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Modules commuting (via Hom) with some limits

by

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Abstract. For every module M we have a natural monomorphism

$$\Phi : \coprod_{i\in \mathrm{I}} \mathrm{Hom}_R(A_i, M) \to \mathrm{Hom}_R\left(\prod_{i\in \mathrm{I}} A_i, M\right)$$

and we focus attention on the case when Φ is also an epimorphism. The corresponding modules M depend on thickness of the cardinal number card(I). Some other limits are also considered.

0. Introduction. Let $\triangle$ be a diagram (i.e., a small category) of modules. Given a module M, we have natural isomorphisms

$$\lim(\mathrm{Hom}_R(M,\triangle)) \cong \mathrm{Hom}_R(M,\lim(\triangle)),$$
$$\lim(\mathrm{Hom}_R(\triangle,M)) \cong \mathrm{Hom}_R(\mathrm{colim}(\triangle),M)$$

and natural (connecting) homomorphisms

$$\Phi : \mathrm{colim}(\mathrm{Hom}_R(\triangle,M)) \to \mathrm{Hom}_R(\lim(\triangle),M),$$
$$\Psi : \mathrm{colim}(\mathrm{Hom}_R(M,\triangle)) \to \mathrm{Hom}_R(M,\mathrm{colim}(\triangle)).$$

It may happen that Φ (resp. Ψ) is an isomorphism whenever $\triangle$ is a diagram of certain type and, in such a case, we shall say that M commutes (via Hom) with limits (resp. colimits) of the diagrams considered.

The present note is concerned with the most important limits: direct products, pull-backs and limits of downwards-directed spectra. The corresponding (commuting) modules are fully characterized in each case and some examples are given (for the direct product case). The easier (and more fashionable) colimit case is not treated here (the reader is referred to [12]).

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1. Preliminaries. Throughout the paper, R stands for a non-zero associative ring with unit and modules are unitary left R-modules. Further, $\mathfrak{a}$ will always denote a cardinal number. The category of modules and homomorphisms will be denoted by R-MOD (for various basic properties of this category, we refer e.g. to [29]). The category of sets (including the empty set $\emptyset$) and mappings will be denoted by SET.

Let S be a (non-empty) ordered set. By an S-*spectrum* (in a given category) we shall mean any diagram of the type $f_{r,s} : A_r \to A_s$, $r, s \in S$, $r \leq s$. An S-spectrum will be called *upwards/downwards-directed* if so is the ordered set S. As a special case, we get $\mathfrak{a}$-*spectra* ($\mathfrak{a}$ with the usual order) and $\widetilde{\mathfrak{a}}$-*spectra* ($\mathfrak{a}$ with the dual order).

Let A be a complete boolean algebra. The following easy observation will be useful in the sequel:

1.1. Lemma. *Let $a_\alpha \in A$, $\alpha < \mathfrak{a}$. Then there are pairwise* (*meet-*) *orthogonal elements $b_\alpha \in A$ such that* $\sup_A(\{a_\alpha : \alpha < \mathfrak{a}\}) = \sup_A(\{b_\alpha : \alpha < \mathfrak{a}\})$ *and $b_\alpha \leq a_\alpha$ for every $\alpha < \mathfrak{a}$.*

An ideal I of A will be called $\mathfrak{a}$-*complete* (in A) if $\sup_A(S) \in I$ for every subset $S \subseteq I$ such that $\mathrm{card}(S) < \mathfrak{a}$. The boolean algebra A will be called $\mathfrak{a}$-*measurable* if A has at least one $\mathfrak{a}$-complete non-principal maximal ideal. This definition may be weakened in the following obvious way:

1.2. Lemma. *A is $\mathfrak{a}$-measurable if and only if A has an $\mathfrak{a}$-complete non-principal ideal I such that the factor algebra A/I is finite.*

A set S will be called $\mathfrak{a}$-*measurable* if so is the boolean algebra $\mathcal{P}(S)$ of subsets of S.

A set S will be called *measurable* if $\mathfrak{a} = \mathrm{card}(S) \geq \aleph_1$ and S is $\mathfrak{a}$-measurable.

2. Modules commuting with direct products—introduction. Let I be a non-empty index set and $A_i, i \in \mathrm{I}$, an idexed family of modules. Put $B = \coprod_{i\in \mathrm{I}} A_i \subseteq \prod_{i\in \mathrm{I}} A_i = A$ and $\mathrm{supp}_\mathrm{I}(a) = \{i \in \mathrm{I} : a(i) \neq 0\}$ for every $a \in A$. If J is a subset of I, then we define $A(\mathrm{J}) = \{a \in A : \mathrm{supp}_\mathrm{I}(a) \subseteq \mathrm{J}\}$ and $B(\mathrm{J}) = \{a \in B : \mathrm{supp}_\mathrm{I}(a) \subseteq \mathrm{J}\}$.

Let M be a module. A homomorphism $\varphi : A \to M$ is called

- *slender* if $\varphi(A(\{i\})) = 0$ for almost all $i \in \mathrm{I}$;
- *completely slender* if $\varphi(B) = 0$;
- *slim* if $\varphi(A(\mathrm{J})) = 0$ for a cofinite subset J of I.

We say that a module M is $\mathfrak{a}$-*slim* if every homomorphism $\prod_{i\in \mathrm{I}} A_i \to M$, $\mathrm{card}(\mathrm{I}) < \mathfrak{a}$, is slim, and we say that M is *slim* if it is $\mathfrak{a}$-slim for every cardinal $\mathfrak{a}$.

Slim modules appear in the following obvious context:

2.1. Proposition. (i) *A module M is $\mathfrak{a}$-slim if and only if M commutes with all direct products $\prod_{i\in \mathrm{I}} A_i$ such that* $\operatorname{card}(\mathrm{I}) < \mathfrak{a}$.

(ii) *A module M is slim if and only if M commutes with direct products.*

$\aleph_1$-slim modules were introduced by J. Łoś under the name of slender modules.

2.2. Lemma ([15, §94]). *The following conditions are equivalent for a module M*:

(i) *M is slender* (*i.e.*, *$\aleph_1$-slim*).

(ii) *Every homomorphism $R^{\aleph_0} \to M$ is slim.*

(iii) *Every homomorphism $R^{\aleph_0} \to M$ is slender.*

(iv) *Every homomorphism $\prod_{i\in \mathrm{I}} A_i \to M$ is slender* (*for any index set* I).

Proof. (iii)⇒(iv). Proceeding by contradiction, we easily get a non-slender homomorphism $\varphi : B = \prod_{i<\aleph_0} B_i \to M$. Now, $\psi : R^{\aleph_0} \to M$ is not slender, where $\psi(a) = \varphi(\sum_{i<\aleph_0} a(i)b_i)$, $b_i \in B(\{i\})$, $\varphi(b_i) \neq 0$ if $\varphi(B(\{i\})) \neq 0$.

(iii)⇒(i). Let $\varphi : A = \prod_{i<\aleph_0} A_i \to M$ be not slim. For every i, there is $a_i \in A$ such that $\varphi(a_i) \neq 0$ and $\operatorname{supp}(a_i) > i$. Now, defining $\psi : R^{\aleph_0} \to A$ by $\psi(a) = \sum a(i)a_i$, we get a non-slender homomorphism $\varphi\psi : R^{\aleph_0} \to M$. ∎

The following result is a basic criterion for slimness:

2.3. Theorem. *Suppose that $\mathfrak{a} \geq \aleph_1$. A module M is $\mathfrak{a}$-slim if and only if every homomorphism $R^{\mathfrak{w}} \to M$ is slim*, *whenever $\mathfrak{w}$ is a cardinal such that $\mathfrak{w} < \mathfrak{a}$ and either $\mathfrak{w} = \aleph_0$ or $\mathfrak{w}$ is measurable.*

Proof. In view of 2.2, let $\mathfrak{a} \geq \aleph_2$ and let M be a slender module such that M is not $\mathfrak{a}$-slim. Consider the smallest cardinal $\mathfrak{w}$ with a non-slim homomorphism $\varrho : A = \prod_{\alpha<\mathfrak{w}} A_\alpha \to M$. Then $\aleph_1 \leq \mathfrak{w} < \mathfrak{a}$ and we can take ϱ to be completely slender.

Let $a \in A$ be such that $\varrho(a) \neq 0$ and define $\varsigma : T = R^{\mathfrak{w}} \to A$ by $(\varsigma(v))(\alpha) = v(\alpha)a(\alpha)$ for all $v \in T$ and $\alpha < \mathfrak{w}$. Then $\varphi = \varrho\varsigma : T \to M$ is a completely slender non-slim homomorphism.

Put $\mathcal{P} = \mathcal{P}(\mathfrak{w})$ and $\mathcal{I} = \mathcal{I}_\varphi = \{P \in \mathcal{P} : \varphi(T(P)) = 0\}$. Then $\mathcal{I}$ is an ideal of the boolean algebra $\mathcal{P}$ and our first aim is to show that the factor algebra $\mathcal{G} = \mathcal{P}/\mathcal{I}$ is finite. Suppose this is not true. Then $\mathcal{G}$ contains an infinite set of non-zero pairwise (meet-) orthogonal elements and consequently we can find pairwise disjoint sets $Q_i \in \mathcal{P} \setminus \mathcal{I}$, $i < \aleph_0$. Now, φ induces a non-slender homomorphism $\prod T(Q_i) \to M$, a contradiction. We have thus proved that $\mathcal{G}$ is finite.

Further, using 1.1 and the fact that M is $\mathfrak{w}$-slim (due to the minimality of $\mathfrak{w}$), we conclude that $\mathcal{I}$ is $\mathfrak{w}$-complete. Finally, one sees directly from the properties of φ that $\mathcal{I}$ is not principal. By 1.2, $\mathfrak{w}$ is a measurable cardinal. ∎

The first part of the next useful result is folklore, while the second one is an improved version of [7, 1.6] (see also [15, 94.4], [21, 3.(4)] and [9, III.3.3]).

2.4. PROPOSITION. *Let* $\mathfrak{w}$ *be a cardinal.*

(i) *If* $\mathfrak{w}$ *is* $\mathfrak{a}$*-measurable*, *then* $\operatorname{card}(M) \geq \mathfrak{a}$ *for every non-zero* $\mathfrak{w}^+$*-slim module* M.

