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Fundamental pro-groupoids and covering projections

by

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Abstract. We introduce a new notion of covering projection $E \to X$ of a topological space X which reduces to the usual notion if X is locally connected. We use locally constant presheaves and covering reduced sieves to find a pro-groupoid $\pi\,\mathrm{crs}(X)$ and an induced category $\mathrm{pro}(\pi\,\mathrm{crs}(X), \mathsf{Sets})$ such that for any topological space X the category of covering projections and transformations of X is equivalent to the category $\mathrm{pro}(\pi\,\mathrm{crs}(X), \mathsf{Sets})$. We also prove that the latter category is equivalent to $\mathrm{pro}(\pi CX, \mathsf{Sets})$, where πCX is the Čech fundamental pro-groupoid of X. If X is locally path-connected and semilocally 1-connected, we show that $\pi\,\mathrm{crs}(X)$ is weakly equivalent to πX, the standard fundamental groupoid of X, and in this case $\mathrm{pro}(\pi\,\mathrm{crs}(X), \mathsf{Sets})$ is equivalent to the functor category $\mathsf{Sets}^{\pi X}$. If $(X,*)$ is a pointed connected compact metrisable space and if $(X,*)$ is 1-movable, then the category of covering projections of X is equivalent to the category of continuous $\check{\pi}_1(X,*)$-sets, where $\check{\pi}_1(X,*)$ is the Čech fundamental group provided with the inverse limit topology.

Introduction. It is well known that if X is a locally path-connected and semilocally 1-connected space then the category $\mathrm{Cov\,proj}\,X$ of covering projections and transformations of X is equivalent to the category of πX-sets, that is, to the functor category $\mathsf{Sets}^{\pi X}$. The aim of this work is to study the category $\mathrm{Cov\,proj}\,X$ for any space X, without local conditions of connectedness.

In 1972–73, Fox [F1, F2] introduced the notion of overlay of a metrisable space. The fundamental theorem of Fox's overlay theory establishes the existence of a bi-unique correspondence between the d-fold overlayings of a connected metrisable space X and the representations of the fundamental trope of X in the symmetric group Σ_d of degree d.

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On the other hand, for a locally connected distributive category $\mathcal{C}$, using the filtered small category of hypercoverings, Artin and Mazur [A-M] constructed a pro-simplicial set $\Pi\mathcal{C}$. In particular, for the category $\mathcal{C}$ induced by a locally connected space, this pro-simplicial set is the Čech pro-simplicial set defined by the Čech nerve of all open coverings $\mathcal{U}$ of the space X. As a consequence of this construction they classify the covering projections of X which are trivial over an open covering $\mathcal{U}$. The construction given by Artin and Mazur cannot be applied to non-locally connected spaces.

The objective of this paper is to solve the classification problem of "covering projections" for a general space. We want to remove the condition of local connectedness considered by Artin and Mazur and the conditions of metrisable space and finite fibre of the d-fold overlayings analysed by Fox. Of course we want to have a classification up to isomorphism of covering projections but we also want to have a classification of morphisms between "covering projections".

For this aim, we consider a new notion of covering projection $E \to X$ using atlases and an equivalence relation between atlases. If X is a connected metrisable space and all the fibres of $E \to X$ have a finite cardinal d, we have Fox's d-fold overlay and if X is a locally connected space we have the usual notion of covering projection given, for instance, in Spanier's book [S] and analysed by Artin and Mazur. To generalise the Fox fundamental trope or the Artin–Mazur fundamental pro-group of a space we consider a fundamental pro-groupoid $\pi\,\mathrm{crs}(X)$ and a category $\mathrm{pro}(\pi\,\mathrm{crs}(X), \mathsf{Sets})$ which is equivalent to the category of covering projections of the space X. This kind of category is also related to the notion of Galois category characterized by Grothendieck [Gro] and to the notion of Galois topos considered by Moerdijk [M].

If G is a pro-finite group, we can consider the category G-$\mathsf{FinSets}$ of continuous finite G-sets. A category $\mathcal{C}$ equivalent to G-$\mathsf{FinSets}$ is said to be a *Galois category*. Grothendieck [Gro] gave an axiomatic description of these categories and proved that the pro-finite group G is unique up to isomorphism. The fundamental group of a pointed connected Grothendieck topos $\mathcal{E}$ can be defined as the group determined by the Galois category $\mathcal{E}_{\mathrm{lcf}}$ of locally constant finite objects in $\mathcal{E}$. For instance, if X is a connected CW-complex, then the category of finite covering projections of X is equivalent to the category of continuous finite $\widehat{\pi_1 X}$-sets, where $\widehat{\pi_1 X}$ is the pro-finite completion of $\pi_1 X$.

We also note that Moerdijk [M] gave a characterization of the toposes of the form BG for G a pro-discrete localic group. He also proved that the category of surjective pro-groups is equivalent to the category of pro-discrete localic groups. For a connected locally connected space this equivalence of categories carries the Artin–Mazur fundamental surjective pro-group to the fundamental localic group considered by Moerdijk. This implies that the category of covering projections of a connected locally connected space is determined either by the Artin–Mazur fundamental pro-group or by the corresponding localic group. Nevertheless, this construction does not characterise the category of covering projections of a non-locally connected space X. In this case the fundamental pro-group(oid) $\pi\,\mathrm{crs}(X)$ that we consider need not be a surjective pro-group(oid). At present, we do not know if for every pro-group G the category $\mathrm{pro}(G, \mathsf{Sets})$ is a Galois topos in the sense of the definition given in [M]. One interesting property of a Galois topos is that the pro-discrete localic group is determined up to isomorphism. However, the category of the form $\mathrm{pro}(G, \mathsf{Sets})$ does not determine the pro-groupoid G up to isomorphism. In §1, we give an example of two non-isomorphic pro-groups G and G' such that $\mathrm{pro}(G, \mathsf{Sets})$ is equivalent to $\mathrm{pro}(G', \mathsf{Sets})$. We do not know if the existence of an equivalence of categories implies that G and G' are weakly equivalent in some sense.

We also analyse the category of covering projections of a compact metrisable space X. In this case, the fundamental pro-groupoid is isomorphic to a tower of groupoids. If we assume that X is connected, then the tower of groupoids reduces to a tower of groups G and if for a given point $x \in X$, (X, x) is 1-movable, then the category $\mathrm{pro}(G, \mathsf{Sets})$ is equivalent to the category of continuous $\lim G$-sets, where $\lim G$ is provided with the inverse limit topology. We note that the condition of pointed 1-movability implies that the Čech fundamental pro-group is isomorphic to a tower $G = \{G(n)\}$ of groups with surjective bonding homomorphisms. Since we are working with a tower we see that $\lim G$ is not trivial and the maps $\lim G \to G(n)$ are surjections. For the more general case of surjective pro-groups, Moerdijk [M] has noted that $\lim G$ can be trivial; he has solved this pathology by considering the inverse limit in the category of localic groups. I suppose that for some notion of "surjective pro-groupoid" G the category $\mathrm{pro}(G, \mathsf{Sets})$ will be equivalent to a category of LG-sets, where LG will be an associated pro-discrete localic groupoid.

This paper illustrates some nice relationships between the étale homotopy developed by Artin and Mazur [A-M], the theory of classifying toposes of localic groupoids of Moerdijk [M] and the methods used by Fox [F1, F2], by Edwards and Hastings [E-H] and by Porter [P] in shape theory and strong shape theory.

We finish this introduction by giving a summary of the main results of the paper. In §1, for a pro-groupoid G we define the category $\mathrm{pro}(G, \mathsf{Sets})$ and we show that a map $f : G \to G'$ in $\mathrm{pro}\,\mathsf{Gpd}$ induces an equivalence of categories $f^* : \mathrm{pro}(G', \mathsf{Sets}) \to \mathrm{pro}(G, \mathsf{Sets})$ if f is an isomorphism in $\mathrm{pro}\,\mathsf{Gps}$, or if f is a (level) weak equivalence, or if G, G' are towers and f is an isomorphism in the category $\mathrm{tow}\,\pi_0\mathsf{Gpd}$. The main results of §2 are the right definition of covering projection, the determination of the pro-groupoid $\pi\,\mathrm{crs}(X)$, and Theorem 2.2, which shows that for any space X the category of covering projections of X is equivalent to the category $\mathrm{pro}(\pi\,\mathrm{crs}(X), \mathsf{Sets})$. In §3, we establish a connection with the Artin–Mazur fundamental pro-groupoid which is isomorphic to the Čech fundamental pro-groupoid, and we find a weak equivalence $\pi\,\mathrm{Sd}\,CX \to \pi\,\mathrm{crs}(X)$ from the fundamental pro-groupoid of the subdivision of the Čech pro-simplicial set to the pro-groupoid $\pi\,\mathrm{crs}(X)$ of reduced covering sieves of X. Therefore the category $\mathrm{pro}(\pi\,\mathrm{crs}(X), \mathsf{Sets})$ is equivalent to $\mathrm{pro}(\pi CX, \mathsf{Sets})$. In §4, we give an easy proof of the standard classification of covering projections of a locally connected and semilocally 1-connected space. Finally, in §5, we show that under some shape conditions, we can obtain surjective towers of groups and in this case the category of covering projections reduces to a category of continuous $\check{\pi}_1(X,*)$-sets, where $\check{\pi}_1(X,*)$ is the Čech fundamental topological group. We also prove as a corollary a version of the fundamental theorem of Fox's overlay theory.

0. Preliminaries. In this section, we introduce some notation and terminology which is frequently used in this paper.

Let C be a small category and C^{op} its opposite. As usual, we denote by $\mathsf{Sets}^{C^{\mathrm{op}}}$ the category whose objects are all functors $P : C^{\mathrm{op}} \to \mathsf{Sets}$ and morphisms $P \to P'$ are all the natural transformations $\theta : P \to P'$ between such functors. A functor $P : C^{\mathrm{op}} \to \mathsf{Sets}$ is also called a *presheaf* on C. A presheaf P on C is said to be *locally constant* if for every arrow $f : A \to B$ in C, $Pf : PB \to PA$ is an isomorphism. We denote by $(\mathsf{Sets}^{C^{\mathrm{op}}})_{\mathrm{lc}}$ the category of locally constant presheaves on C.

For the category C we have the Yoneda embedding $y : C \to \mathsf{Sets}^{C^{\mathrm{op}}}$ defined on objects by $yA(B) = \mathrm{Hom}_C(B, A)$. The following result will be used; for more details we refer the reader to [M-M, Theorem I.5.2].

THEOREM 0.1. *Let $l : C \to \mathcal{D}$ be a functor from a small category C to a cocomplete category $\mathcal{D}$. Then the functor $R : \mathcal{D} \to \mathsf{Sets}^{C^{\mathrm{op}}}$ defined by*

$$RX(C) = \mathrm{Hom}_{\mathcal{D}}(lC, X)$$

has a left adjoint $L : \mathsf{Sets}^{C^{\mathrm{op}}} \to \mathcal{D}$.

Let Δ denote the small category whose objects are finite ordered sets $[n] = \{0 < \ldots < n\}$ and whose morphisms are those $\phi : [n] \to [m]$ which preserve the order. We shall consider the category of simplicial sets as the functor category $\mathsf{SS} = \mathsf{Sets}^{\Delta^{\mathrm{op}}}$. By Theorem 0.1 the functor $y : \Delta \to \mathsf{Sets}^{\Delta^{\mathrm{op}}}$ and the functor $\Delta \to \mathsf{Top}$, $[n] \to \Delta_n$, where Δ_n is the standard n-simplex, induce the *singular functor* $\mathrm{Sing} : \mathsf{Top} \to \mathsf{SS}$ and its left adjoint, the *realization functor* $|-| : \mathsf{SS} \to \mathsf{Top}$, $X \to |X|$. We recall that a map $f : X \to Y$ in SS is a *weak equivalence* if for every $x \in X_0$ and $q \geq 0$ the induced map $\pi_q(f) : \pi_q(X, x) \to \pi_q(Y, fx)$ is an isomorphism, where π_q denotes the homotopy group functor.

Let G be a group. We can view G as a small category with one object $*$ and arrows given by the elements of G. The composition is given by the product of G. In this case Sets^G is the category of left G-sets and $\mathsf{Sets}^{G^{\mathrm{op}}}$ is the category of right G-sets. An object in Sets^G is determined by a homomorphism $\eta : G \to \operatorname{Aut} X$, where $\operatorname{Aut} X$ is the group of automorphisms of a set X. For a given map $\widehat{f} : X \to X'$ in Sets, we denote by $\operatorname{Aut} \widehat{f}$ the group of automorphisms of $\widehat{f}$ in the category Maps(Sets) of maps in Sets. If $\eta : G \to \operatorname{Aut} X$, $\eta' : G \to \operatorname{Aut} X'$ determine two objects in Sets^G, a morphism $f : \eta \to \eta'$ is given by a pair $f = (\theta_f, \widehat{f})$ where $\widehat{f} : X \to X'$ is a map and $\theta_f : G \to \operatorname{Aut} \widehat{f}$ is a group homomorphism such that the following diagram is commutative:

$$\begin{array}{ccc} & & \operatorname{Aut} X \\ & \overset{\eta}{\nearrow} & \uparrow \scriptstyle{\mathrm{pr}_1} \\ G & \xrightarrow{\theta_f} & \operatorname{Aut} \widehat{f} \\ & \underset{\eta'}{\searrow} & \downarrow \scriptstyle{\mathrm{pr}_2} \\ & & \operatorname{Aut} X' \end{array}$$

For G a topological group, we have an analogous category of continuous G-sets, which is denoted by G-Sets. In this case, $\operatorname{Aut} X$ and $\operatorname{Aut} \widehat{f}$ are provided with the discrete topology and we consider continuous homomorphisms η and θ_f.

Given a topological space X, we denote by $\mathcal{O}(X)$ the small category whose objects are all open subsets U of X, and arrows $V \to U$ are inclusions $V \subset U$. In this paper, we will consider the category $\mathsf{Sets}^{\mathcal{O}(X)^{\mathrm{op}}}$ of presheaves on X, and the full subcategory $\mathrm{Sh}(X)$ of sheaves on X. For more properties of sheaves we refer the reader to [M-M] and [J]. We also consider the category $\operatorname{Etale} X$ whose objects are étale maps $p : E \to X$; that is, p is a local homeomorphism in the following sense: For each $e \in E$, there is an open set V such that pV is open in X and $p|V : V \to pV$ is a homeomorphism. A morphism $f : p \to p'$ is given by a continuous map $f : E \to E'$ such that $fp' = p$.

