FUNDAMENTA
MATHEMATICAE
155 (1998)

Self-homeomorphisms of the 2-sphere which fix pointwise a nonseparating continuum

by

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Abstract. We prove that the space of orientation preserving homeomorphisms of the 2-sphere which fix pointwise a nontrivial nonseparating continuum is a contractible absolute neighborhood retract homeomorphic to the separable Hilbert space l_2.

1. Introduction. Mason [10] proved that the space of self-homeomorphisms of the closed unit disk which fix pointwise the disk's boundary is an absolute retract homeomorphic to the separable Hilbert space l_2. In this paper we prove the following generalization:

THEOREM 1.1. *If F is a nondegenerate nonseparating proper subcontinuum of the 2-sphere S^2 and if H denotes the group of orientation preserving homeomorphisms of S^2 which pointwise fix F, then H is an absolute retract homeomorphic to the separable Hilbert space l_2.*

For a closed subset F of an n-manifold M^n let $H(M^n,F)$ and $H(M^n,F,\mathrm{id}_F)$ denote respectively the self-homeomorphisms of M^n which leave F invariant and which leave F pointwise fixed. Let $H_0(M^n,F)$ denote the path component of id in $H(M^n,F)$.

Three natural questions are:

1. Under what conditions is $H(M^n,F), H(M^n,F,\mathrm{id}_F)$, or $H_0(M^n,F)$ an ANR?
2. What is the homotopy type (or homeomorphism type) of $H(M^n,F,\mathrm{id}_F)$, or $H_0(M^n,F)$?
3. What is the mapping class group of M^n relative to F?

When $n=2$ it has long been known that $H(M^2,\emptyset)$ is an ANR [9], and the homeomorphism type [6] and mapping class group [8] of $H(M^2,\emptyset)$ are

1991 *Mathematics Subject Classification*: Primary 58D05, 57S25; Secondary 58B05, 57S05.

understood when M^2 is closed. Other than cases in which F is a finite point set or the finite union of disjoint simple closed curves [12], there has been little attempt to address these questions when $F \neq \emptyset$. In this paper we consider the case where $M^2 = S^2$ and $F \subset S^2$ is a nondegenerate nonseparating continuum and we prove $H(M^n, F, \mathrm{id}_F)$ is an absolute retract.

The paper is divided into 9 sections. We begin in Section 2 with some definitions. The main result, Theorem 1.1, is proved in Section 3. The proof depends on Theorems 3.1, 3.2, and 3.3, which are proved in Sections 7, 8, and 9 respectively. Section 4 consists of lemmas and remarks useful elsewhere in the paper. In Section 5 we examine the behavior of members of $H(M^n, F, \mathrm{id}_F)$ near F. In Section 6 we construct a well behaved contraction of a closed PL disk D and use it to gain control of the associated Alexander isotopy.

I am grateful for the help of Alec Norton and Jim West.

2. Definitions. Let $F \subset S^2$ denote a nondegenerate nonseparating continuum, let $U = S^2 \setminus F$, and let $\partial U = \overline{U} \setminus U$. By an *$l_2$-manifold* we mean a space which is locally homeomorphic to l_2, the Hilbert space of square summable sequences. The space Y is said to *dominate* the space X if there exist maps $\phi : X \to Y$ and $\psi : Y \to X$ such that the map $\psi\phi : X \to X$ is homotopic to the identity map $\mathrm{id} : X \to X$. A homotopy $h_t : X \to X$ is said to be a *deformation* if $h_0 = \mathrm{id}$. If $\{x, y\} \subset X$ then a *path* from x to y is the image of a map $\alpha : [0, 1] \to X$ such that $\alpha(0) = x$ and $\alpha(1) = y$. By an *arc* we mean any space homeomorphic to $[0, 1]$. A *disk* is any space homeomorphic to the planar closed unit disk.

We consider S^2 to be the union of two Euclidean 2-simplices σ_1 and σ_2 attached along their common boundary as follows. Let T be a planar equilateral triangle subdivided into four equiltateral triangles. We define a quotient map $q : T \to S^2$ by folding each of the three corner triangles onto the center triangle. We identify those triples of points (one point from each of the three corner triangles) which touch after folding. Thus σ_2 is the center triangle and σ_1 is the image of each of the three corner triangles under q. Furthermore, q induces a metric on S^2 as follows: $d(x, y) = \min\{|w - z| \mid q(w) = x \text{ and } q(z) = y\}$. Thus both σ_1 and σ_2 inherit the Euclidean geometry of a 2-simplex.

Definition 2.1. If $X \subset S^2$ or $\mathbb{R}^2$ then H_X denotes the self-homeomorphisms of S^2 (resp. $\mathbb{R}^2$) which fix the complement of X pointwise. If d_X is a bounded metric on X we endow H_X with the metric $d_{H_X}(h, g) = \sup_{x \in X} d_X(h(x), g(x))$.

Definition 2.2. Suppose $Y \subset S^2$ is connected and locally path connected with metric d_Y. We define a metric $d^*_Y : Y \times Y \to \mathbb{R}^+$ as follows:

$$d_Y^*(x,y) = \inf_C \{\mathrm{diam}(C) \mid C \text{ is a path in } Y \text{ from } x \text{ to } y\}.$$

We let Y^* denote the set Y together with the metric d_Y^*.

Let G denote the space of continuous endomorphisms of H_U. We endow G with the compact-open topology. If $D \subset \overline{U}$ is a closed disk such that $\mathrm{int}(D) \subset U$ and $\partial U \cap \partial D$ has at least two points, then by $\mathrm{mesh}(D)$ we mean $\sup\{\mathrm{diam}(\alpha) \mid \alpha$ is a component of $\partial D \cap U\}$. If $\alpha \subset \mathbb{R}^2$ is a closed arc and $x \neq y$ and $x, y \in \alpha$, then let α_{xy} denote the closed subarc of α with endpoints $\{x, y\}$. We let ANR and AR denote *absolute neighborhood retract* and *absolute retract* respectively.