(ii) *If* $\mathfrak{w}$ *is not* $\mathfrak{a}$*-measurable*, *then every* $\mathfrak{a}$*-slim module is* $\mathfrak{w}^+$*-slim.*

Proof. (i) Let $\mathcal{I}$ be an $\mathfrak{a}$-complete non-principal maximal ideal of $\mathcal{P}(\mathfrak{w})$ and let M be a non-zero module with less than $\mathfrak{a}$ elements. For every $a \in A = M^{\mathfrak{w}}$ there is just one element $\varphi(a) \in M$ such that $\{\alpha < \mathfrak{w} : a(\alpha) \neq \varphi(a)\} \in \mathcal{I}$ and thus we get a non-slim homomorphism $\varphi : A \to M$.

(ii) We use 2.3; we can assume that $\aleph_1 \leq \mathfrak{a} \leq \mathfrak{w}$. Let M be an $\mathfrak{a}$-slim module and let $\mathfrak{r}$ be a measurable cardinal, $\mathfrak{r} \leq \mathfrak{w}$. Since $\mathfrak{w}$ is not $\mathfrak{a}$-measurable, we must have $\mathfrak{r} < \mathfrak{a}$, and hence every homomorphism $R^{\mathfrak{r}} \to M$ is slim. ∎

2.5. REMARK. Let $\mathfrak{a} \geq \aleph_1$ and let M be a slender module that is not $\mathfrak{a}$-slim. Consider the smallest cardinal $\mathfrak{w}$ with a non-slim homomorphism $R^{\mathfrak{w}} \to M$ (see 2.3 and its proof). Then $\aleph_1 \leq \mathfrak{w} \leq \mathfrak{a}$ and $\mathfrak{w}$ is measurable. Now, we are going to show that there exists a non-zero completely slender (non-slim) homomorphism $\varphi : R^{\mathfrak{w}} \to M$ such that the corresponding ideal $\mathcal{I}_\varphi$ of $\mathcal{P}(\mathfrak{w})$ is non-principal, maximal and $\mathfrak{w}$-complete.

We start with a non-zero completely slender homomorphism $\psi : T = R^{\mathfrak{w}} \to M$. If $\mathcal{I}_\psi$ is not maximal, then there are two disjoint subsets P_0 and Q_0 of $\mathfrak{w}$ such that $\mathfrak{w} = P_0 \cup Q_0$, $P_0 \notin \mathcal{I}_\psi$, $Q_0 \notin \mathcal{I}_\psi$ and $\operatorname{card}(P_0) = \mathfrak{w} = \operatorname{card}(Q_0)$. The restrictions $\psi\lceil T(P_0)$ and $\psi\lceil T(Q_0)$ are non-zero completely slender homomorphisms. If the ideal $\mathcal{I}_\psi \cap \mathcal{P}(P_0)$ is maximal in $\mathcal{P}(P_0)$, then our claim is proved. Otherwise, $P_0 = P_1 \cup Q_1$, $P_1 \cap Q_1 = \emptyset$, $P_1 \notin \mathcal{I}_\psi$, $Q_1 \notin \mathcal{I}_\psi$, $\operatorname{card}(P_1) = \mathfrak{w} = \operatorname{card}(Q_1)$, etc. Proceeding in this way, we arrive at a non-slender homomorphism $\prod_{i<\aleph_0} T(Q_i) \to M$, a contradiction. Thus our procedure yields a maximal ideal $\mathcal{I}_\psi \cap \mathcal{P}(P_n)$ (of $\mathcal{P}(P_n)$) after finitely many steps and the rest is clear.

2.6. PROPOSITION. (i) *The class of* $\mathfrak{a}$*-slim modules is closed under submodules and extensions.*

(ii) *The class of slender modules is closed under submodules and extensions.*

(iii) *The class of slim modules is closed under submodules and extensions.*

Proof. Easy. ∎

2.7. REMARK. It follows readily from 2.6 that ($\mathfrak{a}$-) slim modules are closed under finite direct sums. In fact, these modules are closed under arbitrary direct sums: we shall prove it in the next section.

3. Direct sums of slim modules. E. Lady proved in [21, 3.(2)] that slender modules are closed under taking arbitrary direct sums; now we are going to show the same for $\mathfrak{a}$-slim modules:

3.1. THEOREM. *The class of* $\mathfrak{a}$*-slim modules is closed under direct sums.*

P r o o f. We proceed by contradiction. Let $\mathfrak{r}$ be the smallest cardinal such that there exist $\mathfrak{a}$-slim modules M_β, $\beta < \mathfrak{r}$, with $M = \coprod_{\beta<\mathfrak{r}} M_\beta$ not $\mathfrak{a}$-slim and let $\mathfrak{w}$ be the smallest cardinal such that there exists a non-slim homomorphism $\varphi : A = \prod_{\alpha<\mathfrak{w}} A_\alpha \to M$; we have $\aleph_0 \le \mathfrak{r}$ and $\aleph_0 \le \mathfrak{w} < \mathfrak{a}$.

In order to get our contradiction, we construct two sequences $a_1, a_2, \ldots$ and $b_1, b_2, \ldots$ of elements of A. First, choose $a_1 \in A$ such that $\varphi(a_1) \neq 0$ and put $\mu_1 = \max(\operatorname{supp}_{\mathfrak{r}}(\varphi(a_1)))$ and $\nu_1 = \mu_1 + 1$. If $\pi_1 : M \to N_1 = \coprod_{\beta<\nu_1} M_\beta$ is the natural projection, then $\pi_1\varphi : A \to N_1$ is slim and there is a non-empty finite subset T_1 of $\mathfrak{w}$ such that $\varphi(A(\mathfrak{w} \setminus T_1)) \subseteq M(P_{\nu_1})$, $P_{\nu_1} = \{\beta : \nu_1 \le \beta < \mathfrak{r}\}$ and we put $\sigma_1 = \max(T_1)$ and $\tau_1 = \sigma_1 + 1$. Now, we define $b_1 \in A$ by $b_1(\alpha) = a_1(\alpha)$ for $\alpha < \tau_1$ and $b_1(\alpha) = 0$ for $\tau_1 \le \alpha < \mathfrak{w}$. Clearly, $\operatorname{supp}_{\mathfrak{w}}(b_1) \le \sigma_1, b_1 - a_1 \in A(\mathfrak{w} \setminus T_1), \varphi(b_1 - a_1) \in M(P_{\nu_1})$ and $0 \neq \varphi(a_1) \in M(Q_{\nu_1})$, $Q_{\nu_1} = \{\beta : \beta < \nu_1\}$. Moreover, $\varphi(b_1) \neq 0$ and we put $\varsigma_1 = \max(\mu_1, \operatorname{supp}_{\mathfrak{r}}(\varphi(b_1)))$ and $\xi_2 = \varsigma_1 + 1$ $(\xi_1 = 0)$.

Further, proceeding similarly, we get a non-empty finite subset S_1 of $\mathfrak{w}$ such that $\varphi(A(\mathfrak{w} \setminus S_1)) \subseteq M(P_{\xi_1})$ and we put $\overline{\tau}_1 = \max(S_1, \tau_1)$ and $\varepsilon_2 = \overline{\tau}_1$ $(\varepsilon_1 = 0)$. Using the minimality of $\mathfrak{w}$, we see that there exists $a_2 \in A$ such that $\operatorname{supp}_{\mathfrak{w}}(a_2) \ge \varepsilon_2$ and $0 \neq \varphi(a_2) \in M(P_{\xi_1})$. Again, put $\mu_2 = \max(\operatorname{supp}_{\mathfrak{r}}(\varphi(a_2)))$, $\nu_2 = \mu_2 + 1$ and $N_2 = \coprod_{\beta<\nu_2} M_\beta$. Then $\varphi(A(\mathfrak{w} \setminus T_2)) \subseteq M(P_{\nu_2})$ for a non-empty finite subset T_2 of $\mathfrak{w}$. Now, as usual, set $\sigma_2 = \max(T_2, \overline{\tau}_1), \tau_2 = \sigma_2 + 1$ and define $b_2 \in A$ by $b_2(\alpha) = a_2(\alpha)$ for $\alpha < \tau_2$ and $b_2(\alpha) = 0$ for $\tau_2 \le \alpha < \mathfrak{w}$. Clearly, $\operatorname{supp}_{\mathfrak{w}}(b_2) \subseteq \{\varepsilon_2, \ldots, \sigma_2\}$, $b_2 - a_2 \in A(\mathfrak{w} \setminus T_2)$, $\varphi(b_2 - a_2) \in M(P_{\nu_2})$ and $0 \neq \varphi(a_2) \in M(Q_{\nu_2})$, $Q_{\nu_2} = \{\beta : \xi_1 \le \beta < \nu_2\}$. Moreover, $0 \neq \varphi(b_2) \in M(P_{\xi_2})$ and we put $\varsigma_2 = \max(\mu_2, \operatorname{supp}_{\mathfrak{r}}(\varphi(b_2)))$ and $\xi_3 = \varsigma_2 + 1$.

Proceeding by induction, we get a sequence $b_1, b_2, \ldots$ of elements of A and sequences $\varepsilon_1, \varepsilon_2, \ldots, \sigma_1, \sigma_2, \ldots, \xi_1, \xi_2, \ldots, \varsigma_1, \varsigma_2, \ldots$ of ordinal numbers such that $\operatorname{supp}_{\mathfrak{w}}(b_i) \subseteq \{\varepsilon_i, \ldots, \sigma_i\}$, $\varepsilon_i < \sigma_i < \varepsilon_{i+1}$, $\varphi(b_i) \neq 0$, $\operatorname{supp}_{\mathfrak{r}}(\varphi(b_i)) \subseteq \{\xi_i, \ldots, \varsigma_i\}$ and $\xi_i < \varsigma_i < \xi_{i+1}$ for every $i = 1, 2, \ldots$ Now, we can define $b \in A$ such that $b(\alpha) \neq 0$ iff $b_i(\alpha) \neq 0$ for some $i \ge 1$ and then $b(\alpha) = b_i(\alpha)$.

