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Hyperconvexity of $\mathbb{R}$-trees

by

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Abstract. It is shown that for a metric space (M,d) the following are equivalent: (i) M is a complete $\mathbb{R}$-tree; (ii) M is hyperconvex and has unique metric segments.

1. Introduction. The purpose of this paper is not so much to shed light on $\mathbb{R}$-trees as to provide an example of an interesting class of hyperconvex metric spaces. However, it does show that any complete $\mathbb{R}$-tree can be viewed as a nonexpansive retract of a Banach space, and this in turn suggests a new approach to the study of fixed point theory in $\mathbb{R}$-trees.

For a metric space (M,d) we use $B(x;r)$ to denote the *closed* ball centered at x with radius $r \geq 0$; thus $B(x;r) = \{z \in M : d(x,z) \leq r\}$.

Definition 1.1. A metric space (M,d) is said to be *hyperconvex* if

$$\bigcap_{\alpha} B(x_\alpha; r_\alpha) \neq \emptyset$$

for any collection $\{B(x_\alpha; r_\alpha)\}$ of closed balls in M for which $d(x_\alpha, x_\beta) \leq r_\alpha + r_\beta$.

Hyperconvex metric spaces were introduced by Aronszajn and Panitchpakdi in [1], where it is shown that such spaces are injective. Specifically, M is hyperconvex iff given any metric space Y with subspace X, any nonexpansive mapping $f : X \to M$ has a nonexpansive extension $\widetilde{f} : Y \to M$. (Recall that a mapping $f : X \to M$ is *nonexpansive* if $d(f(x), f(y)) \leq d(x,y)$ for $x, y \in X$.) In particular, a hyperconvex space is a nonexpansive retract of any metric space which contains it metrically. It is also known that every metric space can be isometrically embedded in a hyperconvex space, its so-called *injective hull* (see [4]). For other facts about hyperconvex spaces see, e.g., [2], [10], [16].

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$\mathbb{R}$-trees were introduced by J. Tits in [17]. In an ordinary tree the metric is not often stressed since all the edges are assumed to have the same length. An $\mathbb{R}$-tree is a generalization of an ordinary tree which allows for different length edges, thus enriching the behavior of free actions on $\mathbb{R}$-trees (see, e.g., [13], [12]).

Definition 1.2. An $\mathbb{R}$-tree is a nonempty metric space M satisfying:

(a) Any two points of $p, q \in M$ are joined by a unique metric segment $[p, q]$.

(b) If $p, q, r \in M$ then $[p, q] \cap [p, r] = [p, w]$ for some $w \in M$.

(c) If $p, q, r \in M$ and $[p, q] \cap [q, r] = \{q\}$ then $[p, q] \cup [q, r] = [p, r]$.

In this note we show that a complete $\mathbb{R}$-tree is hyperconvex. Among other things this fact provides a connection between $\mathbb{R}$-trees and the abstract hyperconvexity of [5]. We also show that a hyperconvex metric space which has unique metric segments is an $\mathbb{R}$-tree.

2. Preliminaries. We begin with the relevant notation. Let (M, d) be a metric space and let $A \subseteq M$ be nonempty and bounded. Set

$$\operatorname{cov}(A) = \bigcap \{B : B \text{ is a closed ball and } A \subseteq B\}.$$

Let $\mathcal{A}(M) := \{D \subseteq M : D = \operatorname{cov}(D)\}$. Thus $\mathcal{A}(M)$ denotes the collection of all *admissible sets* (ball intersections) in M.

For $D \in \mathcal{A}(M)$, let $\operatorname{diam}(D) = \sup\{d(x, y) : x, y \in D\}$, and let $r(D) = \inf\{r_x(D) : x \in D\}$ where $r_x(D) = \sup\{d(x, y) : y \in D\}$. Thus $r(D)$ denotes the radius of the smallest ball (if one exists) which contains D and whose center lies in D.

