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Period doubling, entropy, and renormalization

by

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Abstract. We show that in any family of stunted sawtooth maps, the set of maps whose set of periods is the set of all powers of 2 has no interior point. Similar techniques then allow us to show that, under mild assumptions, smooth multimodal maps whose set of periods is the set of all powers of 2 are infinitely renormalizable with the diameters of all periodic intervals going to zero as the period goes to infinity.

1. Introduction. The present work is motivated by the following folklore conjecture (see also [OT]):

CONJECTURE A. *A real polynomial map f with set of periods (of its periodic orbits)*

$$P(f) = \{2^i : i \in \mathbb{N}\}$$

can be approximated by polynomial maps with positive entropy and by polynomial maps with finitely many periodic orbits.

This conjecture is now established for quadratic polynomials (as a consequence of [Su] or [La]) and work is in progress toward generalization for higher degree polynomials [Hu]. The interest in such a conjecture comes from Theorems A and B below (see Section 2.1) and the fact that *topological entropy* (conceived as an invariant of topological conjugacy [AKM]) is also one way to measure the complexity of the dynamics of a map (see Section 2.1): one is trying to describe how maps with simple dynamics can be deformed to maps with complicated dynamics, or, as one says, chaotic maps. Tradition, as well as the availability in this framework of a greater set of techniques, has put some emphasis on the particular case of polynomial maps, as in Conjecture A. However, the problem of the transition to chaos

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is more generally interesting in the category of smooth maps, in particular smooth endomorphisms of the interval, for which we recall the following:

CONJECTURE B. *All endomorphisms of the interval* $f \in C^k(I)$, $k \geq 1$, *with* $P(f) = \{2^i : i \in \mathbb{N}\}$ *are on the boundary of chaos and on the boundary of the interior of the set of zero entropy.*

The C^1 version of this conjecture is formulated in [BH]. The C^0 version has recently been proved to hold true: more precisely, the chaotic side of it is contained in [Kl] (in fact, it is easy to prove that infinite topological entropy is a generic property in $C^0(I)$, compare [Ya]) and the zero entropy side is in [JS2].

We first show that any *stunted sawtooth map* (see Section 2.2) whose set of periods is the set of all powers of 2 can be approximated by stunted sawtooth maps with positive entropy and by stunted sawtooth maps with only finitely many periodic orbits (see Section 2.3). This result verifies the symbolic dynamic version of the above conjectures in the sense that stunted sawtooth maps carry all possible *kneading data* (see Section 2.2) of multimodal maps.

We also make a second step toward Conjecture A by proving that maps with $P(f)$ as above which satisfy some smoothness conditions (and in particular polynomial maps) are *infinitely renormalizable* (see Section 2.4).

Sections 3 and 4 contain proofs of the results formulated in Section 2.

Acknowledgements. After we proved Theorem 1, several colleagues suggested we also provide proofs for the long overdue results in Theorems 2, 3, and 4. These results, which, as we shall see, are reasonably easy consequences of Theorems A to G, do not seem to have appeared in print. To the contrary, most if not all auxiliary results might be found, implicitly in the work of Sharkovskiĭ and Misiurewicz, and in some explicit form in a combination of [BC], [Ge], [JS1], [JS2], and [Sm] (to just mention a few; see also the discussion at the beginning of §2.4). Both authors would like to specially thank Dennis Sullivan and Jean Marc Gambaudo for their constant interest and encouragement in this work. We thank the referee of an early version of this paper (which was circulated as Stony Brook Preprint 95/13) and Louis Block for pointing out some relevant references. We also thank this referee and Michał Misiurewicz for suggesting several improvements. The first author is grateful to the IMS at Stony Brook, the second one to IMPA, for their support and hospitality while part of this work was being done.

2. Preliminary definitions and results

2.1. *Topological entropy of one-dimensional maps.* A point x is a *periodic point* of period n of a map f if $f^i(x) \neq x$, $0 < i < n$, and $f^n(x) = x$. The orbit of x is then called a *periodic orbit* (of period n). If $n = 1$, then x is also called a *fixed point*. If the orbit of x contains a periodic orbit but x is not a periodic point, we say that x is *pre-periodic*.