Finally, let $\gamma < \mathfrak{r}$ be such that $\gamma \in \operatorname{supp}_{\mathfrak{r}}(\varphi(b_j))$ for some $j \ge 1$. We claim that $\gamma \in \operatorname{supp}_{\mathfrak{r}}(\varphi(b))$. To show this, let $\pi : M \to M_\gamma$ denote the natural projection. Then $\pi\varphi : A \to M_\gamma$ is slim and it follows that there is a non-empty finite subset T of $\mathfrak{w}$ such that $\varphi(A(\mathfrak{w} \setminus T)) \subseteq M(W)$, $W = \{\beta : \beta < \mathfrak{r},\ \beta \neq \gamma\}$. Clearly, $j \in Z = \{i \ge 1 : T \cap \operatorname{supp}_{\mathfrak{w}}(b_i) \neq \emptyset\}$. The set Z is finite and we put $c = \sum_{i\in Z} b_i \in A$. Then $b - c \in A(\mathfrak{w} \setminus T)$ and $(\varphi(b))(\gamma) = (\varphi(c))(\gamma)$. But $(\varphi(c))(\gamma) = \sum_{i\in Z}(\varphi(b_i))(\gamma) = (\varphi(b_j))(\gamma) \neq 0$.

We have proved our claim and we immediately conclude that the set $\mathrm{supp}_{\mathfrak{r}}(\varphi(b))$ is not finite, a contradiction with the fact that $\varphi(b) \in \coprod M_\beta$. ∎

3.2. Corollary. (i) ([21]) *The class of slender modules is closed under direct sums.*

(ii) *The class of slim modules is closed under direct sums.*

The original proof of [21, 3.(2)] (i.e., 3.2(i)) makes use of the Baire Category Theorem. A similar idea works also for $\mathfrak{a}$-slim modules and we shall give an alternative proof of 3.1. For this purpose, we need a special generalized version of the B.C.T. as described in the following observation:

3.3. Observation. Suppose that $\mathfrak{a}$ is both infinite and regular and put $A = R^{\mathfrak{a}}$.

(i) For every $\alpha < \mathfrak{a}$, let $A_\alpha = A(\mathfrak{a} \setminus \alpha)$. Then $\mathcal{F} = \{A_\alpha : \alpha < \mathfrak{a}\}$ is a (downwards-directed) filtration of the module A and we have the corresponding closure operator $\mathrm{cls}_{\mathcal{F}}$ on A : $\mathrm{cls}_{\mathcal{F}}(S) = \bigcap_{\alpha<\mathfrak{a}}(A_\alpha + S)$ for every $S \in \mathcal{P}(A)$.

3.3.1. Lemma. *A is $\mathcal{F}$-complete*, i.e., *every Cauchy $\mathcal{F}$-net of elements of A is convergent.*

3.3.2. Lemma. *Let I be a finitely generated right ideal of R and $P \in \mathcal{P}(\mathfrak{a})$. Then $IA(P)$ is an $\mathcal{F}$-closed subgroup of $A(+)$.*

3.3.3. Lemma. *Let T_α, $\alpha < \mathfrak{a}$, be $\mathcal{F}$-closed subsets of A such that* $\mathrm{int}_{\mathcal{F}}(T_\alpha) = \emptyset$ *for every α. Then $\bigcup_{\alpha<\mathfrak{a}} T_\alpha \neq A$.*

Proof. Put $\mathcal{I} = \{P \in \mathcal{P}(\mathfrak{a}) : \mathrm{card}(P) < \mathfrak{a}\}$ and find sets $P_\alpha \in \mathcal{I}$ and elements $a_\alpha \in A$ such that the following three conditions are satisfied:

(a) $P_\beta \subseteq P_\alpha$ and $a_\beta - a_\alpha \in A(\mathfrak{a} \setminus P_\beta)$ for $\beta \le \alpha < \mathfrak{a}$;
(b) $a_\alpha \in A(P_\alpha)$ for every $\alpha < \mathfrak{a}$;
(c) $a_\alpha - b \notin A(\mathfrak{a} \setminus P_\alpha)$ for all $\alpha < \mathfrak{a}$ and $b \in T_\alpha$.

Since $\mathrm{cls}_{\mathcal{F}}(T_0) \neq A$, there are $P_0 \in \mathcal{I}$ and $x \in A \setminus T_0$ such that $x - b \notin A(\mathfrak{a} \setminus P_0)$ for every $b \in T_0$. Now, put $a_0 = x - y$, where $y \in A(\mathfrak{a} \setminus P_0)$ and $a_0 \in A(P_0)$.

Let $1 \le \beta < \mathfrak{a}$ be such that P_α and $a_\alpha, \alpha < \beta$, are already found. Then $P = \bigcup_{\alpha<\beta} P_\alpha \in \mathcal{I}$ and, due to (a), there is $a \in A(P)$ such that $a - a_\alpha \in A(\mathfrak{a} \setminus P_\alpha)$ for every $\alpha < \beta$. We have $a + A(\mathfrak{a} \setminus P) \not\subseteq T_\beta = \mathrm{cls}_{\mathcal{F}}(T_\beta), a + z \notin T_\beta$ for some $z \in A(\mathfrak{a} \setminus P)$ and there exists $Q \in \mathcal{I}$ such that $a + z - b \notin A(\mathfrak{a} \setminus Q)$ for every $b \in T_\beta$. It is sufficient to put $P_\beta = P \cup Q$ and to take $a_\beta \in A(P_\beta)$ and $v \in A(\mathfrak{a} \setminus P_\beta)$ such that $a + z = a_\beta + v$.

Now, according to (a), there is $c \in A$ such that $c - a_\alpha \in A(\mathfrak{a} \setminus P_\alpha)$ for every $\alpha < \mathfrak{a}$. By (c), $c \notin \bigcup T_\alpha$. ∎

(ii) Let I_α, $\alpha < \mathfrak{a}$, be finitely generated right ideals of R such that $I_\alpha \subseteq I_\beta$ for $\beta \le \alpha$. Then $\mathcal{G} = \{I_\alpha A_\alpha : \alpha < \mathfrak{a}\}$ is a filtration of $A(+)$; we have $\mathrm{cls}_{\mathcal{G}} \subseteq \mathrm{cls}_{\mathcal{F}}$.

3.3.4. Lemma. *The subgroups $I_\alpha A_\alpha$ are $\mathcal{F}$-closed and $A(+)$ is $\mathcal{G}$-complete.*

Proof. Use 3.3.1 and 3.3.2. ■

(iii) Let $G_\alpha, \alpha < \mathfrak{a}$, be subgroups of $R(+)$ and let H_α denote the set of $a \in A$ such that $a(\beta) = 0$ for $\beta < \alpha$ and $a(\gamma) \in G_\gamma$ for $\alpha \le \gamma$. Again, $\mathcal{H} = \{H_\alpha : \alpha < \mathfrak{a}\}$ is a filtration of $A(+)$ and $\mathrm{cls}_{\mathcal{H}} \subseteq \mathrm{cls}_{\mathcal{F}}$.

3.3.5. Lemma. *Let T_α, $\alpha < \mathfrak{a}$, be $\mathcal{H}$-closed subsets of A such that $\mathrm{int}_{\mathcal{H}}(T_\alpha) = \emptyset$ for every α. Then $\bigcup_{\alpha\in\mathfrak{a}} T_\alpha \neq A$.*

Proof. Similar to that of 3.1.3. ■

If $G_\alpha = I_\alpha$ (see (ii)), then $H_\alpha \subseteq I_\alpha A_\alpha$, and hence $\mathrm{cls}_{\mathcal{H}} \subseteq \mathrm{cls}_{\mathcal{G}} \subseteq \mathrm{cls}_{\mathcal{F}}$.

3.4. Remark. Now, we present another proof of 3.1 (based on 3.3.3):

Again we argue by contradiction. Let $\mathfrak{r}$ be the smallest cardinal such that $M = \coprod_{\beta<\mathfrak{r}} M_\beta$ is not $\mathfrak{a}$-slim for some $\mathfrak{a}$-slim modules M_β (we have $\mathfrak{r} \ge \aleph_0$ by 2.7) and let $\mathfrak{w}$ be the smallest cardinal such that there exists a non-slim homomorphism $\varphi : A = R^{\mathfrak{w}} \to M$. Then $\mathfrak{w}$ is infinite and it follows from 2.3 that $\mathfrak{w}$ is also regular.

For every $\gamma < \mathfrak{r}$, let $N_\gamma = \coprod_{\beta<\gamma} M_\beta \subseteq M$, $B_\gamma = \varphi^{-1}(N_\gamma) \subseteq A$ and let $\pi_\gamma : M \to N_\gamma$ denote the natural projection. The module N_γ is $\mathfrak{a}$-slim, and hence the composition $\psi_\gamma = \pi_\gamma\varphi : A \to N_\gamma$ is slim. This means that ψ_γ is $(\mathcal{F}, \mathcal{Z})$-continuous, where $\mathcal{F}$ is the filtration in A in 3.3(i) and $\mathcal{Z}$ is the zero filtration of N_γ. Now, $B_\gamma = \bigcap_{\gamma\le\delta<\mathfrak{r}} \psi_\delta^{-1}(N_\gamma)$ is an $\mathcal{F}$-closed submodule of A and it follows from 3.3.3 that $\mathrm{int}_{\mathcal{F}}(B_\varepsilon) \neq \emptyset$ for at least one $\varepsilon < \mathfrak{r}$. Equivalently, $A_\mu \subseteq B_\varepsilon$ and $\varphi(A_\mu) \subseteq N_\varepsilon$ for suitable $\mu < \mathfrak{r}$. But N_ε is $\mathfrak{a}$-slim, and so there is ν such that $\mu \le \nu < \mathfrak{r}$ and $\varphi(A_\nu) = 0$. Finally, $\varphi\lceil A(\nu)$ is slim and we conclude that the homomorphism $\varphi : A \to M$ is also slim, a contradiction.

4. Examples of slender modules. The notion of slenderness and some of the basic results for the non-measurable case (e.g., 2.2 and 2.4 for $\mathfrak{a} = \aleph_1$) are due to J. Łoś and were published in [13]. A generalization to the measurable case can be found in [5]–[7] (see also [9, Chapter III]). The following characterization of slender modules was given by R. Dimitrić in [4] (see also [17] and [3]):

A module M is slender if and only if $\mathrm{Hom}_R(W, M) = 0$, where $W = R^{\aleph_0}/R^{(\aleph_0)}$, and M is not complete (i.e., M is not $\mathcal{E}$-complete, whenever $\mathcal{E} = \{M_i : i < \aleph_0\}$ is a downwards-directed filtration of M such that $\bigcap \mathcal{E} = 0$ and $M_i \neq 0$ for every $i < \aleph_0$).

