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Porosity of Collet–Eckmann Julia sets

by

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Abstract. We prove that the Julia set of a rational map of the Riemann sphere satisfying the Collet–Eckmann condition and having no parabolic periodic point is mean porous, if it is not the whole sphere. It follows that the Minkowski dimension of the Julia set is less than 2.

1. Introduction. Let $f:\widehat{\mathbb{C}}\to\widehat{\mathbb{C}}$ be a rational map. Then f is said to satisfy the *Collet–Eckmann condition* if there are constants $C>0$ and $\lambda>1$ such that

$$|(f^n)'(f(c))|\ge C\lambda^n \tag{CE}$$

for all n and all critical points $c\in J(f)$ of f whose forward orbit does not meet another critical point ($J(f)$ stands for the Julia set of f). Here and in what follows derivatives and distances are always with respect to the spherical metric of $\widehat{\mathbb{C}}$, unless stated otherwise.

A set $E\subset\widehat{\mathbb{C}}$ is called *mean porous* if there are constants $p_1<\infty$ and $p_2>0$ such that for each $z\in E$ the following holds: There is an increasing sequence n_j of integers and points z_j with $\operatorname{dist}(z,z_j)\le 2^{-n_j}$ such that $n_j<p_1 j$ and $\operatorname{dist}(z_j,E)>p_2 2^{-n_j}$. Roughly speaking, the set of scales in which E^{c} contains a disc of size comparable to the scale has a density uniformly bounded away from zero.

Theorem 1.1. *If f satisfies the Collet–Eckmann condition, has no parabolic periodic point and $J(f)\ne\widehat{\mathbb{C}}$, then $J(f)$ is mean porous.*

In [KR] it was proved that mean porous sets on the sphere have Minkowski dimension (other names: box dimension, limit capacity) less than 2 (see also Section 4). As an immediate consequence we obtain

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Corollary 1.2. *Under the assumptions of Theorem* 1.1, *the Minkowski dimension of* $J(f)$ *is less than* 2.

Maps satisfying (CE) were first considered by Collet and Eckmann in [CE]. Benedicks and Carleson showed in [BC, Theorem 1] that the set of (real) parameters c for which z^2+c satisfies (CE) is of positive measure. Nowicki and the first author showed in [NP] Hölder conjugacy with a tent map for Collet–Eckmann interval maps and conjectured that the basin of ∞ is a Hölder domain (for complex quadratic maps with (CE)). A new tool to control distortion of components of preimages of discs, *shrinking neighborhoods* in the terminology of Graczyk and Smirnov, was introduced by the first author in [P1] (see also [PUZ, p. 198]). There it was used to establish (for f as in Theorem 1.1, say) existence of invariant measures absolutely continuous with respect to conformal measures. It was also shown that the Hausdorff dimension, Minkowski dimension and hyperbolic dimension of the Julia set coincide. In [P2], a question of Bishop and Lyubich was answered negatively by showing that the Hausdorff dimension of maps as in Theorem 1.1 is less than 2, provided some additional condition (M. Tsujii condition) holds. Shrinking neighborhoods were used to go from small scale to large scale for $f(c)$, c critical. Other types of counterexamples come from Graczyk's work on real quadratic Fibonacci polynomials [G] and from McMullen's work [McM].

In the summer of 1995 the first author had a discussion with M. Lyubich who suggested trying to estimate the dimension of the Julia set of Collet–Eckmann maps directly (without going through conformal measure as in [P2]), as going from large scale to small scale should "pull back a hole" (i.e. a disc contained in J^{c} on large scale) to small scales. It was realized that some notion of porosity could be involved.

Koskela and the second author introduced in [KR] the notion of mean porosity, showed that the boundary of a Hölder domain is mean porous and proved (based on work of Jones and Makarov [JM] and of Smith and Stegenga [SS]) that the Minkowski dimension of mean porous sets is less than 2.

Graczyk and Smirnov proved in [GS] that the components of the Fatou set of Collet–Eckmann maps are Hölder domains. They concluded that, for polynomial Collet–Eckmann maps, the Minkowski dimension of the Julia set is less than 2 and asked whether it is always less than 2. Nazarov, Popovici and Volberg [NPV] extended the results of a first version of [GS] to disconnected Julia sets of polynomials and raised the question whether the Julia set could always be mean porous.

