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X-minimal patterns and a generalization of Sharkovskiĭ's theorem

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Abstract. We study the law of coexistence of different types of cycles for a continuous map of the interval. For this we introduce the notion of eccentricity of a pattern and characterize those patterns with a given eccentricity that are simplest from the point of view of the forcing relation. We call these patterns X-minimal. We obtain a generalization of Sharkovskiĭ's Theorem where the notion of period is replaced by the notion of eccentricity.

0. Introduction. The question of coexistence of different types of cycles (or periodic orbits) arises in the theory of discrete dynamical systems. In dimension one, pattern seems to be the finest relevant classification of cycles. A law of coexistence of different patterns, now usually called the forcing relation, has been proved (see [B], [ALM]). However, the exact structure of this relation is not known. Better results can be obtained if one defines types as larger collections of patterns. The period of a pattern is the first natural thing that can determine a type. This was used by Sharkovskiĭ in his powerful theorem. To state it we need to introduce the Sharkovskiĭ ordering:

$$3 \succ 5 \succ 7 \succ \ldots \succ 2\cdot 3 \succ 2\cdot 5 \succ 2\cdot 7 \succ \ldots \succ 2^2\cdot 3 \succ 2^2\cdot 5 \succ 2^2\cdot 7 \succ \ldots \succ 2^\infty \succ \ldots \succ 2^3 \succ 2^2 \succ 2 \succ 1.$$

The terms "pattern" and "forces" used here will be defined later.

Sharkovskiĭ's Theorem ([S]). (i) *A pattern with period m forces some pattern with period n for any $n \in \mathbb{N}$ such that $m \succ n$.*

(ii) *For any $m \in \mathbb{N} \cup \{2^\infty\}$ there is a continuous map $f : I \to I$ such that it has a cycle of period $n \in \mathbb{N}$ if and only if $m = n$ or $m \succ n$.*

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The aim of this paper is to get better results about the structure of the forcing relation. In order to achieve this we consider a different notion of "type" of a pattern. We consider the position of a fixed point whose existence is implied by the pattern, or more precisely the ratio of the number of points of the cycle on each side of such a fixed point. Patterns with the same ratio will be said to be of the same type. Our main aim is to find which patterns of a given type are simplest in terms of the forcing relation. Let us state this in a more rigorous way. The terminology used here is mainly the same as in [ALM].

Let $P = \{p_1, \ldots, p_n\} \subset \mathbb{R}$ and $\varphi : P \to P$. Then (P, φ) is a *periodic orbit* (or *cycle*) if φ is a cyclic permutation of P. We will usually omit φ and simply say that P is a cycle. The *period* of a cycle P is $\mathrm{per}(P) = n$.

Two periodic orbits $(P, \varphi), (Q, \psi)$ are *equivalent* if there exists a homeomorphism $h : \mathrm{conv}(P) \to \mathrm{conv}(Q)$ such that $h(P) = Q$ and $\psi \circ h|_P = h \circ \varphi|_P$. An equivalence class of this relation will be called a *pattern*. If A is a pattern and $(P, \varphi) \in A$ we say that the cycle P has pattern A (or P is a representative of A) and we will use the symbol $[P]$ to denote the pattern A. The *period* of the pattern A is $\mathrm{per}(A) = \mathrm{per}(P)$.

We consider the space $C(I, I)$ of all continuous maps $f : I \to I$, where I is a closed interval. A function $f \in C(I, I)$ *has a cycle* (P, φ) if $f|_P = \varphi$. We then say that f *exhibits the pattern* $[P]$. Now we can define the forcing relation between patterns.

Definition. A pattern A *forces* a pattern B if all maps in $C(I, I)$ exhibiting A also exhibit B.

We have the following information about the forcing relation:

Theorem ([B], [ALM]). *The forcing relation is a partial order.*

Now we will define our notion of "type" of a pattern. Let $(P = \{p_1, \ldots, p_n\}, \varphi)$ be a cycle with spatial labeling (so $p_1 < \ldots < p_n$). If

$$(p_i - \varphi(p_i)) \cdot (p_{i+1} - \varphi(p_{i+1})) < 0 \tag{$*$}$$

then any continuous function with cycle P has a fixed point in the open interval (p_i, p_{i+1}). On the other hand, if $(*)$ is not true, then there is a function with cycle P that has no fixed point in (p_i, p_{i+1}). Hence we can give the following

Definition. A cycle (P, φ) has *eccentricity* $r \in \mathbb{Q}$ if for any map $f \in C(I, I)$ with P there is a fixed point $c \in \mathrm{Fix}(f)$ such that

$$\frac{\#\{x \in P : x \le c\}}{\#\{x \in P : x \ge c\}} = r.$$

Note that a cycle $(h(P), h \circ \varphi \circ h^{-1})$ where $h(x) = -x$ has eccentricity $\frac{1}{r}$

and so we define the eccentricity of a pattern $[P]$ as an eccentricity of a representative whose eccentricity is not smaller than one.

Remark. Note that a pattern (or cycle) can have more than one eccentricity (see Fig. 1).

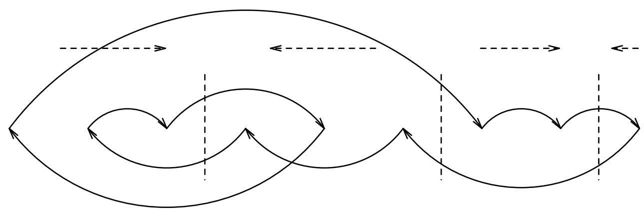


Fig. 1. An example of a cycle P with eccentricities $\frac{1}{2}$, $\frac{2}{1}$ and $\frac{8}{1}$. The pattern $[P]$ has eccentricities $\frac{2}{1}$ and $\frac{8}{1}$.

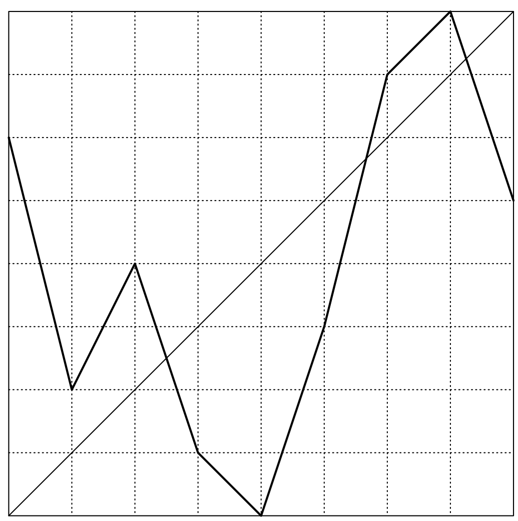


Fig. 2. The graph of the function f_P

In order to quickly demonstrate the connection between Sharkovskiĭ's Theorem and the forcing relation based on the eccentricity of a pattern, take the Sharkovskiĭ ordering on odd numbers

$$3 \succ 5 \succ 7 \succ 9 \succ 11 \succ \ldots \succ 1$$

and rewrite it in the corresponding form

$$\frac{2}{1} \succ \frac{3}{2} \succ \frac{4}{3} \succ \frac{5}{4} \succ \frac{6}{5} \succ \ldots \succ 1.$$

The first sequence gives the order of periods and second the order of eccentricities of Štefan patterns with odd period. The basic idea of the proof of

Sharkovskiĭ's Theorem is to show that every pattern of period $2k+1$ forces a Štefan pattern with the same period which has eccentricity $\frac{k+1}{k}$ and that these patterns form a chain corresponding to the order above.

So we see that the Sharkovskiĭ ordering is defined only on a small part of rational numbers. We of course have the natural order on all rational numbers and in this paper we prove a generalization of Sharkovskiĭ's Theorem for this order.

The crucial role in the proof of Sharkovskiĭ's Theorem was played by so-called Štefan patterns. In this paper a similar role is played by X-minimal patterns defined below.

We fix an $r \in \mathbb{Q}$ and consider the set of all patterns with eccentricity r. We can look at the forcing relation restricted to this set. Some of the patterns may not force any other pattern from this set. These will be called *X-minimal patterns* ([1]) with eccentricity r (*X-minimal r-patterns*) and for their representatives we shall use the term *X-minimal cycles* (*X-minimal r-cycles*).

Remark. Note that in fact X-minimal patterns do not have to exist. Since there are infinitely many patterns with any given rational eccentricity it could be possible that they can be arranged into an infinite chain of patterns each of which forces the next one. We will prove that this is not the case and that there are indeed X-minimal patterns for any given eccentricity. Also note that a pattern with eccentricities $r, q \in \mathbb{Q}$ can theoretically be an X-minimal r-pattern but not an X-minimal q-pattern. Our results show that this is not possible either.

The structure of the paper is as follows.

In Section 1 we give some basic notation, definitions and lemmas used throughout the paper.

In Section 2 we study the forcing relation between patterns with different eccentricity. The main result of this section is Theorem 2.10.

Section 3 is devoted to the characterization of X-minimal patterns. The main result of this section is Theorem 3.7. The reader can also have a look at the easy geometrical condition given in Lemma 3.9, and an important property concerning the period of X-minimal patterns is proved in Lemma 3.4.

In Section 4 we prove the existence of X-minimal patterns and we give a simple algorithm for constructing all X-minimal patterns. The main result proving the existence of X-minimal patterns is Theorem 4.5.

Finally, in Section 5 we define a new notion of "type" of a pattern and using it we prove our generalization of Sharkovskiĭ's Theorem in Theorem 5.1.

([1]) "X" is just a pun on "eccentric".

Remark. Results very similar to those obtained in this paper can also be found in [Bl] and [BM]. They have been obtained independently and by different methods, based on the theory of rotation numbers for maps on the circle, while in this paper we use only elementary combinatorial arguments.

1. Background. By $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Z}$, $\mathbb{N}$ we denote the sets of real, rational, integer and positive integer numbers respectively. By $\mathrm{conv}(X)$ we denote the convex hull of a set X. We will put sets in $\{\}$ brackets; by a set we mean a collection of elements without multiple occurrence. An ordered collection of elements with possible repetitions will be called a sequence and put in $\langle\rangle$ brackets. We denote by f^i the ith iterate of a function f. A point p is a *periodic point* of f if $f^n(p) = p$ for some $n \in \mathbb{N}$. The least such n is called the *period* of p. The *cycle* given by a periodic point p and a function f is (P, φ) where $P = \{f^i(p) : i \in \mathbb{N}\}$ and $\varphi = f|_P$. A point p is a *fixed point* of f if $f(p) = p$. The set of all periodic points of f will be denoted by $\mathrm{Per}(f)$ and the set of all fixed points by $\mathrm{Fix}(f)$. For a cycle $(\{p_1, \ldots, p_n\}, \varphi)$ we will normally use one of the following labelings: the *spatial labeling* when $p_1 < \ldots < p_n$ and the *dynamical labeling* when $\varphi(p_i) = p_{i+1}$ for $i = 1, \ldots, n-1$ and $\varphi(p_n) = p_1$.

We shall use some standard notions and techniques from combinatorial dynamics. The most important is the notion of P-linear map.

Definition. Let (P, φ) be a periodic orbit and $I = \mathrm{conv}(P)$. Then $f_P \in C(I, I)$ is such that $f_P|_P = \varphi$ and $f_P|_J$ is linear for any interval $J \subset I$ such that $J \cap P = \emptyset$. The function f_P is the piecewise linear function given by the cycle P and sometimes it is called the *connect-the-dot map* (see Fig. 2).

Very often we will use the following basic fact.

Lemma 1.1 (Theorem 2.6.13 of [ALM]). *Let (P, φ) be a cycle. If f_P exhibits a pattern B then $[P]$ forces B.*

We say that an interval J *P-covers* an interval L if $L \subset f_P(J)$. We will denote this by $J \overset{P}{\to} L$. A sequence $\mathcal{A} = \langle I_k \rangle_{k=1}^m$ of closed intervals is called *P-cyclic* if $I_1 \overset{P}{\to} I_2 \overset{P}{\to} \ldots \overset{P}{\to} I_m \overset{P}{\to} I_1$. Note that a P-cyclic sequence is in fact a cycle of intervals and therefore we will consider two P-cyclic sequences equal if they form the same cycle and have the same length. This will allow us to start a cyclic sequence wherever we want by simply rotating it. The P-cyclic sequences and cycles of the function f_P are in close relation. Namely we have

Lemma 1.2 ([B2], [BGMY], Lemma 1.2.7 of [ALM]). *Let P be a periodic orbit and $\mathcal{A} = \langle I_k \rangle_{k=0}^{m-1}$ be P-cyclic. Then there is a periodic point $x \in$*

$\mathrm{Per}(f_P)$ *such that* $f_P^k(x) \in I_k$ *for* $k = 0, \ldots, m-1$ *and* $f_P^m(x) = x$. *The period of the cycle given by* x *and* f_P *divides* m.

We will say that a cycle obtained from a P-cyclic sequence $\mathcal{A}$ using Lemma 1.2 is *contained* in $\mathcal{A}$.

