FUNDAMENTA
MATHEMATICAE
155 (1998)

Fine properties of Baire one functions

by

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Abstract. A new theorem in the theory of first return representations of Baire class one functions of a real variable is presented which has as immediate consequences several known characterizations of standard subclasses of the Baire one functions. Further, this theorem yields new insights into how finely Baire one functions can be recovered and yields a characterization of another subclass of Baire one functions.

1. Introduction. Here we shall be dealing with real-valued functions defined on $[0,1]$. During the past several years, five subclasses of the Baire class one functions have been characterized in terms of "first return" notions. These characterizations are as follows: For a function $f:[0,1]\to\mathbb{R}$,

(I) f is Baire one if and only if f is first return recoverable [4] ([1]).

(II) f is Baire one with no isolated points on its graph if and only if f is first return approachable [2].

(III) f is Baire one and Darboux if and only if f is first return continuous [3].

(IV) f is Baire one and quasi-continuous if and only if f is universally first return approachable [2].

(V) f is Baire one, Darboux and quasi-continuous if and only if f is universally first return continuous [5].

Here we use the term "quasi-continuous" in the sense of Kempisty [7], which in our setting can be defined by saying that f is quasi-continuous provided that the graph of $f|C(f)$, f restricted to the set of continuity points of f, is dense in the graph of f. Recently, (I) has been sharpened to read as follows:

1991 *Mathematics Subject Classification*: Primary 26A21.

([1]) This also holds in a more general (metric space) setting [1].

(I*) f is Baire one if and only if f is finely recoverable [6].

Proofs for these results have been rather involved and quite different from one another. For example, (III) entailed the use of the powerful Maximoff–Preiss Theorem ([9], [10]), while (I), (I*) and (II) utilized intricate decomposition arguments, each somewhat different from the others. We have felt that it ought to be possible to formulate and prove one central result which would have all of these results as immediate consequences. Our goal for the present work is to present such a result (Theorem 1) and to use it to obtain new insights into how "finely" Baire one functions can be recovered. Actually, we shall concentrate solely on obtaining the "only if" direction of each of the above statements since the "if" direction is straightforward in each case. Before becoming more precise, we must first review some terminology from the references and define certain "fine" concepts.

2. Terminology. A *trajectory* is any sequence $\{x_n\}_{n=0}^{\infty}$ of distinct points in $[0,1]$, which is dense in $[0,1]$. Any countable dense subset D of $[0,1]$ is called a *support set*. Let $\{x_n\}$ be a fixed trajectory. For a given interval, or finite union of intervals, $H \subseteq [0,1]$, $r(H)$ will be the first element of the trajectory $\{x_n\}$ in H. For $x \in [0,1]$ and $\varrho > 0$,

$$B_\varrho(x) \equiv \{y \in [0,1] : |x-y| < \varrho\}.$$

First return recoverability. Let $x \in [0,1]$ and let $\{x_n\}$ be a fixed trajectory. The *first return route to* x, $\mathcal{R}_x = \{y_k\}_{k=1}^{\infty}$, is defined recursively via

$$y_1 = x_0, \qquad y_{k+1} = \begin{cases} r(B_{|x-y_k|}(x)) & \text{if } x \neq y_k, \\ y_k & \text{if } x = y_k. \end{cases}$$

We say that f is *first return recoverable with respect to* $\{x_n\}$ at x provided that

$$\lim_{k\to\infty} f(y_k) = f(x),$$

and if this happens for each $x \in [0,1]$, we say that f is *first return recoverable with respect to* $\{x_n\}$. Further, we say that f is *first return recoverable* if there exists a trajectory $\{x_n\}$ such that f is first return recoverable with respect to $\{x_n\}$.

First return approachability. For each $x \in [0,1]$ the *first return approach to* x *based on* $\{x_n\}$, $\mathcal{A}_x = \{u_k\}$, is defined recursively via

$$u_1 = r((0,1) \setminus \{x\}), \qquad u_{k+1} = r(B_{|x-u_k|}(x) \setminus \{x\}).$$

We say that f is *first return approachable at* x *with respect to the trajectory* $\{x_n\}$ provided

$$\lim_{\substack{u\to x \\ u\in\mathcal{A}_x}} f(u) = f(x).$$

We say that f is *first return approachable with respect to* $\{x_n\}$ provided it is first return approachable with respect to $\{x_n\}$ at each $x \in [0,1]$. Likewise, f is said to be *first return approachable* provided there exists a trajectory with respect to which f is first return approachable.