In the present paper we give a positive answer to this question. We carry out the program of "pulling back holes from large to small scale". The main ingredient, besides shrinking neighborhoods, is an estimate from [DPU] on the average distance of an orbit $f^n(x)$ from the set of critical points of f, which is our substitute for the Tsujii condition used in [P2]. As a by-product of our proof of mean porosity, we obtain a new approach to the Graczyk and Smirnov Hölder theorem, outlined in Section 3.

Pulling back holes is done with uniformly bounded criticality on surrounding discs, so in an abundance of scales Collet–Eckmann maps behave like semihyperbolic maps (see [CJY], [DU], [U]).

In Section 4 we define a notion of porosity that is slightly stronger than the above mean porosity, show that Collet–Eckmann Julia sets ($\neq \widehat{\mathbb{C}}$, without parabolic periodic points) satisfy this condition and give a simple proof that the Minkowski dimension of such sets is less than 2.

2. Proof of Theorem 1. Consider a disc $B = B(x,\delta)$ of spherical radius δ around $x \in \widehat{\mathbb{C}}$ and a connected component W of $f^{-n}(B)$. We are mainly interested in the case where $f^n|_W$ has at most D critical points (counted with multiplicity), where D is some fixed number. In this situation, say that f^n is *D-critical on W*. Then $f^n|_W$ has distortion properties similar to those of conformal maps. In particular, if W' is a domain compactly contained in W and if $y \in W'$ is arbitrary, then $|(f^n)'(y)|\operatorname{diam}(W') \le C\operatorname{diam}(f(W))$, where the constant C depends on W, W' and D. The method of *shrinking neighborhoods* is based on a careful estimate of this constant $C(W,W',D)$ (see (2.1) below). We collect the distortion estimates needed in our paper as Lemma 2.1 below (see [P1, Section 1]). For the reader's convenience, we have included short proofs of (2.1) and (2.3).

In what follows, the radius δ will always be assumed to be less than $\operatorname{diam}(\widehat{\mathbb{C}})/2$. The assumption that $\widehat{\mathbb{C}}\setminus W$ contains a disc of radius ε made below is of purely technical nature, in that it allows the passage from spherical to Euclidean metric.

Lemma 2.1. *For each $\varepsilon > 0$ and $D < \infty$ there are constants C_1 and C_2 such that the following holds for all rational maps $F : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$, all $x \in \widehat{\mathbb{C}}$ and all r with $1/2 \le r < 1$:*

Assume that W (resp. W') is a simply connected component of $F^{-1}(B(x,\delta))$ (resp. $F^{-1}(B(x,r\delta))$) with $W \supset W'$. Assume further that $\widehat{\mathbb{C}}\setminus W$ contains a disc of radius ε and that F is D-critical on W. Then

$$|F'(y)|\operatorname{diam}(W') \le C_1(1-r)^{-C_2}\delta \tag{2.1}$$

for all $y \in W'$.

Furthermore, if $r = 1/2$ and $0 < \tau < 1/2$, let $B'' = B(z,\tau\delta)$ be any disc contained in $B(x,\delta/2)(= F(W'))$ and let W'' be a component of $F^{-1}(B'')$

contained in W'. Then

$$\operatorname{diam}(W'') \le C_3 \operatorname{diam}(W') \tag{2.2}$$

with $C_3 = C_3(\tau, \varepsilon, D)$ and $C_3 \to 0$ as $\tau \to 0$ (for fixed ε, D). Finally,

$$W'' \text{ contains a disc of radius } \ge C_4 \operatorname{diam}(W') \tag{2.3}$$

around every preimage of $F^{-1}(z)$ that is contained in W''. Here $C_4 = C_4(\tau, \varepsilon, D)$.

Proof. We will give a short proof of (2.1) (which is essentially Lemma 1.4 of [P1]; the statement in the preprint version of [P1] is imprecise) and of (2.3). The inequality (2.2) is [P1, (1.5′) with $\lambda = 1/2$].