$\mathcal{A}(M)$ is said to be *normal* if for each $D \in \mathcal{A}(M)$ for which $\operatorname{diam}(D) > 0$, we have $r(D) < \operatorname{diam}(D)$, and $\mathcal{A}(M)$ is said to be *uniformly normal* if there exists $c \in \left[\frac{1}{2}, 1\right)$ such that for each $D \in \mathcal{A}(M)$ for which $\operatorname{diam}(D) > 0$, we have $r(D) \leq c \operatorname{diam}(D)$.

Finally, $\mathcal{A}(M)$ is said to be *compact* [resp., *countably compact*] if every family [resp., countable family] of nonempty sets in $\mathcal{A}(M)$ which has the finite intersection property has nonempty intersection. (The intersection of such a family is necessarily also a member of $\mathcal{A}(M)$.)

The proof of our main result is rather simple, but one implication hinges on the following somewhat deeper facts. The first is due to Khamsi [6] and the second to Kulesza and Lim [9].

Theorem 2.1. *Let M be a complete metric space for which $\mathcal{A}(M)$ is uniformly normal. Then $\mathcal{A}(M)$ is countably compact.*

Theorem 2.2. *Let M be a metric space for which $\mathcal{A}(M)$ is countably compact and normal. Then $\mathcal{A}(M)$ is compact.*

3. Results. We begin with the following.

PROPOSITION 3.1. *If M is an $\mathbb{R}$-tree, then $\mathcal{A}(M)$ is uniformly normal.*

Proof. Let $\varepsilon \in (0,1)$. For $D \in \mathcal{A}(M)$ with $\delta := \operatorname{diam}(D) > 0$, select $u, v \in D$ such that $d(u,v) > (1-\varepsilon)\delta$, and let $x \in D$ be arbitrary. By (b) of Definition 1.2 there exists $w \in [u,v]$ such that $[u,v] \cap [u,x] = [u,w]$. In particular, $d(x,u) = d(x,w) + d(w,u) \le \delta$ and $d(x,v) = d(x,w) + d(w,v) \le \delta$. Suppose m is the midpoint of $[u,v]$. If $w \in [u,m]$ then

$$\delta \ge d(x,w) + d(w,m) + d(m,v) > d(x,m) + \tfrac{1}{2}(1-\varepsilon)\delta$$

and it follows that $d(x,m) \le \frac{1}{2}(1+\varepsilon)\delta$. Similarly the same conclusion follows if $w \in [v,m]$. Thus $D \subseteq B\big(m; \frac{1}{2}(1+\varepsilon)\delta\big)$. Since any closed ball in M contains the segment joining any two of its points (this also is a simple consequence of (b) of Definition 1.2), and since $D \in \mathcal{A}(M)$, we have $m \in D$. Therefore $r(D) \le \frac{1}{2}(1+\varepsilon)\delta$. Since $\varepsilon > 0$ is arbitrary we conclude that $\mathcal{A}(M)$ is uniformly normal with constant $c = \frac{1}{2}$.

THEOREM 3.2. *For a metric space M the following are equivalent*:

(i) *M is a complete $\mathbb{R}$-tree.*
(ii) *M is hyperconvex and has unique metric segments.*

Proof. (i)$\Rightarrow$(ii). We first show that if $\{B(x_i; r_i) : i = 1, \ldots, n\}$ is an arbitrary finite collection of closed balls in an $\mathbb{R}$-tree M, any two of which intersect, then

$$\bigcap_{i=1}^{n} B(x_i; r_i) \ne \emptyset.$$

We proceed by induction on n. The conclusion is trivial if $n = 2$. Suppose that for fixed $n \ge 2$ each family of n balls, any two of which intersect, has nonempty intersection, and suppose that any two balls of the family $\{B(x_i; r_i) : i = 1, \ldots, n+1\}$ intersect. Then by the inductive hypothesis $S := \bigcap_{i=1}^{n} B(x_i; r_i) \ne \emptyset$. Now suppose $B(x_{n+1}; r_{n+1}) \cap S = \emptyset$ and let $p \in S$. Since $x_{n+1} \notin S$, we have $d(x_{n+1}, p) > r_{n+1}$. Let t be the point of $[x_{n+1}, p]$ for which $d(x_{n+1}, t) = r_{n+1}$ (thus $t \in B(x_{n+1}; r_{n+1})$), and let $i \in \{1, \ldots, n\}$. There are two cases:

