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Correlation dimension for self-similar Cantor sets with overlaps

by

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Abstract. We consider self-similar Cantor sets $\Lambda \subset \mathbb{R}$ which are either homogeneous and $\Lambda - \Lambda$ is an interval, or not homogeneous but having thickness greater than one. We have a natural labeling of the points of Λ which comes from its construction. In case of overlaps among the cylinders of Λ, there are some "bad" pairs (τ, ω) of labels such that τ and ω label the same point of Λ. We express how much the correlation dimension of Λ is smaller than the similarity dimension in terms of the size of the set of "bad" pairs of labels.

1. Introduction. In the literature there are some results (see [Fa1], [PS] or [Si] for a survey) which show that for a family of Cantor sets of overlapping construction on $\mathbb{R}$, the dimension (Hausdorff or box counting) is almost surely equal to the similarity dimension, that is, the overlap between the cylinders typically does not lead to dimension drop. However, we do not understand the cause of the decrease of dimension in the exceptional cases. In this paper we consider self-similar Cantor sets Λ on $\mathbb{R}$ for which either the thickness of Λ is greater than one or Λ is homogeneous and $\Lambda - \Lambda$ is an interval. We prove that the only reason for the drop of the correlation dimension is the size of the set of those pairs of symbolic sequences which label the same point of the Cantor set. We do not know a similar statement for the Hausdorff dimension.

2. Theorem. Consider a family of contractive similarities $S_i : \mathbb{R} \to \mathbb{R}$, $i = 1, \ldots, d$, where $S_i(x) = \lambda_i x + t_i$ for some $\lambda_i \in (0,1)$ and $t_i \in \mathbb{R}$. Let Λ be the attractor of the iterated function system $\{S_i(x)\}_{i=1}^{d}$, that is, the unique non-empty compact set such that

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$$\Lambda = \bigcup_{i=1}^{d} S_i(\Lambda).$$

We may assume, without loss of generality, that $0 = t_1 \le t_i$ for $i > 1$. Then the convex hull of Λ is the interval $[0, b]$ for some $b > 0$. Since we must have

$$[0, b] \supset S_i([0, b]) = [t_i, t_i + \lambda_i b] \quad \text{for all } i,$$

we immediately obtain

$$b = \max_{i \le d} \frac{t_i}{1 - \lambda_i}\,.$$

Suppose that $b = t_d/(1 - \lambda_d)$. It will be assumed that

$$\text{(1)} \qquad 0 = t_1 < t_i,\ i \ne 1; \quad \frac{t_i}{1 - \lambda_i} < b,\ i \ne d.$$

There is a natural labeling of the elements of Λ by symbolic sequences. Let $\Sigma = \{1, \ldots, d\}^{\mathbb{N}}$. For $\tau \in \Sigma$ we let

$$\Pi(\tau) = \lim_{n \to \infty} S_{\tau_1 \ldots \tau_n}(0) = t_{\tau_1} + \lambda_{\tau_1} t_{\tau_2} + \lambda_{\tau_1} \lambda_{\tau_2} t_{\tau_3} + \ldots$$

where $S_{\tau_1 \ldots \tau_n} := S_{\tau_1} \circ \cdots \circ S_{\tau_n}$. Clearly, $\Lambda = \Pi(\Sigma)$. Notice that conditions (1) are equivalent to having a unique symbolic sequence for both 0 and b:

$$\Pi^{-1}(0) = \{111\ldots\}, \quad \Pi^{-1}(b) = \{ddd\ldots\}.$$

The number $s > 0$ such that $\sum_{i=1}^{d} \lambda_i^s = 1$ is called the *similarity dimension* of the iterated function system. Let μ be the product (Bernoulli) measure on Σ with weights $(\lambda_1^s, \ldots, \lambda_d^s)$, and let ν denote the "push-down" measure on Λ, that is, $\nu = \mu \circ \Pi^{-1}$. The measure ν is the "natural" self-similar measure on the attractor. Let $\mu_2 = \mu \times \mu$. We will consider the *correlation dimension* of Λ defined as follows (see [CHY]):

$$D_2(\Lambda) = \sup\{\alpha \ge 0 : I_\alpha(\nu) < \infty\}$$

where

$$I_\alpha(\nu) := \int_\Lambda \int_\Lambda |x - y|^{-\alpha}\, d\nu(x)\, d\nu(y) = \int_{\Sigma^2} |\Pi(\tau) - \Pi(\omega)|^{-\alpha}\, d\mu_2.$$

It immediately follows from the definition and the potential theoretic characterization of the Hausdorff dimension (see [Fa3]) that the correlation dimension is always a lower bound for the Hausdorff dimension. Further, one can estimate the correlation dimension of Λ from a long typical orbit of any point $x \in \mathbb{R}$ as follows.