If $F : \mathcal{O}(X)^{\mathrm{op}} \to \mathsf{Sets}$ is a presheaf on X, we can consider $F_x = \operatorname{colim}_{x \in U} F(U)$ and the map $\mathrm{germ}_x : F(U) \to F_x$. For each $\sigma \in F(U)$, $\dot{s}U = \{\mathrm{germ}_x \sigma \mid x \in U\}$ is a subset of $E = \bigsqcup_{x \in E} F_x$. All the subsets $\dot{s}U$ form the base of a topology on E such that the map $p : E \to X$, $p(\sigma) = x$ if $\sigma \in F_x$, is an étale map, which is also called the *bundle of germs* of F. This construction gives a functor $\Lambda : \mathsf{Sets}^{\mathcal{O}(X)^{\mathrm{op}}} \to \operatorname{Etale} X$ which is left adjoint to the functor $\Gamma : \operatorname{Etale} X \to \mathsf{Sets}^{\mathcal{O}(X)^{\mathrm{op}}}$ defined by

$$\Gamma p(U) = \{s \mid s \text{ is a continuous section of } p \text{ on } U\}.$$

The restriction $\Lambda : \mathrm{Sh}(X) \to \mathrm{Etale}\, X$ is an equivalence of categories with quasi-inverse Γ.

We shall often use categories of fractions and categories of right fractions (see [G-Z]). Let $\mathcal{C}$ be a category and let Σ be a class of morphisms in $\mathcal{C}$. The category of fractions induced by Σ will be denoted by $\mathcal{C}[\Sigma^{-1}]$, and by $\mathcal{C}\Sigma^{-1}$ if Σ admits a calculus of right fractions. In the last case the hom-set can be defined by

$$\mathrm{Hom}_{\mathcal{C}\Sigma^{-1}}(X, Y) = \operatorname*{colim}_{s \in \Sigma,\, \mathrm{codomain}(s) = X} \mathrm{Hom}_{\mathcal{C}}(\mathrm{domain}(s), Y).$$

A category I is said to be *left filtering* if it satisfies the following conditions:

(a) given two objects i, i' in I, there is an object j in I and morphisms $j \to i$, $j \to i'$,

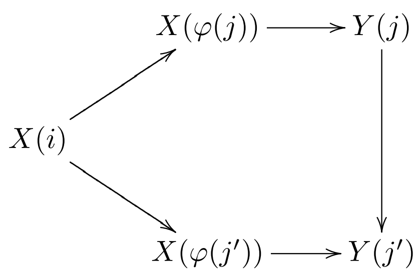
(b) if $u, v : j \to i$ are morphisms in I, there is k in I and a morphism $w : k \to j$ such that $uw = vw$.

A *pro-object* in C is a functor $X : I \to C$, where I is a left filtering small category. An arrow $u : j \to i$ is carried by X to a morphism $X(u) : X(j) \to X(i)$, which is called a *bonding morphism*. In some cases the hom–set $\mathrm{Hom}_I(j, i)$ only has one arrow and we then use the notation $X_i^j : X(j) \to X(i)$. We also use this notation when no confusion is possible.

We are going to consider the category $\mathrm{pro}\, C$ whose objects are pro-objects in C. Given pro-objects $X : I \to C$ and $Y : J \to C$ in C, the morphism set from X to Y is defined by

$$\mathrm{Hom}_{\mathrm{pro}\, C}(X, Y) = \lim_j \operatorname*{colim}_i \mathrm{Hom}_C(X(i), Y(j)).$$

An alternative description of morphisms in $\mathrm{pro}\, C$ can be given as follows: A morphism $u : X \to Y$ is represented by a pair $(\varphi, f(j))$, where $\varphi : |J| \to |I|$ is a map from the object set of J to the object set of I and $f(j) : X(\varphi(j)) \to Y(j)$ is a morphism in C, $j \in |J|$, such that if $j \to j'$ is a morphism in J, then there are $i \in |I|$, $i \to \varphi(j)$ and $i \to \varphi(j')$ such that the following diagram is commutative:



Two pairs $(\varphi, f(j))$, $(\psi, g(j))$ represent the same morphism u if for each

$j \in |J|$, there are $i \in |I|$, $i \to \varphi(j)$ and $i \to \psi(j)$ such that the following diagram commutes:

$$\begin{array}{ccccc} & & X(\varphi(j)) & & \\ & \nearrow & & \searrow & \\ X(i) & & & & Y(j) \\ & \searrow & & \nearrow & \\ & & X(\psi(j)) & & \end{array}$$

One of the more interesting properties of the category $\operatorname{pro} C$ is that if $Y : J \to C$ is a pro-object and $\phi : I \to J$ is a cofinal functor, then $Y\phi : I \to C$ is isomorphic to $Y : J \to C$ in the category $\operatorname{pro} C$.

For each left filtering small category I, we denote by C^I the category whose objects are functors $X : I \to C$ and morphisms are natural transformations; that is, a morphism $f : X \to Y$ is given by a coherent family of morphisms $f(i) : X(i) \to Y(i)$, $i \in |I|$. There is a canonical functor $\gamma : C^I \to \operatorname{pro} C$, and a morphism of the form $\gamma f : \gamma X \to \gamma Y$ is said to be a *level morphism*.

Of particular interest is the full subcategory $\operatorname{tow} C$ of $\operatorname{pro} C$ determined by objects whose indexing category is $\mathbb{N}$, where $\mathbb{N}$ is the category whose objects are non-negative integer numbers and $\operatorname{Hom}_{\mathbb{N}}(n, m)$ has either one element if $n \geq m$ or is the empty set if $n < m$.

1. The category $\operatorname{pro}(G, \mathsf{Sets})$**.** In this section, we define and study the category $\operatorname{pro}(G, \mathsf{Sets})$, where G is a pro-groupoid. Later in §2 we shall prove that the category of covering projections of a space is equivalent to a category of the form $\operatorname{pro}(G, \mathsf{Sets})$.

Recall that a *groupoid* G is a small category where any morphism in G is an isomorphism. Given two groupoids G and G', a groupoid homomorphism is just a functor $f : G \to G'$. Let Gpd denote the category of groupoids.

We denote by $[0, 1]$ the groupoid with two objects $0, 1$ and whose morphisms are the identities and two mutually inverse maps $u : 0 \to 1$ and $u^{-1} : 1 \to 0$. If G is a groupoid, we can consider the product groupoid $G \times [0, 1]$ and the groupoid homomorphisms $\partial_0, \partial_1 : G \to G \times [0, 1]$, where for example ∂_0 carries an arrow $\alpha : U \to U'$ in G to the arrow $\partial_0 \alpha = (\alpha, \mathrm{id}_0) : (U, 0) \to (U', 0)$. Using this cylinder, we can consider homotopies making commutative diagrams of the form

$$\begin{array}{ccc} G + G & \xrightarrow{f+g} & G' \\ {\scriptstyle \partial_0 + \partial_1}\downarrow & \nearrow_{F} & \\ G \times [0,1] & & \end{array}$$

where $G + G$ is the sum groupoid, and F is a groupoid homomorphism.

We note that a homotopy F determines a natural transformation η_F from f to g by $\eta_F(U) = F(\mathrm{id}_U, u)$. Conversely, a natural transformation $\eta : f \to g$ determines a homotopy F_η from f to g by $F_\eta(\mathrm{id}_U, u) = \eta(U)$.

If G and G' are two groupoids, we can consider a groupoid $\mathrm{HOM}_{\mathsf{Gpd}}(G, G')$ whose objects are given by the elements of the set $\mathrm{Hom}_{\mathsf{Gpd}}(G, G')$ and if $f, g : G \to G'$ are objects in $\mathrm{HOM}_{\mathsf{Gpd}}(G, G')$ a morphism $\eta : f \to g$ is a natural transformation from f to g. We denote by $\pi_0\mathrm{HOM}_{\mathsf{Gpd}}(G, G')$ the set of isomorphism classes of the groupoid $\mathrm{HOM}_{\mathsf{Gpd}}(G, G')$. This set is also the set of homotopy classes of groupoid homomorphisms from G to G'. We also consider the category $\pi_0\mathsf{Gpd}$ which has the same objects as Gpd and the hom-set is defined by $\mathrm{Hom}_{\pi_0\mathsf{Gpd}}(G, G') = \pi_0\mathrm{HOM}_{\mathsf{Gpd}}(G, G')$. Denote by $\gamma : \mathsf{Gpd} \to \pi_0\mathsf{Gpd}$ the projection functor which carries an arrow $f : G \to G'$ to the homotopy class $\gamma f : \gamma G \to \gamma G'$. We note that f is an equivalence (of categories) if and only if f is a homotopy equivalence; that is, if γf is an isomorphism in $\pi_0\mathsf{Gpd}$.

For a given pro-groupoid $G : I \to \mathsf{Gpd}$, we consider the category (G, Sets). An object of (G, Sets) is given by a pair $(G(i), F)$ where i is an object in I and $F : G(i) \to \mathsf{Sets}$ is a functor. A morphism α from $(G(i), F)$ to $(G(j), H)$ is a pair $\alpha = (i \to j,\ \theta_\alpha : F \to HG^i_j)$ where $i \to j$ is a morphism in I and $\theta_\alpha : F \to HG^i_j$ is a natural transformation (G^i_j is the corresponding bonding map).

Consider the class

$$\varSigma = \{\alpha \mid \alpha \text{ is a morphism in } (G, \mathsf{Sets}) \text{ and } \theta_\alpha \text{ is an equivalence}\}.$$

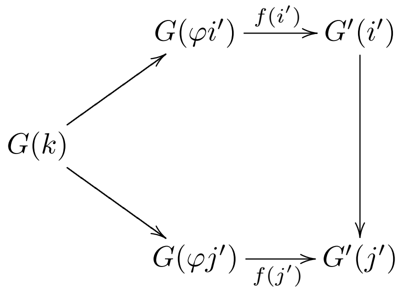
It is easy to check that the class $\varSigma$ admits a calculus of right fractions (see §1 and [G-Z]). Therefore we can consider the category of right fractions $(G, \mathsf{Sets})\varSigma^{-1}$ that will be denoted by $\mathrm{pro}(G, \mathsf{Sets})$.

If I is the indexing category of the pro-groupoid G, and i, j are two objects in I, we consider the category $I{\downarrow}\{i, j\}$ whose objects are pairs (u, v) of maps $u : k \to i$, $v : k \to j$, and a morphism from (u, v) to (u^1, v^1) is given by a map $w : k \to k^1$ such that $u^1 w = u$, $v^1 w = v$. If $(G(i), F)$ and $(G(j), H)$ are two objects in $\mathrm{pro}(G, \mathsf{Sets})$, we can consider the category $I{\downarrow}\{i, j\}$; for an object (u, v) in $I{\downarrow}\{i, j\}$, we write $k = \mathrm{domain}(u) = \mathrm{domain}(v)$. From the definition of the hom-set in a category of right fractions, one has

$$\mathrm{Hom}_{\mathrm{pro}(G,\mathsf{Sets})}((G(i), F), (G(j), H)) \cong \operatorname*{colim}_{I{\downarrow}\{i,j\}} \mathrm{Hom}_{\mathsf{Sets}^{G(k)}}(FG^k_i, HG^k_j).$$

Now assume that $f : G \to G'$ is a morphism in $\mathrm{pro}\,\mathsf{Gpd}$ represented by a pair $(\varphi, f(i'))$. We are going to see how the pair $(\varphi, f(i'))$ induces a functor $(\varphi, f(i'))^* : \mathrm{pro}(G', \mathsf{Sets}) \to \mathrm{pro}(G, \mathsf{Sets})$. First we define a functor from (G', Sets) to $\mathrm{pro}(G, \mathsf{Sets})$. Let $\alpha' = (i' \to j',\ \theta_{\alpha'} : F' \to H'G'^{i'}_{j'})$ be a morphism in (G', Sets) from $(G'(i'), F')$ to $(G'(j'), H')$. Then $(\varphi, f(i'))^*$ carries these objects to $(G(\varphi i'), F' f(i'))$ and $(G(\varphi j'), H' f(j'))$, respectively.

In order to get $(\varphi, f(i'))^*(\alpha')$, we choose k in I and arrows $k \to \varphi i'$ and $k \to \varphi j'$ such that the diagram



is commutative. Then $(\varphi, f(i'))^*(\alpha')$ is the morphism in $\mathrm{pro}(G, \mathsf{Sets})$ represented by the natural transformation $\theta_{\alpha'} * (f(i')G^k_{\varphi i'}) : F'f(i')G^k_{\varphi i'} \to H'f(j')G^k_{\varphi j'}$. It is easy to check that two choices of k represent the same morphism in $\mathrm{pro}(G, \mathsf{Sets})$. The functor $(\varphi, f(i'))^*$ has the property that if α' is in Σ', then $(\varphi, f(i'))^*(\alpha')$ is an isomorphism. Therefore we have an induced functor

$$(\varphi, f(i'))^* : \mathrm{pro}(G', \mathsf{Sets}) \to \mathrm{pro}(G, \mathsf{Sets}).$$

We note that it $(\varphi, f(i'))$ and $(\psi, g(i'))$ represent the same morphism $f : G \to G'$, then the functor $(\varphi, f(i'))^*$ is isomorphic to $(\psi, g(i'))^*$. We will denote by $f^* : \mathrm{pro}(G', \mathsf{Sets}) \to \mathrm{pro}(G, \mathsf{Sets})$ one of these functors.