We may assume that the disc of radius ε contained in $\widehat{\mathbb{C}} \setminus W$ is centered at ∞, hence W is a simply connected planar domain bounded by some constant depending on ε only. We may further assume that $B(x, \delta)$ is the unit disc $\mathbb{D}$ and that $B' = B(x, r\delta)$ is the disc $\{|w| < r\}$. Finally, we assume (translate W' if necessary) that $0 \in W'$ and $F(0) = 0$.

To prove (2.1), we need to show that

$$|F'(y)| \operatorname{diam}(W') \le C_1(1-r)^{-C_2}$$

for each $y \in W'$, where now (and during the rest of the proof of Lemma 2.1) diameters and derivatives are with respect to the *Euclidean* metric (here is where we need the assumption involving ε).

Let $g : \mathbb{D} \to W$ be a conformal map with $g(0) = 0$ and set $h = F \circ g$. Then h is a Blaschke product of degree $\le D$, and $h(0) = 0$. We will show that

$$G := g^{-1}(W') \subset \{|w| < 1 - C_1'(1-r)^{C_2'}\}$$

for constants C_1', C_2' depending only on D. From this (2.1) follows immediately since (writing $d = 1 - C_1'(1-r)^{C_2'}$ for short) $\operatorname{diam}(W') \le 2|g'(0)|/(1-d)^2$ by Koebe distortion [P, Chapter 1.3], and (for $y \in W'$ and $u \in G$ with $g(u) = y$)

$$|F'(y)| = \frac{|h'(u)|}{|g'(u)|} < \frac{\frac{1}{1-|u|^2}}{(1-|u|)|g'(0)|/8}.$$

Here, the numerator is estimated via $|h'(u)| \le (1-|h(u)|^2)/(1-|u|^2)$ (the Schwartz–Pick Lemma), and for the denominator we use the Koebe distortion theorem.

To prove $G \subset \{|w| < d\}$, write

$$h(u) = \prod_{n=1}^{\deg(h)} [(u - a_n)/(1 - \overline{a}_n u)] = \prod_{n=1}^{\deg(h)} T_n(u)$$

and notice that, if $|h(u)| < r$, then at least one of the factors has to be of absolute value $< r^{1/D}$. Denoting the hyperbolic metric of $\mathbb{D}$ by ϱ and using

$\varrho(u,a_n)=\log((1+|T_n(u)|)/(1-|T_n(u)|))$, easy calculation shows

$$\varrho(u,a_n)<\log\left(\frac{1+r^{1/D}}{1-r^{1/D}}\right)\le C(D)+\log(1/(1-r))$$

for such u and n. In other words, every $u\in G$ has hyperbolic distance $\le C(D)+\log(1/(1-r))$ from the set $\{a_1,\dots,a_{\deg(h)}\}$. Since $0\in G$ and $\deg(h)\le D$, it easily follows that

$$\begin{aligned}G&\subset\{w\in\mathbb{D}:\varrho(0,w)<2D(C(D)+\log(1/(1-r)))\}\\&\subset\{|w|<1-C_1'(1-r)^{C_2'}\}\end{aligned}$$

and (2.1) is proven.

Now (2.3) is a simple consequence of the above and Koebe distortion: Using notation as above, we already know $G\subset\{|w|<d\}$, and by the Schwarz Lemma, G contains a hyperbolic disc around every preimage $h^{-1}(z)\in G$, of hyperbolic radius at least the hyperbolic radius of B''. ■

Consider a rational map f with $J(f)\ne\widehat{\mathbb{C}}$ and without parabolic periodic points.

From now on, we will always assume that δ is small enough to guarantee that all components of $f^{-n}(B(y,\delta))$ do not meet a fixed open set that contains all critical points of f in the Fatou set, for all n and all $y\in J$ (pick any small open neighborhood of the critical points in the Fatou set and take the union of its forward orbit under f). This allows us to apply Lemma 2.1 to simply connected D-critical components, with bounds depending only on D.