We will use the following simple notation: If $\mathcal{A} = \langle a_1, \ldots, a_n\rangle$ and $\mathcal{B} = \langle b_1, \ldots, b_m\rangle$ then $\mathcal{A} + \mathcal{B} = \langle a_1, \ldots, a_n, b_1, \ldots, b_m\rangle$.

Finally, if $A, B \subset \mathbb{R}$ then we say that $A < B$ if $A \neq B$ and $a \le b$ for any $a \in A$ and $b \in B$. If $x \in \mathbb{R}$ then we say that $x < A$ ($x > A$) if $\{x\} < A$ ($\{x\} > A$).

2. Unipatterns and forcing between patterns with different eccentricity. A cycle P with unique eccentricity will be called a *unicycle* ([2]). We denote the eccentricity of a unicycle P by $E(P)$. A unicycle P with $E(P) = r$ will be called an r*-unicycle*. Similarly we shall use the terms *unipattern*, $E([P])$ and r*-unipattern*. Note that if a cycle P is not a unicycle then the pattern $[P]$ has at least two different eccentricities. Therefore a representative of a unipattern is a unicycle.

We will show that an X-minimal pattern must be a unipattern.

LEMMA 2.1. *Suppose the cycle* P *is not a unicycle. Then* f_P *has an* r*-unicycle for any positive* $r \in \mathbb{Q}$.

Proof. We will show how to construct an $\frac{m}{n}$-unicycle for f_P for any $m, n \in \mathbb{N}$ where $m \ge n$ (the case $m \le n$ is similar).

Let $z_1 < z_2$ be two rightmost fixed points of f_P. Hence $f_P(x) > x$ for $x \in (z_1, z_2)$ and $f_P(x) < x$ for $x > z_2$. Let $a \in (z_1, z_2)$ be such that $f_P(a) \ge f_P(x)$ for any $x \in (z_1, z_2)$. Clearly $f_P(a) > z_2$ (otherwise the interval $[z_1, z_2]$ would be f_P-invariant, which is impossible because it contains a point from P). Now let $J_1 = [z_1, a]$ and $J_2 = [z_2, f_P(a)]$. We have $J_1 \xrightarrow{P} J_1$ and $J_1 \xrightarrow{P} J_2$. Moreover, $J_2 \xrightarrow{P} J_1$ (otherwise the interval $[z_1, f_P(a)]$ would be f_P-invariant) (see Fig. 3). Hence for any $m \ge n$,

$$\overbrace{\langle J_1, \ldots, J_1\rangle}^{m-n \text{ times } J_1} + \overbrace{\langle J_1, J_2, \ldots, J_1, J_2\rangle}^{n \text{ times } J_1, J_2}$$

is a P-cyclic sequence. Using Lemma 1.2 we obtain a cycle Q for f_P with period $m+n$. But $Q \subset [z_1, f_P(a)]$ and $(z_1, f_P(a)) \cap \mathrm{Fix}(f_P) = \{z_2\}$ so Q is a unicycle. Finally, since $J_1 < z_2 < J_2$ we conclude that Q is an $\frac{m}{n}$-unicycle. ∎

Hence we have the straightforward

([2]) Unfortunately, there is no bicycle. However, according to A. Manning there may be a little comfort for cyclists. Actually, the object we study (a periodic orbit and a fixed point "'in" it) consists of two cycles so it is a bicycle. ☺

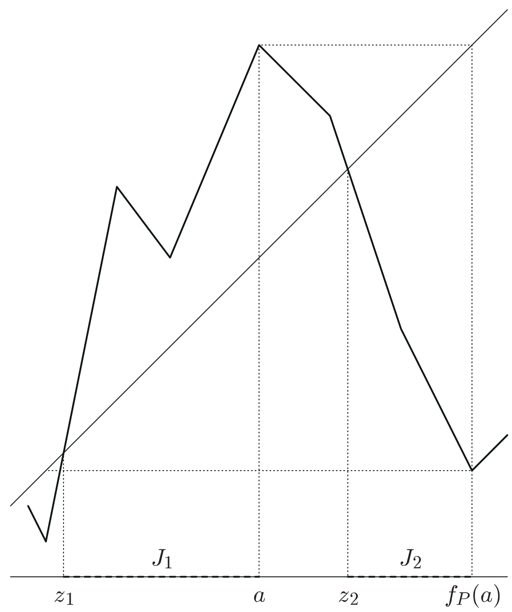


Fig. 3

Corollary 2.2. *An X-minimal pattern is a unipattern.*

Proof. If an r-pattern A is not a unipattern then its representative P is not a unicycle. By Lemma 2.1 the function f_P exhibits an r-unipattern B and by Lemma 1.1 the pattern A forces B. But $A \neq B$ (one is a unipattern and the other is not) and so A is not an X-minimal r-pattern. ■

Now we would like to find all patterns forced by a unipattern $[P]$.

First note that f_P has a unique fixed point and therefore every cycle it has is a unicycle. So a unipattern can only force unipatterns. Later we will often use this fact without mentioning it.

A possible way to find patterns forced by a unipattern $[P]$ is to find all P-cyclic sequences and use Lemmas 1.1 and 1.2 to get some of the patterns forced by $[P]$. But in general if we have a P-cyclic sequence then we have no information about the eccentricity of patterns forced by this sequence. Fortunately, for some special P-cyclic sequences we can get this information.

Definition. Let P be a unicycle and $\mathrm{Fix}(f_P)=\{c\}$. A P-cyclic sequence $\mathcal{A}=\langle I_i\rangle_{i=1}^{a}$ will be called *separated* if $c\notin \mathrm{int}(I)$ for any $I\in\mathcal{A}$. The eccentricity of a separated P-cyclic sequence $\mathcal{A}$ is $E(\mathcal{A})=\#\{i : I_i \le c\}/\#\{i : I_i \ge c\}$.

Lemma 2.3. *Let P be a unicycle, $c \in \mathrm{Fix}(f_P)$ and $\mathcal{A}$ be a separated P-cyclic sequence. Then f_P has an $E(\mathcal{A})$-unicycle contained in the loop $\mathcal{A}$.*

Proof. Assume that $E(\mathcal{A}) \ge 1$ (the case $E(\mathcal{A}) \le 1$ is similar).

If there is $I \in \mathcal{A}$ such that $c \notin I$ then the cycle Q that we get from the sequence $\mathcal{A}$ by using Lemma 1.2 clearly has eccentricity $E(\mathcal{A})$.

Assume that $c \in I$ for all $I \in \mathcal{A}$. If $E(\mathcal{A}) = 1$ then the fixed point c gives such a cycle. So we can assume that $E(\mathcal{A}) > 1$. Then $\mathcal{A} = \langle \ldots, I, J, K \ldots \rangle$ where $I, J < c$ and $K > c$ ($I \neq \{c\}$ because $E(\mathcal{A}, c) \neq 1$). Now there are two possibilities:

1. $f_P(x) \geq c$ for any $x \in J$ or
2. there is an $a \in J$ such that $f_P(a) = c$ and $a \neq c$.

In case 1 we have $J \subset I$. So there is a point $a \in I$ such that $f_P(a) = \inf J$ and a point $b \in I$ such that $b \neq c$ and $f_P(b) = c$. Hence we can replace I by $I^* = \mathrm{conv}(\{a, b\})$ and we have again a separated P-cyclic sequence $\mathcal{B}$.

In case 2 let $b \in J$ be such that $f_P(b) = \sup\{f_P(x) : x \in J\}$. Now we can replace J by $J^* = \mathrm{conv}(\{a, b\})$ to get a separated P-cyclic sequence $\mathcal{B}$.

In both cases we obtain a new separated P-cyclic sequence $\mathcal{B}$ with eccentricity $E(\mathcal{A})$. But now there is an interval in $\mathcal{B}$ (I^* or J^*) that does not contain c. Hence the above argument shows that there is an $E(\mathcal{A})$-unicycle Q in f_P. ∎

Now the question is how we can tell whether we have picked up all possible P-cyclic sequences that can give us some information about patterns forced by $[P]$. We will show that it is enough to examine those P-cyclic sequences that have their elements only from the set of intervals given by the cycle P.

Definition. For a unicycle (P, φ) let $\mathfrak{P}$ be the partition of the interval $I = \mathrm{conv}(P)$ into intervals with endpoints in $P \cup \mathrm{Fix}(f_P)$.

In particular, if $P = \{p_1, \ldots, p_{k(m+n)}\}$ is a unicycle with spatial labeling where $k, m, n \in \mathbb{N}$, m, n are coprime, $E(P) = \frac{m}{n}$ and $\mathrm{Fix}(f_P) = \{c\}$, the partition $\mathfrak{P}$ is $\{J_i\}_{i=1}^{k(m+n)}$ where

$$\begin{aligned} J_i &= [p_i, p_{i+1}] && \text{for } i < km, \\ J_{km} &= [p_{km}, c], \\ J_{km+1} &= [c, p_{km+1}], \\ J_i &= [p_{i-1}, p_i] && \text{for } i > km + 1. \end{aligned}$$

A P-cyclic sequence $\mathcal{A} = \langle I_i \rangle_{i=1}^{a}$ such that each $I_i \in \mathfrak{P}$ will be called a *P-loop*. (Note that any P-loop is separated.)

Now we can prove some kind of converse of Lemma 1.2.

Lemma 2.4. *Let P be a unicycle and $Q = \{q_1, \ldots, q_m\}$ be a cycle of f_P with* $\mathrm{per}(Q) = m > 1$. *Then there is a unique P-loop $\mathcal{A} = \langle I_i \rangle_{i=1}^{m}$ such that $f_P^{i-1}(q_1) \in I_i$ for $1 \leq i \leq m$.*

P r o o f. If $Q \neq P$ then for any q_i there is a unique interval $I_i \in \mathfrak{P}$ such that $q_i \in I_i$. Moreover, because f_P is linear on any interval $I \in \mathfrak{P}$ and $q_i \in \mathrm{int}(I_i)$ we have $I_i \xrightarrow{P} I_{i+1}$ and so the sequence $\mathcal{A} = \langle I_i \rangle_{i=1}^m$ is a P-loop.

So assume that $Q = P = \{p_1, \ldots, p_n\}$ with spatial labeling. Then there is a unique interval $I_1 \in \mathfrak{P}$ such that $p_1 \in I_1$. Assume that $I_j \in \mathfrak{P}$ is such that $f_P^{j-1}(p_1) \in I_j$. There are at most two intervals $I \in \mathfrak{P}$ such that $f_P^j(p_1) \in I$ but only one of them satisfies $I_j \xrightarrow{P} I$ (because f_P is linear on I_j and $f_P^{j-1}(p_1)$ is an endpoint of I_j). Hence there is a unique $I_{j+1} \in \mathfrak{P}$ such that $f_P^j(p_1) \in I_{j+1}$ and $I_j \xrightarrow{P} I_{j+1}$. Therefore there is also a unique P-loop of length $\mathrm{per}(P)$ containing the cycle P. ∎

Remark. Clearly, a cycle Q can be contained in more than one P-loop. But any P-loop containing Q is only a repetition of the unique P-loop $\mathcal{A}$ that has length $\mathrm{per}(Q)$.

Definition. We denote the P-loop containing the cycle P by $\mathcal{A}_P$. We say that a P-loop $\mathcal{A}$ is *simple* if there are no two nonempty P-loops $\mathcal{B}, \mathcal{C}$ such that $\mathcal{A} = \mathcal{B} + \mathcal{C}$.

Lemma 2.5. *A P-loop containing some interval more than once is not simple.*

P r o o f. After rotating we can write our P-loop as $\overbrace{\langle I, \ldots, K \rangle}^{\mathcal{B}} + \overbrace{\langle I, \ldots, L \rangle}^{\mathcal{C}}$ and both $\mathcal{B}$ and $\mathcal{C}$ are nonempty P-loops. ∎

Now we will look at a unicycle P and its loop $\mathcal{A}_P$. There are basically two possibilities. Either $\mathcal{A}_P$ is simple or not. The next lemma shows the importance of simple $\mathcal{A}_P$.

Lemma 2.6. *Let P be a unicycle with* $\mathrm{per}(P) > 2$ *and simple loop $\mathcal{A}_P$. Then for each P-loop $\mathcal{A}$ there is a unique cycle contained in $\mathcal{A}$.*

P r o o f. Assume the contrary. Let $\mathcal{A} = \langle I_i \rangle_{i=1}^m$ be a P-loop and let $x < y \in \mathrm{Per}(f_P)$ be such that $f_P^{i-1}(x), f_P^{i-1}(y) \in I_i$ for $1 \le i \le m$ and $f_P^m(x) = x$ and $f_P^m(y) = y$. Hence $f_P^m|_{[x,y]}$ is linear and therefore the identity. We will take the smallest possible $a \in \mathbb{N}$ such that $f_P^a|_{[x,y]}$ is the identity; that means that x, y have period either a or $a/2$.

Take $x^*, y^* \in I_1$ such that $x^* \le x < y \le y^*$, $f_P^n(x^*), f_P^n(y^*) \in P \cup \mathrm{Fix}(f_P)$ for some $n \in \mathbb{N}$, $f_P^a|_{[x^*,y^*]}$ is linear and $(x^*, y^*) \cap (P \cup \mathrm{Fix}(f_P)) = \emptyset$.