3. Proof of Theorem 1.1

Outline. Let H_U denote the orientation preserving homeomorphisms of S^2 which are supported on $\overline{U}$ where $U = S^2 \setminus F$. Hanner [5], p. 405, has proved the following. Suppose $X_1 \subset X_2 \subset \ldots \subset X$ and for each n, X_n is an AR. Then if $\Psi : X \times [0,1] \to X$ is a deformation such that $\Psi_t(X) \subset X_n$ for $t \le 1/n$ then X is also an AR. Dobrowolski and Toruńczyk [3] have shown that every completely metrizable non-locally-compact separable ANR which admits a group structure is an l_2-manifold. D. Henderson [7] has shown that l_2 manifolds are determined by their homotopy type. Consequently, it suffices to prove that H_U is contractible and dominated by a sequence of ARs in the fashion described above in order to conclude that H_U is homeomorphic to l_2.

We construct a contraction Ψ_s of H_U as follows. We express $\overline{U}$ as the nested union of a sequence of closed disks P_n. With the aid of a conformal map $\psi : U \to \mathrm{int}(D^2)$ we then construct maps $\Phi_n : H_U \to H_{P_n}$ converging to ID_{H_U}. Finally, we string together Alexander-like isotopies between H_{P_n} and $H_{P_{n+1}}$ taking great care that the process is well behaved as $n \to \infty$. The difficulty is that in general ∂U is not locally connected and consequently there is no small homeomorphism from ∂P_n onto ∂P_{n+1}.

The following theorems are essential to the proof of Theorem 1.1.

THEOREM 3.1. *Suppose $V \subset S^2$ and $\phi : U^* \to V$ is a homeomorphism. Then ϕ is uniformly continuous if and only if for each $h \in H_U$ the homeomorphism $g = \phi h \phi^{-1} : V \to V$ can be extended to a homeomorphism $\overline{g} \in H_V$.*

THEOREM 3.2. *There exists a sequence of closed disks $D_1 \subset D_2 \subset \ldots$ together with a uniformly equicontinuous sequence $\phi_n : U^* \hookrightarrow U$ of embeddings such that $\phi_n \to \mathrm{id}$ uniformly on compact sets and such that $\mathrm{im}(\phi_n) = \mathrm{int}(D_n)$.*

THEOREM 3.3. *Suppose $\phi_n : U^* \hookrightarrow U$ is a uniformly equicontinuous sequence of embeddings such that $\phi_n \to \mathrm{id}$ uniformly on compact sets. Then $\Phi_n \in G$ and $\Phi_n \to \mathrm{ID} \in G$ where $\Phi_n : H_U \to H_U$ is defined as follows*:

$$\Phi_n(h)(x) = \begin{cases} \phi_n h \phi_n^{-1}(x) & \text{if } x \in \mathrm{im}(\phi_n), \\ x & \text{otherwise.} \end{cases}$$

Proof of Main Theorem (The space H_U is homeomorphic to l_2). We first show that H_U is contractible. Without loss of generality we may assume that $\mathrm{int}(\sigma_2) \subset U$ and that the vertices of σ_2 belong to ∂U. Construct as in Theorem 3.2 a sequence of closed disks $D_1 \subset D_2 \subset \ldots$ together with a sequence $\psi_n : U^* \hookrightarrow U$ of uniformly equicontinuous embeddings such that $\psi_n \to id$ uniformly on compact sets and such that $\mathrm{im}(\psi_n) = \mathrm{int}(D_n)$ and $\sigma_2 \subset D_1$. Choose a sequence of closed PL disks P_n and homeomorphisms $h_n : D_n \to P_n$ such that $\sigma_2 \subset P_1$, $P_n \subset D_n$, $P_n \subset P_{n+1}$, and $|h_n(x) - x| < 1/n$. Let $\phi_n = h_n(\psi_n)$. Let G denote the space of continuous endomorphisms of H_U. By Theorem 3.3 the sequence $\Phi_n : H_U \to H_U$ defined by

$$\Phi_n(h)(x) = \begin{cases} \phi_n h \phi_n^{-1}(x) & \text{if } x \in \mathrm{im}(\phi_n), \\ x & \text{otherwise,} \end{cases}$$

satisfies $\Phi_n \in G$ and $\Phi_n \to \mathrm{ID} \in G$ in the compact-open topology.

We observe that $\mathrm{im}(\Phi_{n-1}) \subset H_{P^*_{n-1}} \subset H_{P^*_n}$. It follows from Lemma 4.3 and Corollary 6.6 that we may construct a homotopy $F_{n,t} : H_{P^*_n} \times H_{P^*_n} \to H_{P^*_n}$ satisfying $F_{n,0}(f,g) = f$ for $f \in H_{P^*_n}$, $F_{n,1}(f,g) = g$ for $g \in H_{P^*_n}$, and $d_{H_{P^*_n}}(F_{n,t}(f,g),g) \le d_{H_{P^*_n}}(f,g)$ for $(f,g) \in H_{P^*_n} \times H_{P^*_n}$. For $n \in \{2,3,\ldots\}$ and $t \in [0,1]$ define a function $\Psi_{n+t} : H_U \to H_U$ by the rule $\Psi_{n+t}(h) = F_{n,t}(\Phi_{n-1}(h), \Phi_n(h))$. We observe that

$$\begin{aligned} \Psi_{n+1}(h) &= F_{n,1}(\Phi_{n-1}(h), \Phi_n(h)) = \Phi_n(h) \\ &= F_{n,0}(\Phi_n(h), \Phi_{n+1}(h)) = \Psi_{(n+1)+0}(h). \end{aligned}$$

Thus Ψ_{n+t} is well defined and $\Psi_{n+1} = \Phi_n$. Consequently, $\Psi_n \to \mathrm{ID}$ in the compact-open topology. Continuity of Φ_{n-1}, Φ_n, and $F_{n,t}$ ensures that Ψ_{n+t} is continuous. Furthermore, Ψ_s varies continuously with s since the homotopy $F_{n,t}$ varies continuously with t. We observe that for $h \in H_U$ we have by definition

$$d_{H_U}(\Psi_{n+t}(h), \Psi_{n+1}(h)) \le d_{H_{U^*}}(\Psi_{n+t}(h), \Psi_{n+1}(h)).$$

By Lemma 4.2,

$$\begin{aligned} d_{H_{U^*}}(\Psi_{n+t}(h), \Psi_{n+1}(h)) &\le d_{H_{P^*_n}}(\Psi_{n+t}(h), \Psi_{n+1}(h)) \\ &= d_{H_{P^*_n}}(F_{n,t}(\Phi_{n-1}(h), \Phi_n(h)), \Phi_n(h) \\ &\le d_{H_{P^*_n}}(\Phi_{n-1}(h), \Phi_n(h)). \end{aligned}$$

By Lemma 6.1,

$$d_{H_{P_n^*}}(\Phi_{n-1}(h),\Phi_n(h)) \le 4d_{H_{P_n}}(\Phi_{n-1}(h),\Phi_n(h)) = 4d_{H_{P_n}}(\Psi_n(h),\Psi_{n+1}(h)) = 4d_{H_U}(\Psi_n(h),\Psi_{n+1}(h)).$$

Combining these inequalities yields

$$d_{H_U}(\Psi_{n+t}(h),\Psi_{n+1}(h)) \le 4d_{H_U}(\Psi_n(h),\Psi_{n+1}(h)) \quad \text{for all } n,t,h.$$

This establishes that $\Psi_{s_n} \to \mathrm{ID}$ whenever $s_n \to \infty$.