Let $(i_1, \ldots, i_n, \ldots)$ be a typical element of Σ (with respect to the "natural" measure μ on Σ). Then it follows from [Pe] that the limit

$$C(r) := \lim_{n \to \infty} \frac{1}{n^2} \#\{(k, l) : |S_{i_k} \circ \ldots \circ S_{i_1}(x) - S_{i_l} \circ \ldots \circ S_{i_1}(x)| < r \text{ and } l, k < n\}$$

exists and is independent of x. Therefore one can estimate $C(r)$ from a long typical (with respect to μ) orbit. Then using Proposition 2.3 of [SY] we can compute the correlation dimension by the formula

$$D_2(\varLambda) := \liminf_{r\to 0} \frac{\log C(r)}{-\log r}. \tag{2}$$

We are going to express the correlation dimension $D_2(\varLambda)$ in terms of the set of "bad" pairs

$$Z = \{(\tau,\omega) \in \varSigma^2 : \varPi(\tau) = \varPi(\omega)\}.$$

Define a metric ϱ on the symbolic space $\varSigma$ as follows: for τ and ω in $\varSigma$ denote by $\tau\wedge\omega$ their common initial segment and let

$$\varrho(\tau,\omega) = \lambda^{\tau\wedge\omega} \quad\text{where}\quad \lambda^{\tau_1\ldots\tau_n} := \lambda_{\tau_1}\cdot\ldots\cdot\lambda_{\tau_n}.$$

If $\tau_1 \neq \omega_1$ then $\varrho(\tau,\omega) := 1$. The metric on $\varSigma^2$ will be

$$\varrho_2((\tau,\omega),(\tau',\omega')) = \max\{\varrho(\tau,\tau'),\varrho(\omega,\omega')\}.$$

Recall that the *upper* (respectively *lower*) *box dimension* of a compact set K in a metric space is the lim sup (respectively lim inf) of the quantity $\log N(K,\varepsilon)/\log(1/\varepsilon)$ as $\varepsilon\to 0$, where $N(K,\varepsilon)$ is the smallest number of balls of diameter ε needed to cover K. If the lower and upper dimensions coincide, their common value is called the *box dimension*. We write $\overline{\dim}_{\mathrm B} K$, $\dim_{\mathrm B} K$, and $\dim_{\mathrm H} K$ for the upper box dimension, box dimension, and Hausdorff dimension of K respectively.

To state our theorem, we will need the notion of *thickness* (also called Newhouse thickness; see [PT]).

Let $K\subset\mathbb R$ be a compact set and let $\widehat K$ be its convex hull. Then $\widehat K\setminus K = \bigcup_{i=1}^l E_i$, $l\le\infty$, where E_i are complementary intervals (gaps). Enumerate the gaps so that $|E_1|\ge|E_2|\ge\ldots$ For $k\ge 1$ let F_k be the component of $\widehat K\setminus\bigcup_{i=1}^{k-1}E_i$ containing E_k. Then $F_k = F_k^l\cup E_k\cup F_k^r$ where F_k^l and F_k^r are the closed intervals adjacent to E_k. Define $\theta_k = \max\{|F_k^l|/|E_k|, |F_k^r|/|E_k|\}$. The thickness of K is defined as $\theta(K) = \inf\{\theta_k : k\ge 1\}$.