So $f_P^a|_{[x^*,y^*]}$ is the identity and therefore $x^*, y^* \in P \cup \mathrm{Fix}(f_P)$. Hence $f_P^i([x^*, y^*]) \in \mathfrak{P}$ for any $i \ge 0$ and if $\{x^*, y^*\} \cap \mathrm{Fix}(f_P) \neq \emptyset$ then $\mathrm{per}(P) \le 2$. Therefore $x^*, y^* \in P$. Moreover, a is the smallest possible number such that $f_P^a|_{[x^*,y^*]}$ is the identity and therefore $\mathrm{per}(P) = a$.

Take the sequence $\mathcal{A}^* = \langle f_P^i([x^*, y^*]) \rangle_{i=1}^a$. Clearly $\mathcal{A}^*$ is a P-loop which contains the cycle P. Therefore $\mathcal{A}^* = \mathcal{A}_P$ by Lemma 2.4. But it is easy to see that $\mathcal{A}^*$ is not simple (it contains the interval $[x^*, y^*]$ twice: once covering x^* and then y^*)—a contradiction. ∎

Now we will investigate the forcing relation between patterns with different eccentricities.

LEMMA 2.7. *Let P be a unicycle with $E(P) \geq 1$ and $\mathcal{A}_P$ be not simple. Then f_P has a unicycle Q such that* $\operatorname{per}(Q) < \operatorname{per}(P)$ *and* $E(Q) \geq E(P)$.

Proof. Because $\mathcal{A}_P$ is not simple there are P-loops $\mathcal{B}, \mathcal{C}$ such that $\mathcal{A}_P = \mathcal{B} + \mathcal{C}$. But either $E(\mathcal{B}) \geq E(\mathcal{A}_P)$ or $E(\mathcal{C}) \geq E(\mathcal{A}_P)$ and they are both shorter than $\mathcal{A}_P$. Hence we are done by Lemmas 2.3 and 1.2. ∎

LEMMA 2.8. *A unipattern A forces some unipattern B such that $E(A) \leq E(B)$,* $\operatorname{per}(A) \geq \operatorname{per}(B)$ *and a representative Q of B has a simple loop $\mathcal{A}_Q$.*

Proof. If a representative P of A has a simple loop then $B = A$. If not then by Lemma 2.7 the pattern A forces a unipattern A^* such that $E(A) \leq E(A^*)$ and $\operatorname{per}(A) > \operatorname{per}(A^*)$. Since $\operatorname{per}(A)$ is finite, after repeating this finitely many times we must get our unipattern B. ∎

LEMMA 2.9. *Let A be an r-unipattern and let its representative P have a simple loop $\mathcal{A}_P$. Then the pattern A forces some q-unipattern for each $q \in \mathbb{Q}$ such that $r \geq q \geq 1$.*

Proof. We may assume that $E(P) = r > 1$ (the case $r = 1$ is trivial) and $\operatorname{per}(P) = k(m + n)$ where $m/n = r$ (m, n are coprime).

Because $\mathcal{A}_P$ is simple it contains every interval of the partition $\mathfrak{P}$. So we may assume that the loop $\mathcal{A}_P$ starts with the interval J_{km}. Moreover, $J_{km} \xrightarrow{P} J_{km+1}$ and $J_{km+1} \xrightarrow{P} J_{km}$. Hence

$$\mathcal{B} = \langle \overbrace{J_{km}, J_{km+1}, \ldots, J_{km}, J_{km+1}}^{a \text{ times } J_{km}, J_{km+1}} \rangle + \overbrace{\mathcal{A}_P + \ldots + \mathcal{A}_P}^{b \text{ times } \mathcal{A}_P}$$

is a P-loop with eccentricity $\frac{a+bm}{a+bn}$. If $q = r/s$ then we can choose $a = ms - rn$ and $b = r - s$. So $(a + bm)/(a + bn) = q$. Hence from Lemma 2.3 we see that f_P has a q-unicycle and by Lemma 1.1 the pattern A forces a q-unipattern. ∎

Now we can easily get the final statement of this section.

THEOREM 2.10. *Let $r, q \in \mathbb{Q}$ satisfy $r \geq q \geq 1$. Then any r-pattern forces a q-unipattern.*

Proof. Let A be an r-pattern. If A is not a unipattern then, by Lemmas 2.1 and 1.1, A forces a q-unipattern. If A is a unipattern then, by Lemma 2.8, it forces a unipattern B such that $E(B) \geq r$ and a representative P of B has a simple loop $\mathcal{A}_P$. By Lemma 2.9 the pattern B forces a q-unipattern, and so does A because the forcing relation is transitive. ∎

3. X-minimal patterns. First we recall

DEFINITION. An r-pattern is *X-minimal* if it does not force any other r-pattern.

Now we would like to find all X-minimal r-patterns. We already have some information about such patterns. More precisely, we have

LEMMA 3.1. *An X-minimal pattern is a unipattern and its representative P has a simple loop $\mathcal{A}_P$.*

Proof. If $[P]$ is an X-minimal pattern then by Corollary 2.2 it is a unipattern. If it does not have simple loop $\mathcal{A}_P$ then by Lemma 2.8 it forces a unipattern $[Q]$ with simple loop $\mathcal{A}_Q$ and $E(Q) > E(P)$. So $[P] \neq [Q]$ and by Lemma 2.9 the pattern $[Q]$ forces a pattern with eccentricity $E(P)$. Finally, because the forcing relation is antisymmetric we see that $[P]$ is not X-minimal—a contradiction. ∎

DEFINITION. Let P be a unicycle and $c \in \mathrm{Fix}(f_P)$. A sequence $Q = \langle q_i \rangle_{i=0}^{a}$ will be called a *P-semicycle* if

$$q_i \in P, \qquad f_P(q_{i-1}) = q_i \quad \text{for } 1 \le i \le a,$$
$$q_0 \neq q_a, \qquad q_0 \in \mathrm{conv}\{q_a, c\}.$$

The *eccentricity* of the P-semicycle Q is

$$E(Q) = \frac{\#\{i > 0 : q_i < c\}}{\#\{i > 0 : q_i > c\}}.$$

(See Fig. 4.)

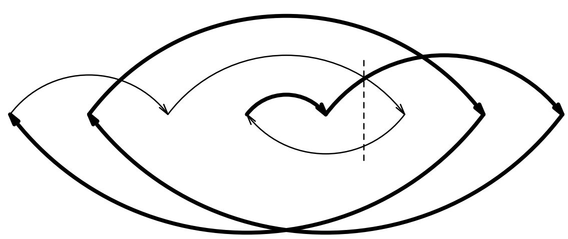


Fig. 4. A cycle P with a semicycle Q (thick lines); $E(Q) = \frac{3}{2}$

LEMMA 3.2. *Let P be a unicycle with a P-semicycle Q. Then f_P has an $E(Q)$-cycle R such that* $\mathrm{per}(R) \neq \mathrm{per}(P)$.

Proof. Let $Q = \langle q_i \rangle_{i=0}^{a}$ and $I_i = \mathrm{conv}\{q_i, c\}$. Clearly $I_0 \xrightarrow{P} I_1 \xrightarrow{P} \ldots \xrightarrow{P} I_a$ and $I_0 \subset I_a$. Therefore $\langle I_i \rangle_{i=1}^{a}$ is a separated P-cyclic sequence with eccentricity $E(Q)$. By Lemma 2.3 the function f_P has an $E(Q)$-cycle R. Moreover, a is not divisible by $\mathrm{per}(P)$ and therefore $\mathrm{per}(R) \neq \mathrm{per}(P)$. ∎

Definition. Let P be an $\frac{m}{n}$-unicycle where $m \geq n \in \mathbb{N}$ are coprime and $c \in \mathrm{Fix}(f_P)$. Define the *coding* $K_P : P \to \mathbb{Z}$ by

$$K_P(p_1) = 0,$$
$$K_P(f_P(p_i)) = \begin{cases} K_P(p_i) + n & \text{for } p_i < c, \\ K_P(p_i) - m & \text{for } p_i > c. \end{cases}$$

We say that P *has monotone code* if either $\mathrm{per}(P) = 1$, or $E(P) > 1$ and for any $p, q \in P$ such that $p \neq q$ and $q \in \mathrm{conv}(\{p, c\})$ we have $K_P(q) > K_P(p)$ (see Fig. 5).

If P has monotone code we also say that the pattern $[P]$ has monotone code.

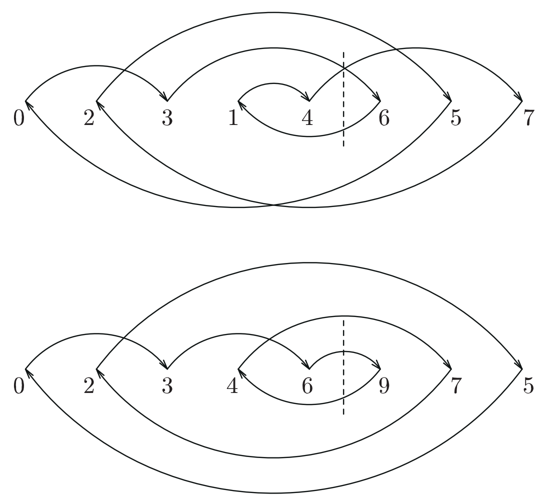


Fig. 5. An example of a cycle without (top) and with (bottom) monotone code

Lemma 3.3. *An X-minimal pattern has monotone code.*

Proof. Let A be an X-minimal $\frac{m}{n}$-pattern (m, n are coprime) and let an $\frac{m}{n}$-cycle P be a representative of A. From Lemma 2.1 we know that P is a unicycle.

Assume that P does not have monotone code. Then there are two different $p, q \in P$ such that $q \in \mathrm{conv}(\{p, c\})$ and $K_P(q) \leq K_P(p)$. Set $Q = \langle q_j \rangle_{j=0}^{a}$ where $q_0 = q$, $q_{j+1} = f_P(q_j)$ and $q_a = p$. Clearly, Q is a semicycle and we can estimate $E(Q)$. From the definition of K_P we have

$$K_P(q_a) = K_P(q_0) + n\#\{j : 0 \leq j < a,\ q_j < c\} - m\#\{j : 0 \leq j < a,\ q_j > c\}.$$

Hence

$$E(Q) = \frac{\#\{j : 0 \leq j < a,\ q_j < c\}}{\#\{j : 0 \leq j < a,\ q_j > c\}} \geq \frac{m}{n}.$$

Using Lemmas 3.2 and 1.1 we find that A forces a pattern $B \neq A$ such that $E(B) \geq m/n$ and from Theorem 2.10 we see that B forces an $\frac{m}{n}$-pattern C. But $A \neq C$ because the forcing relation is antisymmetric and so A is not an X-minimal $\frac{m}{n}$-pattern—a contradiction. ∎

So we have proved that an X-minimal pattern is a unipattern with monotone code. Now we are going to get more information about a unicycle with monotone code.

Let $(P = \{p_1, \ldots, p_{k(m+n)}\}, \varphi)$ be an $\frac{m}{n}$-unicycle with spatial labeling, monotone code ($m \geq n$ are coprime) and $c \in \mathrm{Fix}(f_P)$. From the monotonicity we immediately see that $\varphi(p_i) < c$ for $i > km$.

Hence we can define a new cycle (P^*, ψ) where $P^* = \{p_i\}_{i=1}^{km}$ and

$$\psi(p_i) = \begin{cases} \varphi(p_i) & \text{if } \varphi(p_i) \in P^*, \\ \varphi^2(p_i) & \text{if } \varphi(p_i) \notin P^*. \end{cases}$$

So we can make

Definition. Let $C_P = \langle c_i \rangle_{i=1}^{km}$, where $c_i \in \{0, 1\}$, be a code corresponding to the cycle P in the following way:

$$c_i = \begin{cases} 0 & \text{if } \psi^i(p_1) = \varphi(\psi^{i-1}(p_1)), \\ 1 & \text{if } \psi^i(p_1) = \varphi^2(\psi^{i-1}(p_1)). \end{cases}$$

From the monotonicity of the code K_P it can be seen that the code C_P can also be obtained from the cycle (P^*, ψ) if we start at the point p_1 and following the cycle we write 0 if we move right and 1 if left (see Fig. 6).

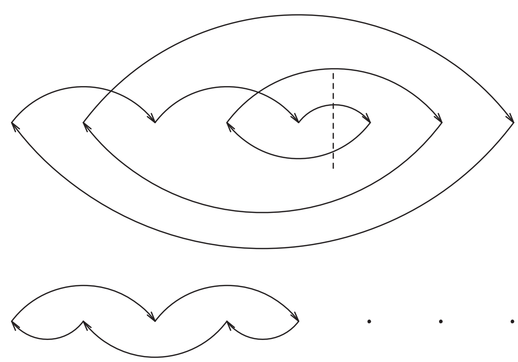


Fig. 6. An example of a cycle (P, φ) (top) and (P^*, ψ) (bottom) with $C_P = \langle 0, 0, 1, 1, 1 \rangle$

Note that C_P contains kn ones and $k(m - n)$ zeros. Moreover,

$$K_P(\psi^i(p_1)) = \begin{cases} K_P(\psi^{i-1}(p_1)) + n & \text{if } c_i = 0, \\ K_P(\psi^{i-1}(p_1)) - m + n & \text{if } c_i = 1. \end{cases}$$

Hence we have the following connection between K_P and C_P:

$$K_P(\psi^i(p_1)) = in - m\sum_{j=1}^{i} c_j.$$

LEMMA 3.4. *Let $m > n$ be coprime and P be an $\frac{m}{n}$-unicycle with monotone code. Then* $\mathrm{per}(P) = m + n$.

