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The fixed-point property for deformations of tree-like continua

by

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Abstract. Let f be a map of a tree-like continuum M that sends each arc-component of M into itself. We prove that f has a fixed point. Hence every tree-like continuum has the fixed-point property for deformations (maps that are homotopic to the identity). This result answers a question of Bellamy. Our proof resembles an old argument of Brouwer involving uncountably many tangent curves. The curves used by Brouwer were originally defined by Peano. In place of these curves, we use rays that were originally defined by Borsuk.

1. Introduction. In 1909, Brouwer [Br] proved that every continuous tangent vector field on a 2-sphere must vanish at some point. This classical theorem has a variety of scientific applications. For example, in electromagnetic-wave theory, it is used to show that there are no isotropic antennas [M]. It also explains why most magnetic plasma containers are tori instead of spheres [C], [A, p. 198]. In many elementary topology books, Brouwer's theorem appears as a corollary to the following fixed-point theorem.

Theorem 1.1. *Let f be a map of a 2-sphere S^2 into S^2 that is homotopic to the identity. Then f has a fixed point.*

The map f is called a *deformation* of S^2. In 1923, Lefschetz [L] generalized Theorem 1.1 to every polyhedron that has a nonzero Euler characteristic. Thus many structurally well-behaved continua (including all the even-dimensional spheres) have the fixed-point property for deformations.

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However, Lefschetz's theory does not extend to general classes of continua that are more pathological than ANR's [Bo1].

A fixed-point theorem for deformations should include at least one continuum that admits a fixed-point-free map. Note that for Theorem 1.1, the antipodal map moves each point of S^2. In 1978, Bellamy [B] constructed a tree-like continuum without the fixed-point property. Shortly thereafter, Bellamy [Le, p. 369] asked if every deformation of a tree-like continuum must have a fixed point.

In 1960, Young [Y] had defined a uniquely-arcwise-connected continuum that admits a fixed-point-free map. The author [H3] in 1986 used a dog-chases-rabbit argument on a Borsuk ray to show that every uniquely-arcwise-connected continuum has the fixed-point property for deformations.

It is not known if every plane continuum that does not separate the plane has the fixed-point property [Bi2]. However, the author [H4], [H8] showed that every deformation of a nonseparating plane continuum has a fixed point. This was accomplished by considering a more general class of maps—those that send each arc-component into itself. We shall use the same strategy to answer Bellamy's question. In Section 3 below, we give a dog-chases-rabbit argument that proves the following theorem.

Theorem 1.2. *Suppose f is a map of a tree-like continuum M that sends each arc-component of M into itself. Then f has a fixed point.*

This result answers a question raised by the author [H4], [H7]. A special case of Theorem 1.2, when M does not contain uncountably many disjoint triods, was proved in [H4]. Since every continuous image of an arc is arcwise connected, deformations send arc-components into themselves. Hence Theorem 1.2 has the following corollary which answers Bellamy's question.

Corollary 1.3. *Every tree-like continuum has the fixed-point property for deformations.*

Special cases of Corollary 1.3 were established in [H5] and [H6]. Corollary 1.3 should also be compared with Minc's example [Mi1] of a tree-like continuum that admits fixed-point-free maps arbitrarily close to the identity (see also [Mi2] and [OR]).

2. Definitions. A *map* is a continuous function.

A map f of a space X is a *deformation* if there exists a map H of $X \times [0,1]$ onto X such that $H(p,0) = p$ and $H(p,1) = f(p)$ for each point p of X.

A space X has the *fixed-point property* (*fixed-point property for deformations*) if for each map (deformation) f of X into X, there is a point p of X such that $f(p) = p$.

A *chain* is a finite collection $\mathcal{A} = \{A_1, \ldots, A_n\}$ of open sets such that $A_i \cap A_j \neq \emptyset$ if and only if $|i-j| \leq 1$. The elements A_1 and A_n are called *end links* of $\mathcal{A}$. Each element of $\mathcal{A} \setminus \{A_1, A_n\}$ is called an *interior link* of $\mathcal{A}$. If $n > 2$ and A_1 also intersects A_n, the collection $\mathcal{A}$ is called a *circular chain*.

A collection $\mathcal{B}$ of sets is *coherent* if, for each nonempty proper subcollection $\mathcal{C}$ of $\mathcal{B}$, there is an element of $\mathcal{C}$ that intersects an element of $\mathcal{B} \setminus \mathcal{C}$.

A finite coherent collection $\mathcal{T}$ of open sets is a *tree chain* if no three elements of $\mathcal{T}$ have a point in common and no subcollection of $\mathcal{T}$ is a circular chain.

A *continuum* is a nondegenerate compact connected metric space.

A continuum M is *tree-like* if for each positive number ε, there is a tree chain with mesh less than ε covering M [Bi1, p. 653].

A continuum is *unicoherent* if it is not the union of two subcontinua whose intersection is disconnected. A continuum is *hereditarily unicoherent* if each of its subcontinua is unicoherent. Every tree-like continuum is hereditarily unicoherent.

3. Proof of Theorem 1.2. Assume f moves each point of M. Let ϱ be a metric on M.

By the compactness of M and the continuity of f, there is a positive number ε such that for each point x of M,

$$\varrho(x, f(x)) > 12\varepsilon. \tag{3.1}$$

For each point x of M, let $A(x)$ denote the arc-component of M that contains x. Since M is hereditarily unicoherent, $A(x)$ does not contain a simple closed curve.

Let y and z be distinct points of $A(x)$. We denote the unique arc, half-open arc, and open arc in $A(x)$ with end points y and z by $[y, z]$, $[y, z)$, and (y, z), respectively. We define $[y, y]$ to be $\{y\}$.

For each point x of M, Borsuk [Bo2] showed there exists a unique sequence $a_1, a_2, \ldots$ of points in $A(x)$ such that $a_1 = x$ and for each positive integer n,

(3.2) $\varrho(a_n, a_{n+1}) = \varepsilon$ [Bo2, p. 19, (4_n)],

(3.3) if $y \in [a_n, a_{n+1})$, then $\varrho(a_n, y) < \varepsilon$ [Bo2, p. 19, (5_n)],

(3.4) $[x, a_n] \cap [a_n, a_{n+1}] = \{a_n\}$ [Bo2, p. 19, (11)], and

(3.5) $a_n \in [x, f(a_n)]$ [Bo2, p. 19, (7_n)].

For each positive integer n, let g_n be a homeomorphism of the half-open real line interval $[n-1, n)$ onto $[a_n, a_{n+1})$. For each nonnegative real number r, let $g(r) = g_n(r)$ if $n - 1 \leq r < n$.

Let $P(x) = \bigcup\{[x, a_n) : n = 2, 3, \ldots\}$. By (3.4), g is a one-to-one map of the nonnegative real line onto $P(x)$. The map g determines a linear ordering $\ll$ of $P(x)$ with x as the first point. The set $P(x)$ is called a *Borsuk ray*.