Theorem 1. *Let $\varLambda$ be the attractor of the iterated function system $\{S_i\}_{i=1}^d$ such that $S_i(x) = \lambda_i x + t_i$ for $0<\lambda_i<1$, and* (1) *is satisfied. If either*

(a) *the set $\varLambda$ has thickness greater than one*, *or*
(b) *$\lambda_1 = \ldots = \lambda_d$ and $\varLambda - \varLambda$ is an interval*,

then

$$D_2(\varLambda) = \dim_{\mathrm B}\varSigma^2 - \overline{\dim}_{\mathrm B} Z. \tag{3}$$

Remarks. 1. It follows that $\dim_{\mathrm H}\varLambda \ge \dim_{\mathrm B}\varSigma^2 - \overline{\dim}_{\mathrm B} Z$. Notice that $\dim_{\mathrm H}\varLambda = \dim_{\mathrm B}\varLambda$ since $\varLambda$ is a self-similar set [Fa2].

2. One can show that, in general, $D_2(\Lambda)$ cannot be replaced by $\dim_{\mathrm{B}} \Lambda$ in (3). The idea is to take a self-similar set such that some cylinder-intervals coincide. Then the measure ν is no longer the natural choice to estimate the dimension $\dim_{\mathrm{H}} \Lambda = \dim_{\mathrm{B}} \Lambda$.

3. If the thickness of Λ is greater than one then $\Lambda - \Lambda$ is an interval but the converse is not true.

4. It is easy to check whether $\Lambda - \Lambda$ is an interval in the homogeneous case $\lambda = \lambda_1 = \ldots = \lambda_m$. Indeed, then $\Lambda - \Lambda$ is the attractor of the iterated function system $\{T_{ij}\}$ where $T_{ij}(x) = \lambda x + (t_i - t_j)$. It follows that $\Lambda - \Lambda$ is an interval if and only if $[-b, b] = \bigcup_{ij} T_{ij}([-b, b])$.

3. Notation and preliminaries. Throughout this paper τ, ω always mean elements of Σ. Further, we write $\widetilde{\tau} := \tau_1 \ldots \tau_n$ and let $[\widetilde{\tau}]$ be the cylinder set of sequences $\tau \in \Sigma$ starting with $\widetilde{\tau}$. We say that $[\widetilde{\tau}]$ is an *ε-cylinder* if

$$\lambda^{\tau_1 \ldots \tau_n} \le \varepsilon \quad \text{and} \quad \lambda^{\tau_1 \ldots \tau_{n-1}} > \varepsilon.$$

The set of ε-cylinders will be denoted by $\mathcal{C}_\varepsilon$. Clearly, Σ is the only 1-cylinder. By the definition of the metric ϱ, an ε-cylinder $[\widetilde{\tau}]$ has diameter $\lambda^{\widetilde{\tau}}$, hence

$$\lambda_{\min} \varepsilon \le \operatorname{diam}[\widetilde{\tau}] \le \varepsilon$$

where $\lambda_{\min} = \min\{\lambda_i : i \le d\}$. Define $[\widetilde{\tau}, \widetilde{\omega}] = [\widetilde{\tau}] \times [\widetilde{\omega}] \in \Sigma^2$. This is a cylinder set in Σ^2 which will be called an ε-cylinder if both $[\widetilde{\tau}]$ and $[\widetilde{\omega}]$ are ε-cylinders in Σ. For any $\varepsilon > 0$, the collection of ε-cylinders $\mathcal{C}_\varepsilon^2 = \mathcal{C}_\varepsilon \times \mathcal{C}_\varepsilon$ provides a disjoint cover of Σ^2 by sets of diameter approximately equal to ε.

For $A \subseteq \Sigma^2$ let $N_\varepsilon(A)$ be the number of ε-cylinders intersecting A. Then a standard argument shows that

$$\overline{\dim}_{\mathrm{B}} A = \limsup_{\varepsilon \to 0} \frac{N_\varepsilon(A)}{\log(1/\varepsilon)}\,. \tag{4}$$

Recall that μ is the product measure on Σ with weights $(\lambda_1^s, \ldots, \lambda_d^s)$ where $\sum_{i=1}^d \lambda_i^s = 1$. Thus, the measure of an ε-cylinder $[\widetilde{\tau}]$ satisfies

$$\lambda_{\min}^s \varepsilon^s < \mu[\widetilde{\tau}] \le \varepsilon^s. \tag{5}$$