Proof. Assume that $P = \{p_1, \ldots, p_{k(m+n)}\}$ with spatial labeling and $k > 1$. We will study the code C_P.

Let i_j be such that $c_{i_j} = 1$ and $\sum_{i=1}^{i_j} c_i = j$ (i_j is the place of the jth unit in the sequence C_P).

Because $k > 1$ we have $\psi^{i_n}(p_1) \neq p_1$ and from the monotonicity of the code we have $K_P(\psi^{i_n}(p_1)) > 0$. But $K_P(\psi^{i_n}(p_1)) = ni_n - mn$ and so $i_n > m$.

Moreover, monotonicity yields that no two points from P^* can have the same value of K_P. If there is a part $C^* = \langle c_i\rangle_{i=j+1}^{j+m}$ of C_P such that $\sum_{i=j+1}^{j+m} c_i = n$ then $K_P(\psi^{j+m}(p_1)) = K_P(\psi^j(p_1)) + (m-n)n + n(n-m) = K_P(\psi^j(p_1))$. But $\psi^{j+m}(p_1) \neq \psi^j(p_1)$ ($k > 1$) contrary to the monotonicity. So no part of C_P of length m contains $m - n$ times 0 and n times 1. Hence $i_n - i_1 \geq m$ (otherwise $\langle c_i\rangle_{i=i_n-m+1}^{i_n}$ contains $m - n$ times 0 and n times 1).

Therefore $i_1 < i_{n+1} - m + 1$ and using the sequence $\langle c_i\rangle_{i=i_{n+1}-m+1}^{i_{n+1}}$ as above we obtain $i_{n+1} - i_2 \geq m$. Inductively, for all $j \leq (k-1)n$,

$$i_{n+j} - i_{1+j} \geq m.$$

We have $c_1 = 0$ because $K_P(\psi(p_1)) \geq 0$ (monotonicity) and so $1 < i_1 < \ldots < i_{kn-1} < i_{kn} \leq kn$. Using the inequalities above we obtain

$$km \geq 1 + \sum_{j=1}^{k}(i_{jn} - i_{(j-1)n+1}) \geq 1 + \sum_{j=1}^{k} m = 1 + km,$$

which is a contradiction. ∎

LEMMA 3.5. *Let P be a unicycle which is not X-minimal. Then f_P has a unicycle R such that* $\mathrm{per}(R) < \mathrm{per}(P)$ *and* $E(R) \geq E(P)$.

Proof. If $\mathcal{A}_P$ is not simple then by Lemma 2.7 the function f_P has a unicycle Q with $\mathrm{per}(Q) < \mathrm{per}(P)$ and $E(Q) \geq E(P)$. By Lemma 2.8, f_P has a unicycle R such that $\mathrm{per}(R) \leq \mathrm{per}(Q)$ and $E(R) \geq E(Q)$ and so we are done.

Assume that $\mathcal{A}_P$ is simple. If $\mathrm{per}(P) = 2$ then the cycle given by a fixed point is our cycle R. So we can assume that $\mathrm{per}(P) > 2$ (if $\mathrm{per}(P) = 1$ then P is X-minimal).

Because P is not X-minimal, f_P contains a cycle $Q \neq P$ with $E(Q) = E(P)$. Let $\mathcal{A}$ be the unique P-loop containing Q (Lemma 2.6). Because $\mathcal{A}_P$ is simple and $P \neq Q$, Lemma 2.6 shows that $\mathcal{A} \neq \mathcal{A}_P + \ldots + \mathcal{A}_P$. Hence

$\mathcal{A}$ can be written as the sum of two P-loops $\mathcal{B}+\mathcal{C}$ ($\mathcal{C}$ may be empty) such that $\mathcal{B}$ is a simple P-loop, $E(\mathcal{B}) \geq E(\mathcal{A}_P)$ and $\mathcal{B} \neq \mathcal{A}_P$.

If the length of $\mathcal{B}$ is smaller than $\operatorname{per}(P)$ then the cycle R given by the P-loop $\mathcal{B}$ (Lemma 1.2) is the one we are looking for (see Lemma 2.3).

So the length of $\mathcal{B}$ is at least $\operatorname{per}(P)$ and by Lemma 2.5 it must be $\operatorname{per}(P)$. Hence both $\mathcal{A}_P$ and $\mathcal{B}$ contain all intervals from $\mathfrak{P}$. Since they are different there are intervals $I, J, K \in \mathfrak{P}$ such that $J \neq K$ and

$$\mathcal{A}_P = \langle \ldots, I, J, \ldots\rangle, \qquad \mathcal{B} = \langle \ldots, I, K, \ldots\rangle.$$

Hence after a suitable rotation we can write

$$\mathcal{A}_P = \overbrace{\langle J, \ldots, L\rangle}^{\mathcal{D}} + \overbrace{\langle K, \ldots, I\rangle}^{\mathcal{E}}, \qquad \mathcal{B} = \langle K, \ldots, I\rangle.$$

Note that $\mathcal{D}$ is nonempty and the loop $\mathcal{E}$ is P-cyclic.

If $E(\mathcal{E}) \geq E(\mathcal{A}_P)$ then $\mathcal{E}$ gives us a cycle R with period smaller than $\operatorname{per}(P)$ and we are done (Lemmas 1.2 and 2.3).

Otherwise $\mathcal{D}+\mathcal{B}$ is a P-cyclic loop with $E(\mathcal{D}+\mathcal{B}) > E(\mathcal{A}_P)$. So it can be written as a sum of two P-loops one of which is a simple P-loop $\mathcal{F}$ such that $E(\mathcal{F}) > E(\mathcal{A}_P)$. This loop has length smaller than $\operatorname{per}(P)$ (all simple P-loops with length $\operatorname{per}(P)$ have eccentricity $E(\mathcal{A}_P)$) and so it will give us a cycle R with period smaller than $\operatorname{per}(P)$ (Lemmas 1.2 and 2.3). ∎

Let P be an $\frac{m}{n}$-unicycle with simple loop $\mathcal{A}_P = \langle I_i\rangle_{i=1}^{k(m+n)}$. We have $f_P^{i-1}(p_1) \in I_i$. So we may define a map $\pi : P \to \mathfrak{P}$ such that $\pi(f_P^{i-1}(p_1)) = I_i$. Since $\mathcal{A}_P$ is simple, π is a bijection. Moreover, if $\pi(p_j) = I_i$ then $p_j \in I_i$. Recall that $\mathfrak{P} = \{J_i\}_{i=1}^{k(m+n)}$ with spatial labeling (see the definition of $\mathfrak{P}$) and $\mathcal{A}_P$ is a simple loop of length $k(m+n)$. Hence there is clearly only one possibility for π:

$$\pi(p_i) = J_i \quad \text{ for } 1 \leq i \leq k(m+n)$$

(see Fig. 7).

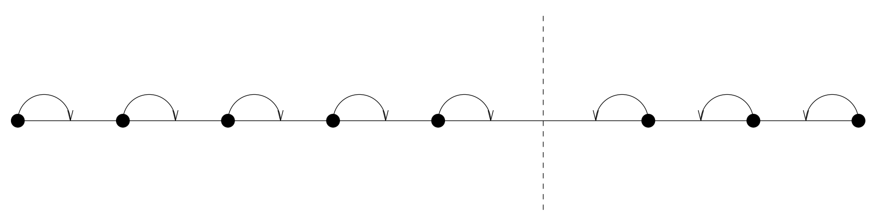


Fig. 7. The arrows show the intervals to which π maps points of the cycle

We know that $\pi(x) \xrightarrow{P} \pi(f_P(x))$ and using the bijection π we can naturally define a coding $K : \mathfrak{P} \to \mathbb{Z}$ similar to $K_P : P \to \mathbb{Z}$:

$$K(J_i) = K(\pi(p_i)) = K_P(p_i).$$

Now we estimate how the code of the intervals in $\mathfrak{P}$ depends on the P-covering property for these intervals.

LEMMA 3.6. *Let P be an $\frac{m}{n}$-unicycle with monotone code*, $\mathfrak{P} = \{J_i\}_{i=1}^{m+n}$ *with spatial labeling and* J_i, $I \in \mathfrak{P}$, *such that* $J_i \xrightarrow{P} I$. *If* $i \le m$ *then* $K(I) \ge K(J_i) + n$ *and if* $i > m$ *then* $K(I) \ge K(J_i) - m$.

Proof. If $i = m$ then $J_i = [p_m, c]$ and so $I \subset [c, f_P(p_m)]$. From the monotonicity of the code we have $K(I) \ge K(\pi(f_P(p_m))) = K(J_i) + n$.

If $i < m$ then $J_i = [p_i, p_{i+1}]$. But f_P is linear on J_i and so if $J_i \xrightarrow{P} J_j$ then $p_j \in \operatorname{conv}\{f_P(p_i), f_P(p_{i+1})\}$. The monotonicity gives $K_P(p_j) \ge \min\{K_P(f_P(p_i)), K_P(f_P(p_{i+1}))\}$. But $K_P(f_P(p_i)) < K_P(f_P(p_{i+1}))$ and so $K_P(p_j) \ge K_P(f_P(p_i)) = K_P(p_i) + n$. Hence $K(I) \ge K(J_i) + n$.

Now let $i > m$. By monotonicity, $f_P(p_{j+1}) < f_P(p_j) < c$ for all $j > m$. Hence $f_P([c, p_i]) = [f_P(p_i), c]$ and so $K(I) \ge K_P(f_P(p_i)) = K(J_i) - m$. ∎

Finally, we are ready to prove

THEOREM 3.7. *Let P be a periodic orbit. Then P is X-minimal if and only if it is a unicycle with monotone code.*

Proof. The necessity is proved in Lemmas 2.1 and 3.3. Now we show the sufficiency.

Let P be an $\frac{m}{n}$-unicycle with monotone code. If P is not X-minimal then by Lemma 3.5 there is a cycle Q of f_P such that $\operatorname{per}(Q) < \operatorname{per}(P)$ and $E(Q) \ge E(P)$. By Lemma 2.4 there is a P-loop $\mathcal{A} = \langle I_i \rangle_{i=1}^{a+b}$ such that $E(\mathcal{A}) = E(Q) = a/b$ where $a = \#\{i : I_i < c\}$ and $b = \#\{i : I_i > c\}$. Finally, by Lemma 3.6 we have

$$K(I_1) = K(I_1) + \sum_{i=1}^{a} n_i - \sum_{i=1}^{b} m_i$$

where $n_i \ge n$ and $m_i \le m$. Therefore $a/b \le m/n$. But Lemma 3.4 shows that $a+b < m+n$ and hence $a/b < m/n$, which contradicts $E(Q) \ge E(P)$. ∎

From this theorem we immediately have

COROLLARY 3.8. *A pattern is X-minimal if and only if it is a unipattern with monotone code.*

Although it is very easy to check if a pattern is a unipattern with monotone code it is still not a "look and see" (geometrical) characterization. We have at least some easy necessary geometrical condition.

LEMMA 3.9. *Let P be a representative of an X-minimal pattern with* $E(P) > 1$. *Then P is unicycle and for any $p, q \in P$ and* $\operatorname{Fix}(f_P) = \{c\}$ *we* *have*

$$\begin{array}{llllll} \textit{if} & p<q<c & \textit{and} & f_P(p), f_P(q) < c & \textit{then} & f_P(p) < f_P(q), \\ \textit{if} & p<q<c & \textit{and} & f_P(p), f_P(q) > c & \textit{then} & f_P(p) > f_P(q), \\ \textit{if} & p>q>c & \textit{then} & f_P(p), f_P(q) < c & \textit{and} & f_P(p) < f_P(q). \end{array}$$

P r o o f. This follows easily from Theorem 3.7 and monotonicity. ■

Unfortunately, these conditions are not sufficient (see Fig. 8) and we do not know if there exists a good geometrical characterization at all.

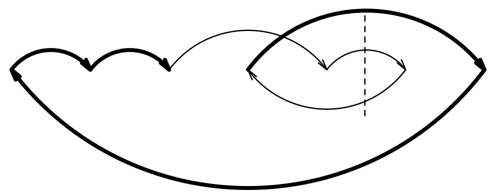


Fig. 8. A cycle P satisfying the conditions of Lemma 3.9 which is not X-minimal. This can be easily checked from the code K_P or by finding a semicycle with eccentricity $\frac{3}{1}$ (thick lines).