The *topological entropy* $h(f)$ of a continuous map f on a compact metric space X with metric d can be defined as follows [Bo]. Given $\varepsilon > 0$, $n \in \mathbb{N}$, we say a subset $S \subset X$ is (n,ε)*-separated* if

$$x, y \in S,\ x \neq y \ \Rightarrow\ \exists m : 0 \le m < n \text{ such that } d(f^m(x), f^m(y)) > \varepsilon.$$

Set $H(f,n,\varepsilon)$ to be the maximal cardinality of (n,ε)-separated sets. Then

$$h(f) = \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log H(f,n,\varepsilon).$$

For maps on an interval, the following result gives a necessary and sufficient condition for the positivity of topological entropy.

Theorem A ([BF], [M1]). *A map $f \in C^0(I)$ has positive topological entropy if and only if it has a periodic point whose period is not a power of* 2.

Remark 1. The "if" part is from [BF], the "only if" part from [M1].

From Theorem A and Theorem 3 of [BH], one gets the following (see also the discussion following Theorem 3 in [BH]):

Theorem B. *In the space $C^k(I)$, $k \ge 1$, of C^k endomorphisms of an interval I, if a map f is on the boundary of positive topological entropy then the set $P(f)$ of its periods is $\{2^i : i \in \mathbb{N}\}$. The same is true for f on the boundary of the interior of the set of maps with zero topological entropy.*

Remark 2. Conjectures A and B are about the converse of Theorem B.

We next give two other necessary and sufficient conditions for the positivity of topological entropy, which will be important tools for us. This requires some more terminology.

So let $f \in C^0(I)$ and p be a fixed point of f. A point x of I belongs to the *unstable manifold* $W^u(p,f)$ of p if, for every neighborhood V of p, $x \in f^n(V)$ for some positive integer n. It is easy to check that $W^u(p,f)$ is connected and invariant under f. A point $x \in I$ is a *homoclinic point* of f if there is a periodic point p of f of period n such that $x \neq p$, $x \in W^u(p,f^n)$ and $f^{mn}(x) = p$ for some $m \in \mathbb{N}$ (see [Bl1]).

Theorem C ([Bl1]). *A map $f \in C^0(I)$ has positive topological entropy if and only if it has a homoclinic point.*

Let O be a periodic orbit of $f \in C^0(I)$ of period $m \ge 2$, with m a power of 2. Block calls O *simple* if for any subset $\{q_1,\ldots,q_n\}$ of O, where n divides

m and $n \geq 2$, and any positive integer r which divides m and is such that $\{q_1, \ldots, q_n\}$ is a periodic orbit of f^r with $q_1 < \ldots < q_n$, we have

$$f^r(\{q_1, \ldots, q_{n/2}\}) = \{q_{n/2+1}, \ldots, q_n\}.$$

Theorem D ([Bl2]). *A map $f \in C^0(I)$ has zero topological entropy if and only if all its periodic orbits are simple.*

Remark 3. For a simple periodic orbit and any $i \in \{1, \ldots, n\}$, if s is a power of 2, then $f^s(q_i)$ cannot belong to the open interval bounded by q_i and $f^{2s}(q_i)$. When $s = 1$ this is a direct consequence of the definition; the general case follows inductively from the fact that any simple periodic orbit of f of period $m \geq 2$ splits into two simple periodic orbits of f^2 of period $m/2$.

2.2. *Multimodal and stunted sawtooth maps.* Consider a continuous map $f : I \to I$ where $I = [c_0, c_{d+1}]$. For $d \geq 0$, assume there are points c_i, $0 < i < d + 1$, with $c_j < c_{j+1}$ for $0 \leq j \leq d$ such that f is monotone on each *lap* $[c_j, c_{j+1}]$, and not monotone on any segment of the form $[c_j, c_{j+2}]$. Such a map is then called *d-modal* or *multimodal* with *modality* d (one says *amodal* if $d = 0$, *unimodal* if $d = 1$, and then *bimodal*, and so on). The maximal interval $[a_i, b_i]$ $(i \in 1, \ldots, d)$ containing c_i on which f is constant is called a *turning interval* and, more precisely, a *plateau* if $a_i < b_i$, and a *turning point* if $a_i = b_i$. The map f being given, one may prefer to choose the points c_i such that, for all $i \in 1, \ldots, d$, $c_i = (a_i + b_i)/2$.