In [Bi2], Bing described $P(x)$ as an endless path on which a dog is chasing a rabbit. When the dog is at a point y of $P(x)$, the rabbit is at $f(y)$. The dog starts at x and constantly moves forward on $P(x)$. The rabbit may start at a point of $M \setminus P(x)$ but eventually goes to $P(x)$. The rabbit may leave $P(x)$ but always returns to its point of departure and moves forward on $P(x)$ before the dog gets to that point. If the rabbit is forced to come within 12ε of the dog during the chase, then f has a fixed point.

By [H3, (3.6)],

$$P(x) = \{y \in A(x) : [x, y] \cap [y, f(y)] = \{y\}\}. \tag{3.6}$$

For each point y of $P(x)$, the Borsuk ray $P(y)$ is the set $\{z \in P(x) : y = z$ or $y \ll z\}$.

We denote the closure of a given set S relative to M by $\mathrm{Cl}\, S$.

Let $L(x) = \bigcap\{\mathrm{Cl}\, P(y) : y \in P(x)\}$. By (3.2), $L(x)$ is not degenerate. Hence $L(x)$ is a subcontinuum of $\mathrm{Cl}\, P(x)$.

Note that

$$L(x) \subset f(L(x)). \tag{3.7}$$

To see this, let z be a point of $L(x)$. By [H2, p. 99, (8)], there exist sequences of points $y_1, y_2, \ldots$ and $z_1, z_2, \ldots$ in $P(x)$ such that

(1) $z_1, z_2, \ldots$ converges to z and

(2) $a_n \ll y_n \ll z_n$ and $f(y_n) = z_n$ for each positive integer n.

Let y be a point of $L(x) \cap \mathrm{Cl}\{y_n : n = 1, 2, \ldots\}$. Then $f(y) = z$. Hence (3.7) is true.

For each point p of $P(x)$,

(3.8) if r is a point of $P(p)$ and the diameter of $[p, r]$ is less than 12ε, then $r \in [p, f(p)]$.

To see this, assume $r \notin [p, f(p)]$. Let q be the last point of $[p, r]$ that belongs to $[p, f(p)]$. By (3.1), $[q, r] \cap f([p, r]) = \emptyset$. By (3.6), $[q, r] \cap [r, f(r)] = \{r\}$. Thus the intersection of $[q, r]$ and the continuum $[q, f(p)] \cup [r, f(r)] \cup f([p, r])$ is the disconnected set $\{q, r\}$, contradicting the hereditary unicoherence of M. Hence (3.8) is true.

Definition 3.9. An arc $[p, q]$ is *directed* if there is a point x of M such that $[p, q] \subset P(x)$ and $p \ll q$.

Definition 3.10. Let δ be a positive number. An arc $[p, s]$ is *δ-folded* if there exist points q and r of $[p, s]$ such that $q \in [p, r)$, $\varrho(p, q) > \varepsilon$, and $\varrho(p, r) < \delta$.

PROPOSITION 3.11. *There does not exist a point x of M such that $x \in L(x)$.*

Proof. Assume there is a point x of M such that $x \in L(x)$. Since M is hereditarily unicoherent, $P(x) \subset L(x)$.

For each positive integer n, let $B(n) = \{p \in L(x) : [p, f(p)]$ is not $(1/n)$-folded$\}$. Note that $L(x) = \bigcup\{B(n) : n = 1, 2, \ldots\}$. By the Baire category theorem, there is a positive integer λ such that $\mathrm{Cl}\, B(\lambda)$ contains a nonempty open subset Ω of $L(x)$.

By (3.1) and (3.7), the diameter of $L(x)$ is greater than 12ε. Thus there exist two points y and z of $P(x)$ such that $y \ll z$, $\varrho(y, z) > 5\varepsilon$, and $z \in \Omega$. It follows from (3.6) that $z \in [y, f(z)]$.

Continuing the proof of Proposition 3.11, we show

(3.12) there exists a tree chain $\mathcal{T}$ with mesh less than ε and $1/(2\lambda)$ covering M that contains a chain $\{C_1, \ldots, C_\nu\}$ such that

(1) $z \in C_1$,
(2) $L(x) \cap \mathrm{Cl}\, C_1 \subset \Omega$,
(3) $f(z) \in C_\nu$,
(4) $[z, f(z)] \subset \bigcup\{C_i : 1 \leq i \leq \nu\}$, and
(5) $[y, z] \cap \bigcup\{C_i : 2 \leq i \leq \nu\} = \emptyset$.

To accomplish this, let κ be the minimum of ε, $1/(2\lambda)$, and $\varrho(z, L(x) \setminus \Omega)$. Let $\{p_1, \ldots, p_n\}$ be a partition of $[z, f(z)]$ $(z = p_1 < \ldots < p_n = f(z))$ such that for each positive integer i less than n, the diameter of $[p_i, p_{i+1}]$ is less than $\kappa/3$. For each positive integer i less than $n-1$, let J_i be the y-component of $[y, f(z)] \setminus (p_i, p_{i+1})$, let K_i be the p_n-component of $[y, f(z)] \setminus (p_i, p_{i+1})$, and let $\delta_i = \varrho(J_i, K_i)$. Let δ be the smallest number in $\{\delta_i : 1 \leq i < n-1\}$. Let $\mathcal{R}$ be a tree chain with mesh less than $\delta/2$ and $\kappa/3$ covering M. Let $\mathcal{S} = \{R \in \mathcal{R} : R \cap [z, f(z)] \neq \emptyset\}$. For each positive integer i less than n, let C_i be the union of all elements of $\mathcal{S}$ that intersect $[p_i, p_{i+1}]$. Let $\mathcal{T}$ be the tree chain $(\mathcal{R} \setminus \mathcal{S}) \cup \{C_1, \ldots, C_{n-1}\}$. Let $\nu = n-1$ and (3.12) is established.

Let $\mathcal{U}$ be the largest tree chain in $\mathcal{T} \setminus \{C_1\}$ such that $C_2 \in \mathcal{U}$.

We denote the boundary of a given set S relative to M by $\mathrm{Bd}\, S$.

