq g$ for every $A \in \mathcal{A}$. Let $b_0 = 0$ and $b_{n+1} = g(b_n)+1$. For $A \in \mathcal{A}$ we have $\forall^\infty n\ b_n \leq e_A(b_n) \leq g(b_n) < b_{n+1}$. It follows that $B = \{b_n : n < \omega\}$ dominates $\mathcal{A}$. Conversely, let $\mathcal{F} \subseteq \omega^\omega$ be a family consisting of strictly increasing functions and suppose that there exists $B \in [\omega]^\omega$ with $\mathrm{ran}(f) \preceq B$ for every $f \in \mathcal{F}$. We shall find g dominating $\mathcal{F}$. Fix $f \in \mathcal{F}$ and choose N such that $\mathrm{ran}(f) \cap [e_B(n), e_B(n+1)) \neq \emptyset$ for every $n \geq N$. Let M be such that $f(M) \in [e_B(N), e_B(N+1))$. Then $f(n) < f(M+n) < e_B(N+n+1)$ for every n. This shows that some shift of enumeration of B dominates f. Let $g_N(n) = e_B(N+n+1)$ and let $g \in \omega^\omega$ dominate every g_N. Then g dominates $\mathcal{F}$. ∎

6. Main result. The following is the main result of this paper. Recall that $\mathrm{P}(\omega)^\omega$ is the product space, where $\mathrm{P}(\omega)$ is identified with the Cantor space.

PROPOSITION 6.1. *Let $\mathcal{A} \subseteq \mathrm{P}(\omega)^\omega$ be analytic* ($\mathbf{\Sigma}^1_1$). *Then either*

($*$) $\exists u \in [\omega]^\omega\ \forall A \in \mathcal{A}\ \exists N > 0\ \bigcup_{i<N} A(i) \preceq u$; *or*

($**$) *there is* $f : \{0,1\}^\omega \to \mathcal{A}$ *such that for all* $x, y \in \{0,1\}^\omega$ *and* $N > 0$, *if* $x \neq y$ *then the set*

$$\bigcup_{i<N} f(x)(i) \cap \bigcup_{i<N} f(y)(i)$$

is finite.

Before we begin the proof let us make two comments. (1) In ($*$) the last condition implies that $\bigcup_{i<N} A(i)$ is infinite. But if for some $A \in \mathcal{A}$ every $\bigcup_{i<N} A(i)$ is finite then ($**$) holds trivially. (2) In ($**$) we cannot replace N by ω. To see this consider the Borel family $\{A_B : B \in [\omega]^\omega\}$, where $A_B(i) = B \cup i$.

Proof (of 6.1). Assume that ($*$) is false. We shall find a function f satisfying ($**$). Consider the set

$$E = \Big\{\langle u, A\rangle \in [\omega]^\omega \times \mathrm{P}(\omega)^\omega : \forall N > 0\ \neg\Big(\bigcup_{i<N} A(i) \preceq u\Big)\Big\}.$$

Easy computation shows that E is a Borel set. Hence, the set $E \cap ([\omega]^\omega \times \mathcal{A})$ is $\mathbf{\Sigma}^1_1$. By our assumption, for every $u \in [\omega]^\omega$ the set $\{A \in \mathcal{A} : \langle u, A\rangle \in E\}$ is nonempty. From the Jankov–von Neumann Uniformization Theorem (see [K]) there exists a $\sigma(\mathbf{\Sigma}^1_1)$-measurable function $\varphi : [\omega]^\omega \to \mathcal{A}$ such that

$$\forall u \forall N > 0 \quad \neg\Big(\bigcup_{i<N} \varphi(u)(i) \preceq u\Big).$$

For $N, n > 0$ let

$$D^N_n = \Big\{u : n \in u \text{ and } \bigcup_{i<N} \varphi(u)(i) \cap [n, n^+) = \emptyset\Big\}.$$

Here n^+ depends on u and denotes the least element of u greater than n. Again, an easy computation shows that D^N_n is in $\sigma(\mathbf{\Sigma}^1_1)$, and therefore, by Silver's Theorem, D^N_n has the Baire property in ET. Note that

$$X \not\preceq u \quad \text{iff} \quad \forall m\ \exists n > m\ (n \in u \text{ and } X \cap [n, n^+) = \emptyset).$$

So we have $\bigcap_m \bigcup_{n>m} D^N_n = [\omega]^\omega$ for all N. Let

$$\mathcal{F}_N = \{(t, B) : (t, B) \subseteq D^N_{\max t}\}.$$

Then each $\mathcal{F}_N$ is dense and semi-open (comp. Definition 4.1 and Lemma 4.2).