First return continuity. For $0 < x \leq 1$, the *left first return path to* x *based on* $\{x_n\}$, $\mathcal{P}_x^l = \{t_k\}$, is defined recursively via

$$t_1 = r(0,x), \qquad t_{k+1} = r(t_k, x).$$

For $0 \leq x < 1$, the *right first return path to* x *based on* $\{x_n\}$, $\mathcal{P}_x^r = \{s_k\}$, is defined analogously. We say that f is *first return continuous from the left* [*right*] *at* x *with respect to the trajectory* $\{x_n\}$ provided

$$\lim_{\substack{t\to x\\ t\in\mathcal{P}_x^l}} f(t) = f(x) \qquad [\lim_{\substack{s\to x\\ s\in\mathcal{P}_x^r}} f(s) = f(x)].$$

We say that for any $x \in (0,1)$, f is *first return continuous at* x *with respect to the trajectory* $\{x_n\}$ provided it is both left and right first return continuous at x with respect to the trajectory $\{x_n\}$. We further adopt the convention of saying that f is first return continuous at zero [one] if it is first return continuous from the right [left] at zero [one].

We say that f is *first return continuous with respect to* $\{x_n\}$ provided it is first return continuous with respect to $\{x_n\}$ at each $x \in [0,1]$. Likewise, f is said to be *first return continuous* provided there exists a trajectory with respect to which f is first return continuous.

Universal notions. If *every* support set D has an ordering with respect to which f is first return continuous [approachable], then f will be called *universally first return continuous* [*approachable*].

With the above notation in place, all of the statements (I)–(V) should make sense, and again, our goal entails finding one result which will yield the "only if" parts of all of these. A first approximation to such a result was obtained in [2], in that it readily yields (I), (II), and (IV). That theorem reads as follows:

(VI) If D is a support set with the property that the graph of $f|D$ is dense in the graph of f, then there is an ordering $\{x_n\}$ of D such that f is first return recoverable with respect to $\{x_n\}$ and if $(x, f(x))$ is not isolated on the graph of f, then f is first return approachable at x with respect to $\{x_n\}$.

Hence, we would like the result we are after to yield (VI) as well. However, we shall seek even more, based on the following concepts.

Fine notions. Type 1 points are those $x \in (0,1)$ for which $(x, f(x))$ is isolated on neither the left nor the right. (The point $x = 0$ [$x = 1$] will be a type 1 point if $(0, f(0))$ $[(1, f(1))]$ is not isolated on the right [left].) Type 2 points are those $x \in (0,1)$ for which $(x, f(x))$ is isolated on exactly one side. (The points 0 and 1 are never considered type 2.) Type 3 points are those $x \in [0,1]$ for which $(x, f(x))$ is isolated. We use the notation $T_i(f)$ to denote the set of type i points of f for $i = 1, 2, 3$. The *good side* of a type 2 point is the side from which it is not isolated.

We say that f is *finely recoverable with respect to the trajectory* $\{x_n\}$ provided that, with respect to $\{x_n\}$, f is first return recoverable *and* first return continuous at each type 1 point as well as first return approachable at each type 2 point. We say that f is *finely recoverable* if there exists a trajectory $\{x_n\}$ such that f is finely recoverable with respect to $\{x_n\}$.

Clearly results (I) and (I*) illustrate that there is no difference between the notions of a function being first return recoverable and being finely recoverable. However, it is not the case that if f is first return recoverable with respect to $\{x_n\}$, then there is some rearrangement $\{y_n\}$ of $\{x_n\}$ such that f is finely recoverable with respect to $\{y_n\}$. This was illustrated by Example 1 of [2]. There a Baire one function f and a trajectory $\{x_n\}$ were exhibited with the properties that f is first return recoverable with respect to $\{x_n\}$ but such that for any rearrangement $\{y_n\}$ of $\{x_n\}$, including the original arrangement $\{x_n\}$ itself, there is a perfectly dense $G_{\delta\sigma}$ set of points at which f is not first return continuous with respect to $\{y_n\}$. (Recall that only countably many points can fail to be type 1 points.) This situation leads us to the following definition.

Always fine recoverability. We say that f is *always finely recoverable* provided that whenever a support set D has the property that the graph of $f|D$ is dense in the graph of f, then there exists an ordering $\{x_n\}$ of D such that f is finely recoverable with respect to $\{x_n\}$.