This result is a useful criterion for slenderness. In particular, when $\operatorname{card}(M) < 2^{\aleph_0}$, we have to check only that $\operatorname{Hom}_R(W, M) = 0$. However, in many particular cases, this is a rather complicated task, and hence various indirect methods are also used (see the following examples).

4.1. Example ([27]). Every reduced torsionfree abelian group containing less than $2^{\aleph_0}$ elements is slender.

4.2. Example ([10]). Let R be a prime ring with less than $2^{\aleph_0}$ elements. Then R is slender if and only if R is not isomorphic to a (full) matrix ring over a division ring.

4.3. Example ([10]). Let R be a strongly regular ring with less than $2^{\aleph_0}$ elements. Then R is slender if and only if $\operatorname{Soc}(R) = 0$.

4.4. Example ([28]). Let R be a countable simple regular ring, not completely reducible. Then all completely reducible modules and all modules of finite length are slender.

4.5. Example ([1], [3], [9], [26]). A Dedekind domain R is slender if and only if R is neither a field nor a complete discrete valuation domain.

4.6. Example ([1], [3], [21], [26]). Let R be a Dedekind domain such that the set $\mathcal{M}$ of maximal ideals is countable. A module M is slender if and only if M is reduced torsionfree, $R^{\aleph_0}$ is not isomorphic to a submodule of M and, for every $P \in \mathcal{M}$, the P-adic completion $\widehat{R}_P$ of R is not isomorphic to a submodule of M.

4.7. Remark. Further results on and examples of slender modules may be found in [8], [11], [14], [16], [22]–[24].

4.8. Remark. (i) As we have seen (2.6(ii), 3.2(i)), the class $\mathcal{S}_R$ of slender modules is closed under submodules, direct sums and extensions. On the other hand, if $\mathcal{S}_R \neq 0$, then $\mathcal{S}_R$ is not closed under direct products and there exist slender modules that are neither finitely generated nor finitely cogenerated. (Conversely, there always exist finitely cogenerated modules that are not slender.)

(ii) The class of rings with $\mathcal{S}_R = 0$ is rather interesting but enigmatic so far. Among these rings we shall certainly find many left semiartinian rings, all right perfect rings and all complete discrete valuation domains.

4.9. Proposition. *Let R be a ring such that* $\operatorname{card}(R) < 2^{\aleph_0}$. *The following conditions are equivalent*:

(i) *R is left noetherian and $\mathcal{S}_R = 0$.*
(ii) *R is left noetherian and $\mathcal{S}_R$ is closed under homomorphic images.*
(iii) *R is left artinian.*

P r o o f. (ii)⇒(iii). Suppose that, on the contrary, R is not left artinian. Then R is not perfect and there is a prime ideal P of R such that $M = R/P$ is not completely reducible. Now, M is slender by 4.2, and hence $N = M^{(\mathfrak{a})}$, where $\mathfrak{a} = \operatorname{card}(M)^{\aleph_0}$, is slender by 3.2(i). On the other hand, the non-slender module $M^{\aleph_0}$ is a homomorphic image of N, a contradiction. ∎

5. Approximation property. Following [7, 2.1], we say that a module M has the $\mathfrak{a}$-*approximation property* if there exist finitely generated right ideals I_α of R and subsets S_α of M, $\alpha < \mathfrak{a}$, such that the following five conditions are satisfied:

(1) $I_\beta \subseteq I_\alpha$ and $S_\alpha \subseteq S_\beta$ for $\alpha \le \beta < \mathfrak{a}$;
(2) $I_\alpha x \ne 0$ for all $\alpha < \mathfrak{a}$ and $x \in M, x \ne 0$;
(3) $I_\alpha M \cap S_\alpha = 0$ for every $\alpha < \mathfrak{a}$;
(4) $0 \in S_\alpha$ and $S_\alpha = -S_\alpha$ for every $\alpha < \mathfrak{a}$;
(5) If $\alpha < \mathfrak{a}$ and $x \in M$, then $x + S_\alpha \subseteq S_\beta$ for some $\beta < \mathfrak{a}$.

5.1. Proposition ([7, 2.2, $\kappa = \aleph_0$]). *Let M be a module with the $\aleph_0$-approximation property. Then M is slender.*

P r o o f. For methodological reasons, we present a proof different from the original one:

Consider the filtration $\mathcal{G}$ of $A(+)$ according to 3.3(ii) and the filtration $\mathcal{E} = \{I_i M : i < \aleph_0\}$ of $M(+)$. It follows easily from (5) that all the sets S_i are $\mathcal{E}$-closed, and hence $T_i = \varphi^{-1}(S_i)$ are $\mathcal{G}$-closed, $\varphi : A = R^{\aleph_0} \to M$ being a homomorphism. Now, $\bigcup T_i = A$ and consequently, by the Baire Category Theorem, we have $\operatorname{int}_{\mathcal{G}}(T_n) \ne \emptyset$ for at least one $n < \aleph_0$. Then there are $a \in A$ and $k \ge m \ge n$ such that $\varphi(a) + I_m\varphi(A[m]) \subseteq S_m$ (where $A[m] = A(\aleph_0 \setminus m)$), $I_k\varphi(A[k]) \subseteq I_m\varphi(A[m]) \subseteq S_m - \varphi(a) \subseteq S_k$ and so $I_k\varphi(A[k]) = 0$. By (2), $\varphi(A[k]) = 0$ and we have proved that φ is slim. ∎

5.2. Remark. It is tempting to formulate the following generalization of 5.1 (see [7, 2.2]):

Let M be a module with the $\mathfrak{a}$-approximation property for a regular cardinal $\mathfrak{a}$. If $\varphi : A = R^{\mathfrak{a}} \to M$ is a homomorphism, then $\varphi(A(\mathfrak{a} \setminus \alpha)) = 0$ for some $\alpha < \mathfrak{a}$.

However, if we try to generalize the proof of 5.1, we find that the analogue of 3.3.3 for the filtration $\mathcal{G}$ is not available (cf. 3.4). Of course, we have 3.3.5, but this assertion is not powerful enough.

There seems to be a gap in the original proof [7, 2.2] (the limit step is not behaving well) and we doubt that the result remains true even for $\kappa = \aleph_1$. At this moment, we do not know any counterexample for $\aleph_1$, but 5.4 or 5.5 might be useful in this respect (possibly some maximal valuation domains should be considered).

5.3. REMARK. Combining 2.3 and [7, 2.2] would easily give the following result:

Let M be a module with the $\mathfrak{w}$-approximation property for every cardinal $\mathfrak{w} < \mathfrak{a}$ such that either $\mathfrak{w} = \aleph_0$ or $\mathfrak{w}$ is measurable. Then M is $\mathfrak{a}$-slim.

Now, this assertion is certainly true for $\mathfrak{a}$ not $\aleph_1$-measurable (see 5.1 and 2.3) but fails for every measurable $\mathfrak{a}$. A counterexample will be constructed in the next section (see 6.9).

5.4. PROPOSITION. *Let $\mathfrak{a}$ be an infinite regular cardinal number and let M be a module such that* $\mathrm{card}(M) \le \mathfrak{a}$ *and there exist finitely generated right ideals K_α of R, $\alpha < \mathfrak{a}$, satisfying the following three conditions*:

(a) $K_\beta \subseteq K_\alpha$ *for* $\alpha \le \beta < \mathfrak{a}$;
(b) $K_\alpha x \ne 0$ *for all* $\alpha < \mathfrak{a}$ *and* $x \in M$, $x \ne 0$;
(c) $\bigcap_{\alpha<\mathfrak{a}} K_\alpha M = 0$.

Then M has the $\mathfrak{a}$-approximation property.

Proof. We have $M = \{x_\alpha : \alpha < \mathfrak{a}\}$. Put $S_0 = 0$ and $\varepsilon_0 = 0$. If $\alpha < \mathfrak{a}$, then $S_{\alpha+1} = (S_\alpha + S_\alpha) \cup \{x_\alpha, -x_\alpha\}$. Since $\mathfrak{a}$ is regular, there is $\varepsilon_{\alpha+1} < \mathfrak{a}$ such that $\varepsilon_\alpha < \varepsilon_{\alpha+1}$ and $K_{\varepsilon_{\alpha+1}} M \cap S_{\alpha+1} = 0$. If $0 < \alpha < \mathfrak{a}$, α limit, then $S_\alpha = \bigcup_{\beta<\alpha} S_\beta$ and $\varepsilon_\alpha = \sup_{\mathfrak{a}}(\{\varepsilon_\beta : \beta < \alpha\})$. Now, it suffices to put $I_\alpha = K_{\varepsilon_\alpha}$. ∎

5.5. PROPOSITION. *Let $\mathfrak{a}$ be an infinite regular cardinal and let R be a prime ring such that* $\mathrm{card}(R) \le \mathfrak{a}$ *and* $\mathrm{Soc}_r(R) = 0$. *Suppose that* $\bigcap_{\alpha<\mathfrak{w}} L_\alpha \ne 0$ *whenever $\mathfrak{w} < \mathfrak{a}$ and L_α, $\alpha < \mathfrak{w}$, are non-zero principal right ideals such that $L_\beta \subseteq L_\alpha$ for $\alpha \le \beta$. Then ${}_R R$ has the $\mathfrak{a}$-approximation property.*

Proof. For every ordinal $\alpha < \mathfrak{a}^+$, we shall find certain non-zero elements $r_\alpha \in R$.