Note that

$$\mathrm{Bd} \bigcup \mathcal{U} \subset C_1. \tag{3.13}$$

LEMMA 3.14. *Let $\mathcal{Z}$ be a tree chain covering M that refines $\mathcal{T}$. Suppose there exist a point p of $L(x)$ and a chain $\mathcal{A} = \{A_1, \ldots, A_n\}$ in $\mathcal{Z}$ such that $p \in A_1 \cap \mathrm{Cl}\, C_1$ and $f(p) \in A_n \subset \bigcup \mathcal{U}$. Suppose A_χ is a link of $\mathcal{A}$ that lies in $\bigcup(\mathcal{T} \setminus \mathcal{U})$. Then $\varrho(A_1, A_\chi) < \varepsilon$.*

Proof. Assume there is a link A_χ of $\mathcal{A}$ in $\bigcup(\mathcal{T} \setminus \mathcal{U})$ and $\varrho(A_1, A_\chi) \geq \varepsilon$. Since $\mathcal{Z}$ refines $\mathcal{T}$ and $A_n \subset \bigcup \mathcal{U}$, there is an integer τ such that $\chi < \tau < n$

and $A_\tau \subset C_1$. Since $p \in \Omega$ and f is continuous, there exists a point b of $A_1 \cap B(\lambda)$ such that $f(b) \in A_n$. Since $\mathcal{Z}$ does not contain a circular chain, $[b, f(b)]$ intersects each link of $\mathcal{A}$. Furthermore, a point c of $A_\chi \cap [b, f(b)]$ precedes a point d of $A_\tau \cap [b, f(b)]$ with respect to the order of $[b, f(b)]$. Since $\varrho(b, c) > \varepsilon$ and $\varrho(b, d) < 1/\lambda$, this contradicts the fact that b belongs to $B(\lambda)$. Hence Lemma 3.14 is true.

LEMMA 3.15. *Let $[p, q]$ be a directed arc in $\bigcup(\mathcal{T} \setminus \mathcal{U})$. Suppose $p \in L(x) \cap \operatorname{Cl} C_1$ and $f(p) \in \bigcup \mathcal{U}$. Then $\varrho(p, q) < 3\varepsilon$.*

Proof. Assume $\varrho(p, q) \geq 3\varepsilon$. Let r be the first point of $[p, q]$ such that $\varrho(p, r) = 3\varepsilon$. By (3.8), $r \in [p, f(p)]$. Let $\mathcal{Z}$ be a tree chain covering M that refines $\mathcal{T}$ and contains a chain $\{A_1, \ldots, A_\chi, \ldots, A_n\}$ such that

(1) $p \in A_1$,
(2) $r \in A_\chi \subset \bigcup(\mathcal{T} \setminus \mathcal{U})$, and
(3) $f(p) \in A_n \subset \bigcup \mathcal{U}$.

Since the mesh of $\mathcal{Z}$ is less than ε, it follows that $\varrho(A_1, A_\chi) > \varepsilon$ and this contradicts Lemma 3.14. Hence Lemma 3.15 is true.

Let $\mathcal{V}$ be a tree chain covering M that refines $\mathcal{T}$ such that

(3.16) z belongs to only one element D of $\mathcal{V}$,

(3.17) C_1 contains D, and

(3.18) the image of each element of $\mathcal{V}$ under f lies in an element of $\mathcal{T}$.

Since $z \in C_1$, $f(z) \in C_\nu$, and $\nu > 2$, it follows from (3.18) that $f(D) \subset \bigcup \mathcal{U}$.

Let $\mathcal{W}$ be the largest tree chain in $\mathcal{V}$ such that $D \in \mathcal{W}$ and $\bigcup \mathcal{W} \subset \bigcup(\mathcal{T} \setminus \mathcal{U})$. Let $\mathcal{E} = \{E \in \mathcal{W} : E \subset C_1\}$.

LEMMA 3.19. *If I is a continuum in $M \setminus \bigcup \mathcal{E}$ that intersects the boundaries of two distinct elements of $\mathcal{E}$, then $I \subset \bigcup \mathcal{W}$.*

Proof. Assume E_i and E_j are distinct elements of $\mathcal{E}$ whose boundaries intersect I. Let $\mathcal{F}$ be the chain in $\mathcal{W}$ that has E_i and E_j as end links. Since $I \cap \bigcup \mathcal{E} = \emptyset$ and $\mathcal{V}$ does not contain a circular chain, $\bigcup \mathcal{F}$ contains $I \cap \operatorname{Bd}(E_i \cup E_j)$. Thus I intersects each interior link of $\mathcal{F}$.

Note that $I \cap \bigcup \mathcal{F} \subset \bigcup \mathcal{W}$. To complete the proof of Lemma 3.19, suppose there is a point p of I that does not belong to $\bigcup \mathcal{F}$. Let $\mathcal{G} = \{G_1, \ldots, G_m\}$ be a chain in $\mathcal{V} \setminus \mathcal{F}$ such that $p \in G_1$ and $G_m \cap \bigcup \mathcal{F} \neq \emptyset$. Let F be the link of $\mathcal{F}$ that intersects G_m. Since $F \in \mathcal{W}$, there is a chain $\mathcal{H} = \{H_1, \ldots, H_n\}$ in $\mathcal{W}$ such that $H_1 = D$ and $H_n = F$. Since $\mathcal{V}$ does not contain a circular chain, I intersects each link of $\mathcal{G}$. Consequently, no link of $\mathcal{G}$ belongs to $\mathcal{E}$.

Therefore $\bigcup\mathcal{G} \subset \bigcup(\mathcal{T}\setminus\mathcal{U})$. It follows that $\mathcal{G}\cup\mathcal{H}$ is a tree chain in $\mathcal{W}$. Thus $p \in \bigcup\mathcal{W}$. Hence $I \subset \bigcup\mathcal{W}$ and Lemma 3.19 is true.

Let $\mathcal{E}^* = \{E \in \mathcal{E} : E \cap L(x) \neq \emptyset\}$.

For each element E of $\mathcal{E}^*$,

$$f(E) \subset \bigcup\mathcal{U}. \tag{3.20}$$

To establish (3.20), first note that $P(z)$ intersects each element of $\mathcal{E}^*$.

Let $E_1, \ldots, E_n$ be a list of the elements of $\mathcal{E}^*$ that has the following order property.

Property 3.21. For each pair of integers i and j with $1 \leq i < j \leq n$, there is a point u of $E_i \cap P(z)$ such that $[z,u] \cap E_j = \emptyset$.

Now assume that (3.20) is false. Let E_m be the first element of $E_1, \ldots, E_n$ such that $f(E_m) \not\subset \bigcup\mathcal{U}$. By (3.16) and (3.17), $D = E_1$. Since $f(D) \subset \bigcup\mathcal{U}$, it follows that $m > 1$. By (3.1) and (3.13), $(\bigcup\mathcal{U})\setminus \operatorname{Cl} C_1$ contains $f(\operatorname{Cl}\bigcup\{E_i : 1 \leq i < m\})$. Note that $\operatorname{Cl} E_m$ and $\operatorname{Cl}\bigcup\{E_i : 1 \leq i < m\}$ are disjoint; for otherwise, it follows from (3.18) that $f(E_m)$ is in an element of $\mathcal{U}$ and this contradicts the definition of E_m.