Now fix $\delta>0$ and $D<\infty$ and consider $B=B(f^n(x),\delta)$ together with the component W of $f^{-n}(B)$ containing x. We call n a *good time* for x, and denote the set of good times by $G(x)$, if f^n is D-critical on W.

LEMMA 2.2 (uniform density of good times). *There exist $\delta>0$ and $D<\infty$ such that the lower density of $G(x)$ in $\mathbb{N}$ is at least $1/2$:*

$$\inf_n\frac{\#(G(x)\cap[1,n])}{n}\ge\frac12.$$

Proof. The proof is a modification of the proof of Lemma 2.1 in [P2]. There the Tsujii condition was used to obtain times n where $f^n:W\to B(f^n(c),\delta)$ has degree one. The main ingredient here is the inequality (3.3) of [DPU], an estimate of the average distance from critical points.

Fix $x\in J$. As in [P2], set

$$\phi(n)=-\log(\mathrm{dist}(f^n(x),\mathrm{Crit}(f,J))),$$

where $\mathrm{Crit}(f,J)$ denotes the set of critical points of f that are contained in J. Then (3.3) of [DPU] asserts that there exists a constant C_f such that for each $n \geq 1$,

$$\sum_{j=0}^{n}{}' \phi(j) \leq nC_f, \tag{2.4}$$

where $\sum'$ denotes summation over all but at most $\#\operatorname{Crit}(f, J)$ indices. Indeed, it follows from the proof in [DPU] that for each critical point $c \in J(f)$ it is sufficient to omit in the summation the index j for which $f^j(x)$ is closest to c.

One could view the "graph" of ϕ as the union of all vertical line segments $\{n\} \times [0, \phi(n)]$ in $\mathbb{R}^2$. Then each segment throws a *shadow* $S_n = (n, n + \phi(n)K_f] \subset \mathbb{R}$, where we set $K_f = 2\nu/\log(\lambda)$ and denote by ν the largest degree of all critical points in J of all iterates of f.

The shadows of the exceptional indices in (2.4) could be infinitely long, but nevertheless (2.4) implies that many of the times n belong to boundedly many shadows. Indeed, set $N_f = 2(\#\operatorname{Crit}(f, J) + C_f K_f)$ and

$$A = \{j \in \mathbb{N} : j \text{ belongs to at most } N_f \text{ shadows } S_n\}.$$

Then for each n we obtain from (2.4),

$$\sum_{j=0}^{n}{}' |S_j| \leq nC_f K_f$$

and conclude that

$$\frac{\#(A \cap [0, n])}{n} \geq \frac{1}{2}.$$

We now show that each $n \in A$ is a good time for x, i.e. $A \subset G(x)$ (with $D = \nu N_f$ and δ suitable). We use the technique of [P1] of "shrinking neighborhoods".

Fix once and for all a subexponentially decreasing sequence $b_j > 0$ with $\prod_{j=1}^{\infty}(1 - b_j) > 1/2$ (for instance, $b_j = c/j^2$ for some suitable constant $c > 0$). Fix $n \in A$ and consider the sequence

$$B_s = B\Big(f^n(x), 2\delta \prod_{j=1}^{s}(1 - b_j)\Big)$$

of neighborhoods of $B = B(f^n(x), \delta)$, together with (compatible) connected components W_s of $f^{-s}(B_s)$ and W'_s of $f^{-s}(B_{s+1})$.

Recall the main idea of shrinking neighborhoods from [P1]. If (along backwards iteration from $f^n(x)$) a critical value is met in W_s but not in W'_s, then it can be ignored (because f^{t-s} maps W_t into W'_s for $t > s$). As W'_s sits "well inside" W_s, distortion on W'_s can be controlled.

We want to show that if W_s contains a critical point, then n belongs to the shadow S_{n-s}. Assume this is not the case. Then there is a smallest such s, a critical point $c \in W_s$, and $f^{s-1} : W_{s-1} \to B_{s-1}$ is at most νN_f-critical (as $n \in A$ and s is smallest). It is not hard to see that W_{s-1} is simply connected. Use induction: The fact that at most N_f of the domains $W_{s-2}, \ldots, W_1$ contain critical points gives control on the diameters of W'_{t-1} for those t for which W_t contains a critical point satisfying (CE), using (2.1) as below. For δ small enough we also control the diameters of W's containing critical points not satisfying (CE) (i.e. whose forward trajectory contains other critical points).