The *shape* of a d-modal map is the alternating sequence of $d + 1$ signs, starting with either $+1$ or -1 according as the map is increasing or decreasing on its initial lap. The i-ordered collection of signs

$$\operatorname{sign}(f^n(c_i) - c_j) \in \{-1, 0, 1\}$$

for $n > 0$ and $1 \leq i, j \leq d$ with j fixed is the jth *kneading sequence* of f. The j-ordered collection of kneading sequences is the *kneading invariant* of f. By the *kneading data* associated with a d-modal map f we mean its shape together with its kneading invariant (for more on kneading theory, we refer to [MiT], [BORT] and [MiTr]). One might wish to first understand at the symbolic level some questions one formulates for polynomials or smooth maps. It is in fact more practical to consider continuous families of d-modal maps rich enough to exhibit all possible kneading data for d-modal maps, yet significantly easier to be studied than smooth maps. Such families exist, and we next recall the construction of one of them.

By the *sawtooth map* of shape $s_1 \ldots s_{d+1}$, $s_i \in \{+1, -1\}$, $s_{i+1} = -s_i$, $d \geq 1$, we mean the unique map $S_d : I \to I$ which is piecewise linear with slopes $s_1(d+1), s_2(d+1), \ldots, s_{d+1}(d+1)$ ($d+1$ alternate values). This is a d-modal map with topological entropy $\log(d+1)$, the largest possible value for d-modal maps.

Given any *critical value vector* $w = (w_1, \ldots, w_d)$ satisfying

$$(w_j - w_{j+1}) \cdot s_{j+1} < 0, \quad w_j \in I,\ j = 1, \ldots, d-1,$$

we obtain the *stunted sawtooth map* S_w from S_d as follows: on the pair of intervals of monotonicity of S_d separated by c_j, for $j = 1, \ldots, d$, we set $S_w = \min(S_d, w_j)$ if c_j is a maximum, and $S_w = \max(S_d, w_j)$ if c_j is a minimum. In particular, every turning point of S_w is also a turning point of S_d, and S_w and S_d coincide at such points.

Remark 4. The endpoints of I are periodic or pre-periodic points of S_d and of S_w, hence all turning points of S_d and of S_w are pre-periodic.

The d-parameter family of stunted sawtooth maps S_w is complete in the following sense:

Theorem E ([DGMT], [MiTr]). *For any d-modal map f there is a canonical d-modal stunted sawtooth map S_w which has exactly the same kneading data as f.*

2.3. *The first main result*

Theorem 1. *Suppose S_w is a stunted sawtooth map with $P(S_w) = \{2^i : i \in \mathbb{N}\}$. Then for any $\varepsilon > 0$, there exist w' and w'' with $|w' - w| < \varepsilon$ and $|w'' - w| < \varepsilon$ such that $h(S_{w'}) > 0$ and $S_{w''}$ has only finitely many periods.*

Corollary 1. *The set $\{w : P(S_w) = \{2^i : i \in \mathbb{N}\}\}$ has no interior point.*

Remark 5. Let f_i, $i \in \{1, 2, 3\}$, be three d-modal maps of same shape with kneading invariants

$$K_i = \{K_{i,1}, \ldots, K_{i,d}\}.$$

Assume that $P(f_2) = \{2^n : n \in \mathbb{N}\}$ and that, with the usual order on kneading sequences (see, e.g., [BORT]),

$$K_{1,j} < K_{2,j} < K_{3,j} \quad \text{if } s_j = +,$$
$$K_{1,j} > K_{2,j} > K_{3,j} \quad \text{if } s_j = -,$$

for $1 \leq j \leq d$. The proof of Theorem 1 implies that f_1 has only finitely many periods and that f_3 has positive topological entropy: this follows from Theorem E and the fact that any permutation allowed by some kneading data is realized by some periodic orbit of any stunted sawtooth map with the same kneading data (such matter is discussed, e.g., in [MiTr]).

2.4. *Renormalization.* Another purpose of this paper is to prove that, under some mild smoothness assumptions, the maps f with $P(f) = \{2^i : i \in \mathbb{N}\}$ are infinitely renormalizable. More precisely, we prove here the infinite renormalizability property needed for the description of the boundary of chaos in [Hu] and the definitions are formulated accordingly.