By Property 3.21, there is a point u of $E_{m-1} \cap P(z)$ such that $[z,u] \cap E_m = \emptyset$. Let w be the first point of $P(u)$ in $\operatorname{Cl} E_m$. By Property 3.21, $[u,w] \cap \bigcup\{E_i : m \leq i \leq n\} = \emptyset$. Let p be the last point of $[u,w]$ in $\operatorname{Cl}\bigcup\{E_i : 1 \leq i < m\}$. Since $[p,w] \subset L(x)$, it follows that $[p,w] \cap \bigcup\mathcal{E} = \emptyset$. Thus, by Lemma 3.19, $[p,w] \subset \bigcup\mathcal{W}$. Hence $[p,w] \subset \bigcup(\mathcal{T}\setminus\mathcal{U})$.

By (3.1), $f(p) \notin C_1$. Let E_i be the element of $\mathcal{E}^*$ such that $p \in \operatorname{Bd} E_i$. Since $f(E_i) \subset \bigcup\mathcal{U}$, it follows from (3.13) that $f(p) \in \bigcup\mathcal{U}$.

Observe that

$$f(w) \in \bigcup\mathcal{U}. \tag{3.22}$$

To see this, assume the contrary. Let q be the first point of $[p,w]$ such that $f(q) \notin \bigcup\mathcal{U}$. By (3.13), $f(q) \in C_1$. By (3.1), $\varrho(q, f(q)) > 12\varepsilon$. Since $p \in \operatorname{Cl} C_1$ and the diameter of C_1 is less than ε, it follows that $\varrho(p,q) > 11\varepsilon$. Since $[p,q]$ is a directed arc in $\bigcup(\mathcal{T}\setminus\mathcal{U})$, this contradicts Lemma 3.15. Hence (3.22) is true.

By (3.1), $f(w) \notin \operatorname{Cl} C_1$. Since $w \in \operatorname{Bd} E_m$, it follows from (3.18) and (3.22) that $f(E_m) \subset \bigcup\mathcal{U}$, and this contradicts the definition of E_m. Hence (3.20) is true.

Since $z \in D$, $[y,z] \cap C_2 = \emptyset$, and $\mathcal{V}$ refines $\mathcal{T}$, it follows that $[y,z] \subset \bigcup\mathcal{W}$. Let W be an element of $\mathcal{W}$ that contains y. Since $y \in L(x)$, there exists a directed arc $[p,q]$ in $\bigcup\mathcal{W}$ such that $p \in \bigcup\mathcal{E}^*$ and $q \in W$. By (3.20), $f(p) \in \bigcup\mathcal{U}$. Since $\varrho(y,z) > 5\varepsilon$ and the diameters of W and C_1 are less than ε, it follows that $\varrho(p,q) > 3\varepsilon$, and this contradicts Lemma 3.15. Hence Proposition 3.11 is true.

Let Σ denote the set of nonempty subsets of M. By the Axiom of Choice, there is a function k of Σ into M such that $k(S) \in S$ for each element S of Σ.

Let $x = k(M)$. Define $L_1 = L(x)$ and $x_1 = k(L_1)$.

Let ω_1 denote the first uncountable ordinal. For each pair of ordinals α and σ ($\alpha < \sigma \leq \omega_1$), we let $Q_{[\alpha,\sigma]}$ denote the continuum $\mathrm{Cl} \bigcup \{P(x_\gamma) : \alpha \leq \gamma < \sigma\}$. We define $Q_{[\alpha,\alpha]} = \{x_\alpha\}$.

For each ordinal σ ($1 < \sigma \leq \omega_1$), we define a continuum L_σ and a point x_σ as follows:

If σ is not a limit ordinal, let $L_\sigma = L(x_{\sigma-1})$ and $x_\sigma = k(L_\sigma)$.

If σ is a limit ordinal, let $L_\sigma = \bigcap \{Q_{[\alpha,\sigma]} : \alpha < \sigma\}$ and $x_\sigma = k(L_\sigma)$.

By Proposition 3.11 and the hereditary unicoherence of M, for each pair of ordinals α and σ ($\alpha < \sigma \leq \omega_1$),

(3.23) every subcontinuum of M that contains $\{x_\beta : \alpha \leq \beta \leq \sigma\}$ contains $Q_{[\alpha,\sigma]}$.

Proposition 3.24. *If α and σ are ordinals and $\alpha < \sigma \leq \omega_1$, then $x_\alpha \notin L_\sigma$.*

Proof. Assume there exist ordinals α and σ such that $\alpha < \sigma \leq \omega_1$ and $x_\alpha \in L_\sigma$. By Proposition 3.11, $\alpha + 1 < \sigma$.

Note that

(3.25) there exist ordinals β and μ such that $\alpha \leq \beta < \mu \leq \sigma$ and $\{x_\gamma : \beta \leq \gamma \leq \mu\} \subset L_\mu$.

To see this, assume the contrary. By (3.23), there is an ordinal $\alpha(1)$ such that

(1) $\alpha \leq \alpha(1) < \sigma$,
(2) $\{x_\gamma : \alpha \leq \gamma \leq \alpha(1)\} \subset L_\sigma$, and
(3) $x_{\alpha(1)+1} \notin L_\sigma$.

By the hereditary unicoherence of M and Proposition 3.11, $\alpha(1) + 2 < \sigma$.

We must show

(3.26) there exists an ordinal $\sigma(1)$ such that $\alpha(1) + 1 < \sigma(1) < \sigma$ and $x_{\alpha(1)+1} \in L_{\sigma(1)}$.

Assume (3.26) is false. For each positive number δ, let $N(\delta) = \{p \in M : \varrho(p, x_{\alpha(1)+1}) < \delta\}$. Since $x_{\alpha(1)+1} \notin L_\sigma$, there exist an ordinal $\gamma(1)$ and a positive number δ_1 such that $\alpha(1) + 1 < \gamma(1) < \sigma$ and $N(\delta_1) \cap Q_{[\gamma(1),\sigma]} = \emptyset$. We repeat this process if $\gamma(1) \neq \alpha(1) + 2$. Since $x_{\alpha(1)+1} \notin L_{\gamma(1)}$, there exist an ordinal $\gamma(2)$ and a positive number δ_2 such that

(1) $\alpha(1) + 1 < \gamma(2) < \gamma(1)$,
(2) $\delta_2 < \delta_1$, and

(3) $N(\delta_2) \cap Q_{[\gamma(2),\gamma(1)]} = \emptyset$.