If $f : G \to G'$ and $g : G' \to G''$ are morphisms in $\mathrm{pro}\,\mathsf{Gpd}$ represented by pairs $(\varphi, f(i'))$ and $(\psi, g(i''))$, then gf can be represented by $(\varphi\psi, g(i'')f(\psi i''))$. If $gf = \mathrm{id}$ and $fg = \mathrm{id}$, then $(\varphi\psi, g(i)f(\psi i))^*$ and $(\psi\varphi, f(i')g(\varphi i'))^*$ are isomorphic to identity functors. Therefore the functor $(\varphi, f(i))^*$ is an equivalence of categories. We restate this fact in the following:

LEMMA 1.1. *If* $f : G \to G'$ *is an isomorphism in* $\mathrm{pro}\,\mathsf{Gpd}$, *then* $f^* : \mathrm{pro}(G', \mathsf{Sets}) \to \mathrm{pro}(G, \mathsf{Sets})$ *is an equivalence of categories.*

The following result will be useful:

LEMMA 1.2. *Let* $f : G \to G'$ *be a level morphism in* $\mathrm{pro}\,\mathsf{Gpd}$ *such that for each* $i \in I$, $f(i) : G(i) \to G'(i)$ *is an equivalence. Then* $f^* : \mathrm{pro}(G', \mathsf{Sets}) \to \mathrm{pro}(G, \mathsf{Sets})$ *is an equivalence of categories.*

Proof. In this case, the functor f^* is defined on objects by $f^*(G'(i), F') = (G(i), F'f(i))$ and for morphisms one has

$$\mathrm{Hom}_{\mathrm{pro}(G', \mathsf{Sets})}((G'(i), F'), (G'(j), H')) = \operatorname*{colim}_{I\downarrow\{i,j\}} \mathrm{Hom}_{\mathsf{Sets}^{G'(k)}}(F'G'^k_i, H'G'^k_j)$$

$$\cong \operatorname*{colim}_{I\downarrow\{i,j\}} \mathrm{Hom}_{\mathsf{Sets}^{G(k)}}(F'G_i'^k f(k), H'G_j'^k f(k))$$

$$\cong \operatorname*{colim}_{I\downarrow\{i,j\}} \mathrm{Hom}_{\mathsf{Sets}^{G(k)}}(F'f(i)G_i^k, H'f(j)G_j^k)$$

$$\cong \mathrm{Hom}_{\mathrm{pro}(G,\mathsf{Sets})}((G(i),F'f(i)),(G(j),H'f(j))).$$

Therefore f^* is a full faithful functor. On the other hand, if $(G(i),F)$ is an object in $\mathrm{pro}(G,\mathsf{Sets})$, we can take a quasi-inverse $g : G'(i) \to G(i)$ of the equivalence $f(i) : G(i) \to G'(i)$. Then $f^*(G'(i),Fg) = (G(i),Fgf(i))$. However, $(G(i),Fgf(i))$ is isomorphic to $(G(i),F)$. Thus we have shown that f^* is an equivalence of categories.

At the beginning of this section we have considered the categories Gpd and $\pi_0\mathsf{Gpd}$ and the functor $\gamma : \mathsf{Gpd} \to \pi_0\mathsf{Gpd}$. This functor γ induces a functor $\gamma = \mathrm{pro}\,\gamma : \mathrm{pro}\,\mathsf{Gpd} \to \mathrm{pro}\,\pi_0\mathsf{Gpd}$. We have shown that two isomorphic objects G, G' in $\mathrm{pro}\,\mathsf{Gpd}$ induce equivalent categories $\mathrm{pro}(G,\mathsf{Sets})$, $\mathrm{pro}(G',\mathsf{Sets})$. Next we analyse this kind of questions for objects in the category $\mathrm{pro}\,\pi_0\mathsf{Gpd}$.

If $G : \mathbb{N} \to \pi_0\mathsf{Gpd}$ is an object in $\mathrm{tow}\,\pi_0\mathsf{Gpd}$, we can choose for each bonding morphism $G(i+1) \to G(i)$ in $\pi_0\mathsf{Gpd}$ a representative map $\overline{G}_i^{i+1}$; in this way we obtain and object $\overline{G} : \mathbb{N} \to \mathsf{Gpd}$ in $\mathrm{tow}\,\mathsf{Gpd}$ such that $\overline{G}(i) = G(i)$ and $\gamma\overline{G} = G$. If we choose different bonding maps $\widetilde{G}_i^{i+1}$, we have a new pro-groupoid $\widetilde{G} : \mathbb{N} \to \mathsf{Gpd}$, but we can prove the following result:

Lemma 1.3. *The category* $\mathrm{pro}(\overline{G},\mathsf{Sets})$ *is equivalent to the category* $\mathrm{pro}(\widetilde{G},\mathsf{Sets})$.

Proof. For each $i \geq 0$, since $\overline{G}_i^{i+1}$ is homotopic to $\widetilde{G}_i^{i+1}$, we can choose a homotopy $L^{i+1} : G(i+1) \times [0,1] \to G(i)$ such that $L^{i+1}\partial_0 = \overline{G}_i^{i+1}$ and $L^{i+1}\partial_1 = \widetilde{G}_i^{i+1}$.

Consider the commutative diagram

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & G(i+1) & \xrightarrow{\overline{G}_i^{i+1}} & G(i) & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle\partial_0} & & \downarrow{\scriptstyle\partial_0} & & \\
\cdots & \longrightarrow & G(i+1)\times[0,1] & \xrightarrow{(L^{i+1},\mathrm{pr}_2)} & G(i)\times[0,1] & \longrightarrow & \cdots \\
 & & \uparrow{\scriptstyle\partial_1} & & \uparrow{\scriptstyle\partial_1} & & \\
\cdots & \longrightarrow & G(i+1) & \xrightarrow[\widetilde{G}_i^{i+1}]{} & G(i) & \longrightarrow & \cdots
\end{array}$$

and denote by $G \times [0,1]$ the "cylinder" pro-groupoid. We conclude that $\partial_0 : \overline{G} \to G \times [0,1]$ and $\partial_1 : \widetilde{G} \to G \times [0,1]$ satisfy the conditions of Lemma 1.2, and therefore $\mathrm{pro}(\overline{G},\mathsf{Sets})$ is equivalent to $\mathrm{pro}(\widetilde{G},\mathsf{Sets})$.

As a consequence of Lemma 1.3, if G is an object in $\operatorname{tow}\pi_0\mathsf{Gpd}$, the category $\operatorname{pro}(\overline{G},\mathsf{Sets})$ will be denoted by $\operatorname{pro}(G,\mathsf{Sets})$.

LEMMA 1.4. *Let* $f=\{f(i):G(i)\to G'(i)\}$ *be a level morphism in* $\operatorname{tow}\pi_0\mathsf{Gpd}$ *and assume that there are maps* $g(i):G'(i+1)\to G(i)$ *such that for each* $i\ge 0$ *the diagram*

$$\begin{array}{ccc} G(i+1) & \xrightarrow{f(i+1)} & G'(i+1) \\ {\scriptstyle G_i^{i+1}}\downarrow & \overset{g(i)}{\swarrow} & \downarrow{\scriptstyle (G')_i^{i+1}} \\ G(i) & \xrightarrow[f(i)]{} & G'(i) \end{array}$$

is commutative in $\pi_0\mathsf{Gpd}$. *Then* $\operatorname{pro}(G,\mathsf{Sets})$ *is equivalent to* $\operatorname{pro}(G',\mathsf{Sets})$.

Proof. Since $(G')_i^{i+1}=f(i)g(i)$ in $\pi_0\mathsf{Gpd}$, from Lemma 1.3 it follows that the towers

$$\ldots\to G'(i+1)\xrightarrow{(G')_i^{i+1}}G'(i)\to\ldots,$$
$$\ldots\to G'(i+1)\xrightarrow{f(i)g(i)}G'(i)\to\ldots$$

determine equivalent categories. Since the towers

$$\ldots\to G'(i+1)\xrightarrow{f(i)g(i)}G'(i)\to\ldots,$$
$$\ldots\to G(i+1)\xrightarrow{f(i+1)}G'(i+1)\xrightarrow{g(i)}G(i)\xrightarrow{f(i)}G'(i)\to\ldots,$$
$$\ldots\to G(i+1)\xrightarrow{g(i)f(i+1)}G(i)\to\ldots$$

are isomorphic in $\operatorname{tow}\mathsf{Gpd}$ and $g(i)f(i+1)=G_i^{i+1}$ in $\pi_0\mathsf{Gpd}$, Lemmas 1.1 and 1.3 show that $\operatorname{pro}(G,\mathsf{Sets})$ is equivalent to $\operatorname{pro}(G',\mathsf{Sets})$.

As a consequence of these lemmas, we have:

PROPOSITION 1.1. *Let* G,G' *be objects in* $\operatorname{tow}\pi_0\mathsf{Gpd}$. *If* G *is isomorphic to* G' *in* $\operatorname{tow}\pi_0\mathsf{Gpd}$, *then* $\operatorname{pro}(G,\mathsf{Sets})$ *is equivalent to* $\operatorname{pro}(G',\mathsf{Sets})$.

Proof. Let $f:G\to G'$ be an isomorphism. The map f can be represented by a pair $(\varphi,f(i))$ such that $\varphi(i)\ge i$, $\varphi(i)>\varphi(j)$ if $i>j$, and for each $i\ge 0$, the diagram

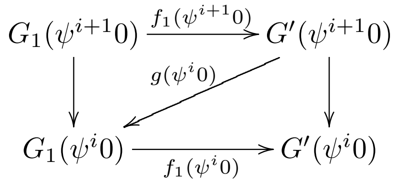
$$\begin{array}{ccc} G(\varphi(i+1)) & \xrightarrow{f(i+1)} & G'(i+1) \\ \downarrow & & \downarrow \\ G(\varphi(i)) & \xrightarrow[f(i)]{} & G'(i) \end{array}$$

is commutative in $\pi_0\mathsf{Gpd}$. Define an object G_1 in $\operatorname{tow}\mathsf{Gpd}$ by $G_1(i)=G(\varphi(i))$

and $(G_1)_i^{i+1} = G_{\varphi(i)}^{\varphi(i+1)}$. We also have $f_1 : G_1 \to G'$ defined by $f_1(i) = f(i) : G_1(i) \to G'(i)$.

We know that G is isomorphic to G_1 in $\operatorname{tow} \mathsf{Gpd}$ and $f_1 : G_1 \to G'$ is a level isomorphism in $\operatorname{tow} \pi_0\mathsf{Gpd}$. By Lemma 1.1, $\operatorname{pro}(G, \mathsf{Sets})$ is equivalent to $\operatorname{pro}(G_1, \mathsf{Sets})$.

Since $f_1 : G_1 \to G'$ is a level isomorphism in $\operatorname{tow} \pi_0\mathsf{Gpd}$, there is a map $g : G' \to G_1$ in $\operatorname{tow} \pi_0\mathsf{Gpd}$ represented by $(\psi, g(i))$ such that $\psi(i) \geq i$, $\psi(i) < \psi(j)$ if $i < j$, and the diagram



is commutative in $\pi_0\mathsf{Gpd}$, where ψ^k denotes the iterated map $\psi \overset{(k)}{\ldots} \psi$. Define

$$G_2(i) = G_1(\psi^i 0), \qquad G_1'(i) = G'(\psi^i 0),$$
$$f_2(i) = f_1(\psi^i 0), \qquad g(i) = g(\psi^i 0).$$

Now we see that G_1 is isomorphic to G_2 and G' is isomorphic to G_1' in $\operatorname{tow} \mathsf{Gpd}$. By Lemma 1.1, $\operatorname{pro}(G_1, \mathsf{Sets})$ is equivalent to $\operatorname{pro}(G_2, \mathsf{Sets})$, and $\operatorname{pro}(G', \mathsf{Sets})$ is equivalent to $\operatorname{pro}(G_1', \mathsf{Sets})$. Since $f_2 : G_2 \to G_1'$ and $g : G_1' \to G_2$ satisfy the conditions of Lemma 1.4, it follows that $\operatorname{pro}(G_2, \mathsf{Sets})$ is equivalent to $\operatorname{pro}(G_1', \mathsf{Sets})$. Thus we conclude that $\operatorname{pro}(G, \mathsf{Sets})$ is equivalent to $\operatorname{pro}(G', \mathsf{Sets})$.

Example. We exhibit two non-isomorphic pro-groups F, F' which are isomorphic in $\operatorname{tow} \pi_0\mathsf{Gpd}$. Therefore the categories $\operatorname{pro}(F, \mathsf{Sets})$ and $\operatorname{pro}(F', \mathsf{Sets})$ are equivalent. For $n \geq 0$ let $F(n)$ be the free group generated by $x_0, x_{n+1}, x_{n+2}, \ldots$ and the bonding morphism $F(n+1) \to F(n)$ is defined to be the inclusion. The pro-group F' is defined by $F'(n) = F(n)$ and the bonding is $(F_n')^{n+1}(a) = x_{n+1} a x_{n+1}^{-1}$. It is easy to check that $\lim F$ is the infinite cyclic group and $\lim F'$ is trivial. This implies that F is not isomorphic to F'. However, the bonding $F(n+1) \to F(n)$ is homotopic to the bonding $F'(n+1) \to F'(n)$. Therefore F is isomorphic to F' in the category $\operatorname{tow} \pi_0\mathsf{Gpd}$ and by Proposition 1.1 above we see that $\operatorname{pro}(F, \mathsf{Sets})$ is equivalent to $\operatorname{pro}(F', \mathsf{Sets})$.

2. Classification of covering projections. In this section, we define a notion of covering projection that for locally connected spaces agrees with the notion given in Spanier's book [S]. The main result of this section is the determination of a pro-groupoid $\pi \operatorname{crs}(X)$ such that the category of covering projections of X is equivalent to the category $\mathrm{pro}(\pi\,\mathrm{crs}(X), \mathsf{Sets})$ defined in Section 1.

Given sets (or spaces) A, F, G a map $\theta : A \times F \to A \times G$ such that $\mathrm{pr}_A\theta = \mathrm{pr}_A$ is of the form $\theta(a, x) = (a, \theta_a(x))$ for $a \in A$ and $x \in F$, where $\theta_a : F \to G$ is a map which depends on $a \in A$. A map $\theta : A \times F \to A \times G$ such that $\mathrm{pr}_A\theta = \mathrm{pr}_A$ is said to be A-*constant* if $\theta_a = \theta_{a'}$ for all $a, a' \in A$.

Let $p : E \to X$ be a continuous map and let $\mathcal{U}$ be an open covering of X. An *atlas* $\mathcal{A}$ *for* $p : E \to X$ *on* $\mathcal{U}$ consists of a family of homeomorphisms $\varphi_U : U \times F(U) \to p^{-1}U$, where $U \in \mathcal{U}$ and $F(U)$ is a discrete space, such that if $U, V \in \mathcal{U}$ and $\emptyset \neq W = U \cap V$, then the induced homeomorphism

$$W \times F(U) \xrightarrow{\varphi_U} p^{-1}W \xrightarrow{\varphi_V^{-1}} X \times F(V)$$

is W-constant, where φ_U, φ_V also denote the corresponding restrictions.

