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A polarized partition relation and failure of GCH at singular strong limit

by

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Abstract. The main result is that for λ strong limit singular failing the continuum hypothesis (i.e. $2^\lambda > \lambda^+$), a polarized partition theorem holds.

1. Introduction. In the present paper we show a polarized partition theorem for strong limit singular cardinals λ failing the continuum hypothesis. Let us recall the following definition.

Definition 1.1. For ordinal numbers α_1, α_2, β_1, β_2 and a cardinal θ, the *polarized partition symbol*

$$\binom{\alpha_1}{\beta_1} \to \binom{\alpha_2}{\beta_2}^{1,1}_\theta$$

means that if d is a function from $\alpha_1 \times \beta_1$ into θ then for some $A \subseteq \alpha_1$ of order type α_2 and $B \subseteq \beta_1$ of order type β_2, the function $d \restriction A \times B$ is constant.

We address the following problem of Erdős and Hajnal:

(∗) if μ is strong limit singular of uncountable cofinality with $\theta < \mathrm{cf}(\mu)$, does

$$\binom{\mu^+}{\mu} \to \binom{\mu}{\mu}^{1,1}_\theta \ ?$$

The particular case of this question for $\mu = \aleph_{\omega_1}$ and $\theta = 2$ was posed by Erdős, Hajnal and Rado (under the assumption of GCH) in [EHR, Problem 11, p. 183]). Hajnal said that the assumption of GCH in [EHR] was not crucial, and he added that the intention was to ask the question "in some, preferably nice, Set Theory".

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Baumgartner and Hajnal have proved that if μ is weakly compact then the answer to $(*)$ is "yes" (see [BH]), also if μ is strong limit of cofinality $\aleph_0$. But for a weakly compact μ we do not know if for every $\alpha < \mu^+$:

$$\binom{\mu^+}{\mu} \to \binom{\alpha}{\mu}^{1,1}_{\theta}.$$

The first time I heard the problem (around 1990) I noted that $(*)$ holds when μ is a singular limit of measurable cardinals. This result is presented in Theorem 2.2. It seemed likely that we could combine this with suitable collapses, to get "small" such μ (like $\aleph_{\omega_1}$) but there was no success in this direction.

In September 1994, Hajnal reasked me the question putting great stress on it. Here we answer the problem $(*)$ using methods of [Sh:g]. But instead of the assumption of GCH (postulated in [EHR]) we assume $2^\mu > \mu^+$. The proof seems quite flexible but we did not find out what else it is good for. This is a good example of the major theme of [Sh:g]:

Thesis 1.2. *Whereas CH and GCH are good* (*helpful*, *strategic*) *assumptions having many consequences*, *and*, *say*, $\neg$*CH is not*, *the negation of GCH at singular cardinals* (*i.e. for μ strong limit singular $2^\mu > \mu^+$*, *or the really strong hypothesis*: $\mathrm{cf}(\mu) < \mu \Rightarrow \mathrm{pp}(\mu) > \mu^+$) *is a good* (*helpful*, *strategic*) *assumption.*

Foreman pointed out that the result presented in Theorem 1.2 below is preserved by μ^+-closed forcing notions. Therefore, if

$$V \models \binom{\lambda^+}{\lambda} \to \binom{\lambda}{\lambda}^{1,1}_{\theta}$$

then

$$V^{\mathrm{Levy}(\lambda^+, 2^\lambda)} \models \binom{\lambda^+}{\lambda} \to \binom{\lambda}{\lambda}^{1,1}_{\theta}.$$

Consequently, the result is consistent with $2^\lambda = \lambda^+$ & λ is small. (Note that although our final model may satisfy the Singular Cardinals Hypothesis, the intermediate model still violates SCH at λ, hence needs large cardinals, see [J].) For λ not small we can use Theorem 2.2.

Before we move to the main theorem, let us recall an open problem important for our methods:

Problem 1.3. (1) *Let* $\kappa = \mathrm{cf}(\mu) > \aleph_0$, $\mu > 2^\kappa$ *and* $\lambda = \mathrm{cf}(\lambda) \in (\mu, \mathrm{pp}^+(\mu))$. *Can we find* $\theta < \mu$ *and* $\mathfrak{a} \in [\mu \cap \mathrm{Reg}]^\theta$ *such that* $\lambda \in \mathrm{pcf}(\mathfrak{a})$, $\mathfrak{a} = \bigcup_{i<\kappa} \mathfrak{a}_i$, $\mathfrak{a}_i$ *bounded in* μ *and* $\sigma \in \mathfrak{a}_i \Rightarrow \bigwedge_{\alpha<\sigma} |\alpha|^\theta < \sigma$? *For this it is enough to show*:

(2) *If* $\mu = \mathrm{cf}(\mu) > 2^{<\theta}$ *but* $\bigvee_{\alpha<\mu} |\alpha|^{<\theta} \geq \mu$ *then we can find* $\mathfrak{a} \in [\mu \cap \mathrm{Reg}]^{<\theta}$ *such that* $\lambda \in \mathrm{pcf}(\mathfrak{a})$. (*In fact*, *it suffices to prove it for the case* $\theta = \aleph_1$.)