We have thus constructed a deformation Ψ_s of H_U (parameterized by $[2,\infty]$) into $H_{P_1^*}$ such that $\Psi_s(H_U) \subset H_{P_n^*}$ if $s \ge n+1$. By the title theorem of Mason [10], $H_{P_n^*}$ is an AR. Thus H_U is dominated by a sequence of ARs in the sense of Theorem 7.2, p. 405 of Hanner [5], and therefore H_U is an AR. Also H_U admits a complete metric since it is a closed subspace of the full homeomorphism group of the compact space S^2, and the homeomorphism group of any compactum admits a complete metric (p. 25 of [13]). Hence H_U is an l_2-manifold by the title theorem of [3] since H_U is a completely metrizable non-locally-compact separable metric space admitting a group structure. Thus H_U is a contractible l_2-manifold, and (Corollary 3, p. 759 of [7]) l_2-manifolds are determined up to homeomorphism by their homotopy type. Therefore H_U is homeomorphic to l_2. ■

4. Lemmas and remarks

REMARK 4.1. If $\phi_n : G \to H$ is a sequence of continuous homomorphisms between metrizable topological groups then ϕ_n converges in the compact-open topology if and only if ϕ_n converges uniformly on each sequence $g_k \to \mathrm{id} \in G$.

LEMMA 4.1. *If $\phi : U^* \to U$ is uniformly continuous then $\phi : U^* \to U^*$ is uniformly continuous.*

Proof. Suppose $\varepsilon > 0$. Choose $\delta > 0$ so that for all $x, y \in U$ if $d^*(x,y) < \delta$ then $d(\phi(x),\phi(y)) < \varepsilon$. Suppose $d^*(x,y) < \delta$. Choose a path $\alpha \subset U$ connecting x and y such that $\mathrm{diam}(\alpha) < \delta$. Suppose $\phi(w),\phi(z) \in \phi(\alpha)$. Because $d^*(w,z) < \delta$ it follows that $d(\phi(w),\phi(z)) < \varepsilon$. Hence $\mathrm{diam}(\phi(\alpha)) < \varepsilon$. Furthermore, $\phi(\alpha)$ is a path which connects $\phi(x)$ and $\phi(y)$. This shows $d^*(\phi(x),\phi(y)) < \varepsilon$. ■

REMARK 4.2. For a fixed map $g \in H_X$ the self-map of H_X defined by the rule $h \mapsto h(g)$ is an isometry.

REMARK 4.3. It is shown in [4] that orientation preserving homeomorphisms of S^2 are isotopic to id. If $h : S^2 \to S^2$ is an orientation preserving homeomorphism which fixes $\{w,y,z\}$ pointwise then there exists an isotopy between h and id which fixes $\{w,y,z\}$ at all times.

LEMMA 4.2. *Suppose $D \subset S^2$ is a closed disk such that* $\mathrm{int}(D) \subset U$. *Then* ID : $H_{D^*} \hookrightarrow H_{U^*}$ *is a contraction mapping in the sense that* $d_{H_{U^*}}(f,g) \le d_{H_{D^*}}(f,g)$ *for* $f,g \in H_{D^*}$.

Because $\mathrm{int}(D) \subset U$ it follows that $H_{D^*} \subset H_{U^*}$ and thus ID is well defined. Suppose $f,g \in H_{D^*}$. Suppose $x \in \mathrm{int}(D)$ and $\varepsilon > 0$. Choose a path $\alpha \subset D$ such that $\mathrm{diam}(\alpha) < d^*_D(f(x),g(x)) + \varepsilon/2$. Choose a path $\beta \subset \mathrm{int}(D)$ such that $\mathrm{diam}(\beta) < \mathrm{diam}(\alpha) + \varepsilon/2$. Thus $d^*_U(f(x),g(x)) \le \mathrm{diam}(\beta) < d^*_D(f(x),g(x)) + \varepsilon$. Hence $d_{H_{U^*}}(f,g) \le d_{H_{D^*}}(f,g)$ for $f,g \in H_{D^*}$.

LEMMA 4.3. *Suppose $D \subset S^2$ is a closed disk such that σ_2 is inscribed in D (i.e. $\sigma_2 \subset D$ and the vertices of σ_2 belong to ∂D). Then D^* can be canonically isometrically embedded in the plane.*

Proof. Recall from our definition of S^2 the quotient map $q : T \to S^2$. We observe that $D = \sigma_2 \cup D_1 \cup D_2 \cup D_3$ where $D_i \subset \sigma_1$ $(i = 1,2,3)$ are the three 2-cells attached to σ_2 along the edges of σ_2. Hence we can "unfold" D along each edge of σ_2. In other words, if σ_2 is inscribed in D then there exists a unique isometric lifting $s : D^* \hookrightarrow T$ such that $q(s) = \mathrm{id}_D$. ■

5. Behavior near ∂U. We examine the behavior of a convergent sequence $\{h_n\} \subset H_U$ near ∂U. We prove in Lemma 5.1 that if $x \in U$ is near ∂U then x and $h_n(x)$ can be connected by a uniformly short arc in U.