Let I be an interval. A map $f : I \to I$ is called *renormalizable* if there exists a proper subinterval J of I and an integer p such that

(1) $f^i(J)$, $i = 0, 1, \ldots, p-1$, have no pairwise interior intersection,
(2) $f^p(J) = J$.

Then $f^p|_J : J \to J$ is called a *renormalization* of f. A map $f : I \to I$ is *weakly infinitely renormalizable* if there exists an infinite sequence $\{I_n\}_{n=1}^{\infty}$ of nested intervals and a strictly increasing sequence $\{u(n)\}_{n=1}^{\infty}$ of integers such that $f^{u(n)}|_{I_n} : I_n \to I_n$ are renormalizations of f. A map $f : I \to I$ is *infinitely renormalizable* if it is weakly infinitely renormalizable and the maximal diameter of the intervals I_n, $f(I_n), \ldots, f^{u(n)-1}(I_n)$ tends to zero as $n \to \infty$.

Chapter VI of [BC] contains a discussion which seems close to what we need to prove, in the general case of continuous maps. With that much generality the diameter condition is not necessarily true. Furthermore, the techniques of Block and Coppel allow one to impose $u(n) = 2^n$ but the equality of (2) is replaced by

(2′) $f^p(J) \supset J$.

To the contrary, the methods in [Su] and [Hu] require

(2″) $f^p(J) \subseteq J$,

and that the diameters of the intervals I_n, $f(I_n), \ldots, f^{u(n)-1}(I_n)$ tend to zero as $n \to \infty$.

The referee of the first version of the present paper suggested that multimodal maps with $P(f) = \{2^i : i \in \mathbb{N}\}$ are weakly infinitely renormalizable with $u(n) = 2^n$ in the above notations; during the Spring of 1997, we became aware that this result appears in Theorem 3.5 of [Sm]. Knowing this result would have spared us from proving Lemma 2 and Corollary 3. Discussions with several colleagues seem to prove that Smítal's result was unfortunately overlooked by many of us: it was called to our attention by Jean Marc Gambaudo after one of his students gave a very elementary proof of it [So] (Smítal, like us, uses the full strength of [M2]). Our methods show the weak infinitely renormalizable property and the diameter condition simultaneously but only allow us to have $u(n) = 2^{n+k}$ for some $k \geq 0$. This is all that is needed in [Su] or [Hu]. Nevertheless, it is worth mentioning that, combined with the result of [Sm] (see also [So]), our Theorems 2, 3, and 4 also remain true with the stronger specification that the sequence $\{u(n)\}_{n=1}^{\infty}$ is taken as $u(n) = 2^n$.

Let $f \in C^0(I)$. An open interval $J \subset I$ is called a *wandering interval* of f if

1. $f^n(J) \cap f^m(J) = \emptyset$ for any $n \neq m$, $n, m \in \mathbb{N}$, and
2. $f^n(J)$ does not converge to a periodic orbit.

We shall get the results we seek for smooth maps as corollaries of the following abstract result:

THEOREM 2. *Assume the multimodal map $f : I \to I$ with $P(f) = \{2^i : i \in \mathbb{N}\}$ has no wandering intervals and no plateaus. Then f is infinitely renormalizable.*

Let $f : I \to I$ be a map which is not constant on any open set. We say that f belongs to $\Gamma(2)$ if

(a) f is C^2 away from the critical points;

(b) for every critical point x_0 of f there exists $\alpha > 1$, a neighborhood $U(x_0)$ of x_0 and a C^2-diffeomorphism $\phi : U(x_0) \to (-1, 1)$ such that $\phi(x_0) = 0$ and

$$f(x) = f(x_0) \pm |\phi(x)|^\alpha, \quad \forall x \in U(x_0).$$

Notice that any map in $\Gamma(2)$ is necessarily multimodal.

THEOREM F ([MMS]). *Any $f \in \Gamma(2)$ has no wandering interval.*

Combining this result with Theorem 2 yields:

THEOREM 3. *Any map $f \in \Gamma(2)$ with $P(f) = \{2^i : i \in \mathbb{N}\}$ is infinitely renormalizable.*

In particular, we also have:

THEOREM 4. *Any real polynomial map f with $P(f) = \{2^i : i \in \mathbb{N}\}$ is infinitely renormalizable.*

REMARK 6. Using results from [HS], the smoothness condition in Theorem F can be relaxed, which allows a proof of Theorem 3 with relaxed smoothness condition for multimodal maps.