For an ε-cylinder $[\widetilde{\tau}, \widetilde{\omega}]$ we have $\lambda_{\min}^{2s} \varepsilon^{2s} < \mu_2[\widetilde{\tau}, \widetilde{\omega}] \le \varepsilon^{2s}$. It is easy to deduce from this that $\dim_{\mathrm{B}} \Sigma^2 = 2s$ and

$$(A \subset \Sigma^2,\ \overline{\dim}_{\mathrm{B}} A < 2s) \Rightarrow \mu_2(A) = 0. \tag{6}$$

The function $f(\tau, \omega) := \Pi(\tau) - \Pi(\omega)$ measures the distance between the projections of two elements of Σ. Observe that $Z = \{(\tau, \omega) \in \Sigma^2 : f(\tau, \omega) = 0\}$. Let

$$H_\varepsilon := \{[\widetilde{\tau}, \widetilde{\omega}] \in \mathcal{C}_\varepsilon^2 : [\widetilde{\tau}, \widetilde{\omega}] \cap Z \ne \emptyset\}.$$

The cardinality of H_ε will be denoted by $N_\varepsilon := N_\varepsilon(Z)$.

4. Upper estimate. Here we prove the upper estimate in (3). This is straightforward and does not use the assumptions (a) or (b) of Theorem 1.

Let $[\widetilde{\tau},\widetilde{\omega}] \in H_\varepsilon$. Then $[\widetilde{\tau},\widetilde{\omega}] \cap Z \neq \emptyset$, hence $S_{\widetilde{\tau}}(\Lambda) \cap S_{\widetilde{\omega}}(\Lambda) \neq \emptyset$, and

$$|f(\tau,\omega)| = |\Pi(\tau) - \Pi(\omega)| \le \operatorname{diam}(S_{\widetilde{\tau}}(\Lambda)) + \operatorname{diam}(S_{\widetilde{\omega}}(\Lambda))$$
$$= (\lambda^{\widetilde{\tau}} + \lambda^{\widetilde{\omega}}) \operatorname{diam}(\Lambda) \le 2b\varepsilon$$

for $\tau \in [\widetilde{\tau}]$ and $\omega \in [\widetilde{\omega}]$. Therefore, by (5),

$$\int\limits_{[\widetilde{\tau},\widetilde{\omega}]} |f(\tau,\omega)|^{-\alpha}\, d\mu_2 \ge (2b)^{-\alpha}\varepsilon^{-\alpha}\mu_2[\widetilde{\tau},\widetilde{\omega}] \ge \lambda_{\min}^{2s}(2b)^{-\alpha}\varepsilon^{2s-\alpha}.$$

We have

$$I_\alpha(\nu) = \int\limits_{\Sigma^2} |f(\tau,\omega)|^{-\alpha}\, d\mu_2 \ge \sum_{H_\varepsilon} \int\limits_{[\widetilde{\tau},\widetilde{\omega}]} |f(\tau,\omega)|^{-\alpha}\, d\mu_2 \ge \text{const} \cdot N_\varepsilon\, \varepsilon^{2s-\alpha}.$$

Thus, if $\limsup_{\varepsilon\to 0} \log N_\varepsilon / \log(1/\varepsilon) > 2s-\alpha$, then $I_\alpha(\nu) = \infty$. This, together with (4), implies

$$D_2(\Lambda) \le 2s - \overline{\dim}_{\mathrm{B}} Z.$$

5. Lower estimate

Lemma 5.1. *Suppose that at least one of the conditions* (a), (b) *of Theorem* 1 *is satisfied. Let* $[\widetilde{\tau},\widetilde{\omega}] \in \mathcal{C}_\varepsilon^2 \setminus H_\varepsilon$. *Then one of the sets* $S_{\widetilde{\tau}}(\Lambda)$ *and* $S_{\widetilde{\omega}}(\Lambda)$ *lies in a connected component of the complement of the other one.*

Proof. The assumption $[\widetilde{\tau},\widetilde{\omega}] \in \mathcal{C}_\varepsilon^2 \setminus H_\varepsilon$ means that $[\widetilde{\tau}]$ and $[\widetilde{\omega}]$ are ε-cylinders in Σ such that $S_{\widetilde{\tau}}(\Lambda) \cap S_{\widetilde{\omega}}(\Lambda) = \emptyset$.