It is clear that every universally first return continuous function must be always finely recoverable. What is less obvious, but true, is that every Baire one, Darboux function is always finely recoverable. This follows from Theorem 1 of [5] and will also follow as a simple consequence of a characterization we shall obtain here for always finely recoverable functions. To these ends, we wish to broaden the goal of finding a result which has all of (I)–(VI) and (I*) as corollaries to one of finding a result which, additionally, yields a characterization of always finely recoverable functions and provides information concerning the sharpness of the aforementioned example from [2]. In particular, if we adopt the notation that for a given function $f : [0,1] \to \mathbb{R}$, and for each $n \in \mathbb{N}$,

$$E_n \equiv \{x : \max\{\liminf_{y\to x^-} |f(y) - f(x)|, \liminf_{y\to x^+} |f(y) - f(x)|\} < 1/2^n\}$$

(where the two-sided condition is reduced to the appropriate one-sided one at the endpoints 0 and 1) and

$$F_n \equiv [0,1] \setminus E_n,$$

then the characterization of always finely recoverable functions which we shall obtain is

(VII) A function $f : [0,1] \to \mathbb{R}$ is always finely recoverable if and only if $F_n \cap T_2(f)$ is a scattered set for each n.

Once again, the "only if" direction is the more difficult to establish, and here again, we shall obtain that direction as a consequence of Theorem 1.

3. Main result and consequences. Due to the wide range of propositions that we intend to glean as immediate consequences of our main result, its statement has taken on a rather technical and perhaps imposing appearance. Thus, we feel it best to first give the statement and see that several clean and crisp observations follow from it in this section; then in the final section we shall present its proof.

THEOREM 1. *Suppose $f : [0,1] \to \mathbb{R}$ is a Baire one function, $D = \{d_n\}_{n=1}^{\infty}$ is a support set such that the graph of $f|D$ is dense in the graph of f, $\{A_n\}$ is a decreasing sequence of F_σ sets, and $\{B_n\}$ is an increasing sequence of F_σ sets such that for each n,*

- $A_n \subseteq E_n \cup D$,
- $d_n \in A_n \cup B_n$,
- $A_n \cap B_n = \emptyset$, *and*
- *for all n we have* $A_{n+1} \cup B_{n+1} \subseteq A_n \cup B_n \subseteq [0,1]$.

Then there is an ordering of D with respect to which f is first return continuous at every point of $T_1(f) \cap (\bigcap_{n=1}^{\infty} A_n)$, and first return approachable at every point of $(\bigcup_{n=1}^{\infty} B_n) \setminus T_3(f)$.

We note four corollaries from which the "only if" parts of all of (I*) and (I)–(VII) clearly follow:

COROLLARY 1. *If $f : [0,1] \to \mathbb{R}$ is a Baire one function and D is any support set with $f|D$ dense in f, then there is an ordering of D with respect to which f is first return approachable at each point of $[0,1] \setminus T_3(f)$.*

Proof. For each n let $A_n = \emptyset$ and $B_n = [0,1]$. Then apply the theorem.

COROLLARY 2. *If $f : [0,1] \to \mathbb{R}$ is a Baire one function and D is a support set such that $T_2(f) \subseteq D$ and $f|D$ is dense in f, then there is an ordering of D with respect to which f is finely recoverable.*

Proof. Let $\{p_n\}$ be an ordering of $T_2(f)$. For each n let $B_n = \{p_1, \ldots, p_n\}$ and $A_n = [0,1] \setminus B_n$. Then apply the theorem.

Corollary 3. *If $f : [0,1] \to \mathbb{R}$ is a Baire one function for which each $F_n \cap T_2(f)$ is scattered and D is a support set such that $f|D$ is dense in f, then there is an ordering of D with respect to which f is finely recoverable.*

Proof. For each n let $B_n = F_n \cap T_2(f)$ and $A_n = [0,1] \setminus B_n$. Since each $F_n \cap T_2(f)$ is scattered, it is both an F_σ and a G_δ. Hence A_n and B_n are both F_σ sets. Now apply the theorem.

Corollary 4. *If $f : [0,1] \to \mathbb{R}$ is a Baire one, Darboux function and D is any support set with $f|D$ dense in f, then there is an ordering of D with respect to which f is first return continuous.*

Proof. This is a corollary to Corollary 3.