If $\mathcal{A} = \{\varphi_U : U \times F(U) \to p^{-1}U\}$ is an atlas on $\mathcal{U}$ and $\mathcal{B} = \{\psi_V : V \times G(V) \to p^{-1}V\}$ is an atlas on $\mathcal{V}$, then $\mathcal{A}$ is said to be *equivalent* to $\mathcal{B}$ if there is an open covering $\mathcal{W}$ which refines $\mathcal{U}$ and $\mathcal{V}$ and such that if $W \subset U \cap V$, then the induced homeomorphism

$$W \times F(U) \xrightarrow{\varphi_U} p^{-1}W \xrightarrow{\psi_V^{-1}} W \times G(V)$$

is W-constant.

Definition 2.1. A *covering projection* $(p : E \to X,\ [\mathcal{A}])$ consists of a continuous map $p : E \to X$ and an equivalence class $[\mathcal{A}]$ of atlases.

Remarks. (1) If X is a metrisable space and $\mathcal{A}$ is an atlas on $\mathcal{U}$ such that for any $U \in \mathcal{U}$, $F(U)$ is a finite set with d elements, then the map $p : E \to X$ is a d-fold overlay in the sense of Fox.

(2) If X is a locally connected space and $\mathcal{A}$ is an atlas for $p : E \to X$ on $\mathcal{U}$ and $\mathcal{B}$ is an atlas for $p : E \to X$ on $\mathcal{V}$, then we can choose an open covering $\mathcal{W}$ which refines $\mathcal{U}$ and $\mathcal{V}$ and such that each $W \in \mathcal{W}$ is connected. If $W \subset U \cap V$, then the homeomorphism

$$W \times F(U) \xrightarrow{\varphi_U} p^{-1}W \xrightarrow{\psi_V^{-1}} W \times G(V)$$

sends the connected components of $W \times F(U)$ into connected components of $W \times G(V)$. Therefore $\psi_V^{-1}\varphi_U$ is W-constant, and $\mathcal{A}$ is equivalent to $\mathcal{B}$. Thus if X is a locally connected space, then a covering projection consists of a continuous map such that there is an open covering $\mathcal{U}$ of X and for each $U \in \mathcal{U}$, $p^{-1}U = \coprod_{\alpha \in F(U)} U_\alpha$, where $F(U)$ is an index set, each U_α is an open subset of E and the restriction $p|U_\alpha : U_\alpha \to U$ is a homeomorphism.

We shall use the following notion of covering transformation:

Definition 2.2. Let $\Phi = (p : E \to X,\ [\mathcal{A}])$ and $\Phi' = (p' : E \to X,\ [\mathcal{A}'])$ be two covering projections. A *covering transformation* $f : \Phi \to \Phi'$ is a continuous map $f : E \to E'$ such that $p'f = p$ and if $\mathcal{A} = \{\varphi_U \mid U \in \mathcal{U}\}$ and $\mathcal{A}' = \{\varphi_{U'} \mid U' \in \mathcal{U}'\}$ are two atlases for Φ and Φ', respectively, then there is an open covering $\mathcal{W}$ which refines $\mathcal{U}$ and $\mathcal{U}'$ and such that if $W \subset U \cap U'$, then the induced map

$$W \times F(U) \xrightarrow{\varphi_U} p^{-1}W \xrightarrow{f} p'^{-1}W \xrightarrow{\varphi_{U'}^{-1}} W \times F'(U')$$

is W-constant. We denote by $\mathrm{Cov\,proj}\, X$ the category of covering projections and covering transformations of X.

REMARK. If X is a locally connected space and $p : E \to X$ and $p' : E' \to X$ are covering projections, then any continuous map $f : E \to E'$ such that $p'f = p$ is a covering transformation. Therefore, in this case the category $\mathrm{Cov\,proj}\, X$ is equivalent to $(\mathrm{Etale}\, X)_{\mathrm{cp}}$, where $(\mathrm{Etale}\, X)_{\mathrm{cp}}$ denotes the full subcategory of $\mathrm{Etale}\, X$ determined by covering projections.

If $\mathcal{U}$ is an open covering X, we denote by $(\mathrm{Cov\,proj}\, X)_{\mathcal{U}}$ the subcategory of $\mathrm{Cov\,proj}\, X$ whose objects are those covering projections Φ which admit an atlas on $\mathcal{U}$. Given covering projections Φ and Φ' with atlases $\mathcal{A} = \{\varphi_U \mid U \in \mathcal{U}\}$ and $\mathcal{A}' = \{\varphi'_U \mid U \in \mathcal{U}\}$, a morphism $f : \Phi \to \Phi'$ in $(\mathrm{Cov\,proj}\, X)_{\mathcal{U}}$ is a covering transformation $f : \Phi \to \Phi'$ in $\mathrm{Cov\,proj}\, X$ such that for any $U \in \mathcal{U}$ the map

$$U \times F(U) \xrightarrow{\varphi_U} p^{-1}U \xrightarrow{f} p'^{-1}U \xrightarrow{(\varphi'_U)^{-1}} U \times F'(U)$$

is U-constant. We note that if $\mathcal{U}$ refines $\mathcal{V}$, then we have a faithful functor $(\mathrm{Cov\,proj}\, X)_{\mathcal{V}} \to (\mathrm{Cov\,proj}\, X)_{\mathcal{U}}$. One has the following result:

PROPOSITION 2.1. *Let Φ and Φ' be two covering projections and let $\mathcal{A} = \{\varphi_U \mid U \in \mathcal{U}\}$ and $\mathcal{A}' = \{\varphi_{U'} \mid U' \in \mathcal{U}'\}$ be two atlases of Φ and Φ', respectively. Then*

$$\mathrm{Hom}_{\mathrm{Cov\,proj}\, X}(\Phi, \Phi') \cong \operatorname*{colim}_{\mathcal{W} \geq \mathcal{U},\, \mathcal{W} \geq \mathcal{U}'} \mathrm{Hom}_{(\mathrm{Cov\,proj}\, X)_{\mathcal{W}}}(\Phi, \Phi').$$

Proof. This follows directly from the definition of covering transformation.

In order to use locally constant sheaves to study the category $\mathrm{Cov\,proj}\, X$, we recall and introduce some notions:

A family $\mathcal{U}$ of open subsets of the space X is called a *sieve* on X if for every $U \in \mathcal{U}$ and every open subset $V \subset U$, we have $V \in \mathcal{U}$. If moreover $X = \bigcup_{U \in \mathcal{U}} U$, then $\mathcal{U}$ is called a *covering sieve* on X. We denote by $\mathcal{O}$ the covering sieve of all open subsets of X.

DEFINITION 2.3. Let $\mathcal{U}$ be a family of non-empty open subsets of X such that if $U \in \mathcal{U}$ and $\emptyset \neq V \in \mathcal{O}$, $V \subset U$, then $V \in \mathcal{U}$. We then say that $\mathcal{U}$ is a *reduced sieve* on X and if $X = \bigcup_{U \in \mathcal{U}} U$, then $\mathcal{U}$ is a *covering reduced sieve* on X.

We note that if $\mathcal{U}$ is a covering sieve on X, then $^*\mathcal{U} = \mathcal{U}\backslash\{\emptyset\}$ is a covering reduced sieve on X. Every open covering $\mathcal{V}$ of X generates a covering sieve $s\mathcal{V} = \{U \in \mathcal{O} \mid \text{there is } V \in \mathcal{V} \text{ such that } U \subset V\}$ and the corresponding covering reduced sieve $^*s\mathcal{V} = \{U \in \mathcal{O} \mid U \neq \emptyset \text{ and there is } V \in \mathcal{V} \text{ such that } U \subset V\}$.

A (reduced) sieve $\mathcal{U}$ can be considered as a small category, denoted again by $\mathcal{U}$, where the set of morphisms from U to V is given by $\mathrm{Hom}_{\mathcal{U}}(U,V) = 1$ if $U \subset V$, and $\mathrm{Hom}_{\mathcal{U}}(U,V) = \emptyset$ otherwise. A functor $P : \mathcal{U}^{\mathrm{op}} \to \mathsf{Sets}$ is said to be a *presheaf* on $\mathcal{U}$.

Definition 2.4. Given a covering reduced sieve $\mathcal{U}$ on X, a presheaf $P : \mathcal{U}^{\mathrm{op}} \to \mathsf{Sets}$ is said to be *locally constant* if P carries any arrow $U \subset V$ in $\mathcal{U}$ into an isomorphism $P(V) \to P(U)$. We denote by $\mathsf{Sets}^{\mathcal{U}^{\mathrm{op}}}$ the category of presheaves on $\mathcal{U}$ and by $(\mathsf{Sets}^{\mathcal{U}^{\mathrm{op}}})_{\mathrm{lc}}$ the category of locally constant presheaves on $\mathcal{U}$.

Given a covering sieve $\mathcal{U}$, the canonical inclusion $\mathcal{U}^{\mathrm{op}} \subset \mathcal{O}^{\mathrm{op}}$ induces a restriction functor $\mathrm{re} : \mathsf{Sets}^{\mathcal{O}^{\mathrm{op}}} \to \mathsf{Sets}^{\mathcal{U}^{\mathrm{op}}}$. We also have an extension functor $\mathrm{ex} : \mathsf{Sets}^{\mathcal{U}^{\mathrm{op}}} \to \mathsf{Sets}^{\mathcal{O}^{\mathrm{op}}}$ which for a given presheaf $P : \mathcal{U}^{\mathrm{op}} \to \mathsf{Sets}$ is defined by

$$\mathrm{ex}\, P(V) = \lim_{U \in \mathcal{U}^{\mathrm{op}}} P(V \cap U).$$

It is routine to check

Proposition 2.2. *The functor* $\mathrm{re} : \mathsf{Sets}^{\mathcal{O}^{\mathrm{op}}} \to \mathsf{Sets}^{\mathcal{U}^{\mathrm{op}}}$ *is left adjoint to* $\mathrm{ex} : \mathsf{Sets}^{\mathcal{U}^{\mathrm{op}}} \to \mathsf{Sets}^{\mathcal{O}^{\mathrm{op}}}$.

If $\mathcal{U}$ is a covering reduced sieve and $P : \mathcal{U}^{\mathrm{op}} \to \mathsf{Sets}$ is a presheaf, we can consider the covering sieve $'\mathcal{U} = \mathcal{U} \cup \{\emptyset\}$ and the presheaf $'P : {'\mathcal{U}}^{\mathrm{op}} \to \mathsf{Sets}$ defined by the unique extension of P such that $P(\emptyset) = 1$. Therefore we also have an extension functor $\mathrm{ex}' : \mathsf{Sets}^{\mathcal{U}^{\mathrm{op}}} \to \mathsf{Sets}^{\mathcal{O}^{\mathrm{op}}}$ defined by $\mathrm{ex}'(P) = \mathrm{ex}('P)$.

Let $\Lambda : \mathsf{Sets}^{\mathcal{O}^{\mathrm{op}}} \to \mathrm{Etale}\, X$ be the functor which carries a presheaf $P : \mathcal{O}^{\mathrm{op}} \to \mathsf{Sets}$ into the bundle ΛP of germs of P (see §1 and [M-M]). For a covering reduced sieve $\mathcal{U}$ on X one has the composite

$$\mathsf{Sets}^{\mathcal{U}^{\mathrm{op}}} \xrightarrow{\mathrm{ex}'} \mathsf{Sets}^{\mathcal{O}^{\mathrm{op}}} \xrightarrow{\Lambda} \mathrm{Etale}\, X.$$

If $P : \mathcal{U}^{\mathrm{op}} \to \mathsf{Sets}$ is a presheaf and $x \in X$ we have the set of germs of P at x,

$$P_x = \operatorname*{colim}_{x \in U \in \mathcal{U}} P(U),$$

and the canonical map $\mathrm{germ}_x^U : P(U) \to P_x$. We note that $\Lambda\, \mathrm{ex}' P = (q(P) : E(P) \to X)$, where

$$E(P) = \bigsqcup_{x \in X} P_x, \qquad (q(P))^{-1}x = P_x$$

and the topology of $E(P)$ is given by the base

$$\mathcal{B} = \{\dot{s}U \mid s \in P(U),\ U \in \mathcal{U}\}, \qquad \dot{s}U = \{\mathrm{germ}_x^U(s) \mid x \in U\}.$$

For each $U \in \mathcal{U}$, we consider the map

$$\varphi_U : U \times P(U) \to (q(P))^{-1}U$$

defined by $\varphi_U(x, s) = \mathrm{germ}_x^U s$ for $x \in U$ and $s \in P(U)$. We note that for a fixed $s \in P(U)$, the restriction $\varphi_U(-, s) : U \times \{s\} \to \dot{s}U$ is a homeomorphism.

If P is a locally constant presheaf, then $\mathrm{germ}_x^U : P(U) \to P_x$ is an isomorphism of discrete spaces. Therefore, in this case, φ_U is a homeomorphism.

Now we check that for a locally constant presheaf $P : \mathcal{U}^{\mathrm{op}} \to \mathsf{Sets}$, $\mathcal{A}(P) = \{\varphi_U \mid U \in \mathcal{U}\}$ is an atlas on $\mathcal{U}$ for $q(P) : E(P) \to X$. If $U, V \in \mathcal{U}$ and $W = U \cap V$, then the map

$$W \times P(U) \xrightarrow{\varphi_U} (q(P))^{-1}W \xrightarrow{\varphi_V^{-1}} W \times P(V)$$

is W-constant, because if $x, y \in W$, $s \in P(U)$ and $t, t' \in P(V)$ are such that

$$\mathrm{germ}_x^U s = \mathrm{germ}_x^V t, \qquad \mathrm{germ}_y^U s = \mathrm{germ}_y^V t'$$

then $t|W = s|W = t'|W$, hence $t = t'$ ($t|W$ denotes the image of $t \in P(V)$ under the restriction map $P(V) \to P(W)$).

Therefore for a locally constant presheaf $P : \mathcal{U}^{\mathrm{op}} \to \mathsf{Sets}$, one has the covering projection $\overline{\Lambda}P = (q(P) : E(P) \to X,\ [\mathcal{A}(P)])$. In order to construct a functor $\overline{\Lambda} : (\mathsf{Sets}^{\mathcal{U}^{\mathrm{op}}})_{\mathrm{lc}} \to (\mathrm{Cov\,proj}\, X)_{\mathcal{U}}$ we recall that if $f : P \to P'$ is a natural transformation of presheaves defined on $\mathcal{U}$, then we have an induced map $\overline{\Lambda}f : E(P) \to E(P')$ defined by $\overline{\Lambda}f(\mathrm{germ}_x^U s) = \mathrm{germ}_x^U(f(U)s)$, where $f(U) : P(U) \to P'(U)$ are the "components" of f.