Thus by (2.1), for $t > 0$ being the smallest integer such that $f^t(c)$ is not critical for iterates of f, applied to $F = f^{s-t}$ (we can assume that $s - t$ is positive because it is sufficient to consider only s large),

$$|(f^{s-t})'(f^t(c))| \operatorname{diam}(W'_{s-t}) \le C_1 b_{s-t+1}^{-C_2} \delta.$$

Now (CE) and the subexponential decay of b_j imply, for every $\theta > 1$, the existence of a constant C_θ with

$$\operatorname{dist}(c, f^{n-s}(x)) \le (C_\theta \lambda^{-s} \theta^s \delta)^{1/\nu} < \lambda^{-s/(2\nu)}.$$

Here the second inequality holds as soon as δ is small enough. We obtain the contradiction $n \in S_{n-s}$ and conclude that (with $D = \nu N_f$) $A \subset G(x)$. ∎

Proof of Theorem 1.1. We need to pass from good times for $x \in J$ to good scales, in which J^c contains some definite disc. The argument we use is similar to the proof of Lemma 1 in [LP].

Let δ be as in Lemma 2.2 and let $W_n(x)$ be the component of $f^{-n}(B(f^n(x), \delta/2))$ containing x.

Denote by $r(W_n(x)) = \operatorname{dist}(x, \partial W_n(x))$ the inradius of $W_n(x)$. We claim that there is an integer N such that the following holds: For all $x \in J$ and for all $n, n' \in G(x)$ with $n - n' \ge N$,

$$\operatorname{diam}(W_n) \le \tfrac{1}{2} r(W_{n'}). \tag{2.5}$$

For $n' = 0$ ($W_0 = B(x, \delta/2)$) this is essentially Mañé's result [M] (see [P1, Lemma 1.1]). In fact, for each $0 < \tau < 1$ there is $N = N(\tau, f, D, \delta)$ such that $\operatorname{diam}(W_n(x)) \le \tau\delta/2$ for $n \ge N$.

For $n' > 0$ use backward iteration: As f^n is D-critical on $W_n(x)$, $f^{(n-n')}$ is D-critical on $W_{n-n'}(f^n(x))$, so that

$$f^{n'}(W_n) = W_{n-n'}(f^{n'}(x)) \subset B(f^{n'}(x), \tau\delta/2)$$

by the first case. Applying $f^{-n'}$ we obtain (2.5) provided τ is small enough, by (2.2).

Consider the increasing sequence g_j of all good times of x, $\{g_j\} = G(x)$, and set $k_j = g_{Nj}$. By Lemma 2.2 we have $k_j \le 2Nj$, and as $k_{j+1} - k_j \ge N$ inequality (2.5) implies

$$\operatorname{diam}(W_{k_{j+1}}(x)) < \tfrac{1}{2} \operatorname{diam}(W_{k_j}(x)). \tag{2.6}$$

On the other hand, as f is Lipschitz continuous, there is a constant L such that $\operatorname{diam}(W_n(x)) > 2^{-nL}$ for all n and x. Hence we obtain an increasing sequence $n_j < p_1 j$ (with $p_1 < 2LN$) such that

$$\operatorname{diam}(W_{k_j}(x)) \sim 2^{-n_j}.$$

As J is nowhere dense, there is $\tau > 0$ such for every $y \in J$ there is a disc $U \subset B(y, \delta/2) \setminus J$ of radius $\tau\delta/2$.