4. Existence of X-minimal orbits. In the previous section we gave a characterization of X-minimal orbits. However, if we have a function $f \in C(I,I)$ with a periodic orbit with eccentricity r it is still not clear whether this map has an X-minimal r-cycle. This is because the set of all patterns with given eccentricity is infinite and so theoretically there may exist a sequence of r-patterns each of which forces the next one and none of which forces an X-minimal r-pattern. In this section we will show that this is impossible. This question has been partially solved in [BK] for patterns with eccentricities of the form $\frac{k+1}{k}$.

LEMMA 4.1. *An r-unipattern A forces an X-minimal q-pattern B such that $q \ge r$ and* $\operatorname{per}(A) \ge \operatorname{per}(B)$.

P r o o f. If A is an X-minimal r-pattern take $B = A$ and we are done.

If A is not an X-minimal pattern then by Lemmas 3.5 and 1.1 we see that A forces a unipattern A^* such that $E(A^*) \ge E(A)$ and $\operatorname{per}(A^*) < \operatorname{per}(A)$. Applying this finitely many times we must get an X-minimal pattern B. ■

Now we investigate X-minimal patterns more closely. Let P be an X-minimal cycle with eccentricity $\frac{m}{n}$. We would like to know what cycles it forces. If we think a little about Lemma 3.1 and the way we proved Lemma 3.5 we can see that among all the cycles forced by P only those with period lower than $\operatorname{per}(P)$ are important. Any other cycle forced by P can be obtained by "gluing" some of these cycles together. Since P is X-minimal, the eccentricities of these cycles depending on the period are bounded above by $[im/n]/i$ for $i = 1, \ldots, n$. We now consider those that give us the maximal possible eccentricity with minimal possible period.

Definition. Let $m, n \in \mathbb{N}$, $m > n$ coprime. The fraction $\frac{[im/n]}{i}$ is called an $\frac{m}{n}$-*extremal fraction* if $1 \le i \le n$ and $[im/n]/i > [jm/n]/j$ for all $j \in \{1, \ldots, i-1\}$.

Remark. Note that $[im/n]$ and i are coprime for an $\frac{m}{n}$-extremal fraction $\frac{[im/n]}{i}$.

Lemma 4.2. *Let*

$$\frac{p_{j-1}}{q_{j-1}} \le \frac{p}{q} \le \frac{p_j}{q_j}$$

where $\frac{p_{j-1}}{q_{j-1}} < \frac{p_j}{q_j}$ *are consecutive* $\frac{m}{n}$*-extremal fractions. There are nonnegative integer numbers* b, c *such that* $p = bp_{j-1} + cp_j$ *and* $q = bq_{j-1} + cq_j$.

Proof. We will use Farey series (see e.g. [HW]). We show that p_{j-1}/q_{j-1}, p_j/q_j are consecutive terms of the Farey series of order q_j. If not then there is a term p^*/q^* such that $p_{j-1}/q_{j-1} < p^*/q^* < p_j/q_j$ and p^*/q^*, p_j/q_j are consecutive terms of the Farey series of order q_j. So we have $q^* \le q_j$.

If $q^* < q_j$ then $\frac{[q^* m/n]}{q^*}$ is an $\frac{m}{n}$-extremal fraction—a contradiction with the assumption that $\frac{p_{j-1}}{q_{j-1}}, \frac{p_j}{q_j}$ are consecutive $\frac{m}{n}$-extremal fractions.

If $q^* = q_j$ then from Theorem 28 of [HW] we have $p_j q^* - p^* q_j = 1$. But this is possible only if $q_j = 1$, contrary to the definition of $\frac{m}{n}$-extremal fractions.

Hence $p_{j-1}/q_{j-1}, p_j/q_j$ are consecutive terms of the Farey series of order q_j and our lemma follows from [HW], 3.3. ∎

Lemma 4.3. *Let P be an X-minimal $\frac{m}{n}$-unicycle* ($m > n$ *coprime*) *and* $\frac{p}{q}$ *be an* $\frac{m}{n}$*-extremal fraction. Then* f_P *has a* $\frac{p}{q}$*-unicycle* Q *with* $\mathrm{per}(Q) = p+q$.

Proof. If $p/q = m/n$ then we can set $Q = P$. So we can assume that $p/q < m/n$ and let $c \in \mathrm{Fix}(f_P)$.

We will show that there is a P-semicycle with eccentricity $\frac{p}{q}$. We will define a code $C = \langle c_i \rangle_{i=1}^{(m+n)q}$ where

$$c_i = \begin{cases} 0 & \text{if } f_P^{i-1}(p_1) < c, \\ 1 & \text{if } f_P^{i-1}(p_1) > c. \end{cases}$$

Note that $c_1 = 0$ and if $c_i = 1$ then $c_{i-1} = 0$ and $c_{i+1} = 0$ (if $i + 1 \le (m+n)q$). There is a close connection between C and K_P:

$$K_P(f_P^i(p_1)) = in - (m+n)\sum_{j=1}^{i} c_j.$$

We will show that there is a piece of code $C^* = \langle c_i \rangle_{i=j+1}^{j+p+q}$ such that $c_{j+1} = 0$, $c_{j+p+q} = 1$ and $\sum_{i=j+1}^{j+p+q} c_i = q$.

Assume to the contrary that there is no such sequence C^*.

We will use a technique very similar to the proof of Lemma 3.4. Let i_j be such that $c_{i_j} = 1$ and $\sum_{i=1}^{i_j} c_i = j$ (i_j is the position of the jth unit in C).

Note that $K_P(p) \geq 0$ for all $p \in P$ because P has monotone code. If $i_q \leq p + q$ then $K_P(f_P^{i_q}(p_1)) < 0$ (because $p/q < m/n$)—a contradiction. Hence $i_q > p + q$.

If $i_q - i_1 < p + q$ then $\langle c_i \rangle_{i=i_q-(p+q)}^{i_q}$ would be our sequence $\mathcal{C}^*$. Hence $i_q - i_1 \geq p + q$ and because $c_{i_q+1} = 0$ we have $i_{q+1} > i_q + 1$ and so $i_{q+1} - (i_1 + 1) > p + q$. Again because there is no sequence C^* we have $i_{q+1} - i_{1+1} \geq p + q$. Repeating this argument we obtain

$$i_{j+q} - i_{j+1} \geq p + q$$

for $1 \leq j \leq (n-1)q$. Now using these inequalities and the fact that

$$1 < i_1 < i_q < i_q + 1 < i_{q+1} < \ldots < i_{jq}$$
$$< i_{jq} + 1 < i_{jq+1} < \ldots < i_{nq} \leq (m+n)q$$

we get the inequality

$$(m+n)q \geq \sum_{j=1}^{n} (1 + i_{jq} - i_{(j-1)q+1}) \geq n + n(p+q).$$

Hence $m/n \geq (p+1)/q$, which contradicts the assumption that $\frac{p}{q}$ is an $\frac{m}{n}$-extremal fraction. Hence we have proved the existence of a sequence C^*.

Now we will show that the sequence $A = \langle f_P^i(p_1) \rangle_{i=j+1}^{j+p+q+1}$ connected with C^* is a P-semicycle.

Since $c_{j+1} = 0$ and $c_{j+p+q} = 1$ we have $f_P^{j+1}(p_1), f_P^{j+p+q+1}(p_1) < c$. Moreover,

$$K_P(f_P^{j+p+q+1}(p_1)) = K_P(f_P^{j+1}(p_1)) + pn - qm < K_P(f_P^{j+1}(p_1))$$

and by monotonicity $f_P^{j+p+q+1}(p_1) < f_P^{j+1}(p_1) < c$. Therefore A is a P-semicycle. Clearly its eccentricity is $\frac{p}{q}$ and so by Lemma 3.2 the function f_P has a $\frac{p}{q}$-unicycle Q. Finally, as p, q are coprime we have $\mathrm{per}(Q) = p + q$. ∎

LEMMA 4.4. *Suppose m, n are coprime, p, q are coprime and $m/n \geq p/q \geq 1$. Then an X-minimal $\frac{m}{n}$-unipattern forces some $\frac{p}{q}$-unipattern with period $p + q$.*

Proof. Let A be an X-minimal $\frac{m}{n}$-unipattern.

If $p/q \leq p_i/q_i < m/n$ where $\frac{p_i}{q_i}$ is an $\frac{m}{n}$-extremal fraction then from Lemmas 4.1 and 4.3 the pattern A forces an X-minimal $\frac{m^*}{n^*}$-pattern A^* such that $p_i/q_i \leq m^*/n^*$ and $\mathrm{per}(A^*) < \mathrm{per}(A)$. But the forcing relation is transitive and so it is enough to prove that A^* forces some $\frac{p}{q}$-unipattern with period $p+q$. We can repeat this reduction and since we decrease the period we must stop after finitely many steps.

Hence we can assume that $p_{a-1}/q_{a-1} < p/q \le p_a/q_a$ where $\frac{p_{a-1}}{q_{a-1}}$, $\frac{p_a}{q_a}$ are two largest $\frac{m}{n}$-extremal fractions.

From Lemma 4.2 we have $p = bp_{a-1} + cp_a$ and $q = bq_{a-1} + cq_a$ for some nonnegative integers b, c. It is clear that $p_a/q_a = m/n$ and $c > 0$ because $p_{a-1}/q_{a-1} < p/q$.

Let P be a representative of A. By Lemma 4.3 the function f_P has a $\frac{p_{a-1}}{q_{a-1}}$-unicycle P_{a-1} with period $p_{a-1} + q_{a-1}$ and by Lemma 2.4 there is a P-loop $\mathcal{A}_{a-1}$ of length $p_{a-1} + q_{a-1}$ and eccentricity $\frac{p_{a-1}}{q_{a-1}}$ connected with this cycle.

The loop $\mathcal{A}_P$ has length $p_a + q_a$ and eccentricity $\frac{p_a}{q_a}$. Moreover, since P is X-minimal by Lemma 3.1 the loop $\mathcal{A}_P$ is simple and therefore it contains all the intervals from $\mathfrak{P}$.

Hence we can connect c times the loop $\mathcal{A}_P$ and b times the loop $\mathcal{A}_{a-1}$ into a single P-loop with length $p+q$ and eccentricity $\frac{p}{q}$. This P-loop gives us a cycle Q (Lemma 2.3) such that $\mathrm{per}(Q) = p + q$ (p, q are coprime). Finally, let $B = [Q]$ and apply Lemma 1.1. ∎

Theorem 4.5. *Any r-pattern forces an X-minimal r-pattern.*

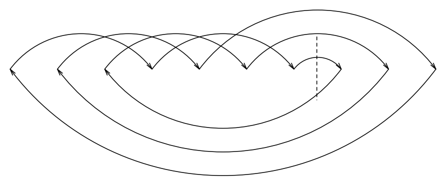
Proof. By Theorem 2.10 an r-pattern forces an r-unipattern and by Lemma 4.1 an r-unipattern forces an X-minimal q-pattern for some $q \ge r$. By Lemma 4.4 the latter forces an r-unipattern with minimal possible period. Because there are only finitely many r-patterns with this period, after repeating this procedure finitely many times we must get an X-minimal r-pattern. ∎

We end this part with an easy algorithm for constructing all X-minimal patterns. Let us consider a cycle P with an eccentricity $\frac{m}{n}$. We have defined a code $K_P : P \to \mathbb{Z}$. Clearly different orbits have different codes. Moreover, from the code $K_P : P \to \mathbb{Z}$ of an X-minimal $\frac{m}{n}$-cycle we can easily reconstruct the function φ of the cycle (P, φ) using the following simple algorithm (assume that $P = \{1, \ldots, m+n\}$).

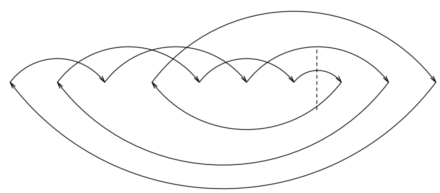
Algorithm 1. *If $K_P(i) - K_P(j) = m$ or $K_P(j) - K_P(i) = n$ then $\varphi(i) = j$.*

We have also defined a code C_P connected with a given X-minimal cycle P. Again, from the code $C_P = \langle c_1, \ldots, c_m \rangle$ connected with an X-minimal $\frac{m}{n}$-cycle P we can easily reconstruct the code $K_P : P \to \mathbb{Z}$ and hence the cycle (P, φ).