Using more language from kneading theory, one could formulate a conjecture corresponding to Theorems 2 to 4 for general renormalizations (not just those at the boundary of chaos). Such a generalization completely escapes the methods of the present paper.

3. Proof of Theorem 1. For any $f \in C^0(I)$ with $P(f) = \{2^i : i \in \mathbb{N}\}$, we let $\Delta_j(f)$ be the set of accumulation points of the periodic points of f with period greater than or equal to 2^j, and set $\Delta(f) = \bigcap_{j=0}^{\infty} \Delta_j(f)$. Clearly, $\Delta(f)$ is not an empty set. Furthermore, we have:

LEMMA 1. *Let $f : I \to I$ be a multimodal map with $P(f) = \{2^i : i \in \mathbb{N}\}$. Then no point in $\Delta(f)$ is periodic and hence $\Delta(f)$ is not a finite set.*

Proof. Let $p \in \Delta(f)$ be a periodic point of period 2^n. Define $g = f^{2^n}$. Then p is a fixed point of g. Look at the map g near p. Because f has isolated turning intervals, so does g. Hence there are only three types of local behaviors for g near p:

1. g is monotone in a small neighborhood of p; if g is monotone reversing then g^2 is monotone preserving in a small neighborhood of p and we rename g the map g^2.
2. p is in the interior or at the end of one of the plateaus of g.
3. p is a turning point of g.

Let $x \neq p$ be close enough to p. Either $g(x) = p$ or $g(x)$ belongs to the open interval bounded by x and $g^2(x)$. From Theorem D and Remark 3 we conclude that x cannot belong to a periodic orbit with period greater than 2^n. ∎

Remark 7. Lemma 1 is false in $C^0(I)$: examples with $\Delta(f)$ reduced to a point are easily provided if f is allowed to have infinitely many turning points.

Let $\Sigma = \{0,1\}^{\mathbb{N}}$ and let σ stand for the *adding machine*, i.e., the map $\sigma : \Sigma \to \Sigma$ defined by $\sigma(x_i)_{i=0}^{\infty} = (y_i)_{i=0}^{\infty}$, where $y_i = 1 - x_i$ if $x_j = 1$ for all $j < i$ and $y_i = x_i$ otherwise. The following result is proved in [M2].

Theorem G ([M2]). *Let $f \in C^0(I)$ be a continuous map with $P(f) = \{2^i : i \in \mathbb{N}\}$. Suppose that K is an infinite closed invariant set of f which supports an ergodic f-invariant non-atomic probability measure. Then there exists a continuous map $h : K \to \Sigma$ such that*

$$h \circ f = \sigma \circ h.$$

Furthermore, $h^{-1}(s)$ contains at most two points for any point $s \in \Sigma$.

Remark 8. The following information about K can be found in the proof of Theorem G in [M2]. The set K can be expressed as a disjoint union $K = K_0 \cup K_1$, where the supporting intervals $[K_0]$ and $[K_1]$ of K_0 and K_1 are disjoint and $f(K_i) = K_{1-i}$, where $i = 0, 1$. Hence K_0 and K_1 are invariant under f^2. Furthermore, K_0 and K_1 have the same bisections under f^2 and so on. Thus one can express K as the disjoint union

$$K = \bigcup_{j=1}^{2^n} K_j^{(n)}$$

for $n \in \mathbb{N}$. Therefore at least one point of each *fiber* $h^{-1}(s)$, $s \in \Sigma$, is recurrent.

Remark 9. Fom the proof of Theorem G in [M2], one also finds out that there exists a fixed point p of f in the gap between $[K_0]$ and $[K_1]$.

Remark 10. It follows from Lemma 1 and Theorem G that, under the assumptions of Lemma 1, there exists a set $K \subset \Delta(S_w)$ as in Theorem G.