We put $r_0 = 1$. Now, $r_\alpha R$ is not a minimal right ideal, and hence $r_\alpha \notin r_{\alpha+1} R$ for some $0 \ne r_{\alpha+1} \in r_\alpha R$. Let $\alpha > 0$ be limit and $K_\alpha = \bigcap_{\beta<\alpha} r_\beta R$. If $K_\alpha \ne 0$, we choose $0 \ne r_\alpha \in K_\alpha$. If γ is the first limit ordinal with $K_\gamma = 0$, then $\mathfrak{a} \le \gamma$. On the other hand, since $\mathrm{card}(R) \le \mathfrak{a}$, we must have $K_\gamma = 0$ for some limit $\gamma < \mathfrak{a}^+$. Then $\mathrm{card}(\gamma) = \mathfrak{a}$ and the rest is clear from 5.4. ∎

In the preceding section, we mentioned various examples of slender (i.e., $\aleph_1$-slim) modules; we now proceed to the $\mathfrak{a}$-slim case, where $\mathfrak{a} \ge \aleph_2$. If $\mathfrak{a}$ is not $\aleph_1$-measurable, then every slender module is $\mathfrak{a}$-slim (2.4(ii)), and so there is nothing to be done in case there exist no measurable cardinals. But what to do when the opposite is true? K. Eda seems to be the first to try to construct examples of $\mathfrak{a}$-slim modules regardless of measurability (see [7, 2.4(4),(5)]; notice that Eda uses the term "$\mathfrak{a}$-slender" for what we call "$\mathfrak{a}$-slim"). In order to check his examples, K. Eda applied [7, 2.2, 2.3] (see 5.2 and 5.3). Although [7, 2.2] is correct for $\kappa = \aleph_0$, we do not know whether it remains true for $\kappa = \aleph_1$ (see 5.2). Of course, [7, 2.3] is true for μ not $\aleph_1$-measurable (in our terminology), but fails for a measurable μ (see 5.3), and so we have to develop new methods for constructing $\mathfrak{a}$-slim modules (7.2, 7.5) or to find better arguments for justifying at least some of Eda's examples (6.3).

6. Examples of $\mathfrak{a}$-slim modules—endomorphism rings. Throughout this section, let $\mathfrak{a}$ be an infinite cardinal, $M = R^{(\mathfrak{a})}$ and $E = \mathrm{End}({}_R M)$, so that M is both a left R-module and a left E-module. The (left R-) module ${}_R M$ is free and its canonical basis is the set $\{e_\alpha : \alpha < \mathfrak{a}\}$ where $e_\alpha(\alpha) = 1$ and $e_\alpha(\beta) = 0$ for every $\beta < \mathfrak{a}$, $\beta \neq \alpha$. The (left E-) module ${}_E M$ is faithful and cyclic.

For every non-zero cardinal $\mathfrak{w} \leq \mathfrak{a}$, write $\mathfrak{a}$ as a disjoint union $\mathfrak{a} = \bigcup_{\alpha<\mathfrak{w}} P(\mathfrak{w}, \alpha)$, where $\mathrm{card}(P(\mathfrak{w}, \alpha)) = \mathfrak{a}$ for $\mathfrak{w} < \mathfrak{a}$ and $P(\mathfrak{a}, \alpha) = \{\alpha\}$. Let $N(\mathfrak{w}, \alpha) = \sum_{\beta \in P(\mathfrak{w},\alpha)} Re_\beta \subseteq M$ be the corresponding (inner) direct sum; then ${}_R N(\mathfrak{w}, \alpha) \cong {}_R R^{(\mathfrak{a})} = {}_R M$ for $\mathfrak{w} < \mathfrak{a}$ and ${}_R N(\mathfrak{a}, \alpha) = {}_R M(\alpha) \cong {}_R R$. Now, $M = S(\mathfrak{w}, \alpha) \oplus L(\mathfrak{w}, \alpha)$, where $S(\mathfrak{w}, \alpha) = \sum_{\beta<\alpha} N(\mathfrak{w}, \beta)$ and $L(\mathfrak{w}, \alpha) = \sum_{\alpha \leq \beta < \mathfrak{w}} N(\mathfrak{w}, \beta)$, and we denote by $p_{\mathfrak{w},\alpha}$ the (uniquely determined) endomorphism of ${}_R M$ such that $\mathrm{Ker}(p_{\mathfrak{w},\alpha}) = S(\mathfrak{w}, \alpha)$ and $p_{\mathfrak{w},\alpha} \lceil L(\mathfrak{w}, \alpha) = \mathrm{id}$. Notice that $p_{\mathfrak{w},\beta} = p_{\mathfrak{w},\alpha} p_{\mathfrak{w},\beta}$ for $\alpha \leq \beta < \mathfrak{w}$.

For $\alpha, \beta < \mathfrak{a}$, define $k_{\alpha,\beta} \in E$ by $k_{\alpha,\beta}(e_\alpha) = e_\beta$ and $k_{\alpha,\beta}(e_\gamma) = 0$ for every $\gamma < \mathfrak{a}$, $\gamma \neq \alpha$.

6.1. Proposition. *Let $\mathfrak{w}$ be an infinite cardinal, $\mathfrak{w} \leq \mathfrak{a}$, and let D be a subring of E such that $p_{\mathfrak{w},\alpha} \in D$ and $k_{\beta,\gamma} \in D$ for all $\alpha < \mathfrak{w}$ and $\beta, \gamma < \mathfrak{a}$. Then the* (*left D-*) *module ${}_D M$ has the $\mathfrak{w}$-approximation property.*

Proof. It is sufficient to put $I_\alpha = p_{\mathfrak{w},\alpha} D$ and $S_\alpha = S(\mathfrak{w}, \alpha)$ for every $\alpha < \mathfrak{w}$. ∎

6.2. Corollary. *${}_E M$ has the $\mathfrak{w}$-approximation property for every* (*infinite*) *cardinal $\mathfrak{w} \leq \mathfrak{a}$.*

For $\alpha < \mathfrak{a}$, let $h_\alpha \in E$ be such that $h_\alpha(e_\beta) = e_\alpha$ for every $\beta < \mathfrak{a}$.

6.3. Theorem. *The* (*left E-*) *module ${}_E M$ is $\mathfrak{a}^+$-slim.*

Proof. By 6.2 and 5.1, ${}_E M$ is a slender module. Assume that, on the contrary, there exists a non-slim homomorphism $\varphi : {}_E G = E^{\mathfrak{a}} \to {}_E M$. We are going to show that φ is not slender.

If $a \in G$ and $\varphi(a) \neq 0$, we put $\sigma(a) = \min(\mathrm{supp}_{\mathfrak{a}}(\varphi(a)))$ and we define $w(a) \in G$ by $(w(a))(\alpha) = k_{\sigma(a),\alpha} k_{\sigma(a),\sigma(a)} a(\alpha)$ for every $\alpha < \mathfrak{a}$. Then $k_{\sigma(a),\sigma(a)} a = h_{\sigma(a)} w(a)$, and therefore $\varphi(w(a)) \neq 0$. Moreover, $\mathrm{supp}_{\mathfrak{a}}(w(w(a))) = \{\sigma(w(a))\}$. Now, proceeding by induction, we find elements $b_i \in G$, $i < \aleph_0$, such that $\varphi(b_i) \neq 0$ and the sets $\mathrm{supp}_{\mathfrak{a}}(b_i) = \{\beta_i\}$ are one-element and pairwise disjoint.

Let $a_0 \in G$ be such that $\varphi(a_0) \neq 0$. Put $b_0 = w(w(a_0))$, $\beta_0 = \sigma(w(a_0))$ and assume that $n < \aleph_0$ is such that the elements $b_0, \dots, b_n$ are already found. Since φ is not slim, there is $a_{n+1} \in G(\mathfrak{a} \setminus \{\beta_0, \dots, \beta_n\})$ such that $\varphi(a_{n+1}) \neq 0$ and we put $b_{n+1} = w(w(a_{n+1}))$ and $\beta_{n+1} = \sigma(w(a_{n+1}))$. ∎

6.4. Remark. Let D be a subring of E containing all the endomorphisms $p_{\aleph_0,i}$, $k_{\alpha,\beta}$ and h_α, $i < \aleph_0$, $\alpha < \mathfrak{a}$, $\beta < \mathfrak{a}$. Then, using 6.1 and proceeding as in the proof of Theorem 6.3, we can show that ${}_DM$ is $\mathfrak{a}^+$-slim (cf. 6.10).

For $f \in E$ and $\beta < \mathfrak{a}$, let $T(f) = \{\alpha < \mathfrak{a} : f(e_\alpha) \notin Re_\alpha\}$ and $T(f,\beta) = \{\alpha < \mathfrak{a} : (f(e_\alpha))(\beta) \neq 0\}$. The following lemma is obvious:

6.5. Lemma. (i) *$T(0) = T(1_E) = T(k_{\alpha,\alpha}) = \emptyset$ for every $\alpha < \mathfrak{a}$.*
(ii) *$T(k_{\alpha,\beta}) = \{\alpha\}$ for all $\alpha, \beta < \mathfrak{a}$, $\alpha \neq \beta$.*
(iii) *$T(h_\alpha) = \mathfrak{a} \setminus \{\alpha\}$ for every $\alpha < \mathfrak{a}$.*
(iv) *$T(p_{\mathfrak{w},\alpha}) = \emptyset$ for all $\alpha < \mathfrak{w} \leq \mathfrak{a}$.*
(v) *$T(-f) = T(f)$ and $T(f+g) \cup T(fg) \subseteq T(f) \cup T(g)$ for all $f, g \in E$.*
(vi) *$T(f,\alpha) \setminus \{\alpha\} \subseteq T(f)$ for all $f \in E$ and $\alpha < \mathfrak{a}$.*

6.6. Corollary. *The set $F = \{f \in E : \operatorname{card}(T(f)) < \mathfrak{a}\}$ is a subring of E and the* (*left F-*) *module ${}_FM$ has the $\mathfrak{w}$-approximation property for every* (*infinite*) *cardinal $\mathfrak{w} \leq \mathfrak{a}$.*

In the remaining part of this section, we assume that $\mathfrak{a}$ is a measurable cardinal and $\operatorname{card}(R) < \mathfrak{a}$. Let $\mathcal{I}$ be an $\mathfrak{a}$-complete non-principal maximal ideal of $\mathcal{P}(\mathfrak{a})$.

For $a \in G = E^{\mathfrak{a}}$, $\beta < \mathfrak{a}$ and $r \in R$, let $Q(a,\beta,r) = \{\alpha < \mathfrak{a} : ((a(\alpha))(e_0))(\beta) = r\}$. It is clear that there exists just one element $t(a,\beta) \in R$ such that $Q(a,\beta) = Q(a,\beta,t(a,\beta)) \notin \mathcal{I}$ and we put $P(a) = \{\beta < \mathfrak{a} : t(a,\beta) \neq 0\}$ and $Q(a) = \bigcap_{\beta \in P(a)} Q(a,\beta)$, while $Q(a) = \mathfrak{a}$ if $P(a) = \emptyset$.