Again we repeat this process if $\gamma(2) \neq \alpha(1)+2$. Since each $\gamma(i)$ is greater than $\gamma(i+1)$, this process can only be repeated finitely many times. Thus there exists a positive number δ such that $N(\delta) \cap Q_{[\alpha(1)+2,\sigma]} = \emptyset$. By Proposition 3.11, we can assume without loss of generality that $N(\delta) \cap L_{\alpha(1)+2} = \emptyset$. Let $[p,q]$ be an arc in $N(\delta) \cap P(x_{\alpha(1)})$. By Proposition 3.11, $[p,q] \cap L_{\alpha(1)+1} = \emptyset$. It follows from the hereditary unicoherence of M that $[p,q] \cap P(x_{\alpha(1)+1}) = \emptyset$. Since $\{x_\gamma : \alpha \le \gamma \le \alpha(1)\} \subset L_\sigma$, by (3.23), $\bigcup\{P(x_\gamma) : \alpha \le \gamma \le \alpha(1)\} \subset L_\sigma$. Thus $[p,q] \cap Q_{[\alpha,\alpha(1)]} = \emptyset$. Consequently, $[p,q]$ and $Q_{[\alpha,\sigma]} \setminus [p,q]$ are continua whose intersection is $\{p,q\}$, and this contradicts the fact that $Q_{[\alpha,\sigma]}$ is unicoherent. Hence (3.26) is true.

We proceed inductively. For each integer n greater than 1, define ordinals $\alpha(n)$ and $\sigma(n)$ such that

(1) $\alpha(n-1) < \alpha(n) < \sigma(n) < \sigma(n-1)$,
(2) $x_{\alpha(n)} \in L_{\sigma(n)}$, and
(3) $x_{\alpha(n)+1} \notin L_{\sigma(n)}$.

Since $\{\sigma(n) : n = 1,2,\ldots\}$ has no first element, this is impossible. Hence (3.25) is true.

It follows from Proposition 3.11 and (3.25) that μ is a limit ordinal.

By (3.23) and (3.25),

$$Q_{[\beta,\mu]} \subset L_\mu. \tag{3.27}$$

For each positive integer n, let $B(n) = \{p \in L_\mu : [p,f(p)] \text{ is not } (1/n)\text{-folded}\}$. Since $L_\mu = \bigcup\{B(n) : n = 1,2,\ldots\}$, there is a positive integer λ such that $\mathrm{Cl}\, B(\lambda)$ contains a nonempty open subset Ω of L_μ.

Note that

(3.28) there exist an ordinal ψ and a subcontinuum Y of L_μ with points y and z such that

(1) $\beta \le \psi < \mu$,
(2) $z \in \Omega \cap P(x_\psi)$,
(3) $\varrho(y,z) > 5\varepsilon$, and
(4) $Y \cap [z,f(z)] = \{z\}$.

To see this, we consider two cases.

Case 1. Suppose $L_{\beta+1} \cap \Omega \neq \emptyset$. By (3.1) and (3.7), the diameter of $L_{\beta+1}$ is greater than 12ε. By (3.27), $P(x_\beta) \subset L_\mu$. Thus there exist two points y and z of $P(x_\beta)$ such that $y \ll z$, $\varrho(y,z) > 5\varepsilon$, and $z \in \Omega$. In this case, (3.28) is established by letting $\psi = \beta$ and $Y = [y,z]$.

Case 2. Suppose $L_{\beta+1} \cap \Omega = \emptyset$. Let Δ be a nonempty open subset of Ω that has diameter less than ε. Since the diameter of $L_{\beta+1}$ is greater than 12ε, there exists a point y of $L_{\beta+1}$ such that $\varrho(y,\Delta) > 5\varepsilon$. Let ψ be the first ordinal larger than β such that $P(x_\psi) \cap \Delta \neq \emptyset$. Let z be a point of $P(x_\psi) \cap \Delta$. Note that $\varrho(y,z) > 5\varepsilon$. For each ordinal γ $(\beta < \gamma \leq \psi)$, it follows from the definition of ψ that $L_\gamma \cap \Delta = \emptyset$. Thus $z \notin L_{\beta+1} \cup Q_{[\beta+1,\psi]}$. Let p be the last point of $P(x_\psi)$ that belongs to $L_{\beta+1} \cup Q_{[\beta+1,\psi]}$. Define $Y = L_{\beta+1} \cup Q_{[\beta+1,\psi]} \cup [p,z]$. By (3.27), $Y \subset L_\mu$. Condition (4) of (3.28) follows from (3.6) and the hereditary unicoherence of M. Hence (3.28) is established.

Let $\mathcal{T}$ be a tree chain with mesh less than ε and $1/(2\lambda)$ covering M that contains a chain $\{C_1,\ldots,C_\nu\}$ such that

(1) $z \in C_1$,
(2) $L_\mu \cap \mathrm{Cl}\, C_1 \subset \Omega$,
(3) $f(z) \in C_\nu$,
(4) $[z,f(z)] \subset \bigcup\{C_i : 1 \leq i \leq \nu\}$, and
(5) $Y \cap \bigcup\{C_i : 2 \leq i \leq \nu\} = \emptyset$.

Let $\mathcal{U}$ be the largest tree chain in $\mathcal{T} \setminus \{C_1\}$ such that $C_2 \in \mathcal{U}$.

Let $\mathcal{V}$ be a tree chain covering M that refines $\mathcal{T}$ and satisfies

(3.29) z belongs to only one element D of $\mathcal{V}$,

(3.30) C_1 contains D, and

(3.31) the image of each element of $\mathcal{V}$ under f lies in an element of $\mathcal{T}$.

Since $z \in C_1$, $f(z) \in C_\nu$, and $\nu > 2$, it follows from (3.31) that $f(D) \subset \bigcup \mathcal{U}$.

Let $\mathcal{W}$ be the largest tree chain in $\mathcal{V}$ such that $D \in \mathcal{W}$ and $\bigcup \mathcal{W} \subset \bigcup(\mathcal{T} \setminus \mathcal{U})$. Let $\mathcal{E} = \{E \in \mathcal{W} : E \subset C_1\}$.

Note that (3.13) and Lemmas 3.14, 3.15, and 3.19, which are stated for $L(x)$, also hold for L_μ.

Let $\mathcal{E}^* = \{E \in \mathcal{E} : E \cap L_\mu \neq \emptyset\}$.

For each element E of $\mathcal{E}^*$,

$$f(E) \subset \bigcup \mathcal{U}. \tag{3.32}$$

To establish (3.32), define $z_\psi = z$. For each ordinal γ $(\psi < \gamma < \mu)$, define $z_\gamma = x_\gamma$.

Note that by the definition of L_μ, the set $\bigcup\{P(z_\gamma) : \psi \leq \gamma < \mu\}$ intersects each element of $\mathcal{E}^*$.

Let $E_1,\ldots,E_n$ be a list of the elements of $\mathcal{E}^*$ that has the following order property.

Property 3.33. *For each integer m $(1 \leq m \leq n)$, let $\gamma(m)$ be the first ordinal such that $\psi \leq \gamma(m) < \mu$ and $P(z_{\gamma(m)}) \cap E_m \neq \emptyset$. Then for each pair* of integers i and j with $1 \le i < j \le n$, either $\gamma(i) < \gamma(j)$ or $\gamma(i) = \gamma(j)$ and there is a point u of $E_i \cap P(z_{\gamma(i)})$ such that $[z_{\gamma(i)}, u] \cap E_j = \emptyset$.