LEMMA 2. *Suppose that S_w is a stunted sawtooth map with $P(S_w) = \{2^i : i \in \mathbb{N}\}$. Let $E_0(S_w)$ be the set consisting of the endpoints of the plateaus of S_w, $E_1(S_w)$ be the set consisting of the endpoints of I and the turning points of S_w, and put $E(S_w) = E_0(S_w) \cup E_1(S_w)$. Then $\Delta(S_w) \cap E(S_w) = \Delta(S_w) \cap E_0(S_w) \neq \emptyset$.*

Proof. Since the endpoints of I are periodic or pre-periodic and all turning points of S_w (if any) are pre-periodic (see Remark 4), by Lemma 1 they are not in $\Delta(S_w)$. Thus $\Delta(S_w) \cap E_1(S_w) = \emptyset$ so that it only remains to prove that $\Delta(S_w) \cap E(S_w) \neq \emptyset$. Since the topological entropy of S_d is $\log(d+1) > 0$, we must have $S_w \neq S_d$ and therefore $E_0(S_w)$ is not empty.

Suppose that the intersection of $\Delta(S_w)$ and $E(S_w)$ is empty and let $\varepsilon > 0$ stand for the distance $d(\Delta(S_w), E(S_w))$. Let $K \subset \Delta(S_w)$ be as in Theorem G (see also Remark 10). Clearly, $d(K, E(S_w)) > \varepsilon$. By Remark 8, we can choose a point $x \in K$ which is recurrent. By Theorem G, there exists $l = 2^k$, $k \geq 1$, such that $|f^l(x) - x| < \frac{1}{8}\varepsilon$. Again by Theorem G and Remark 8, one can choose this point x so that $f^l(x) > x$. Let V be the largest neighborhood of x on which f^l is monotone. The slope of f^l on V is d^l and clearly $V \supset (x - \varepsilon/d^l, x + \varepsilon/d^l)$.

Assume first that f^l is orientation preserving on V. Since $f^l(x) > x$ and $f^l\big(x - \frac{7}{8}\varepsilon/d^l\big) = f^l(x) - \frac{7}{8}\varepsilon < x + \frac{1}{8}\varepsilon - \frac{7}{8}\varepsilon = x - \frac{3}{4}\varepsilon < x - \varepsilon/d^l$, there exists a point $p \in \big(x - \frac{7}{8}\varepsilon/d^l, x\big)$ such that $f^l(p) = p$. The unstable manifold $W^u(p, f^l)$ of f^l at p clearly contains $(p, x + \varepsilon/d^l)$ and $(x - \varepsilon/d^l, p)$, hence also $(p, f^l(x) + \varepsilon)$ and $(f^l(x) - \varepsilon, p)$. Let W be the largest neighborhood of $f^l(x)$ on which f^l is monotone. Then $W^u(p, f^l) \supset W \supset (f^l(x) - \varepsilon/d^l, f^l(x) + \varepsilon/d^l)$. It follows easily from Theorem G that $f^l(f^l(x)) < f^l(x)$ (this is the limiting version of Remark 3), which implies that neither x nor p belongs to W. It is plain that one of $f^l(f^l(x) - \varepsilon/d^l)$ and $f^l(f^l(x) + \varepsilon/d^l)$ is equal to $f^l(f^l(x)) - \varepsilon < f^l(x) - \varepsilon < x + \frac{1}{8}\varepsilon - \varepsilon = x - \frac{7}{8}\varepsilon < p$. Since $f^l(x) > x > p$, this implies that there exists $y \in (x, f^l(x) + \varepsilon/d^l) \subset W^u(p, f^l)$ such that $y \neq p$ and $f^l(y) = p$. This means that y is a homoclinic point, a contradiction. If f^l is orientation reversing on V, one proceeds similarly. ∎

Proof of Theorem 1. For any $\varepsilon > 0$, we push up a little bit all the concave down plateaus and push down a little bit all the concave up plateaus to get another stunted sawtooth map $S_{w'}$ with $|w - w'| < \varepsilon$ (in fact, one can only move the plateaus which have at least one of their endpoints in $E(S_w) \cap \Delta(S_w)$). We will show that $S_{w'}$ has positive topological entropy. Since every periodic orbit of S_w which has no point in the interior of any plateau of S_w is also a periodic orbit of $S_{w'}$, we have $\Delta(S_w) \subset \Delta(S_{w'})$. Clearly, $\Delta(S_w) \cap E(S_{w'}) = \emptyset$. By the proof of Lemma 2, $S_{w'}$ has a homoclinic point, and hence, by Theorem C, it has positive topological entropy.