LEMMA 5.1. *Suppose h_n is a convergent sequence in H_U. Then for each $\varepsilon > 0$ there exists a compact set $D \subset U^*$ such that $d^*_U(x,h_n(x)) < \varepsilon$ for all $n \in \mathbb{Z}^+$ and $x \in U^* \setminus D$.*

Proof. Because S^2 is compact it suffices to prove the lemma for any metric which generates the usual topology on S^2. For convenience we consider S^2 to be the unit sphere in $\mathbb{R}^3$. Suppose $1 > \varepsilon > 0$. By uniform equicontinuity of h_n choose $\delta < \varepsilon/2$ such that if $d(x,y) < 2\delta$ then $|h_n(x) - h_n(y)| < \varepsilon/2$. Choose $\gamma < \delta$ such that if $|x-y| < \gamma$ then $|h_n(x) - h_n(y)| < \delta$. Let $D = \overline{U} \setminus \bigcup_{x \in \partial U} B(x,\gamma)$. Suppose $x \in U \setminus D$ and $n \in \mathbb{Z}^+$. There exists $y \in \partial U$ with $|x-y| < \gamma$. So $|h_n(x) - h_n(y)| = |h_n(x) - y| < \delta$ and hence

$$|h_n(x) - x| \le |h_n(x) - y| + |y - x| < \delta + \gamma < 2\delta.$$

Choose a simple closed curve $\alpha \subset \mathbb{R}^2$ such that $\mathrm{diam}(\alpha) < 2\delta$ and $\{x, h_n(x)\} \subset \alpha$. If $\alpha \cap F = \emptyset$ or if $\alpha \cap F = \{y\}$ then α contains an arc in U connecting x and y and establishes in these cases that $d^*_U(x,h_n(x)) \le \mathrm{diam}(\alpha) < 2\delta < \varepsilon$.

If $\alpha \cap F$ contains at least two points then let β denote the closure of the component of $U \cap \alpha$ which contains x. Let $\{w,z\}$ denote the endpoints of β. Thus $\{w,z\} = F \cap \beta$. Let C denote the curve $\beta \cup h_n(\beta)$. We observe that $\mathrm{diam}(h_n(\beta)) < \varepsilon/2$. Consequently, $\mathrm{diam}(C) = \mathrm{diam}(\beta \cup h_n(\beta)) < 2\delta + \varepsilon/2 = \varepsilon$. If $\beta \cap h_n(\beta) \ne \{w,z\}$ then there is a point $y \in \beta \cap h_n(\beta) \cap U$.

Hence $C \setminus \{w,z\}$ is path connected in U and it follows in this case that $d^*_U(x,h_n(x)) < \varepsilon$.

Finally, suppose that $\beta \cap h_n(\beta) = \{w,z\}$. In this case C is a simple closed curve. Let D denote the closed disk bounded by C. We will show that h_n has no fixed points on $\operatorname{int}(D)$. Suppose in order to obtain a contradiction that h_n has a fixed point y in the interior of D. Let ω denote the space of paths from w to z inside the space $\mathbb{R}^2 \setminus y$. Because C is a simple closed curve, β and $h_n(\beta)$ belong to different homotopy classes in ω. On the other hand, h_n is orientation preserving and hence by Remark 4.3, h_n is isotopic to the identity via an isotopy which leaves w, z, and y fixed at all times. This shows that β and $h_n(\beta)$ belong to the same homotopy class in ω and we have a contradiction. Therefore h_n must be fixed point free on $\operatorname{int}(D)$. Hence $D \setminus \{w,z\}$ is path connected in U and $\{x, h(x)\} \subset D \setminus \{w,z\}$. Furthermore, $\operatorname{diam}(D \setminus \{w,z\}) = \operatorname{diam}(C)$ since the diameter of a small closed disk in S^2 is achieved on its boundary. This establishes that $d^*_U(x,h_n(x)) < \varepsilon$. ∎

COROLLARY 5.2. *If $h_n \in H_U$ and $h_n \to \mathrm{id}$ then $h_n \to \mathrm{id}$ uniformly in H_{U^*}.*

Proof. Suppose $\varepsilon > 0$. By Theorem 5.1 choose a compact set $D \subset U$ such that $d^*_U(x,h_n(x)) < \varepsilon$ for all $n \in \mathbb{Z}^+$ and $x \in U \setminus D$. Choose $\delta < \varepsilon$ such that $\bigcup_{x\in D} B(x,\delta) \subset U$. Choose $N \in \mathbb{Z}^+$ such that if $n \geq N$ then $d(h_n(x),x) < \delta$. Suppose $x \in U^*$ and $n \geq N$. If $x \notin D$ then $d^*_U(x,h_n(x)) < \varepsilon$. If $x \in D$ then $B(x,\delta) \subset U$. Thus $d^*_U(h_n(x),x) = d(h_n(x),x) < \delta < \varepsilon$. ∎

LEMMA 5.3. *Suppose $\phi_n : U^* \hookrightarrow U$ is a uniformly equicontinuous sequence of embeddings. Suppose $h_k \in H_U$ and $h_k \to \mathrm{id}$ uniformly. Then the doubly indexed sequence $\phi_n h_k \phi_n^{-1}$ enjoys the following convergence property: for each $\varepsilon > 0$ there is $N \in \mathbb{Z}^+$ such that if $k, n \geq N$ and $x \in \operatorname{im}(\phi_n)$ then $d(\phi_n h_k \phi_n^{-1}(x), x) < \varepsilon$.*

Proof. Suppose $\varepsilon > 0$. By uniform equicontinuity of $\{\phi_n\}$ choose $\delta > 0$ so that if $d^*(x,y) < \delta$ then $d(\phi_n(x),\phi_n(y)) < \varepsilon$ for all n. By Corollary 5.2 choose $N \in \mathbb{Z}^+$ such that $k \geq N \Rightarrow d^*(x,h_k(x)) < \delta$. Suppose that $k \geq N$ and $x \in \operatorname{im}(\phi_n)$. Let $x = \phi_n(y)$. Thus $d(\phi_n h_k \phi_n^{-1}(x), x) = d(\phi_n h_k(y), \phi_n(y)) < \varepsilon$. ∎

6. The geometry of planar PL disks. Suppose $D \subset \mathbb{R}^2$ is a closed PL disk. We will construct a contraction π_t of D which monotonically shrinks the diameter of each path in D. This is achieved by triangulating D and almost collapsing successive 2-simplices. The contraction π_t is not conjugate to the radial contraction since the orbits are not injective. However, π_t is 1-1 for each $t \in [0,1)$, and the "Alexander isotopy" determined by π_t is well behaved in a sense which does not depend on D. This enables us to canonically connect pairs of points h and g in H_{D^*} by a path whose diameter is controlled only by the distance between h and g.