Since the "if" direction of characterization (VII) has not appeared in print, we shall give it here in the following proposition. In the process we shall point out yet another characterization of always finely recoverable functions, one which again was motivated by Example 1 from [2].

Proposition 1. *Let $f : [0,1] \to \mathbb{R}$ be a Baire one function. The following are equivalent*:

(1) *f is always finely recoverable.*

(2) *There is a support set $D \subset T_1(f) \cup T_3(f)$ and an ordering of D with respect to which f is finely recoverable.*

(3) *Each $F_n \cap T_2(f)$ is a scattered set.*

Proof. That (1) implies (2) is immediate since $f|(T_1(f) \cup T_3(f))$ is dense in the graph of f. That (3) implies (1) follows from Corollary 3. We now show (2) implies (3). Suppose that D is a support set and $\{x_i\}$ is an ordering of D which witnesses statement (2). To obtain a contradiction, assume that there is an n for which $F_n \cap T_2(f)$ is not scattered. Let $L(f)$ $[R(f)]$ denote the collection of $x \in T_2(f)$ for which $(x, f(x))$ is isolated on the left [right] on the graph of f. One of the sets $L(f) \cap F_n$, $R(f) \cap F_n$ must fail to be scattered. Without loss of generality we shall suppose that $R(f) \cap F_n$ is not scattered. Let S be a subset of $R(f) \cap F_n$ which is dense in itself. Since f is Baire one, we may find a perfect subset C of the closure of S for which $S \cap C$ is dense in C and such that for all $x, y \in C$ we have $|f(x) - f(y)| < 1/(3 \cdot 2^n)$. List the elements of $S \cap C$ in a sequence $\{a_j\}$. Since each $a_j \in R(f) \cap F_n$, there is a number $b_j > a_j$ such that for all $x \in (a_j, b_j)$, $|f(x) - f(a_j)| > 2/(3 \cdot 2^n)$. Note that D contains none of the a_j's. For each $x \in [0,1)$, let $\mathcal{P}_x^r$ denote the right first return path to x based on $\{x_i\}$.

For each $j \in \mathbb{N}$, let $\{i(j,k)\}_{k=1}^{\infty}$ be an increasing sequence of positive integers such that $x_{i(j,k)} \in \mathcal{P}_{a_j}^r \cap (a_j, b_j)$. Let $c_{j,k}$ be the closest term of $\{x_i : i < i(j,k)\}$ lying to the left of a_j. Then for each $x \in (c_{j,k}, x_{i(j,k)})$ we have $x_{i(j,k)} \in \mathcal{P}_x^r$ and if $x \in C \cap (c_{j,k}, x_{i(j,k)})$, then $\left|f(x_{i(j,k)}) - f(x)\right| > 1/(3 \cdot 2^n)$.

Let

$$U_k = \bigcup_{j=1}^{\infty} C \cap (c_{j,k}, x_{i(j,k)}) \quad \text{and} \quad G = \bigcap_{k=1}^{\infty} U_k.$$

Then at no point of the dense G_δ subset, G, of C is f first return continuous with respect to the trajectory $\{x_i\}$. Since at most countably many points can fail to be type 1 points, this means that f is not finely recoverable with respect to $\{x_i\}$. This contradiction completes the proof.

The reader might be wondering whether (VII) and Proposition 1 can be simplified by replacing $F_n \cap T_2(f)$ with F_n. The following example shows this is not so. Let C be the standard middle third Cantor set and let $\{r_n\}$ and $\{l_n\}$ be a listing of the set of all right and left endpoints, respectively, of intervals contiguous to C. Let $C_n = C \cap \{x : x > r_n\}$ and $D_n = C \cap \{x : x \geq l_n\}$. Define $f : [0,1] \to \mathbb{R}$ by

$$f = \begin{cases} 3 & \text{on } [0,1] \setminus C, \\ \displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n}(\chi_{C_n} + \chi_{D_n}) & \text{on } C. \end{cases}$$

Then f is finely recoverable since $T_2(f) = \emptyset$. However, $F_1 = \{r_n, l_n : n \in \mathbb{N}\}$ is not scattered.

4. Proof of main result. Having observed that Theorem 1 does, indeed, immediately yield the consequences we promised in the first two sections, we now conclude with its proof.