We next push down a little bit all concave down plateaus and push up a little bit all concave up plateaus to get another stunted sawtooth map $S_{w''}$ with $|w-w''|<\varepsilon$. Suppose that $P(S_{w''})=\{2^i : i\in\mathbb{N}\}$. We can then use the previous argument comparing S_w to $S_{w'}$ to compare now $S_{w''}$ to S_w, and conclude that S_w has positive topological entropy, a contradiction. ∎

4. Proof of Theorem 2

LEMMA 3. *Suppose $f\in C^0(I)$ with $P(f)=\{2^i : i\in\mathbb{N}\}$ and assume there exists a set K as in Theorem* G. *Then with the notations of Remark* 8 *and for each $j\in\{0,1\}$, there exists a periodic point q of f with period $m\in\{1,2\}$ in the gap between $[K_0]$ and $[K_1]$ such that $W^u(q,f^2)$ contains $[K_j]$.*

Proof. By Remark 9, there exists a fixed point p of f in the gap between $[K_0]$ and $[K_1]$. Assume for instance that K_1 is to the right of K_0. For $j=1$, let $s=\inf\{x : x\in K_1\}$ and consider the map f^2 on the interval $[p,s]$. The point p is also a fixed point of f^2. Let q be the largest fixed point of f^2 in $[p,s]$. Because of Lemma 1, $q\neq s$ and then $f^2(y)>y$ for any $y\in(q,s)$. Clearly, $f^2(K_1)\subset K_1$ and $f^2(s)>s$. Therefore s is in $W^u(q,f^2)$ and the same holds true for $[K_1]$ by connectivity. The case $j=0$ is treated similarly. ∎

From Theorem G and Lemma 3, one has

COROLLARY 2. *Suppose $f\in C^0(I)$ with $P(f)=\{2^i : i\in\mathbb{N}\}$ and assume there exists a set K as in Theorem* G, *with $K=\bigcup_{j=1}^{2^n}K_j^{(m)}$, $n\in\mathbb{N}$. Then for any $n\in\mathbb{N}$ there exists a periodic point p of period m, where $m\in\{2^n,2^{n+1}\}$, whose orbit is contained in the set $\bigcup_{j=1}^{2^n}[K_j^{(n)}]\setminus\bigcup_{j=1}^{2^{n+1}}[K_j^{(n+1)}]$ and such that $W^u(p,f^m)$ contains some $K_j^{(n+1)}$, where $1\le j\le 2^{n+1}$.*

LEMMA 4. *Assume a multimodal map $f:I\to I$ with $P(f)=\{2^i : i\in\mathbb{N}\}$ has no wandering interval and no plateau. Let K be as in Theorem* G. *Then the semi-conjugacy h in Theorem* G *is actually a conjugacy.*

Proof. Suppose that h is not a conjugacy. Then by Theorem G there exists a point $s\in\Sigma$ such that $h^{-1}(s)=\{x,y\}$, $x\neq y$. We claim that $h^{-1}(\sigma^n(s))$ contains a single point when n is large enough. Let I_n denote the supporting interval of $h^{-1}(\sigma^n(s))$, $n\ge 0$. Since there are only finitely many turning points and the intervals I_n, $n\ge0$, are pairwise disjoint, there exists $m>0$ such that I_n contains no turning points for any $n\ge m$. If the claim is false, then I_m is a wandering interval, a contradiction. Therefore one can assume that $f(x)=f(y)$, where $x,y\in K$ and $x\neq y$. We separate our considerations into three cases.

CASE 1. Suppose that f is orientation preserving in some neighborhoods of x and y (the proof is similar when f is orientation reversing near both x and y). By Corollary 2, there exists a periodic point p of f of period $n = 2^k$, $k > 0$, such that the unstable manifold $W^u(p, f^n)$ of f^n at p contains the interval $[x, y]$, p is not in $[x, y]$ and p is close enough to x or y. By continuity of f, when p is close enough to x (or q), then $f(p)$ is very close to $f(x) = f(y)$. By the intermediate value theorem, there are at least two points $a, b \in (x, y)$ such that $f(a) = f(b) = f(p)$. Then $f^n(a) = f^n(p) = p$. Hence a is a homoclinic point, a contradiction.