To show $\overline{\Lambda}f$ is a morphism in $(\mathrm{Cov\,proj}\, X)_{\mathcal{U}}$ we have to check that

$$U \times P(U) \xrightarrow{\varphi_U} (q(P))^{-1}U \xrightarrow{\overline{\Lambda}f} (q(P'))^{-1}U \xrightarrow{(\varphi'_U)^{-1}} U \times P'(U)$$

is U-constant. If $(x, s) \in U \times P(U)$, then $(\varphi'_U)^{-1}(\overline{\Lambda}f)\varphi_U(x, s) = (x, f(U)(s))$ and $f(U)(s)$ does not depend on x. This implies that the above map is U-constant.

Thus we have constructed a functor $\overline{\Lambda} : (\mathsf{Sets}^{\mathcal{U}^{\mathrm{op}}})_{\mathrm{lc}} \to (\mathrm{Cov\,proj}\, X)_{\mathcal{U}}$. Now we can prove that the category of covering projections and transformations which trivialise on $\mathcal{U}$ is equivalent to the category of locally constant presheaves on $\mathcal{U}$.

THEOREM 2.1. *Given a covering reduced sieve* $\mathcal{U}$ *on a space* X, *the functor* $\overline{\Lambda} : (\mathsf{Sets}^{\mathcal{U}^{\mathrm{op}}})_{\mathrm{lc}} \to (\mathrm{Cov\,proj}\, X)_{\mathcal{U}}$ *is an equivalence of categories.*

P r o o f. First, we show that $\bar{\Lambda}$ is a faithful functor. Suppose that $f, g : P \to P'$ are natural transformations and P, P' are objects in $(\mathsf{Sets}^{\mathcal{U}^{\mathrm{op}}})_{\mathrm{lc}}$. If $\bar{\Lambda} f = \bar{\Lambda} g$, then for each $U \in \mathcal{U}$ and $s \in P(U)$, we have $\mathrm{germ}_x^U(f(U)s) = \bar{\Lambda} f(\mathrm{germ}_x^U\, s) = \bar{\Lambda} g(\mathrm{germ}_x^U\, s) = \mathrm{germ}_x^U(g(U)s)$. Since P and P' are locally constant, the maps of the form germ_x^U are isomorphisms, hence $f(U)s = g(U)s$ for each $s \in P(U)$. Therefore $f(U) = g(U)$ for all $U \in \mathcal{U}$. Thus $\bar{\Lambda}$ is a faithful functor.

Now assume that $h : E(P) \to E(P')$ is a covering transformation in $(\mathrm{Cov\,proj}\, X)_{\mathcal{U}}$. Then the composite

$$U \times P(U) \xrightarrow{\varphi_U} (qP)^{-1}U \xrightarrow{h} (qP')^{-1}U \xrightarrow{(\varphi'_U)^{-1}} U \times P'(U)$$

is U-constant. Let $f(U) : P(U) \to P'(U)$ be the unique map such that $\mathrm{id}_U \times f(U) = (\varphi'_U)^{-1} h \varphi_U$. It is easy to check that if $U, V \in \mathcal{U}$ and $U \subset V$ then $(P')_U^V f(V) = f(U) P_U^V$. Then $f(U) : P(U) \to P'(U)$, $U \in \mathcal{U}$, is a natural transformation from P to P'. The map $\bar{\Lambda} f$ has the property that for each $U \in \mathcal{U}$, the corresponding restrictions are such that $(\varphi'_U)^{-1}(\bar{\Lambda}) f \varphi_U = (\varphi'_U)^{-1} h \varphi_U$. This implies that for any U, $\bar{\Lambda} f | (qP)^{-1}U = h | (qP)^{-1}U$, and so that $\bar{\Lambda} f = h$. Thus we have shown that $\bar{\Lambda}$ is a full functor.

In order to check that $\bar{\Lambda}$ is an equivalence of categories, it suffices to prove that if $\Phi = (p : E \to X, [\mathcal{A}])$ is a covering projection in $(\mathrm{Cov\,proj}\, X)_{\mathcal{U}}$, then there is a locally constant presheaf $F : \mathcal{U}^{\mathrm{op}} \to \mathsf{Sets}$ such that $\bar{\Lambda} F \cong \Phi$.

Suppose that $\mathcal{A} = \{\psi_U : U \times F(U) \to p^{-1}U\}$ is an atlas on $\mathcal{U}$. If $U' \subset U$, then

$$U' \times F(U') \xrightarrow{\psi_{U'}} p^{-1}U' \xrightarrow{\psi_U^{-1}} U' \times F(U)$$

is U'-constant. Denote by $F_{U'}^U : F(U) \to F(U')$ the unique bijective map such that $\mathrm{id}_{U'} \times F_{U'}^U = \psi_{U'}^{-1} \psi_U$. It is easy to check that $F_U^U = \mathrm{id}_{F(U)}$ and if $U'' \subset U' \subset U$, then $F_{U''}^{U'} F_{U'}^U = F_{U''}^U$. Therefore F is a locally constant functor from $\mathcal{U}^{\mathrm{op}}$ to Sets. Now we can consider the map $h : \bar{\Lambda} F \to \Phi$, where $h : E(F) \to E$ is defined by $h(\mathrm{germ}_x^U\, s) = \psi_U(x, s)$ for $x \in U$ and $s \in F(U)$. We note that if $U' \subset U$ and $\mathrm{germ}_x^U\, s = \mathrm{germ}_x^{U'}\, s'$, then $s' = F_{U'}^U s$ and $\psi_U(x, s) = \psi_{U'}(x, F_{U'}^U s)$. Therefore h is well defined. We also see that the maps

$$U \times F(U) \xrightarrow{\varphi_U} (qF)^{-1}U \xrightarrow{h} p^{-1}U \xrightarrow{\psi_U^{-1}} U \times F(U)$$

satisfy $h\varphi_U(x, s) = h(\mathrm{germ}_x^U\, s) = \psi_U(x, s)$. Then $\psi_U^{-1} h \varphi_U = \mathrm{id}_U \times \mathrm{id}_{F(U)}$ is U-constant and $h | (qF)^{-1}U$ is an isomorphism. Thus $h : \bar{\Lambda} F \to \Phi$ is an isomorphism in $(\mathrm{Cov\,proj}\, X)_{\mathcal{U}}$. Therefore $\bar{\Lambda}$ is an equivalence of categories.

Given a covering reduced sieve $\mathcal{U}$, if we take the class Σ of all morphisms in $\mathcal{U}$, we have the corresponding category of fractions $\pi\mathcal{U} = \mathcal{U}[\Sigma^{-1}]$ which is a groupoid. We note the existence of natural isomorphisms $(\pi\mathcal{U})^{\mathrm{op}} \cong \pi(\mathcal{U}^{\mathrm{op}})$.

Therefore we use the notation $\pi\mathcal{U}^{\mathrm{op}}$. For locally constant presheaves on $\mathcal{U}$, one has:

LEMMA 2.1. *Given a covering reduced sieve $\mathcal{U}$ on X, the category $(\mathsf{Sets}^{\mathcal{U}^{\mathrm{op}}})_{\mathrm{lc}}$ of locally constant presheaves on $\mathcal{U}$ is equivalent to the functor category $\mathsf{Sets}^{\pi\mathcal{U}^{\mathrm{op}}}$.*

Proof. Denote by $\gamma : \mathcal{U}^{\mathrm{op}} \to \pi\mathcal{U}^{\mathrm{op}}$ the projection functor. If $P : \mathcal{U}^{\mathrm{op}} \to \mathsf{Sets}$ is locally constant, then P carries each arrow of $\mathcal{U}^{\mathrm{op}}$ into an isomorphism. Therefore, P factors through $\pi\mathcal{U}^{\mathrm{op}}$ as $P = \overline{P}\gamma$. Conversely, if $F : \pi\mathcal{U}^{\mathrm{op}} \to \mathsf{Sets}$ is a functor, then because $\pi\mathcal{U}^{\mathrm{op}}$ is a groupoid, F carries every arrow of $\pi\mathcal{U}^{\mathrm{op}}$ into an isomorphism. Thus $F\gamma$ is a locally constant functor.

We recall that if $\mathcal{U}$ and $\mathcal{V}$ are open coverings of a space X, then we say that $\mathcal{U}$ *refines* $\mathcal{V}, \mathcal{U} \geq \mathcal{V}$, if for every $U \in \mathcal{U}$, there is $V \in \mathcal{V}$ such that $U \subset V$. We note that for a given U, in general it is possible to find various $V \in \mathcal{V}$ such that $U \subset V$. It would be interesting to have a canonical way of finding a V for each U. We solve this problem if we work only with covering reduced sieves. We note that if $\mathcal{U}$ and $\mathcal{V}$ are two covering reduced sieves then $\mathcal{U}$ refines $\mathcal{V}$ if and only if $\mathcal{U} \subset \mathcal{V}$. If $U \in \mathcal{U}$, then there is $V \in \mathcal{V}$ such that $U \subset V$, but this implies that $U \in \mathcal{V}$. If $\mathcal{U} \subset \mathcal{V}$, then there is an induced functor $\pi^{\mathcal{U}}_{\mathcal{V}} : \pi\mathcal{U}^{\mathrm{op}} \to \pi\mathcal{V}^{\mathrm{op}}$ that again induces a functor $\mathsf{Sets}^{\pi\mathcal{V}^{\mathrm{op}}} \to \mathsf{Sets}^{\pi\mathcal{U}^{\mathrm{op}}}$.

Using the equivalence of categories $\mathsf{Sets}^{\pi\mathcal{U}^{\mathrm{op}}} \xrightarrow{\gamma^*} (\mathsf{Sets}^{\mathcal{U}^{\mathrm{op}}})_{\mathrm{lc}}$ we have a new equivalence Λ' obtained as the composite $\Lambda' = \overline{\Lambda}\gamma^*$:

$$\mathsf{Sets}^{\pi\mathcal{U}^{\mathrm{op}}} \xrightarrow{\gamma^*} (\mathsf{Sets}^{\mathcal{U}^{\mathrm{op}}})_{\mathrm{lc}} \xrightarrow{\overline{\Lambda}} (\mathrm{Cov\,proj}\,X)_{\mathcal{U}}.$$

If $\mathcal{U}$ refines $\mathcal{V}$ one has the following:

PROPOSITION 2.3. *Let $\mathcal{U}$ and $\mathcal{V}$ be two covering reduced sieves on X. If $\mathcal{U} \subset \mathcal{V}$, then the functor diagram*

$$\begin{array}{ccc}
\mathsf{Sets}^{\pi\mathcal{V}^{\mathrm{op}}} & \xrightarrow{\Lambda'} & (\mathrm{Cov\,proj}\,X)_{\mathcal{V}} \\
\downarrow & & \downarrow \\
\mathsf{Sets}^{\pi\mathcal{U}^{\mathrm{op}}} & \xrightarrow[\Lambda']{} & (\mathrm{Cov\,proj}\,X)_{\mathcal{U}}
\end{array}$$

is commutative up to natural isomorphism.

Proof. If F is an object in $\mathsf{Sets}^{\pi\mathcal{V}^{\mathrm{op}}}$, we have the presheaf $P = \gamma F$, which satisfies

$$\operatorname*{colim}_{x\in V\in\mathcal{V}} P(V) \cong \operatorname*{colim}_{x\in U\in\mathcal{U}} P(U).$$

This fact easily gives the existence of an isomorphism of functors in the diagram above.

Given a space X, we denote by $\mathrm{COV}(X)$ the set of open coverings $\mathcal{U}$ of X directed by refinement. We denote by $\mathrm{CRS}(X)$ the set of covering reduce sieves of X directed by refinement or equivalently by "inclusion"; that is, if $\mathcal{U},\mathcal{V}\in \mathrm{CRS}(X)$ then $\mathcal{U}\geq\mathcal{V}$ if and only if $\mathcal{U}\subset\mathcal{V}$. Recall that Gpd denotes the category of groupoids. Using the directed set $\mathrm{CRS}(X)$ as an indexing category we can consider the pro-groupoid

$$\pi\,\mathrm{crs}(X):\mathrm{CRS}(X)\to\mathsf{Gpd}$$

defined by $\pi\,\mathrm{crs}(X)(\mathcal{U})=\pi\mathcal{U}^{\mathrm{op}}$. Associated with the pro-groupoid $\pi\,\mathrm{crs}(X)$ we have the category $\mathrm{pro}(\pi\,\mathrm{crs}(X),\mathsf{Sets})$ defined in §1. The main result of this section is the following:

THEOREM 2.2. *The category* $\mathrm{Cov\,proj}\,X$ *of covering projections and transformations of a topological space* X *is equivalent to the category* $\mathrm{pro}(\pi\,\mathrm{crs}(X),\mathsf{Sets})$.

Proof. For the groupoid $\pi\,\mathrm{crs}(X)$, consider the category $(\pi\,\mathrm{crs}(X),\mathsf{Sets})$ defined in §1. Now we are going to define a functor

$$\Lambda':(\pi\,\mathrm{crs}(X),\mathsf{Sets})\to\mathrm{Cov\,proj}\,X.$$

Suppose that $(\pi\mathcal{U}^{\mathrm{op}},F)$ and $(\pi\mathcal{V}^{\mathrm{op}},G)$ are objects in $(\pi\,\mathrm{crs}(X),\mathsf{Sets})$ and $\alpha=(\mathcal{U}\subset\mathcal{V},\ \theta_\alpha:F\to G\pi^{\mathcal{U}}_{\mathcal{V}})$ is a morphism in $(\pi\,\mathrm{crs}(X),\mathsf{Sets})$. The functor Λ' carries $\alpha:(\pi\mathcal{U}^{\mathrm{op}},F)\to(\pi\mathcal{V}^{\mathrm{op}},G)$ to $\Lambda'\alpha:\Lambda'(\pi\mathcal{U}^{\mathrm{op}},F)\to\Lambda'(\pi\mathcal{V}^{\mathrm{op}},G)$, where $\Lambda'(\pi\mathcal{U}^{\mathrm{op}},F)=\overline{\Lambda}(F\gamma)$, $\Lambda'(\pi\mathcal{V}^{\mathrm{op}},G)=\overline{\Lambda}(G\gamma)$ and if $s\in F\gamma(U)$ and $x\in U\in\mathcal{U}$, then $\Lambda'\alpha(\mathrm{germ}^U_x\, s)=\mathrm{germ}^U_x(\theta_\alpha*\gamma(U)(s))$.