(I) $t \notin [x_i, p]$. In this case $[x_i, t] \cap [x_{n+1}, t] = \{t\}$, so by (c) of Definition 1.2, we have $t \in [x_{n+1}, x_i]$ and therefore t is the point of $B(x_{n+1}; r_{n+1})$ nearest to x_i; hence $t \in B(x_i; r_i)$ by the binary intersection property.

(II) $t \in [x_i, p]$. In this case $d(x_i, t) \le d(x_i, p) \le r_i$ so again $t \in B(x_i; r_i)$.

Therefore $t \in B(x_i; r_i)$ in either case, so $t \in \bigcap_{i=1}^{n+1} B(x_i; r_i)$, completing the induction.

Now suppose M is a complete $\mathbb{R}$-tree. Since M is metrically convex, to see that M is hyperconvex it need only be shown that $\bigcap_{\alpha \in A} B(x_\alpha; r_\alpha) \ne \emptyset$

whenever $\{B(x_\alpha; r_\alpha)\}_{\alpha\in A}$ is any family of closed balls in M any two of which intersect. However, if any two balls in such a family intersect then by what we have seen above the family $\{B(x_\alpha; r_\alpha)\}_{\alpha\in A}$ has the finite intersection property. Also, $\{B(x_\alpha; r_\alpha)\}_{\alpha\in A}$ is a subfamily of $\mathcal{A}(M)$, and by Proposition 3.1, $\mathcal{A}(M)$ is uniformly normal. Thus, since M is complete, $\mathcal{A}(M)$ is compact by Theorems 2.1 and 2.2, so any subfamily of $\mathcal{A}(M)$ which has the finite intersection property must have nonempty intersection. Therefore $\bigcap_{\alpha\in A} B(x_\alpha; r_\alpha) \neq \emptyset$, proving (i)⇒(ii).

We now show (ii)⇒(i). Suppose M is hyperconvex and suppose any two points $p, q \in M$ are joined by a unique metric segment $[p, q]$. We need show that (b) and (c) of Definition 1.2 hold. To see that (b) holds, suppose $w \in [p, q] \cap [p, r]$. Then by uniqueness of metric segments it must be the case that $[p, w] \subseteq [p, q] \cap [p, r]$. It follows that $[p, q] \cap [p, r] = [p, w]$ where w is the point of $[p, q] \cap [p, r]$ which is nearest to q.

To see that (c) holds, suppose $[p, q] \cap [q, r] = \{q\}$, and without loss of generality assume $d(q, r) \le d(q, p)$. Let r' denote the point of $[r, p] \cap [r, q]$ which is nearest to q. If $[p, r'] = [p, q] \cup [q, r']$ then it follows that $[p, r] = [p, q] \cup [q, r]$ (by transitivity of metric betweenness [3, p. 33]) and there is nothing to prove. So we assume $[p, r'] \neq [p, q] \cup [q, r']$. (Thus $r' = r$ is possible, but $r' \neq q$.) It follows that $d(p, r') < d(p, q) + d(q, r')$ and therefore

$$\varrho := d(p, q) + d(q, r') - d(p, r') > 0.$$

Now let m denote the midpoint of $[q, r']$ and consider the family

$$B_1 = B\big(q; \tfrac{1}{2}d(q, r')\big),\ \ B_2 = B\big(r'; \tfrac{1}{2}d(q, r')\big),\ \ B_3 = B\big(p; d(p, q) - \tfrac{1}{2}d(q, r')\big).$$