Let $\{(t_s, B_s) : s \in \bigcup_{n<\omega}\{0,1\}^n\}$ be a sequence from Lemma 4.3. In particular, we have $(t_s, B_s) \subseteq D^{|s|}_{\max t_s}$. This implies that

$$(t_s, B_s) \subseteq \Big\{u : \bigcup_{i<|s|} \varphi(u)(i) \cap [\max t_s, \min B_s) = \emptyset\Big\}$$

because $(\max t_s)^+ \geq \min B_s$ for $u \in (t_s, B_s)$. For $x \in \{0,1\}^\omega$ let $u_x = \bigcup_n t_{x|n}$. Then $u_x \in \bigcap_n (t_{x|n}, B_{x|n})$. Put $f(x) = \varphi(u_x)$. We claim that f is the required function for $(**)$. Let $x \neq y$ and $N > 0$. Then for $n > N$ we have

$$\bigcup_{i<N} f(x)(i) \cap [\max t_{x|n}, \min B_{x|n}) = \emptyset,$$

and similarly for y. Recalling the definition of S_x from Lemma 4.3, we have

$$\bigcup_{i<N} f(x)(i) \cap S_x =^* \emptyset \quad \text{and} \quad \bigcup_{i<N} f(y)(i) \cap S_y =^* \emptyset.$$

Hence the set

$$\bigcup_{i<N} f(x)(i) \cap \bigcup_{i<N} f(y)(i)$$

is finite. This completes the proof. ∎

COROLLARY 6.2. *The conclusion of Proposition* 6.1 *holds if* $\mathcal{A}$ *is* $\mathbf{\Sigma}^1_2$ *and* $\mathbf{b} > \omega_1$.

Proof. Every $\mathbf{\Sigma}^1_2$ set is a union of ω_1 Borel sets (cf. [J]). So write $\mathcal{A} = \bigcup_{\alpha<\omega_1} \mathcal{A}_\alpha$ with $\mathcal{A}_\alpha$ Borel. If the application of 6.1 yields $(**)$ for *some* $\mathcal{A}_\alpha$ we are done. Otherwise, for every $\alpha < \omega_1$ there exists u_α satisfying $(*)$ for $\mathcal{A}_\alpha$. By $\mathbf{b} > \omega_1$ and Lemma 5.1 we can find U dominating every u_α. Put $u = \{e_U(2n) : n < \omega\}$. It is easy to see that u works in $(*)$ for $\mathcal{A}$. ∎

7. Suslin orders

DEFINITION 7.1. Shelah calls a partial order $(P, \leq)$ a $\mathbf{\Sigma}^1_n$ *order* if

1. P is a $\mathbf{\Sigma}^1_n$ subset of $\mathbb{R}$;
2. the set $\{\langle p,q\rangle : p \leq q\}$ is a $\mathbf{\Sigma}^1_n$ subset of $\mathbb{R}\times\mathbb{R}$;
3. the set $\{\langle p,q\rangle : p \perp q\}$ is a $\mathbf{\Sigma}^1_n$ subset of $\mathbb{R}\times\mathbb{R}$.

A *Suslin forcing* is a $\mathbf{\Sigma}^1_1$ order. The following simple lemmata are included for completeness' sake. Recall that the class $\mathbf{\Sigma}^1_n$ is closed under countable unions and intersections.

LEMMA 7.1. *Let* P *be a* $\mathbf{\Sigma}^1_n$ *c.c.c. order and* $B \in \mathrm{RO}(P)$. *Both sets* $\{p \in P : pB > \mathbf{0}\}$ *and* $\{p \in P : pB = \mathbf{0}\}$ *are* $\mathbf{\Sigma}^1_n$.

Proof. First note the abuse of notation here. There exists a canonical dense homomorphism $h : P \to \mathrm{RO}(P)$ which (in case of nonseparative P) may not be one-to-one (comp. [J], p. 154). So we should write $\{p \in P : h(p)B > \mathbf{0}\}$. Nevertheless, h preserves the incompatibility: $p \perp q$ iff $h(p) \perp h(q)$ and this is all we need. For the proof write $B = \sum_n p_n$ where $\{p_n\} \subseteq P$.

This is possible from c.c.c. Now

$$p \in P \wedge pB > \mathbf{0} \leftrightarrow p \in P \wedge \exists n\, \exists r\, (r \le p \wedge r \le p_n)$$
$$p \in P \wedge pB = \mathbf{0} \leftrightarrow p \in P \wedge \forall n\, p \perp p_n$$

are both $\mathbf{\Sigma}^1_n$. ■

Lemma 7.2. *Let P be a $\mathbf{\Sigma}^1_n$ c.c.c. order and let $\{B_{n,k} : n,k < \omega\} \subseteq \mathrm{RO}(P)$. For $p \in P$ let $A_p(n) = \{k : pB_{n,k} > \mathbf{0}\}$. Then the family $\mathcal{A} = \{A_p : p \in P\}$ is a $\mathbf{\Sigma}^1_n$ subset of* $\mathrm{P}(\omega)^\omega$.