Lemma 6.1. *If $h \in H_{D^*}$ then $d_{H_{D^*}}(h, \mathrm{id}) \leq 4 d_{H_D}(h, \mathrm{id})$.*

Proof. Suppose $|h(x) - x| < \varepsilon$ for $x \in D$. Suppose $x \in D$. If the straight line segment $[x, h(x)] \subset D$ then $d^*(x, h(x)) = |h(x) - x| < \varepsilon < 4\varepsilon$. Otherwise let $z \in [x, h(x)] \cap \partial D$. We observe that $[x, z] \cup h([z, x])$ is a path in D connecting x and $h(x)$. Suppose $v, w \in [x, z]$. Then $|h(v) - h(w)| \leq |h(v) - v| + |v - w| + |w - h(w)|$. Thus $\mathrm{diam}(h([x, z])) \leq 3\varepsilon$. Hence $\mathrm{diam}([x, z] \cup h([z, x])) \leq 4\varepsilon$. This shows $d_{H_{D^*}}(h, \mathrm{id}) \leq 4 d_{H_D}(h, \mathrm{id})$. ∎

Definition 6.1. Suppose $D, E \subset \mathbb{R}^2$ are closed PL disks. We endow D and E with the respective metrics d^*_D and d^*_E. Suppose $E \subset D$. By a *careful deformation* of D into E we mean a homotopy $H : D \times [0, 1] \to D$ such that

1. $H(x, 0) = x\ \forall x \in D$. ($H_0 = \mathrm{id}$)
2. $\forall t \in [0, 1]$, $\forall x, y \in D$, $H(x, t) = H(y, t)$ iff $x = y$. (H_t is one-to-one)
3. $H(x, 1) \in E\ \forall x \in D$. ($\mathrm{im}(H_1) \subset E$)
4. $d^*_E(x, y) = d^*_D(x, y)\ \forall x, y \in E$. (minimal paths in E are also minimal in D)
5. $d^*_D(H(x, t), H(y, t)) \leq d^*_D(x, y)\ \forall x, y \in D$, $\forall t \in [0, 1]$. (x and y are never further apart than their initial positions)

Lemma 6.2. *If P is a convex PL disk with a side c such that the interior angles of P are acute at the endpoints of c then for each $\varepsilon > 0$ there exists a careful deformation of P into a convex PL disk W such that $c \subset W$ and the angles in W at each endpoint of c are less than ε. We call such a deformation a* fundamental move of W towards c.

Proof. Embed P in the plane isometrically so that $c \subset x$-axis. For ε sufficiently small and $t \in [0, 1 - \varepsilon]$, the deformation is realized by the linear maps determined by the matrices $\begin{bmatrix} 1 & 0 \\ 0 & 1-t \end{bmatrix}$ acting on P. ∎

Lemma 6.3. *Suppose T is a 2-simplex with vertices x, y, and z. Suppose W_{xz} and W_{yz} are convex PL disks such that $W_{xz} \cap T = [x, z]$, $W_{yz} \cap T = [y, z]$, and $W_{xz} \cap W_{yz} = z$. Suppose furthermore that the interior angles in W_{xz} and W_{yz} are acute at the endpoints of $[x, z]$ and $[y, z]$ respectively. Let $P = T \cup W_{xz} \cup W_{yz}$. Then there exists a careful deformation of P into T such that $H_1(P)$ is convex, $[x, y] \subset H_1(P)$ and the interior angles of $H_1(P)$ are acute at x and y.*

Proof. If the angles of T at x and y are both acute then apply fundamental moves of W_{xz} and W_{yz} respectively towards $[x, z]$ and $[y, z]$ as in Lemma 6.2 until the resulting PL disk is convex with acute angles at x and y. Now apply a fundamental move towards $[x, y]$ until the resulting body is inside T. Suppose on the other hand that the angle of T at x is nonacute.

See Figure 1. Choose $w \in [y, z]$ such that that $|x - z| = |w - z|$. Hence the triangle $[x, w, z]$ has acute angles at x and w, and the triangle $[x, w, y]$ has acute angles at x and y.

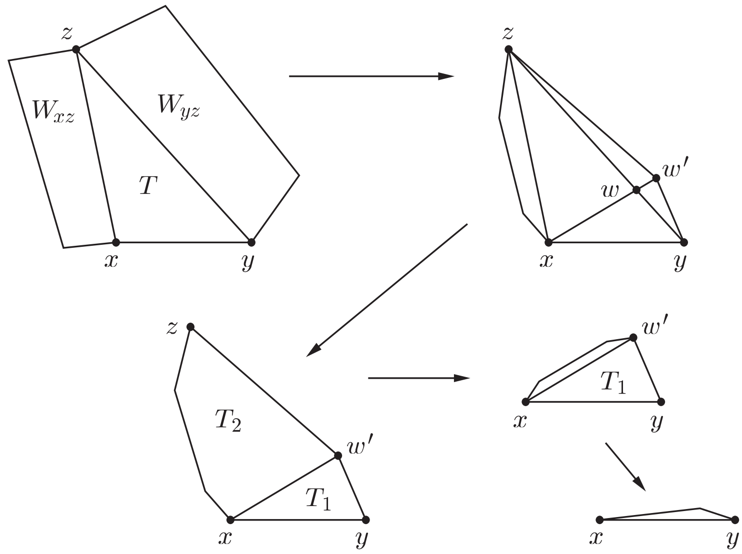

Fig. 1. Carefully deforming $T \cup W_{xz} \cup W_{yz}$ into T

Apply fundamental moves of W_{xz} and W_{yz} respectively towards $[x, z]$ and $[y, z]$ as in Lemma 6.2 until the resulting PL disk is the union of two acute convex disks T_1 and T_2 joined along a common side $[x, w']$ such that $[x, w] \subset [x, w']$, and such that the interior angles of T_1 are acute at x and y and the interior angles of T_2 are acute at x and w'. Now apply a fundamental move of T_2 towards $[x, w']$ until the resulting disk is convex and has convex angles at both x and y. Apply another fundamental move towards $[x, y]$ until the resulting body is contained in T. ■

LEMMA 6.4. *Suppose $D \subset \mathbb{R}^2$ is a closed PL disk. There exists a contraction $\pi_t : D^* \to D^*$ such that $\pi_{t_{[0,1-\varepsilon]}}$ is a careful deformation for each $\varepsilon > 0$.*