To show porosity of J at $x \in J$, apply the last statement to $y = f^{k_j}(x)$ (with the sequence k_j constructed above). By (2.3) we find that a component V of $f^{-k_j}(U)$ in $W_{k_j}(x)$ contains a disc of radius $\geq C_4 2^{-n_j}$. For the center z_j of this disc we have $\operatorname{dist}(z_j, x) < \operatorname{diam}(W_{k_j}(x)) \sim 2^{-n_j}$ and $\operatorname{dist}(z_j, J) > C_4 2^{-n_j}$. We have thus found the desired sequence of points in J^{c} and the proof is finished. ■

3. On Hölder Fatou components. As a by-product, Lemma 2.2 gives a new approach to the result of Graczyk and Smirnov:

THEOREM [GS]. *If f is as in Theorem* 1.1 *and A is a Fatou component, then A is a Hölder domain.*

See [P], [SS], [JM], [KR], [GS] and the references therein for the definition and results about Hölder domains.

The main estimate in our proof of the above theorem, replacing the second Collet–Eckmann condition established and used in [GS], is

PROPOSITION 3.1. *There exist $0 < \xi < 1$ and $\delta_0 > 0$ such that for all n, all $x \in J$ and for every component W of $f^{-n}(B(f^n(x), \delta_0))$,*

$$\operatorname{diam} W \leq \xi^n.$$

Proof. [P1, Remark 3.2] asserts that there exists $\delta_0 > 0$ such that $\operatorname{diam} V \to 0$ as $n \to \infty$ uniformly for all components V of $f^{-n}(B(z, \delta_0))$, $z \in J$. So we find an integer N and $\delta_0 > 0$ (smaller than the δ of Lemma 2.2) with the property that every component of $f^{-m}(B(f^m(y), \delta_0))$ has diameter less than $\delta_0/2$ whenever $m \geq N$ and $y \in J$ (m does not have to be a good time for y).

As in the proof of Theorem 1.1 let $k_j \in G(x)$ be the (Nj)th good time of x ($k_j = g_{Nj}$ with $\{g_j\} = G(x)$). For $n > N$ and W as above, let k be the largest of the k_j with $n - k \geq N$. Then applying the above to $y = f^k(x)$ and $m = n - k$ we obtain

$$f^k(W) \subset B(f^k(x), \delta_0/2)$$

and from (2.6) we obtain

$$\operatorname{diam}(W) \leq \left(\frac{1}{2}\right)^{(n-N)/(2N)}. \ ■$$

From Proposition 3.1 the Hölder property of invariant Fatou components (and thus of all components, [GS, Lemma 5.4]) can be concluded as in [GS, Section 5]: Let F be an attractive (or superattractive) invariant Fatou component and pick $\Omega \Subset F$ open, containing all critical points in F, with $f(\Omega) \Subset \Omega$ and such that $\operatorname{dist}(x, \Omega) < \delta_0$ for all $x \in \partial F$.

Fix a point $z_0 \in \Omega$. Set $n(z) = \min\{n \geq 0 : f^n(z) \in \Omega\}$ for $z \in F$. Then the quasihyperbolic distance satisfies

$$\operatorname{dist}_{qh}(z, z_0) \sim n(z)$$

for $z \in F \setminus \Omega$ by [GS, Lemma 5.2].

For $z \in F$ with $\operatorname{dist}(z, J) < \delta_0$, let $x \in J$ be closest to $f^{n(z)-1}(z)$. Then $\operatorname{dist}(f^{n(z)-1}(z), x) < \delta_0$ and we obtain

$$\operatorname{dist}(z, J) < \xi^{n(z)-1}$$

by Proposition 3.1. It follows that

$$n(z) \sim \operatorname{dist}_{qh}(z, z_0) \lesssim \log \frac{1}{\operatorname{dist}(z, J)},$$

establishing the Hölder property of F.

4. Dimension of porous sets. In this section, we give a simple proof of an estimate of Minkowski dimension that is sufficient for the proof of Corollary 1.2, without using the estimates from [KR].

Let $E \subset \mathbb{R}^d$ be a bounded set. Call E *mean porous in all directions* if for all $\alpha > 0$ there exist $\beta > 0$ and $P > 0$ such that for all $z \in E$ there exists a sequence $n_j \leq Pj$ such that for all j and for all balls $B(z', \alpha 2^{-n_j}) \subset B(z, 2^{-n_j})$, there exists a ball $B(z'', \beta 2^{-n_j}) \subset B(z', \alpha 2^{-n_j}) \setminus E$. It is easy to see that there are sets which are mean porous but not mean porous in all directions.