Proof. For n, k consider the set

$$W_{n,k} = \{\langle A, p\rangle : A \in \mathrm{P}(\omega)^\omega \wedge p \in P \wedge (k \in A(n) \leftrightarrow pB_{n,k} > \mathbf{0})\}.$$

Rewriting the equivalence $a \leftrightarrow b$ as $(\neg a \vee b) \wedge (a \vee \neg b)$ and using Lemma 7.1 we obtain a $\mathbf{\Sigma}^1_n$ definition of $W_{n,k}$. Now $\mathcal{A}$ is the projection of the set $\bigcap_n \bigcap_k W_{n,k}$ into $\mathrm{P}(\omega)^\omega$. ■

8. Adding Cohen reals. Shelah's Theorem

Definition. Let us say that a forcing P *adds a Cohen real* if there exists a P-name c such that $\Vdash c \in \{0,1\}^\omega$, and for every open dense subset G of $\{0,1\}^\omega$ we have $\Vdash c \in G^*$. Here G^* denotes the encoding of G in the Boolean universe (cf. [J]). We say that c is a (*name for a*) *Cohen real*.

By a standard argument, if P adds a Cohen real and $\mathcal{B} = \mathrm{RO}(P)$ then for some $a > \mathbf{0}$ from $\mathcal{B}$, the reduced Boolean algebra $\mathcal{B}|a$ contains $\mathcal{C}$ as a regular subalgebra. Shelah found the following condition $\bigotimes$ on P, which implies that P adds a Cohen real.

Definition. Let f be a P-name such that $\Vdash f \in \omega^\omega$. For $p \in P$ and $s \in \bigcup_n \omega^n$ let

$$C(p,s) = \{k < \omega : p[\![s^\frown k \subseteq f]\!] > \mathbf{0}\}.$$

Consider the *Shelah condition* $\bigotimes$:

$$\exists u \in [\omega]^\omega\ \forall p \in P\ \exists \text{ finite } F \subseteq \bigcup_{n<\omega} \omega^n$$
$$\forall m\ \exists n\ \forall i < m \bigcup_{s \in F} C(p,s) \cap [e_u(n+i), e_u(n+i+1)) \ne \emptyset.$$

Remark. The last line says that $\bigcup_{s\in F} C(p,s)$ intersects an arbitrarily large number of consecutive intervals defined by u. Note that this last fact is obviously true if u *dominates* $\bigcup_{s\in F} C(p,s)$.

Lemma 8.1 (Shelah). *If P satisfies $\bigotimes$ then P adds a Cohen real.*

Proof. Let f and u be given from $\bigotimes$. Without loss of generality $0 \in u$. For $k < \omega$ let $\varrho(k)$ be the binary expansion (¹) of the unique n such that $k \in [e_u(n), e_u(n+1))$. In the Boolean universe define a P-name c as follows:

$$c = \varrho(f(0))^\frown \varrho(f(1))^\frown \ldots$$

We claim that c is a Cohen real. Let G be dense open and let $p \in P$. Then there is a finite $F \subseteq \bigcup_n \omega^n$ from $\bigotimes$. For $s \in F$ let t_s be the concatenation of all $\varrho(s(i))$ for $i < |s|$. By density of G there is a single t such that

$$[t_s{}^\frown t] \subseteq G \quad \text{for every } s \in F.$$

Now fix $m > 2^{|t|}$ and find n from $\bigotimes$. There must be some $i < m$ such that the binary expansion of the number $n+i$ extends t. By $\bigotimes$ there exists $s \in F$ and k such that $p[\![s^\frown k \subseteq f]\!] > \mathbf{0}$ and $\varrho(k) \supseteq t$. Let $q \le p[\![s^\frown k \subseteq f]\!]$. Then $q \Vdash c \supseteq t_s{}^\frown \varrho(k) \supseteq t_s{}^\frown t$. Thus $q \Vdash c \in G^*$ as required. ∎

Definition. Let us say that P *adds an unbounded real* if there exists a P-name f such that $\Vdash f \in \omega^\omega$, and for every $g \in \omega^\omega$ (from the ground model) we have $\Vdash \neg(f \preceq g)$. We say that f is a *name for an unbounded real*.

Lemma 8.2. *Let f be a P-name for an unbounded real. For $p \in P$ consider the following tree*:

$$T_p = \Big\{ s \in \bigcup_{n<\omega} \omega^n : p[\![s \subseteq f]\!] > \mathbf{0} \Big\}.$$

Then T_p is a Miller tree, i.e., for every $t \in T_p$ there is $s \supseteq t$ such that $\{k : s^\frown k \in T_p\}$ is infinite.