Proof of Theorem 1. Note that at any point of $C(f)$, the set of points of continuity of f, f will be first return continuous with respect to every trajectory. Furthermore, since f is Baire one, $C(f)$ is a dense G_δ subset of $[0,1]$. Thus, in light of what we are attempting to prove, there is no loss of generality in replacing each set A_n by $A_n \setminus (C(f) \setminus D)$ and each B_n by $B_n \setminus (C(f) \setminus D)$, with the result that each A_n and B_n now has the additional property of being zero-dimensional.

We use ν, σ, τ etc. to denote elements of $\mathbb{N}^{<\mathbb{N}}$. We shall use $|\nu|$ to denote the length of ν. The kth term of ν is denoted by $\nu(k)$, and if ν has length at least n, then the truncated sequence $\{\nu(1), \ldots, \nu(n)\}$ is denoted by $\nu|_n$. If $\tau = \nu|_n$ for some n, then we say that ν extends τ, or ν is an extension of τ.

Using standard facts from Kuratowski [8], we can obtain a collection $\{M_\nu : \nu \in \mathbb{N}^{<\mathbb{N}}\}$ of closed sets such that

(1) For each $n \in \mathbb{N}$, $\bigcup_{|\nu|=n} M_\nu = A_n \cup B_n$.

(2) M_ν intersects only one of $A_{|\nu|}$ and $B_{|\nu|}$.

(3) If ν and τ are such that neither is an extension of the other, then $M_\nu \cap M_\tau = \emptyset$.

(4) If τ is an extension of ν, then $M_\tau \subseteq M_\nu$.

(5) If $|\nu| = n$, then $\operatorname{osc}(f, M_\nu) < 1/2^n$, where $\operatorname{osc}(f, M_\nu)$ denotes the oscillation of f on M_ν.

(6) For each $n \in \mathbb{N}$ the singleton $\{d_n\}$ is in $\{M_\nu : |\nu| = n\}$.

To see how this can be accomplished, consider the following. Fix an n. Since A_n and B_n are zero-dimensional F_σ sets and f is of Baire class one, we can express each as a countable union of disjoint closed sets on each of which the oscillation of f is less than $1/2^n$, say $A_n = \bigcup_j A_{n,j}$ and $B_n = \bigcup_j B_{n,j}$ with $\operatorname{osc}(f, A_{n,j}) < 1/2^n$ and $\operatorname{osc}(f, B_{n,j}) < 1/2^n$. Next we relabel the countable collection $\{A_{n,j}, B_{n,j} : j \in \mathbb{N}\}$ as $\{C_{n,k} : k \in \mathbb{N}\}$ and refine it, if necessary, so that the singleton $\{d_n\}$ is in $\{C_{n,k} : k \in \mathbb{N}\}$. Then for each $\nu \in \mathbb{N}^{<\mathbb{N}}$ we set

$$M_\nu = C_{1,\nu(1)} \cap C_{2,\nu(2)} \cap \ldots \cap C_{n,\nu(n)},$$

where $n = |\nu|$. It readily follows that $\{M_\nu : \nu \in \mathbb{N}^{<\mathbb{N}}\}$ satisfies conditions (1)–(6).

For each $k \in \mathbb{N}$ and $1 \le l \le k$, let

$$\mathcal{G}^k_l = \{M_\nu : |\nu| = l \text{ and each term of } \nu \text{ is at most } k\} \quad \text{and} \quad \mathcal{G}^k = \bigcup_{l=1}^{k} \mathcal{G}^k_l.$$

Furthermore, set

$$A = \bigcap_{n=1}^{\infty} A_n \quad \text{and} \quad B = \bigcup_{n=1}^{\infty} B_n.$$

Let $\{\mathcal{P}_k\}_{k=1}^\infty$ be a sequence of partitions of $[0,1]$ such that for each k,

(a) $\mathcal{P}_k = \{0 = p^k_0, p^k_1, \ldots, p^k_{j_k} = 1\}$,

(b) no point of $\mathcal{P}_k$, except possibly 0 and 1, belongs to $A_1 \cup B$,

(c) $\mathcal{P}_{k+1}$ strictly refines $\mathcal{P}_k$,

(d) for each $1 \le l \le k+1$, each partition interval of $\mathcal{P}_k$ intersects at most one element of $\mathcal{G}^{k+1}_l$.