Suppose first that condition (a) is satisfied, that is, Λ has thickness greater than one. Then both $S_{\widetilde{\tau}}(\Lambda)$ and $S_{\widetilde{\omega}}(\Lambda)$ have thickness greater than one, since they are similar to Λ. By the Gap Lemma of Newhouse (see [PT, pp. 63–82]), $S_{\widetilde{\tau}}(\Lambda) \cap S_{\widetilde{\omega}}(\Lambda) = \emptyset$ implies that one of these sets lies in a component of the complement of the other one.

Now suppose that condition (b) is satisfied, that is, $\Lambda - \Lambda$ is an interval and $\lambda = \lambda_i$ for all i. Then all ε-cylinders have the same length $n = \lfloor \log(1/\varepsilon)/\log(1/\lambda) \rfloor$. It follows that the sets $S_{\widetilde{\tau}}(\Lambda)$ and $S_{\widetilde{\omega}}(\Lambda)$ are both translated copies of $\lambda^n\Lambda$. Observe that

$$\{a \in \mathbb{R} : \lambda^n\Lambda \cap (\lambda^n\Lambda + a) \neq \emptyset\} = \lambda^n\Lambda - \lambda^n\Lambda$$

is an interval. Therefore, $S_{\widetilde{\tau}}(\Lambda) \cap S_{\widetilde{\omega}}(\Lambda) = \emptyset$ can only happen if the convex hulls of these sets are disjoint. This means that each of these sets lies in the unbounded component of the complement of the other one. ∎

The next lemma is the key part of the proof.

Lemma 5.2. *Suppose that at least one of the conditions* (a), (b) *in Theorem* 1 *is satisfied. Let* $[\widetilde{\tau},\widetilde{\omega}] \in \mathcal{C}^2_\varepsilon \setminus H_\varepsilon$. *Then for every* $\alpha < s$ *there exists* $C > 0$ *such that*

$$\int\limits_{[\widetilde{\tau},\widetilde{\omega}]} |f(\tau,\omega)|^{-\alpha}\, d\mu_2 \le C\varepsilon^{2s-\alpha}.$$

Proof. Using Lemma 5.1, we can assume without loss of generality that $S_{\widetilde{\tau}}(\Lambda)$ lies in a connected component of $\mathbb{R} \setminus S_{\widetilde{\omega}}(\Lambda)$. Write $\overline{1} = 111\ldots$ and $\overline{d} = ddd\ldots$ We have

$$(7)\qquad \begin{aligned} |f(\tau,\omega)| &= |\Pi(\tau) - \Pi(\omega)| \\ &\ge \min\{|f(\tau,\widetilde{\tau}\overline{1})|, |f(\tau,\widetilde{\tau}\overline{d})|\} \quad \text{for } \tau \in \widetilde{\tau} \text{ and } \omega \in \widetilde{\omega} \end{aligned}$$

since $\Pi(\widetilde{\tau}\overline{1}) = \min S_{\widetilde{\tau}}(\Lambda)$ and $\Pi(\widetilde{\tau}\overline{d}) = \max S_{\widetilde{\tau}}(\Lambda)$. Now we have to use condition (1) which implies that $|f(\tau,\widetilde{\tau}\overline{1})| \approx \lambda^{\tau\wedge\widetilde{\tau}\overline{1}}$ and $|f(\tau,\widetilde{\tau}\overline{d})| \approx \lambda^{\tau\wedge\widetilde{\tau}\overline{d}}$, up to a multiplicative constant. Let

$$A_m := [\widetilde{\tau}1^m] \setminus [\widetilde{\tau}1^{m+1}], \quad B_m := [\widetilde{\tau}d^m] \setminus [\widetilde{\tau}d^{m+1}] \quad \text{for } m \ge 1,$$

and

$$A_0 := [\widetilde{\tau}] \setminus ([\widetilde{\tau}1] \cap [\widetilde{\tau}d]), \quad B_0 := \emptyset.$$

Then

$$[\widetilde{\tau}] = \bigcup_{m=0}^{\infty} (A_m \cup B_m).$$

It follows from (1) and the definition of ε-cylinders that

$$|f(\tau,\widetilde{\tau}\overline{1})| \ge \text{const} \cdot \lambda^{\widetilde{\tau}} \lambda_1^m \ge \text{const} \cdot \varepsilon\lambda_1^m \quad \text{for } \tau \in A_m$$

and

$$|f(\tau,\widetilde{\tau}\overline{d})| \ge \text{const} \cdot \lambda^{\widetilde{\tau}} \ge \text{const} \cdot \varepsilon \quad \text{for } \tau \in A_m.$$