THEOREM 4.1. *If f satisfies the Collet–Eckmann condition, has no parabolic periodic point and $J(f) \neq \widehat{\mathbb{C}}$, then $J(f)$ is mean porous in all directions.*

Proof. This is a small modification of the proof of Theorem 1.1. Only the last two paragraphs of the proof need to be changed: The fact that J is nowhere dense guarantees, for each $\overline{\alpha} > 0$, the existence of $\overline{\beta} > 0$ and disks $B(y'', \overline{\beta}\delta/2) \subset B(y', \overline{\alpha}\delta/2) \setminus J$ for all $y \in J$ and for all disks $B(y', \overline{\alpha}\delta/2) \subset B(y, \delta/2)$. Applying this to $\overline{\alpha} = C\alpha$ and taking preimages as in the proof of Theorem 1.1 proves the assertion, with $\beta = C'\overline{\beta}$. ∎

Consider a covering $\mathcal{B}_n$ of $\mathbb{R}^d$ by boxes of the form $[p_1 2^{-n}, (p_1+1)2^{-n}) \times \ldots \times [p_d 2^{-n}, (p_d + 1)2^{-n})$ for all sequences of integers $p_1, \ldots, p_d$. For every $z \in E$ write $Q(z, n)$ for the element of $\mathcal{B}_n$ which contains z. Call $E \subset \mathbb{R}^n$ *box mean porous* if there exist $N, P > 0$ and for all $z \in E$ a sequence $n_j \leq Pj$ and $Q_j \in \mathcal{B}_{n_j+N}$ such that $Q_j \subset Q(z, n_j) \setminus E$.

Of course, *mean porosity in all directions* implies *box mean porosity*, which in turn implies *mean porosity*. Thus the next statement follows also from [KR].

PROPOSITION 4.2. *If a bounded set* $E \subset \mathbb{R}^d$ *is box mean porous*, *then the Minkowski dimension of* E *satisfies* $\mathrm{MD}(E) < d$.

Proof. We may assume E is contained in the unit box $[0,1)^d =: Q_0$. Consider the graph (a tree) $\mathcal{T}$ whose vertices are those elements of all $\mathcal{B}_n$, $n = 0, 1, \ldots$, which intersect E. We join $Q \in \mathcal{B}_n$ to $Q' \in \mathcal{B}_{n+1}$ with an edge if $Q \supset Q'$. For every vertex $Q \in \mathcal{T}$ call all the vertices in the line in $|\mathcal{T}|$ (the body of $\mathcal{T}$) joining Q to Q_0 *ancestors*. If $Q \in \mathcal{B}_n$ and $Q' \in \mathcal{B}_{n+k}$ with $Q' \subset Q$, let us say that Q' a *k-child* of Q.

Set $K = (2^d)^N$. We prove that for any integer n which is a multiple of PN, the number of n-children of Q_0 is at most $C(K-1)^{n/(PN)} K^{n/N - n/(PN)}$, where C depends on d and N only. As this equals $C(2^d)^{\alpha n}$ with $\alpha = 1 - (1 - \log(K-1)/\log K)/P$, the estimate $\mathrm{MD}(E) \le \alpha d$ follows at once.

The definition of box mean porosity implies two properties of $\mathcal{T}$:

(i) The number of N-children of each vertex is at most K;

(ii) For all $v \in \mathcal{B}_n$ there exists a sequence of at least n/P ancestors each of which has at most $K-1$ N-children.

We want to estimate $\#\mathcal{B}_n$ from above, assuming (i) and (ii). The following simple argument is due to Michał Rams.