Proof. Triangulate D with 2-simplices $T_1, \ldots, T_n$ such that T_i has two free edges in the PL disk $\bigcup_{k=1}^{i} T_k$. For the existence of such a triangulation see p. 23 of Bing [1]. Let $c_i = T_i \cap \bigcup_{k=1}^{i-1} T_k$. Starting with $i = n$ and working backwards towards $i = 1$, carefully deform T_i and the attached convex 2-cells towards c_i as in Lemma 6.3. After $n-1$ moves we are left with a starlike disk inside of which T_1 is inscribed. Perform fundamental moves on the convex cells attached to the edges of T_1 until the resulting disk is convex. Now contract radially to a point. ■

COROLLARY 6.5. *There exists a contraction $\Pi_t : H_{D^*} \to H_{D^*}$ such that for all $t \in [0,1]$ and all $g \in H_{D^*}$, $d_{H_{D^*}}(\Pi_t(g), \mathrm{id}) \le d_{H_{D^*}}(g, \mathrm{id})$.*

Proof. Let $\pi_t : D^* \to D^*$ be a contraction as in Lemma 6.4. Define $\Pi_t : H_{D^*} \to H_{D^*}$ as follows:

$$\Pi_t(g)(x) = \begin{cases} \pi_t g \pi_t^{-1}(x) & \text{if } x \in \mathrm{im}(\pi_t), \\ x & \text{otherwise.} \end{cases}$$

It follows from Theorem 3.1 that Π_t is well defined. Continuity of π_t ensures continuity of Π_t. If $x \notin \mathrm{im}(\pi_t)$ then $d^*_D(\Pi_t(g)(x), \mathrm{id}(x)) = d^*_D(x,x) = 0$. If $x \in \mathrm{im}(\pi_t)$ then $x = \pi_t(y)$. Thus

$$d^*_D(\Pi_t(g)(x), \mathrm{id}(x)) = d^*_D(\pi_t g \pi_t^{-1}(x), x) = d^*_D(\pi_t g(y), \pi_t(y)).$$

But by Lemma 6.4,

$$d^*_D(\pi_t g(y), \pi_t(y)) \le d^*_D(g(y), y) \le d_{H_{D^*}}(g, \mathrm{id}).$$

Thus $d_{H_{D^*}}(\Pi_t(g), \mathrm{id}) \le d_{H_{D^*}}(g, \mathrm{id})$. ∎

COROLLARY 6.6. *There exists a homotopy $F_t : H_{D^*} \times H_{D^*} \to H_{D^*}$ satisfying*

1. $F_0(f,g) = f \ \forall f \in H_{D^*}$.
2. $F_1(f,g) = g \ \forall g \in H_{D^*}$.
3. $d_{H_{D^*}}(F_t(f,g), g) \le d_{H_{D^*}}(f,g) \ \forall (f,g) \in H_{D^*} \times H_{D^*}$.

Proof. Let $F_t(f,g) = \Pi_t(fg^{-1})g$ where $\Pi_t : H_{D^*} \to H_{D^*}$ is constructed as in Corollary 6.5. Thus

$$\begin{aligned} d_{H_{D^*}}(F_t(f,g), g) &= d_{H_{D^*}}(\Pi_t(fg^{-1})g, g) = d_{H_{D^*}}(\Pi_t(fg^{-1})g, g) \\ &= d_{H_{D^*}}(\Pi_t(fg^{-1}), \mathrm{id}) \end{aligned}$$

(by Remark 4.2). But $d_{H_{D^*}}(\Pi_t(fg^{-1}), \mathrm{id}) \le d_{H_{D^*}}(fg^{-1}, \mathrm{id}) = d_{H_{D^*}}(f,g)$ (by Corollary 6.5). Thus $d_{H_{D^*}}(F_t(f,g), g) \le d_{H_{D^*}}(f,g)$. ∎

7. Proof of Theorem 3.1. $\Rightarrow$ It suffices to check continuity of $\overline{g}$ at ∂V. Suppose $x_n \to x$ where $x \in \partial V$ and $x_n \in V$. Let $y_n = \phi^{-1}(x_n)$. We will first observe that $\lim_{n\to\infty} d(y_n, \partial U) = 0$. Otherwise for some $\varepsilon > 0$ and for some subsequence $\{z_n\}$ of $\{y_n\}$ we would have $d(z_n, \partial U) \ge \varepsilon$. Hence by compactness of $\overline{U}$, $\{z_k\}$ has a convergent subsequence $\{w_n\}$ such that $\lim_{n\to\infty} w_n \in U$. It follows that $\lim_{n\to\infty} \phi(w_n) \in V$. This contradicts $\lim_{n\to\infty} \phi(w_n) = x \in \partial V$, and establishes that $\lim_{n\to\infty} d(y_n, \partial U) = 0$.

We will next observe that $\lim_{n\to\infty} d^*(y_n, h(y_n)) = 0$. Suppose, in order to obtain a contradiction, that $d^*(z_n, h(z_n) \ge \gamma$ for some $\gamma > 0$ and some subsequence $\{z_n\}$ of $\{y_n\}$. Because $\overline{U}$ is compact, $\{z_n\}$ has a convergent subsequence $\{w_n\}$. Let $w = \lim_{n\to\infty} w_n$. We have $w \in \partial U$ since $\lim_{n\to\infty} d(y_n, \partial U) = 0$. Hence by Lemma 5.1, $\lim_{n\to\infty} d^*(w_n, h(w_n)) = 0$.

This contradicts $d^*(w_n, h(w_n) \geq \gamma$ and establishes that $\lim_{n\to\infty} d^*(y_n, h(y_n)) = 0$.

Suppose $\varepsilon > 0$. By uniform continuity of ϕ choose $\delta > 0$ such that if $\{w, z\} \subset U^*$ and $d^*(w, z) < \delta$ then $d(\phi(w), \phi(z)) < \varepsilon/2$. Choose $N \in \mathbb{Z}^+$ such that if $n \geq N$ then $d^*(y_n, h(y_n)) < \delta$ and $d(x_n, x) < \varepsilon/2$. Suppose $n \geq N$. Then

$$\begin{aligned} d(\overline{g}(x), \overline{g}(x_n)) = d(x, g(x_n)) &\leq d(x, x_n) + d(x_n, g(x_n)) \\ &= d(x, x_n) + d(x_n, \phi h \phi^{-1}(x_n)). \end{aligned}$$

Furthermore,

$$d(x_n, (\phi h \phi)^{-1}(x_n)) = d(\phi\phi^{-1}(x_n), \phi h \phi^{-1}(x_n)) = d(\phi(y_n), \phi h(y_n)) < \varepsilon/2.$$

Thus $d(\overline{g}(x), \overline{g}(x_n)) < \varepsilon/2 + \varepsilon/2$.