6.7. Lemma. *$P(a)$ is a finite set and $Q(a) \notin \mathcal{I}$.*

Proof. Assume that there are pairwise different elements $\beta_i \in P(a)$, $i < \aleph_0$. Then $V = \bigcup(\mathfrak{a} \setminus Q(a,\beta_i)) \in \mathcal{I}$ and we take $\gamma \in \mathfrak{a} \setminus V$. Now, $\gamma \in \bigcap Q(a,\beta_i)$ and consequently the set $\operatorname{supp}_{\mathfrak{a}}((a(\gamma))(e_0))$ is infinite, a contradiction. ∎

Due to the preceding lemma, we can define a mapping $\varphi : G \to M$ by $\varphi(a) = \sum_{\beta < \mathfrak{a}} t(a,\beta) e_\beta$ and check easily that φ is additive.

6.8. Lemma. *$\varphi : {}_FG \to {}_FM$ is a homomorphism of F-modules.*

Proof. Let $a \in G$, $f \in F$, $\sigma < \mathfrak{a}$ and

$$Q = Q(a) \cap \bigcap_{\alpha \in T(f,\sigma)} Q(a,\alpha).$$

We have $Q \notin \mathcal{I}$ and if $\gamma \in Q$ and $P = \operatorname{supp}_{\mathfrak{a}}((a(\gamma))(e_0))$, then

$$f\varphi(a) = (fa(\gamma))(e_0) - f(x),$$

where

$$x = \sum_{\beta \in P \setminus P(a)} (((a(\gamma))(e_0))(\beta)) e_\beta.$$

But $(f(x))(\sigma) = 0$, $(f\varphi(a))(\sigma) = ((fa(\gamma))(e_0))(\sigma)$ and we have proved that $Q \subseteq Q(fa, \sigma, (f\varphi(a))(\sigma)) \notin \mathcal{I}$. Thus $(f\varphi(a))(\sigma) = t(fa, \sigma) = (\varphi f(a))(\sigma)$. ∎

6.9. Theorem. *The (left F-) module ${}_F M$ has the $\mathfrak{w}$-approximation property for every (infinite) cardinal $\mathfrak{w} \le \mathfrak{a}$, nevertheless, ${}_F M$ is not $\mathfrak{a}^+$-slim.*

Proof. Define $a_\sigma \in G$, $\sigma < \mathfrak{a}$, by $a_\sigma(\alpha) = 0$ and $a_\sigma(\beta) = k_{0,0}$ for all $\alpha, \beta < \mathfrak{a}$, $\alpha < \sigma \le \beta$. Then $\varphi(a_\sigma) = e_0$, and hence φ is not slim. (Notice that $a_\sigma \in F^{\mathfrak{a}}$.) ∎

6.10. Remark. Let F_1 designate the set of $f \in E$ such that $\varphi(fa) = f\varphi(a)$ for every $a \in G$. Then $F \subseteq F_1$ and F_1 is a subring of E. Now, if D_1 is a subring of F_1, then $\varphi : {}_{D_1}G \to {}_{D_1}M$ is not slim, and therefore ${}_{D_1}M$ is not $\mathfrak{a}^+$-slim.

6.11. Remark. There is another way to exploit a measurable cardinal to construct a non-slim homomorphism $\varphi : {}_F G \to {}_F M$ (see 6.8, 6.9). First, the existence of a measurable cardinal is equivalent to the existence of a non-identical elementary embedding $\iota : \mathcal{V} \to \mathcal{M}$, where $\mathcal{V}$ is the class of all sets and $\mathcal{M} (\subseteq \mathcal{V})$ is an inner model of set theory (ZFC, see e.g. [19]). Next, let $\mathfrak{a} = \mathrm{crit}(\iota)$ be the first ordinal such that $\iota(\mathfrak{a}) \ne \mathfrak{a}$; then $\mathfrak{a}$ is a measurable cardinal. Moreover, $\overline{M} = \iota(R^{(\mathfrak{a})}) = R^{(\iota(\mathfrak{a}))} \in \mathcal{M}$, $\iota(E) = \overline{E}$ is an endomorphism ring of ${}_R\overline{M}$ and $\overline{G} = \iota(E^{\mathfrak{a}}) = (\overline{E})^{\iota(\mathfrak{a})}$.

Now, we are going to construct φ:

$$\begin{tikzcd} \overline{G} \arrow[r, "\pi"] & \overline{E} \arrow[r, "\psi"] & \overline{M} \arrow[d, "\varrho"] \\ G \arrow[u, "\iota"] & E \arrow[u, "\iota"] & M \end{tikzcd}$$

Here, π is the $\mathfrak{a}$th projection, ψ is the $\overline{e}_0$-substitution, ϱ is the natural projection of the free R-modules and the desired F-homomorphism $\varphi : {}_F G \to {}_F M$ is just the composition $\varphi = \varrho\psi\pi\iota$; use the fact that $\iota : E \to \overline{E}$ can be viewed as a ring homomorphism yielding an F-module structure.

7. Examples of $\mathfrak{a}$-slim modules—boolean rings. We say that an ordered set S is

- *downwards-$\mathfrak{a}$-inductive* if every non-empty chain containing less than $\mathfrak{a}$ elements of S has a lower bound in S;
- *strongly downwards-$\mathfrak{a}$-inductive* if every non-empty downwards-directed set containing less than $\mathfrak{a}$ elements of S has a lower bound in S.

Let R be a boolean ring (or algebra). We say that R is (*strongly*) $\mathfrak{a}$-*inductive* if the ordered set $R \setminus \{0\}$ is (strongly) downwards-$\mathfrak{a}$-inductive.

7.1. Proposition. *Let R be an $\aleph_1$-inductive boolean ring such that* $\mathrm{Soc}(R) = 0$. *Then R is slender.*

Proof. Suppose that, on the contrary, R is not slender. One may easily see that there exists a (module) homomorphism $\varphi : T = R^{\aleph_0} \to R$ such that $s_n = \varphi(e_n)$, $n < \aleph_0$, are non-zero and pairwise orthogonal. Define $a \in T$ by $a(n) = s_n$. Then $s = \varphi(a) = \sup_R(\{s_n : n < \aleph_0\})$ and $t_0 \dots t_m \neq 0$ for every $m < \aleph_0$, where $t_n = s + s_n$. Consequently, there is $t \in R$ such that $t \neq 0$ and $t \leq t_n$ for every n. Now, $ts_n = tt_ns_n = 0$, and hence $t = tt_n = t(s+s_n) = ts$. On the other hand, $(1+t)s_n = s_n$, and so $s \leq 1+t$. Thus $ts = 0$ and $t = 0$, a contradiction. ∎

7.2. Theorem. *Let $\mathfrak{a}$ be an uncountable cardinal and let R be an $\mathfrak{a}$-inductive boolean ring such that* $\mathrm{Soc}(R) = 0$. *Then R is $\mathfrak{a}$-slim.*

Proof. The case $\mathfrak{a} = \aleph_1$ is settled by 7.1 and we now assume that $\mathfrak{a} \geq \aleph_2$ and R is not $\mathfrak{a}$-slim (notice that R is slender by 7.1).

According to 2.5, there are a measurable cardinal $\mathfrak{w}$ and a non-zero completely slender (module) homomorphism $\varphi : T = R^{\mathfrak{w}} \to R$ such that $\mathfrak{w} < \mathfrak{a}$ and the ideal $\mathcal{I}_\varphi$ corresponding to φ is a maximal ideal of $\mathcal{P}(\mathfrak{w})$. Now, we need to show that $\varphi(b_r) = r$ for at least one non-zero element $r \in R$, where $b_r \in T$ is such that $b_r(\alpha) = r$ for every $\alpha < \mathfrak{w}$.

First, take $a \in T$ such that $\varphi(a) \neq 0$ and $a(\alpha) \leq \varphi(a)$ for every ordinal $\alpha < \mathfrak{w}$ and denote by $\mathcal{M}$ the set of ordered triples (γ, f, Φ), where γ is an ordinal such that $\gamma < \mathfrak{w}^+$, $f : \gamma \to \mathfrak{w}$ is an injective mapping and $\Phi : f(\gamma) \to R\setminus\{0\}$ is a mapping such that $\Phi(f(\beta_1)) \leq a(f(\beta_1))$ and $\Phi(f(\beta_2)) \leq \Phi(f(\beta_1))$ for all $\beta_1 \leq \beta_2 < \gamma$. The set $\mathcal{M}$ is ordered in the obvious way and we take a maximal element of $\mathcal{M}$, say (δ, g, Ψ). Since $\delta < \mathfrak{a}$, there is $r \in R$ such that $r \neq 0$ and $r\Psi(g(\beta)) = r$ for every $\beta < \delta$; then $(ra)(g(\beta)) = r$. On the other hand, the maximality of (δ, g, Ψ) yields that $(ra)(\varepsilon) = 0$ for every $\varepsilon \in \mathfrak{w} \setminus g(\delta)$. Clearly, $\mathfrak{w} \setminus g(\delta) \in \mathcal{I}_\varphi$, and hence $\varphi(b_r) = r$.

We have thus found our element r. Further, taking into account that the boolean ring R is $\mathfrak{a}$-inductive and contains no atoms and that $\mathfrak{w} < \mathfrak{a}$, we can easily find pairwise different elements $s_\alpha \in R$, $\alpha < \mathfrak{w}$, such that $s_0 \leq r$ and $s_\gamma \leq s_\beta$ whenever $\beta \leq \gamma < \mathfrak{w}$. Let $c \in T$ be such that $c(\alpha) = s_\alpha$. We now check $\varphi(c) = \inf_R(\{s_\alpha : \alpha < \mathfrak{w}\})$.

Indeed, if $\alpha < \mathfrak{w}$, then

$$\{\beta : \beta < \mathfrak{w},\ c(\beta) \neq (s_\alpha c)(\beta)\} = \{\beta : \beta < \alpha\} \in \mathcal{I}_\varphi,$$

and hence $\varphi(c) = \varphi(s_\alpha c) = s_\alpha \varphi(c)$. On the other hand, if s is a lower bound of the set considered, then $s = \varphi(b_s) = \varphi(sc) = s\varphi(c)$.