Inductively by stages we shall select the required ordering of D which will serve as our trajectory. At the kth stage we will select a non-negative integer m_k and utilize $\mathcal{P}_k$ to assist us in selecting points $\{x_l\}_{l=m_{k-1}+1}^{m_k}$ from D. At the end of the kth stage we want $\{x_l\}_{l=0}^{m_k}$, the ordering of trajectory selected through this stage, to satisfy the following conditions:

(i) If $x \in M_\tau \cap T_1(f) \cap A_{|\tau|} \cap [p^k_i, p^k_{i+1}]$ for some $M_\tau \in \mathcal{G}^k$ and $s \ne x$ is a point of $\{x_l\}_{l=0}^{m_k}$ nearest to x from the left or nearest to x from the right, then $s \in [p^k_i, p^k_{i+1}]$.

(ii) If $x \in (M_\tau \cap B_{|\tau|} \cap [p_i^k, p_{i+1}^k]) \setminus T_3(f)$ for some $M_\tau \in \mathcal{G}^k$ and $s \neq x$ is a point of $\{x_l\}_{l=0}^{m_k}$ nearest to x, then $B_{|s-x|}(x) \subseteq [p_i^k, p_{i+1}^k]$.

(iii) If $x \in M_\tau \cap T_1(f) \cap A_{|\tau|}$ for some $M_\tau \in \mathcal{G}^{k-1}$ and x_l is in the first return path to x with $m_{k-1} < l \leq m_k$, then $|f(x_l) - f(x)| < 2/2^{|\tau|}$.

(iv) If $x \in (M_\tau \cap B_{|\tau|}) \setminus T_3(f)$ for some $M_\tau \in \mathcal{G}^{k-1}$ and x_l is in the first return approach to x with $m_{k-1} < l \leq m_k$, then $|f(x_l) - f(x)| < 2/2^{|\tau|}$.

The construction at the first stage is analogous to the construction at a general stage so we proceed with the general case. Assume that the kth stage has been completed, that the points $x_0, x_1, \ldots, x_{m_k}$ in D have been selected, and that conditions (i)–(iv) are satisfied at this stage. We move to stage $k+1$. Let $D^{k+1} = D \setminus \{x_0, x_1, \ldots, x_{m_k}\}$. We now describe how to select the points to be added to the trajectory at this stage and then we shall explain how to order these newly selected points.

Fix any partition interval $[p_i^{k+1}, p_{i+1}^{k+1}]$ of $\mathcal{P}_{k+1}$ which intersects the union of those M_ν's belonging to $\mathcal{G}^{k+1}$. First, by using property (d) of $\mathcal{P}_k$ and condition (3) note that if both M_ν and M_τ in $\mathcal{G}^{k+1}$ intersect this interval, then either ν is an extension of τ or τ is an extension of ν. Let $M_{\nu_i^{k+1}}$ be the unique element of $\mathcal{G}^{k+1}$ having the longest ν_i^{k+1} such that $M_{\nu_i^{k+1}}$ intersects this interval. Note that $|\nu_i^{k+1}| \leq k+1$. Set

$$H_i^{k+1} = \{\min(M_\sigma \cap [p_i^{k+1}, p_{i+1}^{k+1}]), \max(M_\sigma \cap [p_i^{k+1}, p_{i+1}^{k+1}]) : \nu_i^{k+1} \text{ extends } \sigma\}.$$

Next, choose $\varepsilon_i^{k+1} > 0$ so small that no two points of $H_i^{k+1} \cup \{p_i^{k+1}, p_{i+1}^{k+1}\}$ are within $2\varepsilon_i^{k+1}$ of one another, and so small that for each $1 \leq l \leq k+2$ no two sets in $\mathcal{G}_l^{k+2}$ are within $2\varepsilon_i^{k+1}$ of one another. (We are assuming that ε's at previous stages satisfy analogous appropriate separation conditions.) Now, let

$$\begin{aligned} w_i^{k+1}(l,-1) &= \min([p_i^{k+1}, p_{i+1}^{k+1}] \cap M_{\nu_i^{k+1}|l}), \\ w_i^{k+1}(l,+1) &= \max([p_i^{k+1}, p_{i+1}^{k+1}] \cap M_{\nu_i^{k+1}|l}) \quad \text{for } 1 \leq l \leq |\nu_i^{k+1}|. \end{aligned}$$

Choose $S_i^{k+1}(l)$ according to the following scheme for $1 \leq l \leq |\nu_i^{k+1}|$.