It is easy to check that if α is in Σ, then $\Lambda'\alpha$ is an isomorphism, hence there is an induced functor

$$\Lambda':\mathrm{pro}(\pi\,\mathrm{crs}(X),\mathsf{Sets})\to\mathrm{Cov\,proj}\,X.$$

We note that if $\mathcal{U},\mathcal{V}\in\mathrm{CRS}(X)$, then $\mathcal{U}\cap\mathcal{V}=\{W\mid W\in\mathcal{U}$ and $W\in\mathcal{V}\}\in\mathrm{CRS}(X)$. We also see that the inclusion $\mathrm{CRS}(X)\to\mathrm{COV}(X)$ of directed sets is cofinal.

If $(\pi\mathcal{U}^{\mathrm{op}},F)$ and $(\pi\mathcal{V}^{\mathrm{op}},G)$ are objects in $\mathrm{pro}(\pi\,\mathrm{crs}(X),\mathsf{Sets})$, then

$$\begin{aligned}
&\mathrm{Hom}_{\mathrm{pro}(\pi\,\mathrm{crs}(X),\mathsf{Sets})}((\pi\mathcal{U}^{\mathrm{op}},F),(\pi\mathcal{V}^{\mathrm{op}},G))\\
&\quad\cong \operatorname*{colim}_{\mathcal{W}\in\mathrm{CRS}(X),\,\mathcal{W}\subset\mathcal{U}\cap\mathcal{V}}\mathrm{Hom}_{\mathsf{Sets}^{\pi\mathcal{W}^{\mathrm{op}}}}(F\pi^{\mathcal{W}}_{\mathcal{U}},G\pi^{\mathcal{W}}_{\mathcal{V}})\\
&\quad\cong \operatorname*{colim}_{\mathcal{W}\in\mathrm{CRS}(X),\,\mathcal{W}\subset\mathcal{U}\cap\mathcal{V}}\mathrm{Hom}_{(\mathrm{Cov\,proj}\,X)_{\mathcal{W}}}(\Lambda'(F\pi^{\mathcal{W}}_{\mathcal{U}}),\Lambda'(G\pi^{\mathcal{W}}_{\mathcal{V}}))\\
&\quad\cong \operatorname*{colim}_{\mathcal{W}\in\mathrm{CRS}(X),\,\mathcal{W}\subset\mathcal{U}\cap\mathcal{V}}\mathrm{Hom}_{(\mathrm{Cov\,proj}\,X)_{\mathcal{W}}}(\Lambda'F,\Lambda'G)\\
&\quad\cong \mathrm{Hom}_{\mathrm{Cov\,proj}\,X}(\Lambda'F,\Lambda'G).
\end{aligned}$$

Thus we have shown that $\Lambda':\mathrm{pro}(\pi\,\mathrm{crs}(X),\mathsf{Sets})\to\mathrm{Cov\,proj}\,X$ is a full faithful functor.

On the other hand, if $\Phi = (p : E \to X,\ [\mathcal{A}])$ is an object in $\operatorname{Cov}\operatorname{proj} X$ and $\mathcal{A}$ is an atlas on a covering reduced sieve $\mathcal{U}$, then by Theorem 2.1 and Lemma 2.1, there is $F : \pi\mathcal{U}^{\mathrm{op}} \to \mathsf{Sets}$ such that $\Lambda'(\pi\mathcal{U}^{\mathrm{op}}, F) \cong \Phi$. Therefore $\Lambda' : \operatorname{pro}(\pi \operatorname{crs}(X), \mathsf{Sets}) \to \operatorname{Cov}\operatorname{proj} X$ is an equivalence of categories.

3. The subdivision of the Čech nerve. The *Čech nerve* $CX(\mathcal{U})$ of an open covering $\mathcal{U}$ of X is defined to be the simplicial set whose q-simplexes are given by

$$CX(\mathcal{U})_q = \{(U_0, \ldots, U_q) \mid U_0, \ldots, U_q \in \mathcal{U},\ U_0 \cap \ldots \cap U_q \neq \emptyset\}.$$

The face and degeneracy operators are defined in the usual way. We note that if $\mathcal{U} \geq \mathcal{V}$, then we can choose a map $\varphi : \mathcal{U} \to \mathcal{V}$ such that $U \subset \varphi U$. This induces a simplicial map $(U_0, \ldots, U_q) \to (\varphi U_0, \ldots, \varphi U_q)$, which is denoted by $C\varphi : CX(\mathcal{U}) \to CX(\mathcal{V})$, and we also have the corresponding realization $|C\varphi| : |CX(\mathcal{U})| \to |CX(\mathcal{V})|$. If we choose a different map $\psi : \mathcal{U} \to \mathcal{V}$ such that $U \subset \psi U$, we have

$$\emptyset \neq U_0 \cap \ldots \cap U_q \subset \varphi U_0 \cap \ldots \cap \varphi U_q \cap \psi U_0 \cap \ldots \cap \psi U_q,$$

hence there is a simplex $(\varphi U_0, \ldots, \varphi U_q, \psi U_0, \ldots, \psi U_q)$ having as faces $(\varphi U_0, \ldots, \varphi U_q)$ and $(\psi U_0, \ldots, \psi U_q)$. Then $|C\varphi|$ is contiguous to $|C\psi|$. Therefore $|C\varphi|$ is homotopic to $|C\psi|$, and we have $C\varphi = C\psi$ in the category $\mathrm{Ho}(\mathsf{SS})$ obtained by inverting the weak equivalences of the category of simplicial sets. As a consequence of these facts, we have a functor

$$CX : \mathrm{COV}(X) \to \mathrm{Ho}(\mathsf{SS}).$$

It is interesting to observe that if $\mathcal{U}$ and $\mathcal{V}$ are covering reduced sieves and $\mathcal{U} \geq \mathcal{V}$, then there is a canonical map $\mathcal{U} \subset \mathcal{V}$. In this case we have an induced map $CX(\mathcal{U}) \to CX(\mathcal{V})$ and then we obtain a functor

$$CX : \mathrm{CRS}(X) \to \mathsf{SS}.$$

Let Cat denote the category of small categories and functors. There is a functor $l : \Delta \to \mathsf{Cat}$ which carries an ordered set $[p] = \{0 \leq 1 \leq \ldots \leq p\}$ to the small category $l[p] = \{0 \leftarrow 1 \leftarrow \ldots \leftarrow p\}$. On the other hand, we consider the Yoneda embedding $y : \Delta \to \mathsf{SS}$, $[p] \to \Delta[p] = \mathrm{Hom}_\Delta(-, [p])$. Since Cat is a cocomplete category, we can apply [M-M, Th. I.5.2] to obtain a pair of functors $L : \mathsf{SS} \to \mathsf{Cat}$ and $\mathrm{Ner} : \mathsf{Cat} \to \mathsf{SS}$ such that L is left adjoint to Ner and the diagram

$$\begin{array}{ccc} \Delta & & \\ {\scriptstyle l}\downarrow & \searrow{\scriptstyle y} & \\ \mathsf{Cat} & \xleftarrow[L]{} & \mathsf{SS} \end{array}$$

is commutative up to isomorphism.

Given a small category C, $\operatorname{Ner} C$ is called the *nerve* of C; we note that $\operatorname{Ner} C_0 \cong C_0 \cong$ Objects of C and for $q > 0$,

$$(\operatorname{Ner} C)_q = \{(f_0, \ldots, f_{q-1}) \mid f_i \text{ is an arrow of } C \text{ and } \operatorname{domain}(f_i) = \operatorname{codomain}(f_{i+1})\}.$$

If $\mathcal{C}$ is a category and X is an object of $\mathcal{C}$, we denote by $\mathcal{C}{\downarrow}X$ the category which has as objects those morphisms u of $\mathcal{C}$ such that $\operatorname{codomain}(u) = X$. A morphism from $u : A \to X$ to $v : B \to X$ is a morphism $f : A \to B$ in $\mathcal{C}$ such that $vf = u$. We note that a morphism $g : X \to X'$ induces a functor $\mathcal{C}{\downarrow}g : \mathcal{C}{\downarrow}X \to \mathcal{C}{\downarrow}X'$ defined by $\mathcal{C}{\downarrow}g(u) = gu$.

In particular, for the category Δ, we have the functor $\Delta{\downarrow}- : \Delta \to \mathsf{Cat}$, $[q] \to \Delta{\downarrow}[q]$, and we can consider the composite

$$\operatorname{sd} = \operatorname{Ner}(\Delta{\downarrow}-) : \Delta \to \mathsf{Cat} \to \mathsf{SS}.$$

Now we can apply [M-M, Th. I.5.2] to the functors $\operatorname{sd} : \Delta \to \mathsf{Cat}$ and $y : \Delta \to \mathsf{SS}$ to obtain a pair of adjoint functors $\operatorname{Sd} : \mathsf{SS} \to \mathsf{SS}$, $\operatorname{Ex} : \mathsf{SS} \to \mathsf{SS}$ such that the diagram

$$\begin{array}{ccc} \Delta & & \\ {\scriptstyle \mathrm{sd}}\downarrow & \searrow^{y} & \\ \mathsf{SS} & \xleftarrow[\mathrm{Sd}]{} & \mathsf{SS} \end{array}$$

is commutative up to isomorphism. The left adjoint $\operatorname{Sd} : \mathsf{SS} \to \mathsf{SS}$ is called the *subdivision functor*.

For each space X, we consider the functor $NX : \operatorname{CRS}(X) \to \mathsf{SS}$ defined by $NX(\mathcal{U}) = \operatorname{Ner}(\mathcal{U}^{\mathrm{op}})$, where $\mathcal{U}^{\mathrm{op}}$ is the opposite category of $\mathcal{U}$ considered as a small category. A typical q-simplex of $NX(\mathcal{U})_q$ is of the form $U_0 \subset U_1 \subset \ldots \subset U_q$ with $U_0, \ldots, U_q \in \mathcal{U}$.

Next we prove that the subdivision $\operatorname{Sd} CX$ of the Čech nerve is weakly equivalent to NX.

THEOREM 3.1. *There is a natural transformation ψ from the functor $\operatorname{Sd} CX : \operatorname{CRS}(X) \to \mathsf{SS}$ to the functor $NX : \operatorname{CRS}(X) \to \mathsf{SS}$ such that for each $\mathcal{U}$ in $\operatorname{CRS}(X)$, $\psi(\mathcal{U}) : \operatorname{Sd} CX(\mathcal{U}) \to NX(\mathcal{U})$ is a weak equivalence.*

Proof. Using the notation of [M-M] (see also §0), we recall that for an object $\mathcal{U}$ of $\operatorname{CRS}(X)$,

$$\operatorname{Sd} CX(\mathcal{U}) = \operatorname{colim}\Big(\int_{\Delta} CX(\mathcal{U}) \overset{\mathrm{pr}_1}{\to} \Delta \overset{\mathrm{sd}}{\to} \mathsf{SS}\Big).$$

Given an object $([n], (V_0, \ldots, V_n))$ in $\int_\Delta CX(\mathcal{U})$, we have $\operatorname{sd} \operatorname{pr}_1([n], (V_0, \ldots, V_n)) = \operatorname{sd}[n]$. An element of $(\operatorname{sd} \operatorname{pr}_1([n], (V_0, \ldots, V_n)))_q$ is determined by a diagram in Δ of the form

$$\begin{array}{ccccccc} [p_0] & \xleftarrow{\alpha_0} & [p_1] & \xleftarrow{\alpha_1} & \ldots [p_{q-1}] & \xleftarrow{\alpha_{q-1}} & [p_q] \\ {\scriptstyle\alpha}\downarrow & & & & & & \\ [n] & & & & & & \end{array}$$

and it will be denoted by ${}_{\alpha}(\alpha_0, \ldots, \alpha_{q-1})$.

For each $([n], (V_0, \ldots, V_n))$ we have the map

$$\psi([n], (V_0, \ldots, V_n))_q : (\operatorname{sd} \operatorname{pr}_1([n], (V_0, \ldots, V_n)))_q \to NX(\mathcal{U})_q$$

defined by

$$\begin{aligned} &\psi([n], (V_0, \ldots, V_n))_q \ {}_{\alpha}(\alpha_0, \ldots, \alpha_{q-1}) \\ &\qquad = (V_{\alpha(0)} \cap \ldots \cap V_{\alpha(p_0)} \subset V_{\alpha\alpha_0(0)} \cap \ldots \cap V_{\alpha\alpha_0(p_1)} \subset \ldots \\ &\qquad\qquad \subset V_{\alpha\alpha_0\ldots\alpha_{q-1}(0)} \cap \ldots \cap V_{\alpha\alpha_0\ldots\alpha_{q-1}(p_q)}). \end{aligned}$$

These maps induce a map

$$\psi(\mathcal{U}) : \operatorname{Sd} CX(\mathcal{U}) \to NX(\mathcal{U}).$$

Now we define a transformation $\varphi(\mathcal{U}) : NX(\mathcal{U}) \to \operatorname{Sd} CX(\mathcal{U})$. Given an element $U_0 \subset \ldots \subset U_q$ of $NX(\mathcal{U})_q$, $([q], (U_0, \ldots, U_q))$ is an object of $\int_{\Delta} CX(\mathcal{U})$. Consider the element ${}_{\beta}(\beta_0, \ldots, \beta_{q-1})$ of $(\operatorname{sd} \operatorname{pr}_1([q], (U_0, \ldots \ldots, U_q)))_q$ determined by the diagram

$$\begin{array}{ccccccccc} [q] & \xleftarrow{\beta_0} & [q-1] & \xleftarrow{\beta_1} & [q-2] & \longleftarrow & \ldots [1] & \xleftarrow{\beta_{q-1}} & [0] \\ {\scriptstyle\beta}\downarrow & & & & & & & & \\ [q] & & & & & & & & \end{array}$$

where $\beta = \mathrm{id}_{[q]}$ and $\beta_i : [q-i-1] \to [q-i]$ is defined by $\beta_i(j) = j+1$ for $0 \le j \le q-i-1$. The element ${}_{\beta}(\beta_0, \ldots, \beta_{q-1})$ represents an element $[{}_{\beta}(\beta_0, \ldots, \beta_{q-1})]$ of $\operatorname{Sd} CX(\mathcal{U})_q$. We define $\varphi(\mathcal{U})_q : NX(\mathcal{U})_q \to \operatorname{Sd} CX(\mathcal{U})_q$ by

$$\varphi(\mathcal{U})_q \ (U_0 \subset \ldots \subset U_q) = [{}_{\beta}(\beta_0, \ldots, \beta_{q-1})].$$

Because

$$\psi([q], (U_0, \ldots, U_q))_q({}_{\beta}(\beta_0, \ldots, \beta_{q-1})) = (U_0 \subset U_1 \subset \ldots \subset U_q)$$

we have $\psi(\mathcal{U})\varphi(\mathcal{U}) = \mathrm{id}_{NX(\mathcal{U})}$.