Now assume that (3.32) is false. Let E_m be the first element of $E_1, \ldots, E_n$ such that $f(E_m) \not\subset \bigcup \mathcal{U}$. By (3.29) and (3.30), $D = E_1$. Since $f(D) \subset \bigcup \mathcal{U}$, it follows that $m > 1$. By (3.1) and (3.13), $(\bigcup \mathcal{U}) \setminus \mathrm{Cl}\, C_1$ contains $f(\mathrm{Cl} \bigcup \{E_i : 1 \le i < m\})$. Note that $\mathrm{Cl}\, E_m$ and $\mathrm{Cl} \bigcup \{E_i : 1 \le i < m\}$ are disjoint; for otherwise, by (3.31), $f(E_m)$ is in an element of $\mathcal{U}$ and this contradicts the definition of E_m.

We must eliminate two cases.

Case 1. Suppose $\gamma(m-1) = \gamma(m)$. Then $P(z_{\gamma(m)})$ intersects both E_{m-1} and E_m. By Property 3.33, there is a point u of $E_{m-1} \cap P(z_{\gamma(m)})$ such that $[z_{\gamma(m)}, u] \cap E_m = \emptyset$. As in the argument for (3.20), let w be the first point of $P(u)$ in $\mathrm{Cl}\, E_m$. By Property 3.33, $[u, w] \cap \bigcup \{E_i : m < i \le n\} = \emptyset$. Let p be the last point of $[u, w]$ in $\mathrm{Cl} \bigcup \{E_i : 1 \le i < m\}$. By the argument for (3.20), both $f(p)$ and $f(w)$ belong to $\bigcup \mathcal{U}$. By (3.1), $f(w) \notin \mathrm{Cl}\, C_1$. Since $w \in \mathrm{Bd}\, E_m$, it follows from (3.31) that $f(E_m) \subset \bigcup \mathcal{U}$, and this contradicts the definition of E_m. Hence Case 1 is impossible.

Case 2. Suppose $\gamma(m-1) < \gamma(m)$. Since $z_{\gamma(m)} \in L_{\gamma(m)}$, by Property 3.33, $z_{\gamma(m)}) \notin E_m$. Let w be the first point of $P(z_{\gamma(m)})$ in $\mathrm{Cl}\, E_m$. Let v be the last point of $[z_{\gamma(m)}, w]$ in $L_{\gamma(m)}$.

For each pair of ordinals θ and ξ ($\psi \le \theta < \xi \le \mu$), let $R_{[\theta,\xi]}$ denote the continuum $\mathrm{Cl} \bigcup \{P(z_\gamma) : \theta \le \gamma < \xi\}$ and let $R_{[\theta,\theta]}$ be $\{z_\theta\}$. Note that $R_{[\theta,\xi]} = Q_{[\theta,\xi]}$ if θ is greater than ψ.

By Property 3.33, $R_{[\psi,\gamma(m)]} \cup [v, w]$ and $\bigcup \{E_i : m \le i \le n\}$ are disjoint. Furthermore, by the argument given in Case 1, $[v, w]$ and $\bigcup \{E_i : 1 \le i < m\}$ are disjoint.

Let θ be the first ordinal such that $\gamma(m-1) < \theta \le \gamma(m)$ and $R_{[\theta,\gamma(m)]} \cap \bigcup \{E_i : 1 \le i < m\} = \emptyset$.

For each ordinal γ ($\gamma(m-1) < \gamma < \theta$),

$$(3.34) \qquad R_{[\gamma,\theta]} \cap \bigcup \{E_i : 1 \le i < m\} \ne \emptyset.$$

Let I denote the w-component of $(R_{[\gamma(m-1),\gamma(m)]} \cup [v, w]) \setminus \bigcup \{E_i : 1 \le i < m\}$. By (3.34), $I \cap \mathrm{Bd} \bigcup \{E_i : 1 \le i < m\} \ne \emptyset$ [W, p. 16, (10.1)]. Since $w \in \mathrm{Bd}\, E_m$, by Lemma 3.19, $I \subset \bigcup \mathcal{W}$. Let J denote the continuum $I \cap L_\theta$.

For a given set S and a positive number δ, we denote $\{p \in M : \varrho(p, S) < \delta\}$ by $N(S, \delta)$.

Note that

$$(3.35) \qquad J \subset \mathrm{Cl}\, N(C_1, 5\varepsilon) \text{ if } \theta \text{ is not a limit ordinal.}$$

To see this, assume θ is not a limit ordinal and there is a point t of J such that $\varrho(t, C_1) \ge 5\varepsilon$. By (3.34) and the definition of L_θ, there exists a directed arc $[p,q]$ in $P(z_{\theta-1}) \setminus \bigcup\{E_i : 1 \le i < m\}$ such that

(1) $p \in \operatorname{Bd} \bigcup\{E_i : 1 \le i < m\}$,
(2) $f(p) \in \bigcup \mathcal{U}$,
(3) $\varrho(q,t) < \varepsilon$, and
(4) $[p,q] \subset \bigcup \mathcal{W}$.

Since $\operatorname{Bd} \bigcup\{E_i : 1 \le i < m\} \subset \operatorname{Cl} C_1$, it follows that $\varrho(p,q) > 3\varepsilon$, and this contradicts Lemma 3.15. Hence (3.35) is true.

Observe that

$$J \cap \operatorname{Bd} \bigcup\{E_i : 1 \le i < m\} \ne \emptyset. \tag{3.36}$$

To see this, assume the contrary. Then $J = L_\theta$. Note that θ is a limit ordinal; for otherwise, it follows from (3.1) and (3.7) that the diameter of J is greater than 12ε and this contradicts (3.35). Therefore $L_\theta = \bigcap\{R_{[\gamma,\theta]} : \gamma(m-1) < \gamma < \theta\}$. Thus, by (3.34), $L_\theta \cap \operatorname{Bd} \bigcup\{E_i : 1 \le i < m\} \ne \emptyset$. Since $J = L_\theta$, this is a contradiction. Hence (3.36) is true.

Next we show

(3.37) $J \subset \operatorname{Cl} N(C_1, 5\varepsilon)$ if θ is a limit ordinal.

To accomplish this, let $\mathcal{Y}$ be the largest tree chain in $\mathcal{W} \setminus \mathcal{E}^*$ such that $J \subset \bigcup \mathcal{Y}$. Let $K = (\bigcup \mathcal{Y}) \setminus (\bigcup \mathcal{E}^*)$. By the definition of θ, there exists an ordinal η less than θ such that $P(z_\eta) \not\subset K$. Since $J \subset K$, there exist an ordinal γ and points p and q such that

(1) $\eta \le \gamma < \theta$,
(2) $\{p,q\} \subset P(z_\gamma)$,
(3) $p \ll q$,
(4) $p \in \operatorname{Bd} \bigcup\{E_i : 1 \le i < m\}$, and
(5) $q \in K$.