First, we consider the case where $M_{\nu_i^{k+1}|l} \subseteq A_l$.

• Suppose that $w_i^{k+1}(l,-1) \in E_l$. We may pick a point $u \in D^{k+1}$ such that $u \in (w_i^{k+1}(l,-1) - \varepsilon_i^{k+1}, w_i^{k+1}(l,-1))$ and $|f(u) - f(w_i^{k+1}(l,-1))| < 2^{-l}$. We put this point in $S_i^{k+1}(l)$. Similarly, if $w_i^{k+1}(l,+1) \in E_l$ we pick a point $v \in D^{k+1}$ such that $v \in (w_i^{k+1}(l,+1), w_i^{k+1}(l,+1) + \varepsilon_i^{k+1})$ and $|f(v) - f(w_i^{k+1}(l,+1))| < 2^{-l}$ and we put this point in $S_i^{k+1}(l)$. In addition, if either $w_i^{k+1}(l,-1)$ or $w_i^{k+1}(l,+1)$ is in D, we put it in $S_i^{k+1}(l)$ as well.

• If one of $w_i^{k+1}(l,-1)$ or $w_i^{k+1}(l,+1)$ is not in E_l, then it has to be in D by hypothesis. In this case, we simply put it in $S_i^{k+1}(l)$.

Now we consider the case where $M_{\nu_i^{k+1}|l} \subseteq B_l$.

• If $z = w_i^{k+1}(l,-1)$ or $z = w_i^{k+1}(l,+1)$ is in $T_1(f) \cup T_2(f)$, then we pick a point $u \neq z$ in D^{k+1} within ε_i^{k+1} of z such that $|f(u) - f(z)| < 2^{-(k+1)}$ and put this point in $S_i^{k+1}(l)$. In addition, if $z \in D$, we put it in $S_i^{k+1}(l)$ as well.

• If either $w_i^{k+1}(l,-1)$ or $w_i^{k+1}(l,+1)$ does not belongs to $T_1(f) \cup T_2(f)$, then it has to be in D and we simply put it in $S_i^{k+1}(l)$.

We also want the sets $S_i^{k+1}(l)$ to be chosen carefully enough so that the following additional property is satisfied as well:

$$[\min S_i^{k+1}(l_1), \max S_i^{k+1}(l_1)] \subseteq [\min S_i^{k+1}(l_2), \max S_i^{k+1}(l_2)]$$

for $l_2 \leq l_1 \leq |\nu_i^{k+1}|$. Finally, we set $S_i^{k+1}(l) = \emptyset$ for $|\nu_i^{k+1}| < l \leq k+1$.

Now repeat this process for each interval of the partition $\mathcal{P}_{k+1}$ which intersects the union of those M_ν's belonging to $\mathcal{G}^{k+1}$. If a partition interval misses the union of those M_ν's belonging to $\mathcal{G}^{k+1}$, we do not select any points from that interval at this stage, i.e., we set all the corresponding S^{k+1} sets equal to $\emptyset$. Let $B_l^{k+1} = \bigcup_i S_i^{k+1}(l)$ for $1 \leq l \leq k+1$. We observe that the B_l^{k+1}'s are not necessarily disjoint.

Now we are ready to explain in what order these new points will be added to the trajectory as x_l's, beginning with x_{m_k+1}. First, append those from B_k^{k+1} which have not already been labeled, ordering them from left to right. Next, append those from B_{k-1}^{k+1} which have not already been labeled, ordering them from left to right, and continue this for $B_{k-2}^{k+1}, B_{k-3}^{k+1}, \ldots, B_1^{k+1}$ in that order. Finally, append those from B_{k+1}^{k+1} which have not been labeled from left to right, labeling the last as $x_{m_{k+1}}$.

Now we must show that conditions (i)–(iv) are satisfied at the end of stage $k+1$. It is easy to check that conditions (i) and (ii) hold from the method of construction at stage $k+1$.