$\Leftarrow$ Suppose that ϕ is not uniformly continuous. We will construct a homeomorphism $h \in H_U$ such that $\phi h \phi^{-1}$ cannot be extended to a homeomorphism in H_V. Choose $\varepsilon > 0$ together with a sequence $\delta_n \to 0$ and points $x_n, y_n \in U^*$ such that $d^*(x_n, y_n) < \delta_n$ but $d(\phi(x_n), \phi(y_n)) \geq \varepsilon$. Because $\overline{U}$ is compact $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\}$ which converges in $\overline{U}$ to some point $x \in \partial U$. (Otherwise if $x \in U$ it would follow from continuity of ϕ over U that $d(\phi(x_{n_k}), \phi(y_{n_k})) \to 0$.)

Let α_k be a closed arc in U^* which connects x_{n_k} and y_{n_k} such that $\operatorname{diam}(\alpha_k) \to 0$. Let α_{k_j} be a subsequence of α_k of disjoint closed arcs. In order to avoid so many indices we may assume to begin with that we have obtained a sequence of disjoint closed arcs α_n in U with endpoints x_n and y_n such that $\operatorname{diam}(\alpha_n) \to 0$ and $d(\phi(x_n), \phi(y_n)) \geq \varepsilon$. Choose a sequence of disjoint closed disks D_n such that $\alpha_n \subset \operatorname{int}(D_n) \subset U$ and $\operatorname{diam}(D_n) \to 0$. Choose a sequence of closed disks E_n such that $x_n \in E_n \subset \operatorname{int}(D_n)$ and $\operatorname{diam}(\phi(E_n)) \to 0$. Let $z_n \in E_n$ such that $z_n \neq x_n$. Let $h_n : D_n \to D_n$ be a homeomorphism fixing ∂D_n pointwise such that $h_n(x_n) = x_n$ and $h_n(z_n) = y_n$. Define $h : \overline{U} \to \overline{U}$ as follows:

$$h(x) = \begin{cases} h_n(x) & \text{if } x \in D_n, \\ x & \text{otherwise.} \end{cases}$$

Let $w_n = \phi(x_n)$ and let $v_n = \phi(z_n)$.

We know that $d(w_n, v_n) \to 0$ since $\operatorname{diam}(\phi(E_n)) \to 0$. On the other hand, $d(\phi h \phi^{-1}(w_n), \phi h \phi^{-1}(v_n)) = d(\phi h(x_n), \phi h(z_n)) = d(\phi(x_n), \phi(y_n)) \geq \varepsilon$. Thus $\phi h \phi^{-1}$ is not uniformly continuous over V and consequently does not admit a continuous extension to $\overline{V}$. ∎

8. Proof of Theorem 3.2. Construct a sequence of closed disks $D_1 \subset D_2 \subset \ldots \subset \overline{U}$ such that $\operatorname{int}(D_n) \subset U$, $U = \bigcup_{n\in\mathbb{Z}^+} \operatorname{int}(D_n)$, $D_n \cap \partial U$ is finite and contains at least three points, and $\operatorname{mesh}(D_n) < 1/n$.

Let $E_n \subset D_n$ be a closed disk such that $\partial E_n \cap \partial D_n = D_n \cap \partial U$ and such that each component of $D_n \setminus E_n$ has diameter less than $1/n$. Let $\psi : U^* \to \operatorname{int}(D^2)$ be conformal. It is well known ([2], p. 634) that if h is a self-homeomorphism of $\overline{U}$ such that $h(U) = U$, then $\psi^{-1} h \psi$ extends to a self-homeomorphism of D^2. In particular, by Theorem 3.1, ψ is uniformly continuous. It is also known ([11], p. 29) that ψ admits a continuous extension $\overline{\psi} : U \cup D_n$ such that $\psi_{|D_n}$ is one-to-one. Let $K_n = \overline{\psi}(E_n)$. Extend $(\overline{\psi}^{-1})_{|K_n}$ to a homeomorphism $\kappa_n : D^2 \to D_n$. Let $\phi_n = \kappa_n \psi$. Thus ϕ_n is one-to-one and uniformly continuous since it is the composition of uniformly continuous one-to-one functions.

To see that $\{\phi_n\}$ is uniformly equicontinuous suppose $\varepsilon > 0$. Choose $N > 3/\varepsilon$. Choose $\delta < \varepsilon/3$ so that if $d^*(x, y) < \delta$ and $n < N$ then $d(\phi_n(x), \phi_n(y)) < \varepsilon$. Suppose $n \ge N$ and $d^*(x, y) < \delta$. Choose a closed arc $\alpha \subset U$ connecting x and y such that $\operatorname{diam}(\alpha) < \delta$.

If $\alpha \cap E_n = \emptyset$ then α is contained in some component of $U \setminus E_n$. It follows that $\phi_n(\alpha)$ is contained in some component of $D_n \setminus E_n$ since $\phi_n(E_n) = E_n$. Thus $d(\phi_n(x), \phi_n(y)) < \operatorname{diam}(\phi_n(\alpha)) \le 1/n \le 1/N < \varepsilon/3 < \varepsilon$.

Finally, suppose that $\alpha \cap E_n \ne \emptyset$. Let w and z be the first and last points respectively on $E_n \cap \alpha$ starting from x. Since $w, z \in E_n$ we have $d(\phi_n(w), \phi_n(z)) = d(w, z) \le \operatorname{diam}(\alpha) < \delta < \varepsilon/3$. If $x \ne w$ then $\operatorname{int}(\alpha_{xw})$ is contained in some component of $U \setminus E_n$. Hence

$$d(\phi_n(x), \phi_n(w)) \le \operatorname{diam}(\phi_n(\alpha_{xw})) \le 1/n < \varepsilon/3.$$

Similarly it follows that $d(\phi_n(y), \phi_n(z)) < \varepsilon/3$. Thus by the triangle inequality we have

$$d(\phi_n(x), \phi_n(y)) \le d(\phi_n(x), \phi_n(w)) + d(\phi_n(w), \phi_n(z)) + d(\phi_n(z), \phi_n(y)) < \varepsilon.$$

Suppose $C \subset U$ is compact and $\varepsilon > 0$. Choose $N > 1/\varepsilon$ so that if $n \ge N$ then $C \subset D_n$. Suppose $n \ge N$ and $x \in C$. If $x \in E_n$ then $\phi_n(x) = x$. If $x \in D_n \setminus E_n$ then $|\phi_n(x) - x| < 1/n < \varepsilon$ since each component of $D_n \setminus E_n$ maps into itself under ϕ_n and has diameter less than $1/n$. This shows that $\phi_n \to \mathrm{id}$ uniformly on compact sets.