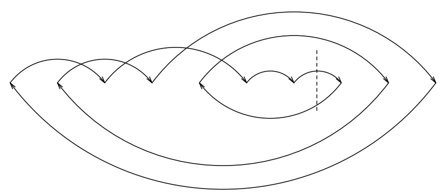
$C_P = \langle 0,0,1,0,1,0,1 \rangle$



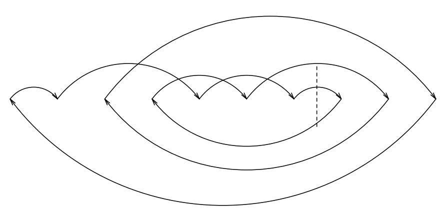
$C_P = \langle 0,0,1,0,0,1,1 \rangle$



$C_P = \langle 0,0,0,1,1,0,1 \rangle$



$C_P = \langle 0,0,0,1,0,1,1 \rangle$



$C_P = \langle 0,0,0,0,1,1,1 \rangle$

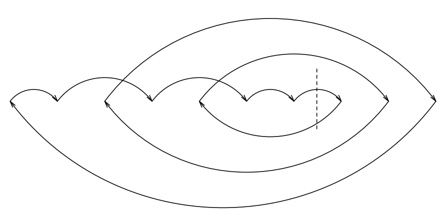


Fig. 9. The list of all X-minimal $\frac{7}{3}$-patterns

ALGORITHM 2. *The function $K_P : P \to \mathbb{Z}$ is*

- *increasing on* $\{1, \dots, m\}$
  with values $\{\sum_{j=1}^{k}(n - c_j m) : k = 1, \dots, m\}$,

- *decreasing on* $\{m+1,\ldots,m+n\}$ *with values* $\{m+\sum_{j=1}^{k}(n-c_j m) : c_k = 1\}$.

Now we take some code C^* which is only a rotation of C_P. The function K^* obtained from C^* using Algorithm 2 is nothing but K_P shifted by some negative multiple of n. Therefore if we apply Algorithm 1 to K^* we again obtain the cycle (P,φ).

Hence we can get any X-minimal $\frac{m}{n}$-cycle by choosing a sequence $C \in C(m,n)$ where $C(m,n) = \{\{c_i\}_{i=1}^m \in \{0,1\}^m : \sum_{i=1}^m c_i = n\}$, then using Algorithm 2 to get a code K^* and finally Algorithm 1 to get an X-minimal cycle. (See Fig. 9.)

Note that we get different patterns if and only if we start from $C_1, C_2 \in C(m,n)$ such that C_1 is not a rotation of C_2. So we have the following simple

Corollary 4.6 (Proposition 4.4 of [BM]). *There are* $m!/(mn!(m-n)!)$ *different X-minimal* $\frac{m}{n}$*-patterns* (m,n *coprime*).

Remark. According to this corollary there is a unique X-minimal $\frac{n+1}{n}$-pattern. This is of course the pattern of the Štefan cycle because this is the only pattern that does not force any other pattern with period $2n+1$ (see eg. [ALM]) and clearly any pattern with period $2n+1$ has eccentricity at least $\frac{n+1}{n}$.

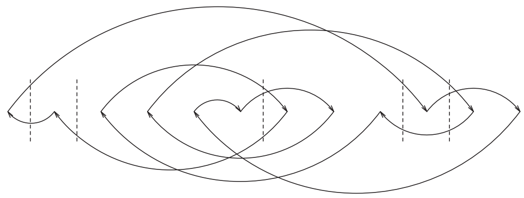
5. A generalization of Sharkovskiĭ's theorem. If we look back at Lemma 3.4 and Theorems 3.7, 4.5 and 2.10 then we can see that they in fact give a generalization of a part of Sharkovskiĭ's Theorem for odd periods. Indeed, a pattern with period $2k+1$ has eccentricity at least $\frac{k+1}{k}$, by Lemma 3.4 it forces a pattern with eccentricity $\frac{k+2}{k+1}$, which by Theorems 3.7, 4.5 and 2.10 forces a pattern with period $2k+3$.

The part of Sharkovskiĭ's Theorem concerning even periods is somehow hidden in eccentricity 1. So in order to get a full generalization we need to define a better type of patterns that will make a finer division of the set of all patterns with eccentricity 1.

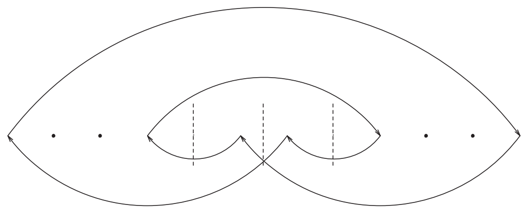
Let us look a bit closer at a periodic orbit (P,φ) with $\mathrm{per}(P) > 1$ and $E(P) = 1$. It is clear that $\mathrm{per}(P)$ is even. So $P = P_1 \cup P_2$ such that (P_i,φ^2) is a periodic orbit with period $\mathrm{per}(P)/2$ for $i = 1,2$.

We shall say that $[(P,\varphi)]$ is a $(2,r)$-*pattern* if $E([(P,\varphi)]) = 1$, $\mathrm{per}(P) > 1$ and $E([(P_i,\varphi^2)]) = r$ for some $i \in \{1,2\}$. Inductively we say that $[(P,\varphi)]$ is a $(2^k,r)$-*pattern* for $k > 1$ if $E([(P,\varphi)]) = 1$ and $[(P_i,\varphi^2)]$ is a $(2^{k-1},r)$-pattern for some $i \in \{1,2\}$. Finally, we say that an r-pattern is a $(1,r)$-*pattern*. (See Fig. 10.)

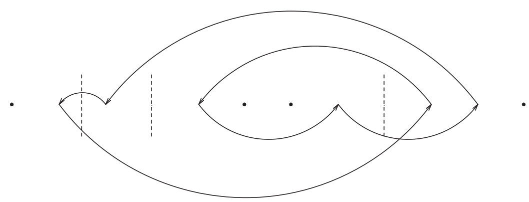
A cycle (P, φ) with eccentricities $\frac{1}{11}$, $\frac{1}{5}$, $\frac{1}{1}$, $\frac{3}{1}$ and $\frac{5}{1}$:



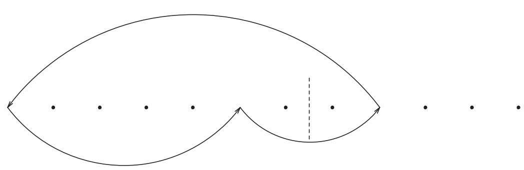
(P, φ) is a 1-cycle so we look at (P_1, φ^2) and (P_2, φ^2).
The cycle (P_1, φ^2) has eccentricities $\frac{1}{2}$, $\frac{1}{1}$ and $\frac{2}{1}$:



The cycle (P_2, φ^2) has eccentricities $\frac{1}{5}$, $\frac{1}{2}$ and $\frac{2}{1}$:



(P_1, φ^2) is a 1-cycle so we look at $(P_{1,1}, \varphi^4)$ and $(P_{1,2}, \varphi^4)$.
The cycle $(P_{1,1}, \varphi^4)$ has eccentricity $\frac{2}{1}$:



The cycle $(P_{1,2}, \varphi^4)$ has eccentricity $\frac{2}{1}$:

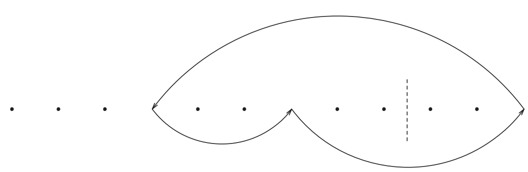


Fig. 10. The pattern $[(P, \varphi)]$ has types $(1, \frac{11}{1})$, $(1, \frac{5}{1})$, $(1, \frac{3}{1})$, $(1, \frac{1}{1})$, $(2, \frac{5}{1})$, $(2, \frac{2}{1})$, $(2, \frac{1}{1})$ and $(4, \frac{2}{1})$.

We define a space $\mathcal{X} = \{(2^k, a) : k \in \mathbb{N} \cup \{0\},\ a \in \mathbb{R} \cup \{\infty\},\ a \geq 1\} \cup \{(2^\infty, 1)\}$ and a total ordering relation on $\mathcal{X}$ such that

$$(2^{k-1}, a) > (2^{k-1}, b) > (2^k, a) > (2^\infty, 1) > (2^k, 1) > (2^{k-1}, 1)$$

for any $a, b \in \mathbb{R} \cup \{\infty\}$ such that $a > b > 1$ and $k \in \mathbb{N}$.

For any $(a, b) \in \mathcal{X}$ we define

$$\mathcal{X}(a, b) = \{(c, d) \in \mathcal{X} : (a, b) \geq (c, d),\ c \in \mathbb{N},\ d \in \mathbb{Q}\},$$
$$\mathcal{X}_0(a, b) = \{(c, d) \in \mathcal{X} : (a, b) > (c, d),\ c \in \mathbb{N},\ d \in \mathbb{Q}\}.$$

Now we may state

THEOREM 5.1 (Generalized Sharkovskiĭ's Theorem). (i) *Any (a, b)-pattern forces a (c, d)-pattern for any $(c, d) \in \mathcal{X}(a, b)$.*

(ii) *For any $(a, b) \in \mathcal{X}$ there is a function $f \in C(I, I)$ such that f exhibits a (c, d)-pattern if and only if $(c, d) \in \mathcal{X}(a, b)$.*

(iii) *For any $(a, b) \in \mathcal{X}$ there is a function $f \in C(I, I)$ such that f exhibits a (c, d)-pattern if and only if $(c, d) \in \mathcal{X}_0(a, b)$.*

(iv) *A $\left(2^k, \frac{m}{n}\right)$-pattern with $\frac{m}{n} > 1$ (m, n coprime) forces a pattern with period $2^k(m + n)$. A $(2^k, 1)$-pattern forces a pattern with period 2^k.*

Proof. Part (i) will be proved as Lemma 5.9 and part (iv) will be proved as Lemma 5.10. Part (iii) will be proved as Lemma 5.16 and finally part (ii) will follow directly from Lemmas 5.15, 5.16 and Claim 5.17. ∎

In order to prove the theorem above we need the notion of block structure.

Let (P, φ) be a cycle of period n and $B = [(\{1, \ldots, m\}, \psi)]$ be a pattern of period m. Let $P = \{p_1, \ldots, p_n\}$ have the spatial labeling. We say that (P, φ) *has a block structure* over B if $n = sm$, $P = P_1 \cup \ldots \cup P_m$ with $P_i = \{p_{(i-1)s+1}, \ldots, p_{(i-1)s+s}\}$ for all $i = 1, \ldots, m$ and $\varphi(P_i) = P_{\psi(i)}$. Each of the sets P_i will be called a *block* of P. In other words, we could consider each block as a "fat" point and P as a "fat" cycle with pattern B (see Fig. 11).

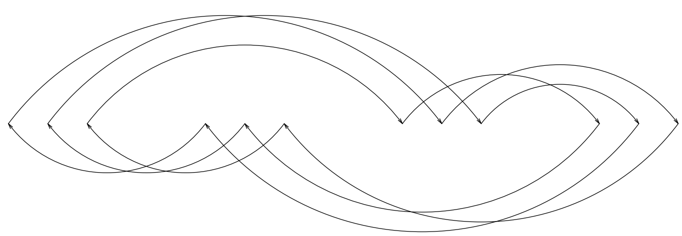


Fig. 11. A pattern which has a block structure over a pattern with period 4

Assume that P has a block structure over B and (Q, ψ) is a cycle with pattern B. Then we also say that P has a block structure over (Q, ψ). If

P has a block structure over B (respectively over Q) we also say that the pattern $[P]$ has a block structure over B (respectively over Q).

Note that if (P,φ) has a block structure over a pattern of period m, then (P_i,φ^m) is a cycle of period $\mathrm{per}(P)/m$ for all i.

If a cycle (pattern) has a block structure over a pattern with period 2 then we say that it *has a division.*

We already have enough information about patterns of type $(1,r)$ but we have no information about the forcing relation for patterns of type $(2^k,r)$ where $k\in\mathbb{N}$. So take a $(2^k,r)$-pattern and let (P,φ) be its representative. Because $k\ge 1$ our pattern is also a $(2,q)$-pattern (either $q=1$ or $q=r$). Hence $\mathrm{per}(P)=2n$ where $n\in\mathbb{N}$. There are two possibilities. Either P has a division or not. In the latter case we can use the following lemma.

LEMMA 5.2 (Proposition 3.4 of [LMPY]). *Let A be a pattern with* $\mathrm{per}(A)=2n$ *that does not have a division* (*so* $n>1$). *Then if n is odd*, *the pattern A forces a pattern with period n. If n is even it forces a pattern with period $n+1$.*

Hence we have the following simple

COROLLARY 5.3. *A pattern with period greater than* 1 *which does not have a division forces a $(1,q)$-pattern for some $q>1$.*

Proof. This is straightforward from Lemma 5.2. ∎

Now we will look closely at the patterns that have a division. Let (P,φ) be a representative of such a pattern. Obviously P is a unicycle, $E(P)=1$ and $\mathrm{per}(P)>1$. We can look at the two cycles (P_1,φ^2) and (P_2,φ^2). If we have information about the types of patterns forced by $[P_1]$ and $[P_2]$ we can deduce information about the patterns forced by $[P]$. More precisely, we have

LEMMA 5.4. *Let (P,φ) be a representative of a pattern with division and $P_1\subset P$ such that (P_1,φ^2) is a cycle. If $[P_1]$ forces an (a,b)-pattern A then $[P]$ forces a $(2a,b)$-pattern with division and period* $2\,\mathrm{per}(A)$.