To show that condition (iii) holds, let $x \in M_\tau \cap A_{|\tau|} \cap T_1(f)$, where $M_\tau \in \mathcal{G}^k$, and let $t \in \{x_l\}_{l=m_k+1}^{m_{k+1}}$ belong to the first return path to x. Let I be the partition interval from $\mathcal{P}_k$ which contains x, and let $J = [p_i^{k+1}, p_{i+1}^{k+1}]$ be that partition interval in $\mathcal{P}_{k+1}$ containing x. From (i) of the induction hypothesis at stage k, it follows that $t \in I$. Let j be such that $t \in K = [p_j^{k+1}, p_{j+1}^{k+1}] \subseteq I$. Hence $t \in S_j^{k+1}(l_1)$ for some l_1. As $x \in M_\tau \cap A_{|\tau|} \cap T_1(f)$, we know that $S_i^{k+1}(|\tau|) \neq \emptyset$, $S_i^{k+1}(|\tau|) \subseteq J$, and there is a point of $S_i^{k+1}(|\tau|)$ to the right of x and to the left of x. From the method in which these sets S were chosen and their points ordered, we see that $|\tau| \le l_1 < k+1$ if $i \ne j$. If $i = j$, then $|\tau| \le l_1 \le k+1$. In either case, from property (d) of the partitions $\mathcal{P}_k$ and $\mathcal{P}_{k+1}$ and properties (3) and (4), it follows that ν_j^{k+1} is an extension of τ. Also recall that $|f(t) - f(z)| < 2^{-l_1} \le 2^{-|\tau|}$ for at least one of the substitutions $z = w_j^{k+1}(l_1, -1)$ or $z = w_j^{k+1}(l_1, +1)$. Since in either case we have $z \in M_{\nu_j^{k+1}|l_1}$, ν_j^{k+1} is an extension of τ, and $l_1 \ge |\tau|$, we conclude that $z \in M_\tau$ as well.

Hence, we have

$$|f(t) - f(x)| \le |f(t) - f(z)| + |f(z) - f(x)| < \frac{1}{2^{|\tau|}} + \frac{1}{2^{|\tau|}} = \frac{2}{2^{|\tau|}},$$

completing the proof that condition (iii) holds. The argument for condition (iv) is analogous to the one for condition (iii) and is left to the reader.

This completes the selection of the trajectory $\{x_l\}$. It is clear that the range of $\{x_l\}_{l=0}^\infty$ is contained in D; to see that it is all of D, fix any $d_n \in D$. Let $\nu \in \mathbb{N}^{<\mathbb{N}}$ be such that the singleton $\{d_n\}$ equals M_ν and $|\nu| = n$. Let $j = \max\{\nu(1), \ldots, \nu(n), n\}$. Then $d_n \in \{x_l\}_{l=0}^{m_j}$.

To see that f is first return continuous [approachable] at each $x \in A \cap T_1(f)$ $[x \in B \setminus T_3(f)]$, consider the following: Let $\varepsilon > 0$ and let n and ν be such that $2^{-n} < \varepsilon/2$ and $x \in M_\nu \subseteq A_n$ $[x \in M_\nu \subseteq B_n]$. Let $N = \max\{v(1), \ldots, v(n), n\}$. Then, by conditions (iii) and (iv), for all x_j in the first return path [approach] to x with $j > m_N$ we have $|f(x_j) - f(x)| \le \varepsilon$.

References

[1] U. B. Darji and M. J. Evans, *Recovering Baire one functions*, Mathematika 42 (1995), 43–48.

[2] U. B. Darji, M. J. Evans, and P. D. Humke, *First return approachability*, J. Math. Anal. Appl. 199 (1996), 545–557.

[3] U. B. Darji, M. J. Evans, and R. J. O'Malley, *First return path systems: differentiability, continuity, and orderings*, Acta Math. Hungar. 66 (1995), 83–103.

[4] —, —, —, *A first return characterization of Baire one functions*, Real Anal. Exchange 19 (1993–94), 510–515.

[5] —, —, —, *Universally first return continuous functions*, Proc. Amer. Math. Soc. 123 (1995), 2677–2685.

[6] M. J. Evans and R. J. O'Malley, *Fine tuning the recoverability of Baire one functions*, Real Anal. Exchange 21 (1995-96), 165–174.

[7] S. Kempisty, *Sur les fonctions quasicontinues*, Fund. Math. 19 (1932), 184–197.

[8] K. Kuratowski, *Topology*, Vol. I, Academic Press, 1966.

[9] I. Maximoff, *Sur la transformation continue de quelques fonctions en dérivées exactes*, Bull. Soc. Phys. Math. Kazan (3) 12 (1940), 57–81.
[10] D. Preiss, *Maximoff's theorem*, Real Anal. Exchange 5 (1979), 92–104.

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Received 16 April 1997