9. Proof of Theorem 3.3. It follows from Theorem 3.1 that Φ_n is well defined. It is immediate from the definition that Φ_n is a homomorphism. Thus we need only check that each Φ_n is a continuous function, and that $\Phi_n \to \mathrm{ID}$ uniformly on compact sets. To verify that Φ_n is a continuous function it suffices to check that Φ_n is continuous at $\mathrm{id} \in H_U$ since Φ_n is a homomorphism between topological groups. Suppose $\varepsilon > 0$, $n \in \mathbb{Z}^+$ and $h_n \to \mathrm{id}$. By uniform continuity of ϕ_n choose $\delta > 0$ such that $d_U^*(a, b) < \delta \Rightarrow d(\phi_n(a), \phi_n(b)) < \varepsilon$. By Corollary 5.2 choose $N \in \mathbb{Z}^+$ such that if $n \ge N$ then $d_U^*(y, h_n(y)) < \delta$ for all $y \in U^*$. If $x \notin \operatorname{im}(\phi_n)$ then

$d(\Phi_n(h_n(x)), \Phi_n(\mathrm{id}(x))) = d(x,x) = 0 < \varepsilon$. Suppose $x \in \mathrm{im}(\phi_n)$ and $n \geq N$. Let $x = \phi_n(y)$. We observe that

$$d(\Phi_n(h_n(x)), \Phi_n(\mathrm{id}(x))) = d(\phi_n h_n \phi_n^{-1}(x), x) = d(\phi_n h_n(y), \phi_n(y)) < \varepsilon.$$

Thus Φ_n is a continuous function for all $n \in \mathbb{Z}^+$.

Now we show $\Phi_n \to \mathrm{ID}$ pointwise. Suppose $h \in H_U$ and $\varepsilon > 0$. By uniform continuity of h and uniform equicontinuity of $\{\phi_n\}$ choose $\delta < \varepsilon/2$ such that if $d(y,x) < \delta$ then $d(h(x), h(y)) < \varepsilon/2$ and such that if $d_U^*(x,y) < \delta$ then $d(\phi_n(x), \phi_n(y)) < \varepsilon/2$. By Lemma 5.1 choose a compact set $D \subset U$ such that $d_U^*(x, h(x)) < \delta$ for $x \in U \setminus D$. Choose a compact set E such that $D \cup h(D) \subset \mathrm{int}(E) \subset E \subset U$. By uniform continuity of $\{\phi_n\}$ on $E \cup h(E)$ choose $N \in \mathbb{Z}^+$ such that if $n \geq N$ then $\phi_n^{-1}(D) \subset E \subset \mathrm{im}(\phi_n)$ and $d(\phi_n(x), x) < \delta$ for all $x \in E \cup h(E)$. Suppose $N \geq n$ and $x \in \overline{U}$. If $x \notin \mathrm{im}(\phi_n)$ then $x \notin D$. Hence

$$d(\Phi_n h(x), \mathrm{ID}\, h(x)) = d(x, h(x)) \leq d_U^*(x, h(x)) < \varepsilon/2 < \varepsilon.$$

Suppose $x \in \mathrm{im}(\phi_n)$. Let $\phi_n(y) = x$. Then

$$d(\Phi_n h(x), \mathrm{ID}\, h(x)) = d(\phi_n h \phi_n^{-1}(x), h(x)) = d(\phi_n h(y), h\phi_n(y)).$$

If $y \in E$ then $h(y) \in E \cup h(E)$ and $d(y, \phi_n(y)) < \delta$. Thus

$$d(\phi_n h(y), h\phi_n(y)) \leq d(\phi_n h(y), h(y)) + d(h(y), h\phi_n(y)) < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

If $y \notin E$ then $y \notin D$, and hence $d_U^*(x, h(x)) < \delta$. Furthermore, $\phi_n(y) \notin D$ since $\phi_n^{-1}(D) \subset E$. Thus

$$d(\phi_n h(y), h\phi_n(y)) \leq d(\phi_n h(y), \phi_n(y)) + d(\phi_n(y), h\phi_n(y)) < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

This establishes that $\Phi_n \to \mathrm{ID}$ pointwise. We now show that $\Phi_n \to \mathrm{ID}$ uniformly on compact sets. By Remark 4.1 it suffices to show Φ_n converges uniformly on each sequence $h_k \to \mathrm{id} \in H_U$. Suppose $\varepsilon > 0$. By uniform convergence of $\{h_k\}$ choose $K \in \mathbb{Z}^+$ so that if $k \geq K$ then $d(h_k(x), x) < \varepsilon$ for all $x \in \overline{U}$. By Lemma 5.3 choose $M \geq K$ such that if $k, n \geq M$ and $x \in \mathrm{im}(\phi_n)$ then $d(\phi_n h_k \phi_n^{-1}(x), x) < \varepsilon$. By pointwise convergence of $\{\Phi_n\}$ on $\{h_1, \ldots, h_{M-1}\}$ choose $N \geq M$ such that if $n \geq N$ and $k < M$ then $d(\Phi_n h_k(x), \mathrm{ID}\, h(x)) < \varepsilon$ for all $x \in \overline{U}$. Suppose $n \geq N$ and $x \in \overline{U}$. If $k < M$ then $d(\Phi_n h_k(x), \mathrm{ID}\, h(x)) < \varepsilon$. If $k \geq M$ and $x \in \mathrm{im}(\phi_n)$ then $d(\Phi_n h_k(x), \mathrm{ID}\, h(x)) = d(\phi_n h_k \phi_n^{-1}(x), x) < \varepsilon$. If $k \geq M$ and $x \notin \mathrm{im}(\phi_n)$ then $d(\Phi_n h_k(x), \mathrm{ID}\, h_k(x)) = d(x, h_k(x)) < \varepsilon$. Thus $\Phi_n \to \mathrm{ID}$ uniformly on compact sets. ∎

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Received 6 November 1995;
in revised form 12 November 1996