CASE 2. Suppose that f is orientation preserving in a neighborhood of x and orientation reversing in a neighborhood of y (the proof is similar when the situation is reversed). Define $x_t = f^t(x) = f^t(y)$, $t \in \mathbb{N}$. Let $\{c_i\}_{i\in\alpha}$ denote the set of turning points of f, where α is a finite set. Clearly there exists $t_0 > 0$ such that $K_{(t,i)} := (x_t, c_i) \cap K \neq \emptyset$ for any $t \geq t_0$. In the interval (x, y) we select v and w near x and y respectively with $f(v) = f(w)$. Because of the nonexistence of wandering intervals, the itineraries of $f(p) = f(q)$ and $f(v) = f(w)$ under f are eventually different. Let $v_t = f^t(v)$, $t \in \mathbb{N}$. Then there exists $t > t_0$ such that there is at least one $c_i \in (x_t, v_t)$. Clearly, $K_{(t,i)} \subset (x_t, v_t)$ and $f^t((x, v)) = f^t((w, y)) \supset (x_t, v_t)$. From Theorem G and Corollary 2, there exists a periodic point p of period $n = 2^k$, $k > 0$, where p is not in $[x, y]$, such that p is very near x (or y), $f^j(p)$ is in the supporting interval $[K_{(t,i)}]$ and the unstable manifold $W^u(p, f^n) \supset [x, y]$. Clearly there exists a point $u \in (x, v)$ (or $u \in (w, y)$) such that $f^t(u) = f^j(p)$. Hence $u \in W^u(p, f^n)$, $u \neq p$ and $f^{nt}(u) = f^n(f^j(p)) = p$. So u is a homoclinic point, a contradiction.

CASE 3. If at least one of x and y is a turning point, the proof is just a slight modification of that in Case 1 or Case 2. ■

Proof of Theorem 2. By Remark 10, we know there exists a set K as in Theorem G. By Lemma 4, $f : K \to K$ is conjugate to $\sigma : \Sigma \to \Sigma$ by h. Separate the turning points of f into two parts. Let S_1 (resp. S_2) denote the turning points of f contained (resp. not contained) in K. Define $K = \bigcup_{j=1}^{2^n} K_j^{(n)}$, $n \in \mathbb{N}$. We claim that there exists $n_0 \in \mathbb{N}$ such that for any $n > n_0$, $S_2 \cap \bigcup_{j=1}^{2^n} [K_j^{(n)}] = \emptyset$. Suppose to the contrary that, for any $x \in S_2$, there exists n_0 such that for any $n > n_0$, x is in $\bigcup_{j=1}^{2^n} [K_j^{(n)}]$. Then there exists an infinite sequence of nested intervals $[K_{k(n)}^{(n)}]$ containing x. Hence $x \in \bigcap_{n\in\mathbb{N}} [K_{k(n)}^{(n)}]$. Since the endpoints of $\bigcap_{n\in\mathbb{N}} [K_{k(n)}^{(n)}]$ are in the same fiber of h, which has to be a point, x is in K, a contradiction. Now let $n > n_0$. Then each critical point contained in some $[K_i^{(n)}]$ is indeed in K. Since both endpoints of any $[K_j^{(n)}]$ are also in K and f restricted to K is a homeomorphism,

f maps each $[K_j^{(n)}]$ onto another $[K_k^{(n)}]$. The remaining thing is to show that the maximal diameter $D^{(n)}$ of $[K_j^{(n)}]$ with $j \in \{1, \ldots, 2^n\}$ goes to zero when $n \to \infty$, and therefore f is infinitely renormalizable. Notice that $\{D^{(n)}\}_{n=1}^{\infty}$ is a decreasing sequence. Let $\lim_{n\to\infty} D^{(n)} = D$. Clearly, $D \geq 0$. Let us call $[K_j^{(n)}]$, $j \in \{1, \ldots, 2^n\}$, the nth level intervals. Suppose that $D > 0$. Then for each n, there exists at least one nth level interval with diameter $\geq D$, and only those nth level intervals with diameter $\geq D$ can contain deeper level intervals with diameter $\geq D$. Therefore one can have an infinite sequence of nested intervals $[K_j^{(n)}]$ with diameter $\geq D$, and hence $\bigcap_{n=1}^{\infty}[K_j^{(n)}]$ has diameter $\geq D$. This contradicts the fact that the intersection is a point (Lemma 4). ∎

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