By Lemma 3.15, every directed arc in $P(p) \cap K$ that starts at a point of $\operatorname{Bd} \bigcup\{E_i : 1 \le i < m\}$ is in $N(C_1, 5\varepsilon)$.

Therefore

$$L_{\gamma+1} \cap K \subset \operatorname{Cl} N(C_1, 5\varepsilon). \tag{$3.38_{\gamma+1}$}$$

We must show

$$P(z_{\gamma+1}) \cap K \subset \operatorname{Cl} N(C_1, 5\varepsilon). \tag{$3.39_{\gamma+1}$}$$

If $P(z_{\gamma+1}) \subset L_{\gamma+1}$, then ($3.39_{\gamma+1}$) follows immediately from ($3.38_{\gamma+1}$). Thus we assume $P(z_{\gamma+1}) \not\subset L_{\gamma+1}$. Let $p_{\gamma+1}$ be the last point of $P(z_{\gamma+1})$ that belongs to $L_{\gamma+1}$.

Note that

($3.40_{\gamma+1}$) if $p_{\gamma+1}$ belongs to K, then every directed arc in $P(p_{\gamma+1}) \cap K$ that starts at $p_{\gamma+1}$ is in $N(C_1, 5\varepsilon)$.

To see this, assume the contrary. Let r be the first point of $P(p_{\gamma+1})$ such that $\varrho(r, C_1) = 5\varepsilon$. Then $[p_{\gamma+1}, r] \subset K$. Let L denote the $p_{\gamma+1}$-component of $L_{\gamma+1} \cap K$. By the argument for (3.36), $L \cap \mathrm{Bd} \bigcup\{E_i : 1 \le i < m\} \neq \emptyset$. Let b be a point of $L \cap \mathrm{Bd} \bigcup\{E_i : 1 \le i < m\}$. Since $L \subset L_{\gamma+1}$, by $(3.38_{\gamma+1})$, $L \subset \mathrm{Cl}\, N(C_1, 5\varepsilon)$. Since the diameter of C_1 is less than ε, the diameter of $[p_{\gamma+1}, r]$ is less than 11ε. Thus, by (3.8), $r \in [p_{\gamma+1}, f(p_{\gamma+1})]$.

Let $\mathcal{Z}$ be a tree chain covering M that refines $\mathcal{V}$ and contains a chain $\mathcal{A} = \{A_1, \ldots, A_{m'}, \ldots, A_{n'}\}$ such that

(1) $b \in A_1$,
(2) $p_{\gamma+1} \in A_{m'}$,
(3) $f(p_{\gamma+1}) \in A_{n'}$, and
(4) $[p_{\gamma+1}, f(p_{\gamma+1})] \subset \bigcup\{A_i : m' \le i \le n'\}$.

There is an integer χ such that $m' \le \chi < n'$ and $r \in A_\chi$. Since $r \in K$, it follows that $A_\chi \subset \bigcup(\mathcal{T} \setminus \mathcal{U})$. Since $L \subset \mathrm{Cl}\, N(C_1, 5\varepsilon)$, by (3.1), $\varrho(f(L), C_1) > \varepsilon$. Since $b \in \mathrm{Bd} \bigcup\{E_i : 1 \le i < m\}$, it follows that $f(b) \in \bigcup \mathcal{U}$. By the connectivity of L and (3.13), $f(L) \subset \bigcup \mathcal{U}$. Let $\mathcal{B}$ be the tree chain consisting of the elements of $\mathcal{Z}$ that intersect $f(L)$. Since $\bigcup \mathcal{B} \subset \bigcup \mathcal{U}$ and $L \cup [p_{\gamma+1}, r] \subset K$, there is a chain $\mathcal{C}$ in $\mathcal{A} \cup \mathcal{B}$ such that

(1) A_1 is one end link of $\mathcal{C}$,
(2) the other end link of $\mathcal{C}$ contains $f(b)$, and
(3) A_χ is an interior link of $\mathcal{C}$.

Since $\varrho(r, C_1) = 5\varepsilon$, it follows that $\varrho(A_1, A_\chi) > \varepsilon$, and this contradicts Lemma 3.14. Hence $(3.40_{\gamma+1})$ is true.

By Lemma 3.15,

$(3.41_{\gamma+1})$ every directed arc in $P(p_{\gamma+1}) \cap K$ that starts at a point of $\mathrm{Bd} \bigcup\{E_i : 1 \le i < m\}$ is in $N(C_1, 5\varepsilon)$.

Statement $(3.39_{\gamma+1})$ follows from $(3.40_{\gamma+1})$ and $(3.41_{\gamma+1})$.

Note that $(3.39_{\gamma+1})$ implies $(3.38_{\gamma+2})$. Statement $(3.39_{\gamma+2})$ follows from $(3.38_{\gamma+2})$ and the arguments given for $(3.40_{\gamma+1})$ and $(3.41_{\gamma+1})$. In fact, if φ $(\gamma < \varphi \le \theta)$ is an ordinal and, for each ordinal ζ $(\gamma < \zeta < \varphi)$, statements (3.38_ζ) and (3.39_ζ) are true, then statements (3.38_φ) and (3.39_φ) are true. Thus, by transfinite induction, for each ordinal φ $(\gamma < \varphi \le \theta)$, statements (3.38_φ) and (3.39_φ) are true. Since $J \subset L_\theta \cap K$, it follows from (3.38_θ) that (3.37) is true.

By (3.35) and (3.37), $J \subset \mathrm{Cl}\, N(C_1, 5\varepsilon)$. Thus, by (3.1), $f(J) \cap C_1 = \emptyset$. By (3.36), there is a point q of J in $\mathrm{Bd} \bigcup\{E_i : 1 \le i < m\}$. Since $f(\bigcup\{E_i : 1 \le i < m\}) \subset \bigcup \mathcal{U}$, it follows from (3.1) and (3.13) that $f(q) \in \bigcup \mathcal{U}$. By (3.13), $f(J) \subset \bigcup \mathcal{U}$.

Note that

$$\theta \neq \gamma(m). \tag{3.42}$$

To see this, assume the contrary. Then $v \in L_\theta$. Since $[v,w] \cap \bigcup\{E_i : 1 \leq i \leq m\} = \emptyset$, it follows that $v \in J$. By the argument for $(3.40_{\gamma+1})$, $[v,w] \subset N(C_1, 5\varepsilon)$. Since $f(v) \in \bigcup\mathcal{U}$, by (3.1) and (3.13), $f(w) \in \bigcup\mathcal{U}$. Thus, by (3.13), $f(E_m) \subset \bigcup\mathcal{U}$, and this contradicts the definition of E_m. Hence (3.42) is true.