For every $0 \le b < N$, define a measure μ_b on $\mathcal{B}_n$ by inductively distributing mass from the top of $\mathcal{T}$ down to $\mathcal{B}_n$ as follows: Start with equidistributing unit mass on $\mathcal{B}_b$. Given a mass at each vertex $v \in \mathcal{B}_{kN+b}$, equidistribute it on $\mathcal{C}_N(v)$, where $\mathcal{C}_N(v)$ denotes the set of N-children of v. At the last step, if $n - (kN + b) = k' < N$, we equidistribute on $\mathcal{C}_{k'}(v)$. Set

$$\mu = \frac{1}{N} \sum_{b=0}^{N-1} \mu_b.$$

Then for every $v \in \mathcal{B}_n$ there is at least one $b = b(\nu)$ with

$$\mu_b(v) \ge (K-1)^{n/(PN) - N} K^{n/N - n/(PN) + N},$$

which gives the required upper bound on $\#\mathcal{B}_n$. ∎

References

[BC] M. Benedicks and L. Carleson, *The dynamics of the Hénon maps*, Ann. of Math. 133 (1991), 73–169.

[CE] P. Collet and J.-P. Eckmann, *Positive Lyapunov exponents and absolute continuity for maps of the interval*, Ergodic Theory Dynam. Systems 3 (1983), 13–46.

[CJY] L. Carleson, P. Jones and J.-C. Yoccoz, *Julia and John*, Bol. Soc. Brasil. Mat. 25 (1994), 1–30.

[DPU] M. Denker, F. Przytycki and M. Urbański, *On the transfer operator for rational functions on the Riemann sphere*, Ergodic Theory Dynam. Systems 16 (1996), 255–266.

[DU] M. Denker and M. Urbański, *On Hausdorff measures on Julia sets of subexpanding rational maps*, Israel J. Math. 76 (1992), 193–214.

[G] J. Graczyk, *Hyperbolic subsets, conformal measures and Hausdorff dimension of Julia sets*, preprint, 1995.

[GS] J. Graczyk and S. Smirnov, *Collet, Eckmann, & Hölder*, Invent. Math., to appear.

[JM] P. Jones and N. Makarov, *Density properties of harmonic measure*, Ann. of Math. 142 (1995), 427–455.

[KR] P. Koskela and S. Rohde, *Hausdorff dimension and mean porosity*, Math. Ann., to appear.

[LP] G. Levin and F. Przytycki, *When do two rational functions have the same Julia set?*, Proc. Amer. Math Soc. 125 (1997), 2179–2190.

[M] R. Mañé, *On a theorem of Fatou*, Bol. Soc. Brasil. Mat. 24 (1993), 1–12.

[McM] C. McMullen, *Self-similarity of Siegel discs and Hausdorff dimension of Julia sets*, preprint, 1995.

[NPV] F. Nazarov, I. Popovici and A. Volberg, *Domains with quasi-hyperbolic boundary condition, domains with Greens boundary condition, and estimates of Hausdorff dimension of their boundary*, preprint, 1995.

[NP] T. Nowicki and F. Przytycki, *The conjugacy of Collet–Eckmann's map of the interval with the tent map is Hölder continuous*, Ergodic Theory Dynam. Systems 9 (1989), 379–388.

[P] C. Pommerenke, *Boundary Behaviour of Conformal Maps*, Springer, 1992.

[P1] F. Przytycki, *Iterations of holomorphic Collet–Eckmann maps: conformal and invariant measures*, Trans. Amer. Math. Soc., to appear.

[P2] —, *On measure and Hausdorff dimension of Julia sets for holomorphic Collet–Eckmann maps*, in: Internat. Conf. on Dynamical Systems, Montevideo 1995—a Tribute to Ricardo Mañé (F. Ledrappier, J. Lewowicz and S. Newhouse, eds.), Pitman Res. Notes Math. 362, Longman, 1996, 167–181.

[PUZ] F. Przytycki, M. Urbański and A. Zdunik, *Harmonic, Gibbs and Hausdorff measures on repellers for holomorphic maps, II*, Studia Math. 97 (1991), 189–225.

[SS] W. Smith and D. Stegenga, *Exponential integrability of the quasihyperbolic metric in Hölder domains*, Ann. Acad. Sci. Fenn. 16 (1991), 345–360.

[T] M. Tsujii, *Positive Lyapunov exponents in families of one dimensional dynamical systems*, Invent. Math. 111 (1993), 113–137.

[U] M. Urbański, *Rational functions with no recurrent critical points*, Ergodic Theory Dynam. Systems 14 (1994), 391–414.

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