Proof. Suppose $[P_1]$ forces an (a,b)-pattern A. Consider the function f_P^2 and the interval $I=\mathrm{conv}(P_1)$. Because P has a division we have $f_P(P_1)=P_2$, $f_P(P_2)=P_1$ and $\mathrm{conv}(P_1)\cap\mathrm{conv}(P_2)=\emptyset$. Hence $f_P^2|_I\in C(I,I)$. But $f_P^2|_I$ exhibits the pattern $[P_1]$ (it has the cycle P_1) and therefore it has a cycle Q_1 which is a representative of the pattern A. Let $f_P(Q_1)=Q_2$ and $Q=Q_1\cup Q_2$. Clearly $(Q,f_P|_Q)$ is a cycle. We have $Q_2\subset\mathrm{conv}(P_2)$ and therefore Q has a division, $E(Q)=1$ and $\mathrm{per}(Q)>1$. Hence $[Q]$ is a $(2a,b)$-pattern with division. Clearly $\mathrm{per}(Q)=2\,\mathrm{per}(A)$. ∎

LEMMA 5.5. *A $(1,r)$-pattern with $r>1$ forces a $(2,q)$-pattern with division for each $q\ge 1$.*

Proof. Let A be a $(1,r)$-pattern with $r > 1$. There is a $k \in \mathbb{N}$ such that $r > (k+1)/k$. By Theorems 2.10 and 4.5, A forces an X-minimal $(1, \frac{k+1}{k})$-pattern (which must be a Štefan pattern). So we may assume that A is a Štefan pattern and $(P = \{p_1, \ldots, p_{2k+1}\}, \varphi)$ is its representative. We have

$$\varphi(p_1) = p_{k+1},$$
$$\varphi(p_i) = \begin{cases} p_{2k+3-i} & \text{for } i = 2, \ldots, k+1, \\ p_{2k+2-i} & \text{for } i = k+2, \ldots, 2k+1, \end{cases}$$

and $p_{k+1} < c < p_{k+2}$ for $c \in \mathrm{Fix}(f_P)$. So $\mathfrak{P} = \{J_i\}_{i=1}^{2k+1}$ and

$$\begin{aligned} J_1 &\xrightarrow{P} J_j && \text{for } j = k+1, \ldots, 2k+1, \\ J_j &\xrightarrow{P} J_{2k+3-j} && \text{for } j = 2, \ldots, k+1, \\ J_j &\xrightarrow{P} J_{2k+2-j} && \text{for } j = k+2, \ldots, 2k+1, \\ J_{k+2} &\xrightarrow{P} J_{k+1}. \end{aligned}$$

Note that only J_1 and J_{k+2} P-cover more than one interval. Hence

$$\mathcal{A} = \overbrace{\langle J_{k+2}, J_{k+1}, \ldots, J_{k+2}, J_{k+1}\rangle}^{s+1 \text{ times } J_{k+2}, J_{k+1}} + \langle J_{k+2}, J_k, J_{k+3}, J_{k-1}, \ldots, J_{2k+1}, J_1\rangle$$

is a P-loop of length $2(k+s+1)$. Because the interval J_1 is only once in the loop $\mathcal{A}$ and $\mathcal{A} \neq \mathcal{A}_P$ the cycle Q given by the P-loop $\mathcal{A}$ has period $2(k+s+1)$. We can write $Q = \{q_1, \ldots, q_{2(k+s+1)}\}$ with spatial labeling. By the alternating structure of $\mathcal{A}$, whenever $f_P(q_i) = q_j$ we have $c \in \mathrm{conv}(\{q_i, q_j\})$ and so Q has a division. Moreover, because f_P is monotone on $[p_2, p_{2k+1}]$ we have

$$f_P(q_1) = q_{k+s+2},$$
$$f_P(q_i) = \begin{cases} q_{2(k+s+2)-i} & \text{for } i = 2, \ldots, k+s+1, \\ q_{2(k+s+1)+1-i} & \text{for } i = k+s+2, \ldots, 2(k+s+1). \end{cases}$$

Take $Q_1 = \{q_1, \ldots, q_{k+s+1}\}$. Then $(Q_1, f_P^2|_{Q_1})$ is a cycle and

$$f_P^2(q_1) = q_{k+s+1},$$
$$f_P^2(q_i) = q_{i-1} \quad \text{for } i = 2, \ldots, k+s+1.$$

Hence the cycle Q_1 is a $\frac{k+s}{1}$-unicycle. Since we can choose s arbitrarily large we are done by Theorem 2.10 and Lemma 5.4. ∎

In order to be able to use Lemma 5.5 effectively we must use patterns with a special structure.

Definition. Let A be a $(2^k, r)$-pattern and (P_1^0, φ) be a representative of A. From the definition of $(2^k, r)$-pattern we see that for $1 \le j \le k$

there are sets P_1^j, P_2^j such that $P_1^{j-1} = P_1^j \cup P_2^j$, $\varphi^{2^{j-1}}(P_1^j) = P_2^j$ and (P_1^j, φ^{2^j}) is a $(2^{k-j}, r)$-cycle. The sequence $\langle P_1^j \rangle_{j=0}^k$ of sets will be called $(2^k, r)$-*determining*. Moreover, if the cycle (P_1^j, φ^{2^j}) has a division for all $j < k$ then the sequence $\langle P_1^j \rangle_{j=0}^k$ will be called *splitting*. In this case we say that the pattern A has a splitting $(2^k, r)$-determining sequence (see Fig. 12).

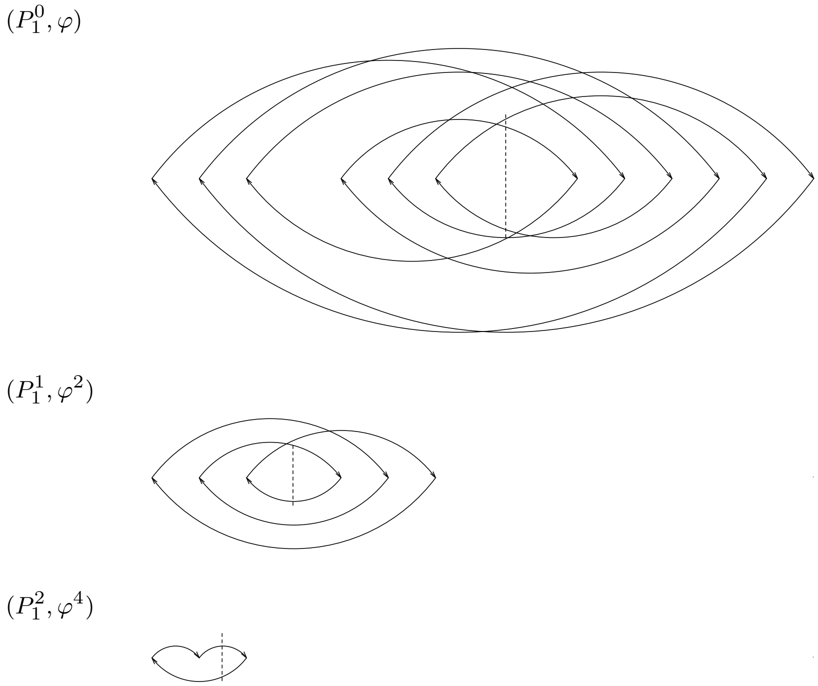


Fig. 12. A $(2^2, \frac{2}{1})$-pattern with a splitting $(2^2, \frac{2}{1})$-determining sequence $\langle P_1^j \rangle_{j=0}^2$

Lemma 5.6. *An (a, b)-pattern with an (a, b)-determining sequence which is not splitting forces a (c, d)-pattern A for some $c < a$ and $d > 1$ such that any (c, e)-determining sequence of A is splitting.*

Proof. Let (P_1^0, φ) be a representative of our (a, b)-pattern and $\langle P_1^j \rangle_{j=0}^k$ be an (a, b)-determining sequence which is not splitting. Take the smallest $j < k$ such that the cycle (P_1^j, φ^{2^j}) does not have a division. Using Corollary 5.3 and repeatedly Lemma 5.4 we find that our pattern forces a (c, d)-pattern such that $c = 2^j < a$ and $d > 1$. We can repeat the same procedure for the new (c, d)-pattern and after finitely many steps we must get a pattern A with splitting determining sequences. ∎

Lemma 5.7. *A $(2^k, r)$-pattern with $r > 1$ and a splitting $(2^k, r)$-determining sequence forces a $(2^k, q)$-pattern for any $q \in \mathbb{Q}$ such that $r \geq q \geq 1$.*

Proof. This follows easily from Theorem 2.10 and Lemma 5.4. ∎

Lemma 5.8. *A $(2^k, r)$-pattern with $r > 1$ and a splitting $(2^k, r)$-determining sequence forces a $(2^{k+1}, q)$-pattern for any $1 \le q \in \mathbb{Q}$.*

Proof. This follows easily from Lemmas 5.5 and 5.4. ∎

Now we are quite ready to prove two parts of Theorem 5.1.

Lemma 5.9. *An (a,b)-pattern forces a (c,d)-pattern for any $(c,d) \in \mathcal{X}(a,b)$.*

Proof. Lemma 5.6 shows that an (a,b)-pattern A forces an (a^*, b^*)-pattern B with splitting (a^*, b^*)-determining sequence for some $(a^*, b^*) \ge (a,b)$. If $b^* > 1$ then using Lemmas 5.7 and 5.8 inductively we see that B forces a (c,d)-pattern for any $(c,d) \le (a^*, b^*)$ such that $c \ge a^*$.

Now if $(c,d) \le (a^*, b^*)$ and $c < a^*$ then $d = 1$. But from the definition it is clear that an (a^*, b^*)-pattern is a $(c,1)$-pattern for any $(c,1) \le (a^*, b^*)$ such that $c < a^*$.

So B forces a (c,d)-pattern for each $(c,d) \in \mathcal{X}(a^*, b^*)$. But A forces B and $(a^*, b^*) \ge (a,b)$. Hence we are done because the forcing relation is transitive. ∎

Lemma 5.10. *A $\left(2^k, \frac{m}{n}\right)$-pattern with $m/n > 1$ (m, n coprime) forces a $\left(2^k, \frac{m}{n}\right)$-pattern with period $2^k(m+n)$. A $(2^k, 1)$-pattern forces a $(2^k, 1)$-pattern with period 2^k.*

Proof. Using Lemmas 5.6 and 5.5 as in the proof of Lemma 5.9 we see that a $\left(2^k, \frac{m}{n}\right)$-pattern forces a $\left(2^k, \frac{m}{n}\right)$-pattern A with a splitting $\left(2^k, \frac{m}{n}\right)$-determining sequence. Let (P_1^0, φ) be a representative of A and $\langle P_1^j \rangle_{j=0}^k$ be the splitting $\left(2^k, \frac{m}{n}\right)$-determining sequence. So $[P_1^k]$ is an $\frac{m}{n}$-pattern. Using Theorems 4.5 and 3.7 together with Lemma 3.4 we deduce that $[P_1^k]$ forces an $\frac{m}{n}$-pattern with period $m+n$ or period 1 if $m/n = 1$. Finally, repeatedly applying Lemma 5.4 we are done. ∎

So we have only two parts of Theorem 5.1 left to prove. As you may already have guessed, knowledge of X-minimal (a,b)-patterns will be very useful for proving them. So first

Definition. An (a,b)-pattern which does not force any other (a,b)-pattern will be called an *X-minimal (a,b)-pattern*.

We will try to prove that some patterns are X-minimal (a,b)-patterns. For this we need to define a special type of block structure.

Let (P, φ) be a cycle and A, B be patterns. We say that (P, φ) is an *A-extension* of B if P has a block structure over B, φ is monotone on each block of P except at most one and, with the notation from the definition of block structure, (P_i, φ^m) has pattern A for some $i \in \{1, \ldots, m\}$ (in fact, this does not depend on i). As above, if P is an A-extension of B and (Q, ψ)

is a cycle with pattern B, then we say that P is an A-extension of (Q, ψ). We also say that $[P]$ is an A-extension of B (respectively of Q) if P is an A-extension of B (respectively of Q).

We define two special types of A-extension. An A-extension where $\mathrm{per}(A)$
$= 2$ will be called a 2-*extension*. An A-extension where A is an X-minimal r-pattern will be called an r-*extension*.

A cycle will be called *simple* if it can be obtained from a cycle of period 1 by making 2-extensions k times and then one r-extension for some $k \in \mathbb{N}$ and $r \in \mathbb{Q}$. A pattern of a simple cycle will be called a *simple pattern*.

Note that a simple pattern obtained from a cycle of period 1 by making 2-extensions k times and then one r-extension will be a $(2^k, r)$-pattern. Moreover, if A is a simple $(2^k, r)$-pattern and B is a simple pattern of period 2^s then an A-extension of B will be a simple $(2^{k+s}, r)$-pattern (see Fig. 13).