Now, the set $\{s_\alpha+\varphi(c):\alpha<\mathfrak{a}\}$ is a chain of non-zero elements of R and it has a non-zero lower bound t in R. We have $t\varphi(c)=t(s_\alpha+\varphi(c))\varphi(c)=0$ and $t=t(s_\alpha+\varphi(c))=ts_\alpha, \alpha<\mathfrak{w}$. Consequently, $0\neq t=t\varphi(c)=0$, a contradiction. ■

7.3. Remark. A crucial step in the preceding proof is to show the existence of $0\neq r\in R$ with $\varphi(b_r)=r$. We can proceed in an easier way in case the ring R is strongly $\mathfrak{a}$-inductive:

Put $R_1=\{a(\alpha):\alpha<\mathfrak{w}\}\subseteq R$. As R_1 contains some non-zero elements, consider $R_2\subseteq R_1$ such that R_2 is maximal with respect to the property that all finite products of elements of R_2 are non-zero. Since R is strongly $\mathfrak{a}$-inductive, there is $0\neq r\in R$ such that $r\leq R_2$. Now, $(ra)(\alpha)=r$ for $\alpha\in P=\{\alpha<\mathfrak{w}:a(\alpha)\in R_2\}$ and $(ra)(\alpha)=0$ for $\alpha\in Q=\mathfrak{w}\setminus P$. Consequently, $\varphi\lceil T(P)\neq 0$, $Q\in\mathcal{I}_\varphi$ and $\varphi\lceil T(Q)=0$. Thus $r=\varphi(ra)=\varphi(b_r)$.

The following simple construction is a convenient source of examples of atomless (strongly) $\mathfrak{a}$-inductive boolean rings (resp. algebras):

7.4. Example. (i) Let T be a non-empty linearly ordered set. We denote by $\mathcal{R}$ the set of equivalence relations r defined on T and having just two blocks, say V_r and W_r, such that $V_r\leq W_r$. We choose pairwise disjoint two-element sets $\{v_i,w_i\}$, $i=0,1$, $\{v_r,w_r\}$, $r\in\mathcal{R}$, and we denote by P their union (we also assume that $T\cap P=\emptyset$). Further, we extend the linear order of T to a linear order of $\varrho(T)=T\cup P$ by means of the following rules: $V_r\leq v_r\leq w_r\leq W_r$ for every $r\in\mathcal{R}$; $w_r\leq v_s$ for all $r,s\in\mathcal{R}$ such that $V_r\subseteq V_s$; $v_0\leq w_0\leq\varrho(T)\setminus\{v_0\}$; $\varrho(T)\setminus\{w_1\}\leq v_1\leq w_1$.

(ii) Let $\mathfrak{a}$ be an infinite regular cardinal. For every ordinal $\alpha<\mathfrak{a}$, define a linearly ordered set S_α in the following way: S_0 is a one-element set; $S_{\alpha+1}=\varrho(S_\alpha)$ (see (i)); if $\alpha>0$ is limit, then $S_\alpha=\bigcup_{\beta<\alpha}S_\beta$. Now, put $S=S_{\mathfrak{a}}=\bigcup_{\alpha<\mathfrak{a}}S_\alpha$. For $a,b\in S$ let $(a,b\rangle=\{x\in S:a<x\leq b\}, (-\infty,a\rangle=\{x\in S:x\leq a\}, (a,+\infty\rangle=\{x\in S:a<x\}$ and $(-\infty,+\infty\rangle=S$. Finally, let $\mathcal{A}$ be the subalgebra of the boolean algebra $\mathcal{P}(S)$ generated by all $(x,y\rangle$, $x,y\in S\cup\{\pm\infty\}$ (cf. the so-called interval algebra, see e.g. [20, 1.1.11, §6.15]). Then the algebra $\mathcal{A}$ contains no atoms and is strongly $\mathfrak{a}$-inductive.

7.5. Remark. (i) The factor algebra $\mathcal{P}(\aleph_0)/\mathrm{Fin}$ is atomless and (strongly) $\aleph_1$-inductive, and hence the equivalent boolean ring is slender.

(ii) If R is a slender boolean ring, then the boolean algebra equivalent to R is not ($\aleph_1$-) complete.

(iii) If $\mathfrak{a}\geq\aleph_1$ and if A is an $\mathfrak{a}$-inductive boolean algebra without atoms, then A is not complete (this is easy to see directly but also follows from (ii) and 7.1).

8. Slim modules and measurable sets. This section summarizes results on slim modules obtained in the preceding parts.

8.1. THEOREM. *Let* $\mathfrak{a}$ *be a cardinal number.*

(i) *If* $\mathfrak{w}$ *is an* $\mathfrak{a}$*-measurable cardinal*, *then* $\mathrm{card}(M) \geq \mathfrak{a}$ *for every non-zero* $\mathfrak{w}^+$*-slim module* M.

(ii) *If* $\mathfrak{a}$ *is measurable*, *then* $\mathrm{card}(M) \geq \mathfrak{a}$ *for every non-zero* $\mathfrak{a}^+$*-slim module* M.

(iii) *If* $\mathfrak{a}$ *is infinite*, *then there exists an* $\mathfrak{a}^+$*-slim module* M (*over a suitable ring*) *such that* $\mathrm{card}(M) = \mathfrak{a}$.

(iv) *If* $\mathfrak{w}$ *is a non-*$\mathfrak{a}$*-measurable cardinal*, *then every* $\mathfrak{a}$*-slim module is* $\mathfrak{w}^+$*-slim.*

(v) *If* $\mathfrak{m}$ *is the smallest* $\mathfrak{a}$*-measurable cardinal* (*provided it exists*), *then every* $\mathfrak{a}$*-slim module is* $\mathfrak{m}$*-slim.*

Proof. See 2.4 and 6.3. ■

8.2. THEOREM. *The following conditions are equivalent*:

(i) *There exists at least one non-zero slim module* (*over at least one ring*).

(ii) *There exists a cardinal number* $\mathfrak{z}$ *such that* $\mathfrak{m} \leq \mathfrak{z}$ *whenever* $\mathfrak{m}$ *is a measurable cardinal.*

Proof. (i) implies (ii) by 8.1(ii) and (ii) implies (i) by 8.1(iii), (iv). ■

8.3. THEOREM. *Suppose that the equivalent conditions of* 8.2 *are satisfied and denote by* $\mathfrak{z}$ *the smallest infinite cardinal such that* $\mathfrak{m} \leq \mathfrak{z}$ *for every measurable cardinal* $\mathfrak{m}$. *Then*:

(i) *A module is slim if and only if it is* $\mathfrak{z}^+$*-slim.*

(ii) *If* $\mathfrak{z} \neq \aleph_0$ *and* $\mathfrak{z}$ *is not measurable*, *then a module is slim if and only if it is* $\mathfrak{z}$*-slim.*

(iii) $\mathrm{card}(M) \geq \mathfrak{z}$ *for every non-zero slim module* M.

(iv) *There exists a slim module* M *such that* $\mathrm{card}(M) = \mathfrak{z}$.

Proof. See 8.1 and 2.3. ■

8.4. REMARK. The conditions of 8.2 are also equivalent to the following assertion: Every concretizable category is algebraic. (We refer to [25] for details and further related material and references.)

9. Modules commuting with pull-backs. The following result is just a routine observation:

9.1. PROPOSITION. *A module* M *commutes with pull-backs if and only if* M *is injective.*

9.2. Corollary. *The following conditions are equivalent for a module M:*

(i) *M commutes with limits of all diagrams.*
(ii) *M commutes with pull-backs and (countable) direct products.*
(iii) *$M = 0$.*

10. Various downwards-directed spectra. Let S be a (non-empty) ordered set and $\triangle : S \to \text{SET}$ an S-spectrum, $f_{r,s} : \triangle(r) \to \triangle(s)$, $r, s \in S, r \le s$. Now, for every $r \in S$, let $F_r = \mathcal{F}(\triangle)(r)$ be a free module over $\triangle(r)$. The maps $f_{r,s}$ can be uniquely extended to homomorphisms $\varphi_{r,s} : F_r \to F_s$ and we get an S-spectrum $\mathcal{F}(\triangle) : S \to$ R-MOD; this spectrum is formed by free modules.

10.1. Example. Consider the following transformation τ of $\aleph_0 : \tau(0) = 0$, $\tau(1) = 0$, $\tau(2) = 1$, $\tau((k^2 + k + 4)/2) = 1$, $\tau((k^2 + k + 6)/2) = (k^2 + k + 4)/2, \ldots, \tau((k^2 + 3k + 4)/2) = (k^2 + 3k + 2)/2$ for $k = 1, 2, \ldots$ Now, let F be a free module over $X = \{x_1, x_2, \ldots\}$ and let φ be the uniquely determined endomorphism of F such that $\varphi(x_1) = 0$ and $\varphi(x_j) = x_{\tau(j)}$ for $j \ge 2$. Consider the following $\widetilde{\aleph}_0$-spectrum:

$$F_0 \xleftarrow{\varphi_0} F_1 \xleftarrow{\varphi_1} F_2 \longleftarrow \ldots,$$

where $F_i = F$ and $\varphi_i = \varphi$ for every $i < \aleph_0$. Let $\psi_i : A \to F_i$ be a limit of this spectrum. Then $A = 0$, and so $x_1 \ne \text{Im}(\psi_0)$. On the other hand, $x_1 \in \bigcap_{n<\aleph_0} \text{Im}(\varphi_n \varphi_{n-1} \ldots \varphi_1 \varphi_0)$.

10.2. Example. Let X be the set of ordered pairs (i, j) of integers such that either $i \ge 2$ or $0 \le i \le 1$ and $j \ge 0$. Define two transformations f and g of X as follows:

$$f(i,j) = \begin{cases} (i,j) & \text{if } i \in \{0,1\} \text{ and } j = 0, \\ (i, j-1) & \text{otherwise;} \end{cases}$$

$$g(i,j) = \begin{cases} (i,j) & \text{if } i = 0, \\ (1,0) & \text{if } i = 2 \text{ and } j \le 0, \\ (i-1, j) & \text{otherwise.} \end{cases}$$

Then $fg = gf$ and $f(X) = X = g(X)$. Now, let F be a free module over X and let φ and ψ be the endomorphisms of F extending f and g, resp. Then $\varphi\psi = \psi\varphi$ and both φ and ψ are epimorphisms. Finally, consider the $\widetilde{\aleph}_0$-spectrum

$$F_0 \xleftarrow{\varphi_0} F_1 \xleftarrow{\varphi_1} F_2 \longleftarrow \ldots,$$

where $F_i = F$ and $\varphi_i = \varphi$. Let $\psi_i : A \to F_i$ be a limit of this spectrum. There exists a uniquely determined endomorphism ξ of A such that $\psi\psi_i = \psi_i\xi$ for every $i < \aleph_0$. The point of this example is that $\xi(A) \ne A$ in spite of the fact that ψ and all ψ_i are epimorphisms.