Proof. Easy. Otherwise some $q \le p$ forces that f is dominated. ∎

Now we can formulate and prove Shelah's Theorem.

Theorem 8.3 (Shelah). *If P is a Suslin c.c.c. forcing and P adds an unbounded real then P adds a Cohen real.*

Proof. Let f be a P-name for an unbounded real. It suffices to prove $\bigotimes$ (with the same f). In fact, we prove the stronger form of $\bigotimes$ where u dominates the unions considered (see the remark after the definition of $\bigotimes$). We use Proposition 6.1 where we replace the exponent ω in $\mathrm{P}(\omega)^\omega$ by $W = \bigcup_n \omega^n$. For $p \in P$ and $s \in W$ let

$$A_p(s) = \{k < \omega : p[\![s^\frown k \subseteq f]\!] > \mathbf{0}\}.$$

By Lemma 7.2 the family $\mathcal{A} = \{A_p : p \in P\}$ is $\mathbf{\Sigma}^1_1$. Now we apply Proposition 6.1 to $\mathcal{A}$. If $(*)$ holds we are done. Let us show that assuming $(**)$ we

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(¹) With the least significant digits *first*.

get a contradiction. From $(**)$ we obtain $\pi : \{0,1\}^\omega \to P$ such that for any distinct $x, y \in \{0,1\}^\omega$ and for every $s \in W$ the set

$$A_{\pi(x)}(s) \cap A_{\pi(y)}(s)$$

is finite. To obtain a contradiction we show that $\{\pi(x) : x \in \{0,1\}^\omega\}$ is an antichain in P. Fix distinct $x, y \in \{0,1\}^\omega$ and assume that $p \le \pi(x), \pi(y)$. Then $T_p \subseteq T_{\pi(x)} \cap T_{\pi(y)}$. Let s be such that $\{k : s^\frown k \in T_p\} = H$ is infinite. But $H \subseteq A_{\pi(x)}(s) \cap A_{\pi(y)}(s)$. Hence the last intersection is infinite. A contradiction. ∎

COROLLARY 8.4. *If P is a $\mathbf{\Sigma}^1_2$ c.c.c. order, $\mathbf{b} > \omega_1$ and P adds an unbounded real then P adds a Cohen real.*

Proof. The same as the proof of 8.3. The family $\mathcal{A}$ is now $\mathbf{\Sigma}^1_2$ but we can use Corollary 6.2. ∎

There are also some cardinality versions of the above facts. Namely, consider the following definition suggested by Shelah's condition $\bigotimes$.

DEFINITION. For infinite $A, B \subseteq \omega$ write $A \preceq^* B$ if

$$\forall m\ \exists n\ \forall i < m \quad A \cap [e_B(n+i), e_B(n+i+1)) \ne \emptyset.$$

If $\mathcal{A} \subseteq [\omega]^\omega$ then write $\mathcal{A} \preceq^* B$ if $A \preceq^* B$ for every $A \in \mathcal{A}$. Finally, let

$$\mathbf{b}^* = \min\{|\mathcal{A}| : \mathcal{A} \subseteq [\omega]^\omega \text{ and for no } B\ (\mathcal{A} \preceq^* B)\}.$$

LEMMA 8.5. *If $|P| < \mathbf{b}^*$ and P adds an unbounded real then P adds a Cohen real.*

Proof. Let f be a name for an unbounded real. For every $p \in P$ fix s_p such that $C(p, s_p)$ is infinite. This is possible by Lemma 8.2. Now let $u \in [\omega]^\omega$ be such that $C(p, s_p) \preceq^* u$ for every $p \in P$. This clearly proves the condition $\bigotimes$ and consequently P adds a Cohen real. ∎

The remark after the definition of $\bigotimes$ says now that $\mathbf{b} \le \mathbf{b}^*$. Let us show that $\mathbf{b}^*$ may be large independently of $\mathbf{b}$. Let $\mathbf{cov}(\mathcal{M})$ be the least cardinal κ such that the real line can be covered by κ meagre sets.

LEMMA 8.6. $\mathbf{cov}(\mathcal{M}) \le \mathbf{b}^*$.

Proof. Just note that for a Cohen real C (treated as a subset of ω) we have $A \preceq^* C$ for every set A from the ground model. ∎

From this we get the following corollary which simplifies the proof from [RS].

COROLLARY 8.7. *If $|P| < \max\{\mathbf{b}, \mathbf{cov}(\mathcal{M})\}$ and P adds an unbounded real then P adds a Cohen real.*

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Department of Mathematics
University of Wrocław
Pl. Grunwaldzki 2/4
50-384 Wrocław, Poland
E-mail: akamb@math.uni.wroc.pl

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