By uniqueness of segments $B_1 \cap B_2 = \{m\}$. For the same reason $B_1 \cap B_3$ consists of exactly one point which lies on $[p, q]$. It follows that $B_2 \cap B_3 = \emptyset$ for otherwise, by the binary intersection property, $m \in \bigcap_{i=1}^3 B_i$; hence $m \in [p, q]$ and since $m \neq q$ this contradicts $[p, q] \cap [q, r] = \{q\}$. Therefore

$$d(p, r') > \tfrac{1}{2}d(q, r') + \big(d(p, q) - \tfrac{1}{2}d(q, r')\big) = d(p, q) \ge d(q, r').$$

Now let

$$B_1' = B(q; \varrho), \quad B_2' = B(r'; d(p, r') - d(p, q)), \quad B_3' = B(p; d(p, q)).$$

Since $d(p, r') > d(p, q)$, we have $d(p, r') - d(p, q) > 0$ and therefore $B_2' \cap B_3' = \{z_1\}$ where $z_1 \in [p, r']$. Also, $B_1' \cap B_2' = \{z_2\}$ where $z_2 \in [q, r']$. Therefore (since $q \in B_1' \cap B_3'$) the family $\{B_1', B_2', B_3'\}$ has the binary intersection property. Thus $\bigcap_{i=1}^3 B_i' \neq \emptyset$, which implies $z_1 = z_2$. This in turn implies $z_1 \in [r, p] \cap [r, q]$, contradicting the definition of r'.

Several facts about $\mathbb{R}$-trees can now be derived from known facts about hyperconvex spaces. For example if S is a closed subtree of a complete $\mathbb{R}$-tree M then it is easy to see that for each point $x \in M$ there is a unique point

$p(x) \in S$ which is nearest to x, and moreover that the mapping $x \mapsto p(x)$ is nonexpansive. However, since S itself is hyperconvex, this can now be viewed as a special case of the well-known fact that a hyperconvex space is a nonexpansive retract of *any* space in which it is isometrically embedded. (See, e.g., [16] for a discussion.)

The next fact has been known for some time (cf. Mańka [11]), but the retraction approach via hyperconvexity seems to be entirely new.

COROLLARY 3.3. *Let M be a compact $\mathbb{R}$-tree. Then every continuous mapping $f : M \to M$ has a fixed point.*

Proof ([7]). It is well known that any complete metric space is isometric with a subset of a Banach space, and any hyperconvex space is a nonexpansive retract of any space in which it is isometrically imbedded. Thus, regarding M as a closed subset of a Banach space, there is a nonexpansive retraction r of $\overline{\mathrm{conv}}(M)$ onto M. By the Schauder fixed point theorem $f \circ r$ has a fixed point which necessarily lies in M and must therefore be a fixed point of f.

The following is a special case of known results for nonexpansive mappings.

COROLLARY 3.4. *Let M be a bounded and complete $\mathbb{R}$-tree and let $f : M \to M$ be nonexpansive. Then f has a nonempty fixed point set which is a closed subtree of M.*

Proof. It is known ([14], [15]) that every bounded hyperconvex metric space M has the fixed point property for nonexpansive self-mappings. (Indeed, any commuting family of nonexpansive self-mappings of M has a nonempty common fixed point set [2].) The final assertion of the corollary is a consequence of the fact that if p and q are in the fixed point set of f then $[p, q]$ is as well.

In fact a little more can be said. (Here ∂S denotes the boundary of S in the usual topological sense.)

COROLLARY 3.5. *Let M be a complete $\mathbb{R}$-tree and let S be a closed and bounded subtree of M. Suppose $f : S \to M$ is nonexpansive and suppose $f(\partial S) \subseteq S$. Then f has a nonempty fixed point set* (*which is a closed subtree of S*).

Proof. The analog holds for hyperconvex spaces [8].

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