By (3.42), $\theta < \gamma(m)$. It follows from the definition of θ that $P(z_\theta) \cap \bigcup\{E_i : 1 \leq i \leq m\} = \emptyset$. Therefore $P(z_\theta) \subset I$. Thus $z_\theta \in J$. By (3.1) and (3.7), the diameter of $L_{\theta+1}$ is greater than 12ε. Note that $P(z_\theta) \not\subset J$; for otherwise, $L_{\theta+1} \subset J$ and this contradicts (3.35) and (3.37). Let p be the last point of $P(z_\theta)$ that belongs to J. By the argument for $(3.40_{\gamma+1})$, $P(p) \subset N(C_1, 5\varepsilon)$. Since the diameter of $L_{\theta+1}$ is greater than 12ε, this is a contradiction. Thus Case 2 is impossible. Hence (3.32) is true.

Since $z \in D$, $Y \cap C_2 = \emptyset$, and $\mathcal{V}$ refines $\mathcal{T}$, it follows that $Y \subset \bigcup\mathcal{W}$. Let W be an element of $\mathcal{W}$ that contains y. According to (3.28), $y \in L_\mu$. Thus there exists a directed arc $[p,q]$ in $\bigcup\mathcal{W}$ such that $p \in \bigcup\mathcal{E}^*$ and $q \in W$. By (3.32), $f(p) \in \bigcup\mathcal{U}$. Since $\varrho(y,z) > 5\varepsilon$ and the diameters of W and C_1 are less than ε, it follows that $\varrho(p,q) > 3\varepsilon$, and this contradicts Lemma 3.15. Hence Proposition 3.24 is true.

By Proposition 3.24, $\{x_\beta : 1 \leq \beta < \omega_1\}$ is uncountable. Thus $\{x_\beta : 1 \leq \beta < \omega_1\}$ contains a condensation point x_α [W, p. 4, (3.8)]. Let $\gamma_1, \gamma_2, \ldots$ be an increasing sequence of ordinals such that $\alpha < \gamma_1$ and $x_{\gamma_1}, x_{\gamma_2}, \ldots$ converges to x_α. Let σ be the first ordinal larger than every ordinal in $\gamma_1, \gamma_2, \ldots$ It follows that $x_\alpha \in L_\sigma$ and this contradicts Proposition 3.24. Hence Theorem 1.2 is true.

4. Disk-like continua. It follows from results of Borsuk [Bo2], Cook [Co], and Young [Y] that every arcwise connected tree-like continuum has the fixed-point property. Note that this theorem also follows from Theorem 1.2.

A map f of a continuum M is an *ε-map* if for each point y of $f(M)$, the diameter of $f^{-1}(y)$ is less than ε. A continuum M is *disk-like* if for each positive number ε, there exists an ε-map of M onto a disk. Every tree-like continuum is disk-like.

The following question was asked in [H1].

Question 4.1. Does every arcwise connected disk-like continuum have the fixed-point property?

Bennett [Be] proved that no locally connected disk-like continuum admits a fixed-point-free map. He accomplished this by showing that all such continua are embeddable in the plane. Simple examples of nonplanar arcwise connected disk-like continua indicate that Bennett's strategy cannot be applied to Question 4.1.

QUESTION 4.2. Let f be a map of a disk-like continuum M that sends each arc-component of M into itself. Must f have a fixed point?

QUESTION 4.3. Does every disk-like continuum have the fixed-point property for deformations?

References

[A] M. A. Armstrong, *Basic Topology*, McGraw-Hill, London, 1979.

[B] D. P. Bellamy, *A tree-like continuum without the fixed point property*, Houston J. Math. 6 (1979), 1–13.

[Be] R. Bennett, *Locally connected* 2*-cell and* 2*-sphere-like continua*, Proc. Amer. Math. Soc. 17 (1966), 674–681.

[Bi1] R. H. Bing, *Snake-like continua*, Duke Math. J. 18 (1951), 653–663.

[Bi2] —, *The elusive fixed point property*, Amer. Math. Monthly 76 (1969), 119–132.

[Bo1] K. Borsuk, *Sur un continu acyclique qui se laisse transformer topologiquement en lui même sans points invariants*, Fund. Math. 24 (1935), 51–58.

[Bo2] —, *A theorem on fixed points*, Bull. Acad. Polon. Sci. 2 (1954), 17–20.

[Br] L. E. J. Brouwer, *On continuous vector distributions on surfaces*, Proc. Konink. Akad. Wetensch. (Amsterdam) 11 (1909), 850–858.

[C] R. W. Conn, *The engineering of magnetic fusion reactors*, Scientific American 249 (4) (October, 1983), 60–71.

[Co] H. Cook, *Tree-likeness of dendroids and λ-dendroids*, Fund. Math. 68 (1970), 19–22.

[H1] C. L. Hagopian, *Fixed-point problems for disk-like continua*, Amer. Math. Monthly 83 (1976), 471–473.

[H2] —, *Uniquely arcwise connected plane continua have the fixed-point property*, Trans. Amer. Math. Soc. 248 (1979), 85–104.

[H3] —, *The fixed-point property for deformations of uniquely arcwise connected continua*, Topology Appl. 24 (1986), 207–212.

[H4] —, *Fixed points of arc-component-preserving maps*, Trans. Amer. Math. Soc. 306 (1988), 411–420.

[H5] —, *Fixed points of tree-like continua*, in: Contemp. Math. 72, Amer. Math. Soc., 1988, 131–137.

[H6] —, *A fixed-point theorem for tree-like continua*, Topology Proc. 16 (1991), 57–62.

[H7] —, *Fixed-point problems in continuum theory*, in: Contemp. Math. 117, Amer. Math. Soc., 1991, 79–86.

[H8] —, *The fixed-point property for simply-connected plane continua*, Trans. Amer. Math. Soc. 348 (1996), 4525–4548.

[L] S. Lefschetz, *Continuous transformations of manifolds*, Proc. Nat. Acad. Sci. U.S.A. 9 (1923), 90–93.

[Le] I. W. Lewis, *Continuum theory problems*, Topology Proc. 8 (1983), 361–394.

[M] H. F. Mathis, *A short proof that an isotropic antenna is impossible*, Proc. Institute Radio Engineers 39 (1951), 970.

[Mi1] P. Minc, *A tree-like continuum admitting fixed point free maps with arbitrarily small trajectories*, Topology Appl. 46 (1992), 99–106.

[Mi2] P. Minc, *A periodic point free homeomorphism of a tree-like continuum*, Trans. Amer. Math. Soc. 348 (1996), 1487–1519.
[OR] L. G. Oversteegen and J. T. Rogers, Jr., *Fixed-point-free maps on tree-like continua*, Topology Appl. 13 (1982), 85–95.
[W] G. T. Whyburn, *Analytic Topology*, rev. ed., Amer. Math. Soc. Colloq. Publ. 28, Amer. Math. Soc., Providence, R.I., 1963.
[Y] G. S. Young, *Fixed-point theorems for arcwise connected continua*, Proc. Amer. Math. Soc. 11 (1960), 880–884.

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