10.3. Example. For $n < \aleph_0$, let $X_n = \{x_n, x_{n+1}, x_{n+2}, \ldots\}$ and let $f_n : X_{n+1} \to X_n$ be the natural injection. In this way, we get an $\widetilde{\aleph}_0$-spectrum

$$X_n \xleftarrow{f_n} X_{n+1}$$

formed by monomorphisms and with empty limit (in SET). Now, consider the corresponding $\widetilde{\aleph}_0$-spectrum $\mathcal{F}(\triangle)$:

$$F_n \xleftarrow{\varphi_n} F_{n+1}.$$

This spectrum is formed by free modules and monomorphisms and $\lim(\mathcal{F}(\triangle)) = 0$.

Let M be a non-zero module and

$$\varphi_n^* : \mathrm{Hom}_R(F_n, M) \to \mathrm{Hom}_R(F_{n+1}, M),$$

$\varphi_n^* = \mathrm{Hom}_R(\varphi_n, \mathrm{id}_M)$, the corresponding Hom-$\aleph_0$-spectrum of abelian groups. Finally, let $\psi_n^* : \mathrm{Hom}_R(F_n, M) \to G$ be a colimit and $\Phi : G \to \mathrm{Hom}_R(\lim(\mathcal{F}(\triangle)), M) = 0$ the connecting homomorphism. Then $G \neq 0$ and $\Phi = 0$ and consequently Φ is not a monomorphism.

The foregoing examples show that limits (in R-MOD) do not always behave dually to colimits. In this respect, we mention another example, even more drastic:

10.4. Example. According to [18], there exists an $\widetilde{\aleph}_1$-spectrum $\triangle : \widetilde{\aleph}_1 \to$ SET such that $\triangle$ is formed by non-empty sets and projective mappings and the empty set is a limit of $\triangle$. Then $\mathcal{F}(\triangle)$ is an $\widetilde{\aleph}_1$-spectrum (in R-MOD) formed by non-zero free modules and epimorphisms and $\lim(\mathcal{F}(\triangle)) = 0$.

11. Modules commuting with limits of downwards-directed spectra

11.1. Proposition. *A module M commutes with limits of all epimorphic $\widetilde{\aleph}_0$-spectra if and only if M is slender* (*i.e.*, $\aleph_1$-*slim*).

Proof. Only the converse implication is (perhaps) not immediate. Suppose that M is slender and let A ($\subseteq \prod_{n<\aleph_0} A_n$), together with the natural projections $p_n : A \to A_n$, be the limit of an epimorphic $\widetilde{\aleph}_0$-spectrum $f_n : A_{n+1} \to A_n$, $n < \aleph_0$. Let $\varphi : A \to M$ be a homomorphism such that $\varphi \notin \mathrm{Im}(\Phi)$, $\Phi : \mathrm{colim}(\mathrm{Hom}_R(A_n, M)) \to \mathrm{Hom}_R(A, M)$ being the connecting homomorphism.

Let $m < \aleph_0$ be such that for every $a \in A$, $a(m) = 0$ implies $a \in \mathrm{Ker}(\varphi)$. Then we can define a homomorphism $\psi : A_m \to M$ by $\psi(x) = \varphi(b_x)$, $x \in M$, $b_x(m) = x$, and we get $\varphi = \psi p_m$. Thus $\varphi \in \mathrm{Im}(\Phi)$, a contradiction.

We have proved that, for every $m < \aleph_0$, there is an element $a_m \in A$ such that $a_m(m) = 0$ and $\varphi(a_m) \neq 0$. Now, we define a homomorphism

$\sigma : R^{\aleph_0} \to A$ by $(\sigma(c))(n) = \sum_{m<\aleph_0} c(m)a_m(n)$ for every $n < \aleph_0$. Then $\varphi\sigma : R^{\aleph_0} \to M$ is not slender, again a contradiction. ∎

11.2. Remark. (i) Let M be a slender module. One sees easily from 11.1 (and its proof) that M is injective with respect to the natural injection $A \hookrightarrow \prod_{n<\aleph_0} A_n$, where A is the limit of an epimorphic $\widetilde{\aleph}_0$-spectrum $A_{n+1} \to A_n$.

(ii) Denote by σ the class of short exact sequences $0 \to B \to C \to D \to 0$ such that all slender modules are injective with respect to the monomorphisms $B \to C$. Then σ is a purity and induces a closed subfunctor $\mathrm{Pext}_{R,\sigma}$ of Ext_R (details on purities and related topics may be found e.g. in [2]). The purity σ is injectively generated and it is injectively rich, since slender modules are closed under submodules. Furthermore, σ is closed under arbitrary direct sums and countable direct products (under arbitrary direct products provided there are no measurable cardinals). A module P is *σ-coprojective* iff $\mathrm{Ext}_R(P, M) = 0$ for every slender M; if R is left hereditary, then every σ-coprojective module is projective. A module Q is *σ-coinjective* iff $\mathrm{Hom}_R(Q, M) = 0$ for every slender M (among such modules Q we find all subinjective modules and also the module $R^{\aleph_0}/R^{(\aleph_0)}$).

11.3. Proposition. *The following conditions are equivalent for a module M:*

(i) *The connecting homomorphism*

$$\Phi : \mathrm{colim}(\mathrm{Hom}_R(\triangle, M)) \to \mathrm{Hom}_R(\lim(\triangle), M)$$

is an epimorphism for every monomorphic $\widetilde{\aleph}_0$-spectrum.

(ii) *M is injective.*

Proof. We only prove (i)⇒(ii), the other implication being trivial.

Let K be a submodule of a module N. We put $A_i = K$, $B_i = N$ and $C_n = A_1 \oplus \ldots \oplus A_n \oplus B_n \oplus B_{n+1} \oplus \ldots$ for every $n < \aleph_0$. Let $f_{n+1} : C_{n+1} \to C_n$ denote the natural injection. In this way, we get an $\widetilde{\aleph}_0$-spectrum and it is clear that $D = K^{(\aleph_0)}$ (together with the natural injections $j_n : D \to C_n$) is a limit of the spectrum.

Let $\varphi : K \to M$ be a homomorphism and let $\psi : D \to M$ be defined by $\psi(a) = \sum_{i<\aleph_0} \varphi(a(i))$, $a \in D$. Then $\psi \in \mathrm{Im}(\Phi)$, and hence there are $m < \aleph_0$ and a homomorphism $\xi : C_m \to M$ such that $\psi = \xi j_m$. Consequently, if $\iota_m : N \to C_m$ denotes the mth natural injection, then $\xi\iota_m : N \to M$ is a homomorphism extending φ. ∎

11.4. Proposition. *The following conditions are equivalent for a module M:*

(i) *M commutes with limits of all diagrams.*

(ii) *M commutes with limits of $\widetilde{\aleph}_0$-spectra.*

(iii) *M commutes with limits of monomorphic $\widetilde{\aleph}_0$-spectra.*

(iv) $M = 0$.

Proof. (iii)⇒(iv). See 10.3. ■

11.5. THEOREM. *The following conditions are equivalent for a module M:*

(i) *M commutes with limits of all diagrams.*

(ii) *M commutes with limits of all epimorphic downwards-directed (linear) spectra.*

(iii) *M commutes with limits of epimorphic $\widetilde{\aleph}_1$-spectra (formed by free modules).*

(iv) $M = 0$.

Proof. (iii)⇒(iv). Let $\varphi_{\alpha,\beta} : F_\alpha \to F_\beta$, $\beta \le \alpha < \aleph_1$, be an epimorphic $\widetilde{\aleph}_1$-spectrum formed by non-zero free modules and having a zero limit (see 10.4), and let G be a colimit of the (monomorphic) $\aleph_1$-spectrum $\mathrm{Hom}_R(\varphi_{\alpha,\beta}, \mathrm{id}_M) : \mathrm{Hom}_R(F_\beta, M) \to \mathrm{Hom}_R(F_\alpha, M)$. Now, the connecting homomorphism Φ equals 0 and, since it is an isomorphism, we have $G = 0$. Consequently, $\mathrm{Hom}_R(F_\alpha, M) = 0$ for at least one $\alpha < \aleph_1$ and $M = 0$, since $F_\alpha \ne 0$. ■

12. Summary

12.1. THEOREM. (i) *A module M commutes with all direct products $\prod_{i\in \mathrm{I}} A_i$, $\mathrm{card}(\mathrm{I}) < \mathfrak{a}$, if and only if M is $\mathfrak{a}$-slim.*

(ii) *A module M commutes with direct products if and only if M is slim.*

(iii) *The following conditions are equivalent for a module M:*

- (iii1) *M commutes with countable direct products.*
- (iii2) *M commutes with limits of epimorphic downwards-directed spectra with countable cofinality.*
- (iii3) *M commutes with limits of epimorphic $\widetilde{\aleph}_0$-spectra.*
- (iii4) *M is $\aleph_1$-slim* (alias *slender*).

(iv) *A module M commutes with pull-backs if and only if M is injective.*

(v) *The following conditions are equivalent for a module M:*

- (v1) *M commutes with limits of all diagrams.*
- (v2) *M commutes with limits of all downwards-directed spectra.*
- (v3) *M commutes with limits of monomorphic $\widetilde{\aleph}_0$-spectra.*
- (v4) *M commutes with limits of epimorphic $\widetilde{\aleph}_1$-spectra.*
- (v5) *M commutes with pull-backs and limits of epimorphic $\widetilde{\aleph}_0$-spectra.*
- (v6) *M commutes with pull-backs and countable direct products.*
- (v7) $M = 0$.

P r o o f. See 2.1, 9.1, 11.1, 11.4 and 11.5. ∎

12.2. Remark. In case there exist no (resp., too many) measurable cardinals, the conditions of 12.1(iii) (resp., 12.1(v)) are also equivalent to the additional one saying that M commutes with all direct products.

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