To see that $\psi(\mathcal{U})$ is a weak equivalence, it is sufficient to show that $|\varphi(\mathcal{U})|\,|\psi(\mathcal{U})|$ is homotopic to the identity. We note the following facts:

If we consider the realization functor $|-| : \mathsf{SS} \to \mathsf{Top}$, we find that for every object Y in SS, $|\operatorname{sd} Y|$ is homeomorphic to $|Y|$, and if y is an n-simplex of Y and ${}_{\alpha}(\alpha_0,\ldots,\alpha_{q-1})$ is a q-simplex of $(\operatorname{sd}\operatorname{pr}_1([n],y))_q$ which represents a q-simplex $[{}_{\alpha}(\alpha_0,\ldots,\alpha_{q-1})]$ of $\operatorname{sd} Y$, then $|[{}_{\alpha}(\alpha_0,\ldots,\alpha_{q-1})]| \subset |y|$, where $|[{}_{\alpha}(\alpha_0,\ldots,\alpha_{q-1})]|$ and $|y|$ denote the realizations of the corresponding simplexes.

For $Y = CX(\mathcal{U})$, if $[{}_{\alpha}(\alpha_0,\ldots,\alpha_{q-1})]$ is a q-simplex of $\operatorname{Sd} CX(\mathcal{U})$ represented by the q-simplex ${}_{\alpha}(\alpha_0,\ldots,\alpha_{q-1})$ of $\operatorname{sd}\operatorname{pr}_1([n],(V_0,\ldots,V_n))$, then we have

$$\psi(\mathcal{U})[{}_{\alpha}(\alpha_0,\ldots,\alpha_{q-1})] = (U_0 \subset \ldots \subset U_q),$$
$$\varphi(\mathcal{U})(U_0 \subset \ldots \subset U_q) = [{}_{\beta}(\beta_0,\ldots,\beta_{q-1})].$$

Consider the $(n+q+2)$-simplex $(V_0,\ldots,V_n,U_0,\ldots,U_q)$ of $CX(\mathcal{U})$. The realizations $|[{}_{\alpha}(\alpha_0,\ldots,\alpha_{q-1})]|$ and $|[{}_{\beta}(\beta_0,\ldots,\beta_{q-1})]|$ are contained in $|(V_0,\ldots, V_n, U_0,\ldots,U_q)|$. This implies that $|\varphi(\mathcal{U})|\,|\psi(\mathcal{U})|$ is contiguous to $\mathrm{id}_{|\operatorname{Sd} CX(\mathcal{U})|}$ with respect to the simplicial decomposition given by $|CX(\mathcal{U})|$ (recall that $|CX(\mathcal{U})|$ is homeomorphic to $|\operatorname{Sd} CX(\mathcal{U})|$). Therefore $|\varphi(\mathcal{U})|\,|\psi(\mathcal{U})|$ is homotopic to $\mathrm{id}_{|\operatorname{Sd} CX(\mathcal{U})|}$.

Finally, we note that if $\mathcal{U},\mathcal{V} \in \mathrm{CRS}(X)$ and $\mathcal{U} \subset \mathcal{V}$, then the diagram

$$\begin{array}{ccc} \operatorname{Sd} CX(\mathcal{U}) & \xrightarrow{\psi(\mathcal{U})} & NX(\mathcal{U}) \\ \downarrow & & \downarrow \\ \operatorname{Sd} CX(\mathcal{V}) & \xrightarrow[\psi(\mathcal{V})]{} & NX(\mathcal{V}) \end{array}$$

is commutative. Therefore $\psi : \operatorname{Sd} CX \to NX$ is a natural transformation.

LEMMA 3.1. *Let $\mathcal{C}$ be a small category. Then the fundamental groupoid $\pi \operatorname{Ner}\mathcal{C}$ is isomorphic to the category of fractions $\mathcal{C}[\Sigma^{-1}]$, where Σ is the set of all morphisms of $\mathcal{C}$.*

Proof. The arrow of $\pi \operatorname{Ner}\mathcal{C}$ represented by an arrow $A_0 \leftarrow A_1$ in $\mathcal{C}$ is carried by the equivalence functor to the arrow of $\mathcal{C}[\Sigma^{-1}]$ induced by $A_0 \leftarrow A_1$. For details we refer the reader to [Go].

REMARK. For a given covering reduced sieve $\mathcal{U}$, the groupoid $\pi N\mathcal{U}^{\mathrm{op}}$ is isomorphic to the category of fractions $\pi\mathcal{U}^{\mathrm{op}}$.

COROLLARY 3.1. *Given a space X, the category* $\operatorname{Cov}\operatorname{proj} X$ *of covering projections of X is equivalent to the category* $\operatorname{pro}(\pi CX, \mathsf{Sets})$.

Proof. For each $\mathcal{U} \in \mathrm{CRS}(X)$,

$$\psi(\mathcal{U}) : \operatorname{Sd} CX(\mathcal{U}) \to NX(\mathcal{U})$$

is a weak equivalence. Since $|CX(\mathcal{U})|$ is homeomorphic to $|\operatorname{Sd} CX(\mathcal{U})|$, we also have the weak equivalences

$$CX(\mathcal{U}) \to \operatorname{Sing}|CX(\mathcal{U})| \to \operatorname{Sing}|\operatorname{Sd} CX(\mathcal{U})| \leftarrow \operatorname{Sd} CX(\mathcal{U}).$$

Therefore, one has the zig-zag weak equivalences of pro-groupoids

$$\pi CX(\mathcal{U}) \to \pi \operatorname{Sing}|CX(\mathcal{U})| \to \pi \operatorname{Sing}|\operatorname{Sd} CX(\mathcal{U})| \leftarrow \pi \operatorname{Sd} CX(\mathcal{U}).$$

Now, by Lemma 2.1, we conclude that $\mathrm{pro}(\pi NX, \mathsf{Sets})$ is equivalent to $\mathrm{pro}(\pi CX, \mathsf{Sets})$.

4. Covering projections for locally path-connected and semilocally 1-connected spaces. In this section we prove that under "good" local conditions of connectedness the pro-groupoid $\pi\,\mathrm{crs}(X)$ is equivalent to the standard fundamental groupoid πX. In this way we obtain the well known equivalence of the category of covering projections of a "good" space and the functor category $\mathsf{Sets}^{\pi X}$ (see [God]).

Given a space X, we denote by πX the fundamental groupoid of X. The groupoid πX has as objects the points of X and for $x_0, x_1 \in X$ an arrow from x_0 to x_1 is represented by a path from x_0 to x_1 (up to relative homotopy). If α is an arrow from x_0 to x_1 and β is an arrow from x_1 to x_2 the composite will be denoted by $\beta\alpha$. Let S be a subspace of X; then the inclusion $S \subset X$ induces a canonical groupoid homomorphism $\pi S \to \pi X$. We say that $\pi S \to \pi X$ is *trivial* if any arrow α of πS such that $\mathrm{domain}(\alpha) = \mathrm{codomain}(\alpha) = s \in S$ is carried to the identity arrow by the functor $\pi S \to \pi X$.

We will consider open coverings $\mathcal{U}$ of X such that

(1) if $U \in \mathcal{U}$, then $U \neq \emptyset$, U is path-connected and $\pi U \to \pi X$ is trivial,

(2) if V is a non-empty open subset of X such that V is path-connected, $\pi V \to \pi X$ is trivial and there is $U \in \mathcal{U}$ such that $V \subset U$, then $V \in \mathcal{U}$.

An open covering $\mathcal{U}$ of X which satisfies conditions (1) and (2) is said to be a *trivial covering* of X. We denote by $\mathrm{TCOV}(X)$ the family of all trivial coverings of X directed by refinement. We note that if $\mathcal{U}, \mathcal{V} \in \mathrm{TCOV}(X)$, then $\mathcal{U} \geq \mathcal{V}$ if and only if $\mathcal{U} \subset \mathcal{V}$.

LEMMA 4.1. *Let X be a locally path-connected and semilocally 1-connected space. Then the map $^*s : \mathrm{TCOV}(X) \to \mathrm{CRS}(X)$ defined by $^*s\mathcal{U} = \{V \mid V \neq \emptyset,\ V$ is open, $\exists U \in \mathcal{U}$ such that $V \subset U\}$ is cofinal.*

Proof. Let $\mathcal{V}$ be a covering reduced sieve on X. For each $V \in \mathcal{V}$ and $x \in V$ there is an open subset $U(V,x)$ such that $x \in U(V,x) \subset V$, $U(V,x)$ is path-connected and $\pi U(V,x) \to \pi X$ is trivial. Let $\mathcal{U} = \{U \in \mathcal{O}(X) \mid U \neq \emptyset$, $\exists U(V,x)$ such that $U \subset U(V,x)$, U is path-connected and $\pi U \to \pi X$ is trivial$\}$. Since X is locally connected and semilocally 1-connected, it follows that $\mathcal{U} \in \mathrm{TCOV}(X)$, and from the definition of $\mathcal{U}$ we have $^*s\mathcal{U} \subset \mathcal{V}$.

COROLLARY 4.1. *Let X be a locally path-connected and semilocally 1-connected space. Then the category* $\operatorname{Cov}\operatorname{proj} X$ *is equivalent to the category*

$$\operatorname{pro}(\{\pi({}^*s\mathcal{U}^{\mathrm{op}}) \mid \mathcal{U} \in \mathrm{TCOV}(X)\}, \mathsf{Sets}).$$

P r o o f. By Lemma 4.1, $\pi\operatorname{crs}(X) = \{\pi\mathcal{U}^{\mathrm{op}} \mid U \in \mathrm{CRS}(X)\}$ is isomorphic to $\{\pi({}^*s\mathcal{U}^{\mathrm{op}}) \mid U \in \mathrm{TCOV}(X)\}$ in $\operatorname{pro}\mathsf{Gpd}$. By Lemma 1.1, $\operatorname{pro}(\pi\operatorname{crs}(X), \mathsf{Sets})$ is equivalent to $\operatorname{pro}(\{\pi({}^*s\mathcal{U}^{\mathrm{op}}) \mid U \in \mathrm{TCOV}(X)\}, \mathsf{Sets})$. Now from Theorem 2.2, it follows that $\operatorname{Cov}\operatorname{proj} X$ is equivalent to $\operatorname{pro}(\{\pi({}^*s\mathcal{U}^{\mathrm{op}}) \mid U \in \mathrm{TCOV}(X)\})$.

If a space X satisfies the usual connectedness conditions of this section, we can consider the following "maximal" open covering in $\mathrm{TCOV}(X)$:

$$\mathcal{U}_0 = \{U \in \mathcal{O}(X) \mid U \neq \emptyset,\ U \text{ is path-connected and } \pi U \to \pi X \text{ is trivial}\}.$$

Now we can take a map $\eta : {}^*s\mathcal{U}_0 \to X$ such that $\eta(U) = x_U \in U \in {}^*s\mathcal{U}_0$. If $U, V \in \mathcal{U}_0$ and $U \subset V$, we can take in V a path α from x_V to x_U. We note that a different path α' determines the same arrow in πX. It is routine to check that this construction gives an equivalence $\eta : \pi({}^*s\mathcal{U}_0^{\mathrm{op}}) \to \pi X$. If $\mathcal{U} \in \mathrm{TCOV}(X)$, the inclusion $\mathcal{U} \subset \mathcal{U}_0$ induces the composite $\pi({}^*s\mathcal{U}^{\mathrm{op}}) \to \pi({}^*s\mathcal{U}_0^{\mathrm{op}}) \xrightarrow{\eta} \pi X$, which is also an equivalence of groupoids. Since the pro-groupoid $\{\pi({}^*s\mathcal{U}^{\mathrm{op}}) \mid \mathcal{U} \in \mathrm{TCOV}(X)\}$ and the constant pro-groupoid πX satisfy the conditions of Lemma 1.2, we deduce that $\operatorname{pro}(\{\pi({}^*s\mathcal{U}^{\mathrm{op}}) \mid \mathcal{U} \in \mathrm{TCOV}(X)\}, \mathsf{Sets})$ is equivalent to $\operatorname{pro}(\pi X, \mathsf{Sets})$. Therefore one has:

THEOREM 4.1. *Let X be a locally path-connected and semilocally* 1*-connected space. Then the category of covering projections of X is equivalent to* $\mathsf{Sets}^{\pi X}$, *the category of functors from πX to* Sets.

5. Covering projections of compact metrisable spaces. The objective of this section is to reduce the pro-groupoid G of the category $\operatorname{pro}(G, \mathsf{Sets})$ to a pro-group, a tower of groups or even a pro-discrete topological group. For this purpose, we use compact metrisable spaces in order to obtain tower of groupoids, connectedness conditions to have a tower of groups, and finally pointed movability conditions to have a surjective tower of groups or a topological group.

In order to reduce a pro-groupoid to a tower of groupoids, we suppose that X is a compact metrisable space. Using the Lebesgue Lemma, we can construct a sequence

$$\ldots \geq \mathcal{V}_{n+1} \geq \mathcal{V}_n \geq \ldots \geq \mathcal{V}_0$$

of open coverings which is cofinal in $\mathrm{COV}(X)$. This implies that

$$\ldots \subseteq {}^*s\mathcal{V}_{n+1} \subseteq {}^*s\mathcal{V}_n \subseteq \ldots \subseteq {}^*s\mathcal{V}_0$$

is cofinal in $\mathrm{CRS}(X)$. Therefore:

(1) πCX is isomorphic to $\{\pi CX(^*s\mathcal{V}_n)\}$ in pro Gpd,

(2) $\{\pi CX(^*s\mathcal{V}_n)\}$ is isomorphic to $\{\pi CX(\mathcal{V}_n)\}$ in tow $\pi_0\mathsf{Gpd}$.