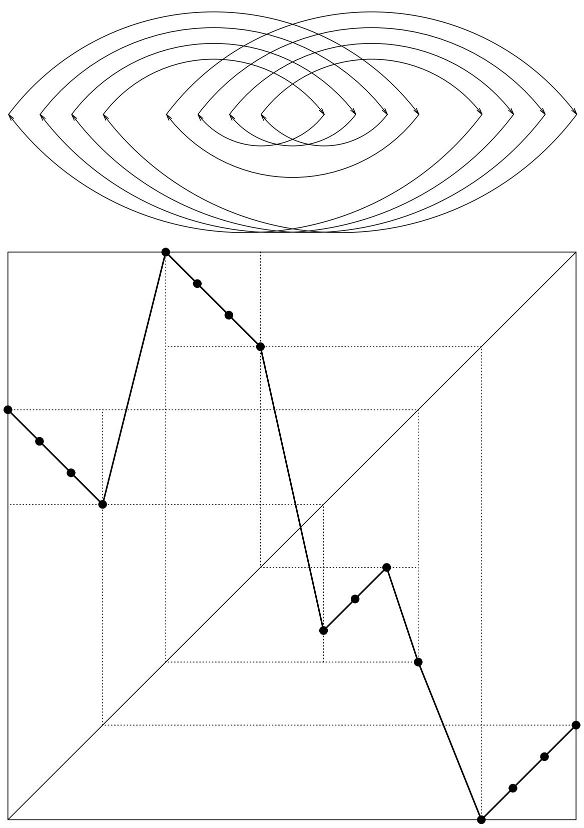


Fig. 13. An example of a simple $(4, \frac{3}{1})$-cycle P and the function f_P

Lemma 5.11 (Proposition 2.10.6 of [ALM]). *Let A, B, C, D be patterns such that C is an A-extension of B and C forces D. Then either B forces*

D or D is an A^*-extension of B for some pattern A^* forced by A. If $C \neq D$ then in the last case $A^* \neq A$.

LEMMA 5.12 (Lemma 2.11.4 of [ALM]). *Let C, B, D be patterns such that C is a 2-extension of B and C forces D. Then either $C = D$ or B forces D.*

The next lemma is only a slight modification of Lemma 2.11.5 of [ALM].

LEMMA 5.13. *Let C be a simple pattern of period 2^k, $k \geq 0$. If C forces a pattern D where $D \neq C$ then D is a simple pattern of period 2^i, $i < k$.*

Proof. We use induction. For $k = 0$ this is obvious. Assume that we know it for simple patterns of period 2^{k-1}. If C is a simple pattern of period 2^k, then C is a 2-extension of a simple pattern of period 2^{k-1}. If C forces D then we are done by Lemma 5.12 and the induction hypothesis. ■

THEOREM 5.14. *A simple pattern is X-minimal.*

Proof. Let C be a simple $(2^k, r)$-pattern. Then there is an X-minimal r-pattern A and a simple pattern B of a period 2^k such that C is an A-extension of B. If $r = 1$ then we are done by Lemma 5.13. Assume now that $r > 1$ and D is a $(2^k, r)$-pattern such that C forces D. Then by Lemmas 5.11 and 5.13, D is an A^*-extension of B where A forces A^*. But $E(A^*) = r$ because D is a $(2^k, r)$-pattern. Finally, $A = A^*$ since A is an X-minimal r-pattern, and $D = C$ from Lemma 5.11. ■

Now we are ready to prove the remaining two parts of Theorem 5.1.

LEMMA 5.15. *Let $(a, b) \in \mathcal{X}(1, \infty)$ and P be a representative of a simple (a, b)-pattern. The function f_P exhibits some (c, d)-pattern if and only if $(c, d) \in \mathcal{X}(a, b)$.*

Proof. Lemmas 1.1 and 5.9 show that f_P exhibits a (c, d)-pattern for any $(c, d) \in \mathcal{X}(a, b)$. Moreover, exactly as in the proof of Theorem 5.14 we can prove that f_P does not exhibit a (c, d)-pattern if $(c, d) \notin \mathcal{X}(a, b)$ (see also Lemma 5.18). ■

LEMMA 5.16. *For any $(a, b) \in \mathcal{X}$ there is a function $f \in C(I, I)$ such that f exhibits some (c, d)-pattern if and only if $(c, d) \in \mathcal{X}_0(a, b)$.*

Proof. Clearly there is a sequence $\langle A_i \rangle_{i=1}^{\infty}$ of patterns such that A_i is a simple (a_i, b_i)-pattern, $(a, b) > (a_{i+1}, b_{i+1}) > (a_i, b_i)$ and $(a, b) = \sup\{(a_i, b_i) : i \geq 1\}$.

Let P_i be a representative of A_i such that $\operatorname{conv}(P_i) = [x_i, y_i] \subset (0, 1)$ and $y_i < x_{i+1}$ for all $i \in \mathbb{N}$.

Define a function $f \in C(I, I)$ where $I = [0, 1]$ as follows. Let $f(0) = 0$, $f(1) = 1$, $f|_{[x_i, y_i]} = f_{P_i}$ for $i \geq 1$ and $f|_J$ be linear for any interval $J \subset I$ such that $J \cap [x_i, y_i] = \emptyset$ for each $i \geq 1$ (see Fig. 14).

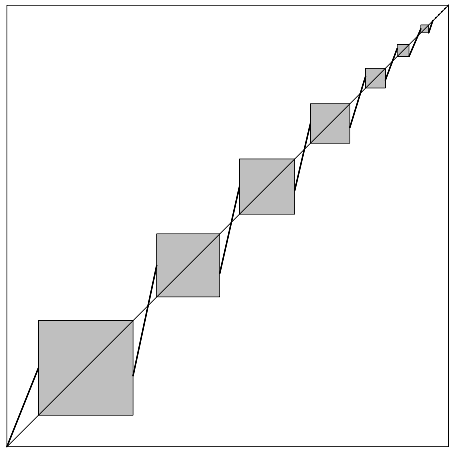


Fig. 14. The graph of a function f. Inside the filled squares are the functions f_{P_i}.

It is easy to see that outside the intervals $[x_i, y_i]$ the function f has only fixed points. Hence if f has a (c,d)-cycle P then there is $i \geq 1$ such that $P \subset [x_i, y_i]$. So f_{P_i} exhibits $[P]$ and from Lemma 5.15 we have $(c,d) \in \mathcal{X}(a_i, b_i) \subset \mathcal{X}_0(a,b)$. Moreover, for any $(c,d) \in \mathcal{X}_0(a,b)$ there is an $i \geq 1$ such that $(a,b) > (a_i, b_i) \geq (c,d)$. By Lemma 5.15, f_{P_i} exhibits a (c,d)-pattern and therefore so does f. ∎

Finally, it suffices to realize

CLAIM 5.17. *If* $(a,b) \in \mathcal{X} \setminus \mathcal{X}(1,\infty)$ *then* $\mathcal{X}(a,b) = \mathcal{X}_0(a,b)$.

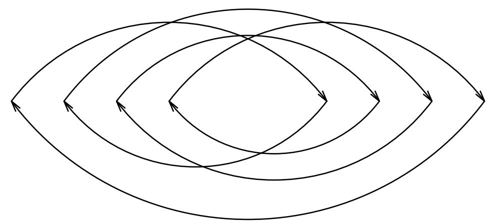
Now let us study X-minimal (a,b)-patterns more closely. The first natural question is to characterize all X-minimal (a,b)-patterns. Unfortunately, it is not true, as one might expect, that the only X-minimal patterns are the simple ones (this idea seems to be natural for someone who knows the characterization of "primary" patterns—see [ALM]) (see Fig. 15).

We can still get some more information about X-minimal (a,b)-patterns. A first question is: what types of patterns does an X-minimal (a,b)-pattern force? Can an X-minimal (a,b)-pattern which is not simple force a (c,d)-pattern for $(c,d) \notin \mathcal{X}(a,b)$?

LEMMA 5.18. *An X-minimal* (a,b)*-pattern forces a* (c,d)*-pattern if and only if* $(c,d) \in \mathcal{X}(a,b)$.

Proof. The "if" part follows from Lemma 5.9. Assume that an X-minimal (a,b)-pattern A forces a (c,d)-pattern for some $(c,d) > (a,b)$. Clearly A has only a finite set of types (fewer than $\mathrm{per}(A)$) and $\mathcal{X}(c,d) \setminus \mathcal{X}(a,b)$ has infinitely many elements. Using this fact, Lemma 5.9 and the fact that the forcing relation is transitive we see that A forces an (e,f)-pattern B such that $(e,f) > (a,b)$ and $A \neq B$. Lemma 5.9 shows that B

A cycle P



The function f_P and an important part of the function f_P^2 (dotted)

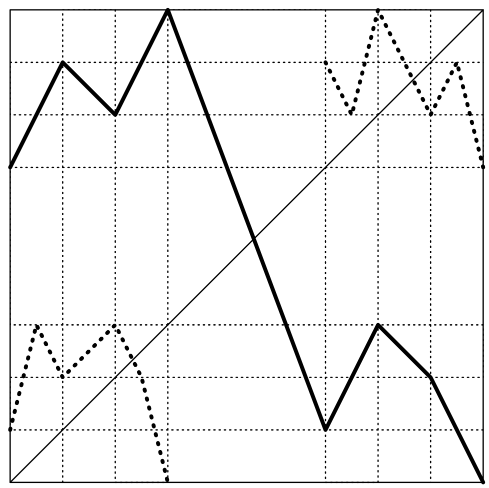


Fig. 15. An example of an X-minimal $(2, \frac{3}{1})$-pattern which is not simple

forces an (a, b)-pattern C. So A forces C and because the forcing relation is antisymmetric we have $A \neq C$—a contradiction. ∎

LEMMA 5.19. *An X-minimal* $(2^k, r)$*-pattern has a block structure over a simple* $(2^k, 1)$*-pattern.*

Proof. It suffices to realize that if a $(2^k, r)$-pattern does not have a block structure over a simple $(2^k, 1)$-pattern then either it has a type $(2^n, s)$ for $n < k$ and $s > 1$ or there is a $(2^k, q)$-determining sequence that is not splitting (note that q does not have to be equal to r). Because we have an X-minimal $(2^k, r)$-pattern, Lemma 5.18 shows that the first case is not possible. In the second case, by Lemma 5.6, our pattern forces an (a, b)-pattern for some $(a, b) > (2^k, r)$—again a contradiction with Lemma 5.18. ∎

LEMMA 5.20. *An X-minimal* $(2^k, \frac{m}{n})$*-pattern* (m, n *coprime*) *has period* $2^k(m + n)$ *if* $m > n$ *and period* 2^k *if* $m = n$.

Proof. This follows immediately from Lemma 5.10. ∎

So we immediately have the following simple

COROLLARY 5.21. *An X-minimal* $(2^k,1)$*-pattern is a simple* $(2^k,1)$*-pattern.*

Proof. This is straightforward from Lemmas 5.19 and 5.20. ∎

Now let A be an X-minimal $(2^k,r)$-pattern and (P,φ) be a representative of A. By Lemma 5.19, P consists of 2^k blocks P_i. Lemmas 2.1, 5.4 and 5.18 show that (P_i,φ^{2^k}) is a unicycle and $E(P_i)\le r$. Clearly, the pattern $[P_i]$ need not force a pattern with eccentricity greater than r. So if $E(P_i)=r$ then $[P_i]$ is an X-minimal r-pattern (see Lemma 3.5).

Now assume that A is an X-minimal $\left(2^k,\frac{n+1}{n}\right)$-pattern. Lemma 5.20 shows that the cycle (P_i,φ^{2^k}) has period $2n+1$. But the minimal possible eccentricity of a pattern with this period is $\frac{n+1}{n}$. Hence in this case for every i the pattern $[(P_i,\varphi^{2^k})]$ must be an X-minimal $\frac{n+1}{n}$-pattern. But there is only one X-minimal $\frac{n+1}{n}$-pattern and that is exactly the pattern of the Štefan cycle of period $2n+1$.

In this special case we can prove even more. We need the following

LEMMA 5.22 ([B], Theorem 2.11.1 of [ALM]). *Let A be a pattern with* $\operatorname{per}(A)=2^k(2n+1)$ *and B be the pattern of the Štefan cycle of period* $2n+1$. *If A is not a B-extension of a simple* $(2^k,1)$*-pattern then A forces another pattern with period* $2^k(2n+1)$.

Now we can prove

LEMMA 5.23. *An X-minimal* $(2^k,\frac{n+1}{n})$*-pattern is simple.*

Proof. By Lemma 5.20, an X-minimal $(2^k,\frac{n+1}{n})$-pattern A has period $2^k(2n+1)$. If it is not simple then by Lemma 5.22 it forces another pattern B with the same period. It is easy to see that if (a,b) is a maximal (in the sense of ordering on $\mathcal{X}$) type of the pattern B then $(a,b)\ge(2^k,\frac{n+1}{n})$. Hence by Lemma 5.9, B forces a $(2^k,\frac{n+1}{n})$-pattern. So A is not an X-minimal $\left(2^k,\frac{n+1}{n}\right)$-pattern—a contradiction. ∎

We end this section and the whole paper by a conjecture.

CONJECTURE. *If A is an X-minimal* $(2^k,r)$*-pattern and* (P,φ) *is a representative of A such that* (P_i,φ^{2^k}) *is an r-cycle for every block P_i then A is simple.*

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