By (7) this implies

$$|f(\tau,\omega)| \ge \text{const} \cdot \varepsilon\lambda_1^m \quad \text{for } \tau \in A_m.$$

Similarly, using (1) and (7) we deduce

$$|f(\tau,\omega)| \ge \text{const} \cdot \varepsilon\lambda_d^m \quad \text{for } \tau \in B_m.$$

Clearly,

$$\mu(A_m) \le \mu[\widetilde{\tau}1^m] \le \varepsilon^s \lambda_1^{ms} \ \text{ and } \ \mu(B_m) \le \varepsilon^s \lambda_d^{ms} \quad \text{for } m \ge 0.$$

Putting everything together, we obtain

$$\int_{[\widetilde{\tau},\widetilde{\omega}]} |f(\tau,\omega)|^{-\alpha}\,d\mu_2$$
$$= \int_{[\widetilde{\omega}]} \sum_{m=0}^{\infty} \int_{A_m\cup B_m} |f(\tau,\omega)|^{-\alpha}\,d\mu(\tau)\,d\mu(\omega)$$
$$\le \text{const} \int_{[\widetilde{\omega}]} \sum_{m=0}^{\infty} (\varepsilon^{-\alpha}\lambda_1^{-\alpha m}\varepsilon^s\lambda_1^{sm} + \varepsilon^{-\alpha}\lambda_d^{-\alpha m}\varepsilon^s\lambda_d^{sm})\,d\mu(\omega)$$
$$= \text{const}\cdot\varepsilon^{s-\alpha} \int_{[\widetilde{\omega}]} \sum_{m=0}^{\infty} (\lambda_1^{(s-\alpha)m} + \lambda_d^{(s-\alpha)m})\,d\mu(\omega)$$
$$\le C\varepsilon^{s-\alpha}\mu[\widetilde{\omega}] \le C\varepsilon^{2s-\alpha},$$

for some $C > 0$. Here we used (5) and the fact that $\alpha < s$. ∎

Now we can conclude the proof of the lower estimate in (3). It is enough to show that $I_\alpha(\nu) = 0$ for every $\alpha < 2s - \overline{\dim}_B Z$. Fix such an α. Notice that Z contains the diagonal in Σ^2, hence $\overline{\dim}_B Z \ge \dim_B \Sigma = s$, and therefore, $\alpha < s$.

Let A_ε be the union of ε-cylinders in Σ^2 intersecting Z, in other words, the union of cylinders from H_ε. Let $\varepsilon_n := 2^{-n}$. Clearly, $Z = \bigcap_{n=0}^{\infty} A_{\varepsilon_n}$. Since $A_1 = \Sigma^2$ we have

$$\Sigma^2 = \bigcup_{n=0}^{\infty} (A_{\varepsilon_n} \setminus A_{\varepsilon_{n+1}}) \cup Z.$$

By Lemma 5.2,

$$I_\alpha(\nu) = \sum_{n=0}^{\infty} \int_{A_{\varepsilon_n}\setminus A_{\varepsilon_{n+1}}} |f(\tau,\omega)|^{-\alpha}\,d\mu_2 + \int_Z |f(\tau,\omega)|^{-\alpha}\,d\mu_2$$
$$\le C\sum_{n=0}^{\infty} N_{\varepsilon_{n+1}}\varepsilon_{n+1}^{2s-\alpha} + \int_Z |f(\tau,\omega)|^{-\alpha}\,d\mu_2.$$

We can assume that $\overline{\dim}_B Z < 2s$ since otherwise the estimate $D_2(\Lambda) \ge 2s - \overline{\dim}_B Z$ is obvious. Then $\mu_2(Z) = 0$ by (6). Thus,

$$I_\alpha(\nu) \le C\sum_{n=0}^{\infty} N_{2^{-(n+1)}} 2^{-(n+1)(2s-\alpha)}.$$

By assumption,

$$\limsup_{n\to\infty} \frac{\log N_{2^{-n}}}{n\log 2} = \overline{\dim}_B Z < 2s - \alpha,$$

so the above series converges. This completes the proof of Theorem 1. ∎

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