From Lemma 1.1, Proposition 1.1 and Corollary 3.1, we obtain the following:

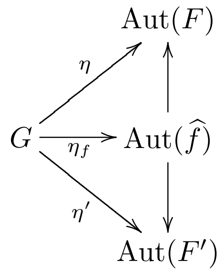
Theorem 5.1. *Let X be a compact metrisable space and suppose that $\ldots \geq \mathcal{V}_{n+1} \geq \mathcal{V}_n \geq \ldots \geq \mathcal{V}_0$ is a cofinal sequence in* $\mathrm{COV}(X)$. *Then the category* $\mathrm{Cov\,proj}\, X$ *is equivalent to* $\mathrm{pro}(\{\pi CX(\mathcal{V}_n)\}, \mathsf{Sets})$.

If X is a connected compact metrisable space and $\mathcal{U}$ is an open covering, then $|CX(\mathcal{U})|$ is 0-connected. We can check this fact as follows: For $x, y \in X$, a *$\mathcal{U}$-path from x to y* is a finite family $U_0, \ldots, U_m$ such that $x \in U_0$, $y \in U_m$ and $U_0 \cap U_1 \neq \emptyset, \ldots, U_{m-1} \cap U_m \neq \emptyset$. If $U, U' \in \mathcal{U}$, a *$\mathcal{U}$-path from U to U'* is a finite family $U_0, \ldots, U_m$ such that $U = U_0$, $U' = U_m$ and $U_0 \cap U_1 \neq \emptyset, \ldots, U_{m-1} \cap U_m \neq \emptyset$. For $x_0 \in X$ it is easy to check that $C_0 = \{x \mid \text{there is a } \mathcal{U}\text{-path from } x_0 \text{ to } x\}$ is open and closed in X, and since X is connected it follows that $C_0 = X$. Now for a given $U_0 \in \mathcal{U}$ if U is also in $\mathcal{U}$ we can choose $x_0 \in U_0$ and $x \in U$. Because $C_0 = X$ there is a $\mathcal{U}$-path $V_0, \ldots, V_m$ from x_0 to x. Therefore $U_0, V_0, \ldots, V_m, U$ is a $\mathcal{U}$-path from U_0 to U. This implies that the groupoid $\pi CX(\mathcal{U})$ is connected. For a connected groupoid π and an object U in π let π_1 be the group of endomorphisms of U. Then if π_1 is considered as a groupoid with one object, the inclusion $\pi_1 \to \pi$ is an equivalence of groupoids or equivalently an isomorphism in $\pi_0\mathsf{Gpd}$.

Suppose that X is a connected compact metrisable space and we have a given point $x \in X$. Then there exists a sequence $\ldots \geq \mathcal{V}_{n+1} \geq \mathcal{V}_n \geq \ldots$ of open coverings; we can also assume that there are open subsets $V_n \in \mathcal{V}_n$ such that $x \in V_n \subset V_{n+1}$. Denote by $\pi_1 CX(\mathcal{V}_i, V_i)$ the fundamental group of the groupoid $\pi CX(\mathcal{V}_i)$ based at the object V_i. In this way, the sequence of pointed open coverings induced by the pointed space (X, x) determines a tower $\{\pi_1 CX(\mathcal{V}_i, V_i)\}$ of groups that will be denoted by $\pi_1 C(X, x)$. Then $\pi_1 C(X, x)$ is isomorphic to $\{\pi CX(\mathcal{V}_i)\}$ in tow $\pi_0\mathsf{Gpd}$. From Proposition 1.1 and Theorem 5.1, we have:

Theorem 5.2. *Let (X, x) be a pointed connected compact metrisable space. Then* $\mathrm{Cov\,proj}\, X$ *is equivalent to* $\mathrm{pro}(\pi_1 C(X, x), \mathsf{Sets})$.

The category G-Sets for G a group (see §0) can be generalised for a pro-group G as follows: Objects are morphisms $\eta : G \to \mathrm{Aut}(F)$ in pro Gps, where F is a set and $\mathrm{Aut}(F)$ is the group of automorphisms, which can be considered as a pro-group. A morphism $f : \eta \to \eta'$ is given by a map $\widehat{f} : F \to F'$ and a morphism $\eta_f : G \to \mathrm{Aut}(\widehat{f})$ such that the diagram



is commutative.

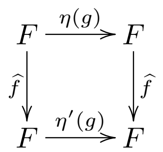
Recall that in §1 for a pro-group G we have constructed the category $\mathrm{pro}(G, \mathsf{Sets})$ as a category of right fractions $(G, \mathsf{Sets})\Sigma^{-1}$. An object $(G(i), F)$ in (G, Sets) is determined by a homomorphism $F : G(i) \to \mathrm{Aut}(F(*))$, where $F(*)$ is the set obtained by the functor $F : G(i) \to \mathsf{Sets}$ when it is applied to the unique object $*$ of the category $G(i)$ (we denote by F both the functor and the homomorphism). It is clear that F represents a unique morphism $\eta_F : G \to \mathrm{Aut}(F(*))$, which is an object in G-Sets. On the other hand, a morphism $\alpha = (i \to j,\ \theta_\alpha : F \to HG^i_j)$ from $(G(i), F)$ to $(G(j), H)$ induces a map $(\theta_\alpha)_* : F(*) \to H(*)$ and a homomorphism $(F, HG^i_j) : G(i) \to \mathrm{Aut}((\theta_\alpha)_*)$ which represents a morphism $\eta_\alpha : \eta_F \to \eta_H$. This functor factorizes to $\mathrm{pro}(G, \mathsf{Sets})$ and one can check the following result:

PROPOSITION 5.1. *For a given pro-group G the category* $\mathrm{pro}(G, \mathsf{Sets})$ *is equivalent to the category G-Sets.*

As a consequence of this description and Theorem 5.2, for a finite set $\{1, \ldots, d\}$ one has the following version of the fundamental theorem of overlay theory proved by Fox [F1, F2].

COROLLARY 5.1. *Let X be a connected compact metrisable space X and let x be a point of X. Then the d-fold covering projections X are in bi-unique correspondence with the representations up to conjugation of the fundamental pro-group $\pi_1 C(X, x)$ in the symmetric group Σ_d of degree d.*

P r o o f. Two d-fold covering projections p, p' are isomorphic in the category $\mathrm{Cov\,proj}\, X$ if and only if the corresponding objects $\eta, \eta' : \pi_1 C(X, x) \to \Sigma_d$ are isomorphic in the category $\pi_1 C(X, x)$-Sets. If $f : \eta \to \eta'$ is an isomorphism given by $\widehat{f} \in \Sigma_d$ and $\eta_f : \pi_1 C(X, x) \to \mathrm{Aut}(\widehat{f})$, then for some pointed open covering $(\mathcal{V}_i, V_i)$ and each element $g \in \pi_1 CX(\mathcal{V}_i, V_i)$, we have $\eta_f(g) = (\eta(g), \eta'(g))$; that is, the following diagram commutes, where $F = \{1, \ldots, d\}$:



This implies that the representations η and η' are conjugate. Conversely, if η and η' are conjugate, then they are isomorphic in the category G-Sets.

Remark. Notice that in this corollary we have considered connected compact metrisable spaces instead of connected separable metrisable spaces of Fox's overlay theorem (see [F1]). The notion of fundamental pro-group in the corollary corresponds to the one of the fundamental tropes of the space considered by Fox. It is not hard to give a version of our main Theorem 2.2 for a connected space with a base point. In this case the pro-groupoid reduces to a pro-group and a version of Fox's theorem can be obtained for covering projections over a pointed connected space with a fibre F. That is, we can avoid the separable metrisable condition and we can consider infinite fibres.

For an object G in tow Gps, $\lim G$ can be provided with the inverse limit topology and we have the category of continuous $\lim G$-sets defined in §0. The canonical map $\theta : \lim G \to G$ induces a functor $\theta^* : G\text{-Sets} \to \lim G\text{-Sets}$ as follows:

If $f : \eta \to \eta'$ is a morphism in G-Sets represented by a map $\widehat{f} : F \to F'$ and a commutative diagram

$$\begin{array}{ccc} & & \mathrm{Aut}(F) \\ & \overset{\eta}{\nearrow} & \uparrow \\ G & \xrightarrow[\eta_f]{} & \mathrm{Aut}(\widehat{f}) \\ & \underset{\eta'}{\searrow} & \downarrow \\ & & \mathrm{Aut}(F') \end{array}$$

we consider $\theta^*(f) = f\theta : \eta\theta \to \eta'\theta$ represented by the diagram

$$\begin{array}{ccc} & & \mathrm{Aut}(F) \\ & \overset{\eta\theta}{\nearrow} & \uparrow \\ \lim G & \xrightarrow[\eta_f\theta]{} & \mathrm{Aut}(\widehat{f}) \\ & \underset{\eta'\theta}{\searrow} & \downarrow \\ & & \mathrm{Aut}(F') \end{array}$$

If the bonding maps, $G(i+1) \to G(i)$, are surjective, the map $\theta : \lim G \to G$ is an epimorphism in tow Gps. Then if $\theta^*(f) = \theta^*(g)$, we have $\eta_f\theta = \eta_g\theta$. Therefore $\eta_f = \eta_g$ and this implies that $f = g$.

On the other hand, if $\eta : \lim G \to \mathrm{Aut}(\widehat{f})$ is a continuous homomorphism, since $\mathrm{Aut}(\widehat{f})$ has the discrete topology, $\eta^{-1}\{1\}$ is an open neighbourhood of 1, and so there is $\theta(i) : \lim G \to G(i)$ such that $\operatorname{Ker}\theta(i) \subset \eta^{-1}\{1\}$. Hence η factors as

$$
\begin{array}{ccc}
\lim G & \xrightarrow{\quad\eta\quad} & \operatorname{Aut}(\widehat{f}) \\
 & {\scriptstyle\theta}\searrow \quad \nearrow{\scriptstyle\overline{\eta}_f} & \\
 & G(i) &
\end{array}
$$

And $\overline{\eta}_f$ defines a morphism $(\overline{\eta}_f, \widehat{f}) : \eta \to \eta'$ such that $\theta^*(\overline{\eta}_f, \widehat{f}) = (\eta, \widehat{f})$. Therefore one has:

PROPOSITION 5.2. *Let G be an object in* tow Gps *such that the bonding morphisms of G are surjective maps. Then the category G*-Sets *is equivalent to the category* $\lim G$-Sets, *where* $\lim G$ *is provided with the inverse limit topology.*

REMARK. The proof given above only works for surjective towers of groups. For a more general surjective pro-group G, $\lim G$ can be trivial (see [M]). In this case, Moerdijk notes that a similar result is obtained by taking the inverse limit in the category of localic groups.

In order to obtain pro-groups it is convenient to work with pointed spaces and if we want to have surjective pro-groups, it will be useful to recall some notion of pointed movability. A *pointed open covering* is a pair $(\mathcal{U}, U_0)$ where $U_0 \in \mathcal{U}$; $(\mathcal{V}, V_0)$ refines $(\mathcal{U}, U_0)$ if $\mathcal{V} \geq \mathcal{U}$ and $V_0 \subset U_0$.

DEFINITION 5.2. A pointed space (X, x) is said to be 1-*movable* if for every pointed open covering $(\mathcal{U}, U_0)$ there is a finer pointed open covering $(\mathcal{V}, V_0) \geq (\mathcal{U}, U_0)$ such that for any pointed open covering $(\mathcal{W}, W_0) \geq (\mathcal{U}, U_0)$, and for any pointed map $h : (P, *) \to (|CX(\mathcal{V})|, *)$, where P is a CW-complex with $\dim P \leq 1$, there exists a pointed map $r : (P, *) \to (|CX(\mathcal{W})|, *)$ such that $|CX|_{\mathcal{U}}^{\mathcal{W}} r$ is homotopic to $|CX|_{\mathcal{U}}^{\mathcal{V}} h$ relative to the base point.

Now we refer the reader to Theorem 2 of [M-S, Ch. II, §8.1] to deduce that if (X, x) is 1-movable, then the pro-group $\pi_1 C(X, x)$ satisfies the Mittag-Leffler condition, and by Theorem 7 of [M-S, Ch. II, §6.2], $\pi_1 C(X, x)$ is isomorphic to a pro-group G whose bonding maps are surjective. Moreover, if $\pi_1 C(X, x)$ is isomorphic to a tower, then we can suppose that G is a tower.

THEOREM 5.3. *Let X be a connected compact metrisable space and let x be a point of X. Suppose that (X, x) is* 1*-movable and denote by $\check{\pi}_1(X, x)$ the Čech fundamental group of (X, x), $\check{\pi}_1(X, x) = \lim \pi_1 C(X, x)$, provided with the inverse limit topology. Then the category of covering projections of X is equivalent to the category of continuous $\check{\pi}_1(X, x)$-sets.*

P r o o f. This follows from Theorem 5.2 and Propositions 5.1 and 5.2.

Remarks. (1) Let X be a pointed connected, compact and metrisable space and let $\pi_1 CX$ denote its fundamental pro-group. For a set F, the Brown $\mathcal{P}$ functor (see [H]) carries a morphism $\eta : \pi_1 CX \to \operatorname{Aut} F$ to a morphism of the form $\mathcal{P}\eta : \mathcal{P}\pi_1 CX \to \mathcal{P} \operatorname{Aut} F$ and then we have the composite $\mathcal{P}\pi_1 CX \to \mathcal{P} \operatorname{Aut} F \to \operatorname{Aut} \mathcal{P}F$. Since $\mathcal{P}\pi_1 CX$ is isomorphic to the Quigley inward group ${}^I\pi_1^Q(X)$, the category of covering projections of X with fibre F is also equivalent to a category of "distinguished" representations of the Quigley inward group ${}^I\pi_1^Q(X)$ with "fibre" $\mathcal{P}F$. In [H], we have constructed a space $\overline{\mathcal{P}}^R CX$ whose fundamental group is isomorphic to the Quigley inward group ${}^I\pi_1^Q(X)$. Thus the category of covering projections of X with fibre F is equivalent to a category of "distinguished" covering projections of $\overline{\mathcal{P}}^R CX$ with fibre $\mathcal{P}F$.

(2) If the pointed space X of remark (1) is also 2-movable (see [M-S]) then the fundamental group of the homotopy limit $\lim^R CX$ is isomorphic to the Čech fundamental group of X. Therefore the category of covering projections of X is equivalent to a category of "distinguished" covering projections of $\lim^R CX$.

(3) Some additional results about classification of principal G-bundles of a space X can be obtained in terms of morphisms of the form $\pi CX \to G$, or $\lim \pi CX \to G$, even in the case where G is a pro-discrete group. This is also connected with the first cohomology set of a pro-groupoid or a topological group with coefficients in a discrete or pro-discrete group.

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