As shown in [Sh:g] we have

THEOREM 1.4. *If* μ *is strong limit singular of cofinality* $\kappa > \aleph_0$ *and* $2^\mu > \lambda = \mathrm{cf}(\lambda) > \mu$ *then for some strictly increasing sequence* $\langle \lambda_i : i < \kappa \rangle$ *of regulars with limit* μ, $\prod_{i<\kappa} \lambda_i / J^{\mathrm{bd}}_\kappa$ *has true cofinality* λ. *If* $\kappa = \aleph_0$, *this still holds for* $\lambda = \mu^{++}$.

[More fully, by [Sh:g, II, §5], we know $\mathrm{pp}(\mu) =^+ 2^\mu$ and by [Sh:g, VIII, 1.6(2)], we know $\mathrm{pp}^+(\mu) = \mathrm{pp}^+_{J^{\mathrm{bd}}_\kappa}(\mu)$. Note that for $\kappa = \aleph_0$ we should replace J^{bd}_κ by a possibly larger ideal, using [Sh 430, 1.1, 6.5] but there is no need here.]

REMARK 1.5. Note that the problem is a pp = cov problem (see more in [Sh 430, §1]); so if $\kappa = \aleph_0$ and $\lambda < \mu^{+\omega_1}$ the conclusion of 1.4 holds; we allow J^{bd}_κ to be increased, even "there are $< \mu^+$ fixed points $< \lambda^+$" suffices.

2. Main result

THEOREM 2.1. *Suppose* μ *is strong limit singular satisfying* $2^\mu > \mu^+$. *Then*:

(1) $\begin{pmatrix} \mu^+ \\ \mu \end{pmatrix} \to \begin{pmatrix} \mu + 1 \\ \mu \end{pmatrix}^{1,1}_\theta$ *for any* $\theta < \mathrm{cf}(\mu)$.

(2) *If* d *is a function from* $\mu^+ \times \mu$ *to* θ *and* $\theta < \mu$ *then for some sets* $A \subseteq \mu^+$ *and* $B \subseteq \mu$ *we have* $\mathrm{otp}(A) = \mu + 1$, $\mathrm{otp}(B) = \mu$ *and the restriction* $d \restriction A \times B$ *does not depend on the first coordinate.*

Proof. (1) This follows from part (2) (since if $d(\alpha, \beta) = d'(\beta)$ for $\alpha \in A$, $\beta \in B$, where $d' : B \to \theta$, and $|B| = \mu$, $\theta < \mathrm{cf}(\mu)$ then there is $B' \subseteq B$ with $|B'| = \mu$ such that $d' \restriction B$ is constant and hence $d \restriction A \times B'$ is constant as required).

(2) Let $d : \mu^+ \times \mu \to \theta$. Let $\kappa = \mathrm{cf}(\mu)$ and $\overline{\mu} = \langle \mu_i : i < \kappa \rangle$ be a continuous strictly increasing sequence such that $\mu = \sum_{i<\kappa} \mu_i$, $\mu_0 > \kappa + \theta$. We can find a sequence $\overline{C} = \langle C_\alpha : \alpha < \mu^+ \rangle$ such that:

(A) $C_\alpha \subseteq \alpha$ is closed, $\mathrm{otp}(C_\alpha) < \mu$,

(B) $\beta \in \mathrm{nacc}(C_\alpha) \Rightarrow C_\beta = C_\alpha \cap \beta$,

(C) if C_α has no last element then $\alpha = \sup(C_\alpha)$ (so α is a limit ordinal) and any member of $\mathrm{nacc}(C_\alpha)$ is a successor ordinal,

(D) if $\sigma = \mathrm{cf}(\sigma) < \mu$ then the set

$$S_\sigma := \{\delta < \mu^+ : \mathrm{cf}(\delta) = \sigma \;\&\; \delta = \sup(C_\delta) \;\&\; \mathrm{otp}(C_\delta) = \sigma\}$$

is stationary

(possible by [Sh 420, §1]); we could have added

(E) for every $\sigma \in \mathrm{Reg} \cap \mu^+$ and a club E of μ^+, for stationary many $\delta \in S_\sigma$, E separates any two successive members of C_δ.

Let c be a symmetric two-place function from μ^+ to κ such that for each $i < \kappa$ and $\beta < \mu^+$ the set

$$a_i^\beta := \{\alpha < \beta : c(\alpha, \beta) \le i\}$$

has cardinality $\le \mu_i$ and $\alpha < \beta < \gamma \Rightarrow c(\alpha, \gamma) \le \max\{c(\alpha, \beta), c(\beta, \gamma)\}$ and

$$\alpha \in C_\beta \;\&\; \mu_i \ge |C_\beta| \Rightarrow c(\alpha, \beta) \le i$$

(as in [Sh 108], easily constructed by induction on β).

Let $\overline{\lambda} = \langle \lambda_i : i < \kappa \rangle$ be a strictly increasing sequence of regular cardinals with limit μ such that $\prod_{i<\kappa} \lambda_i / J_\kappa^{\mathrm{bd}}$ has true cofinality μ^{++} (exists by 1.4 with $\lambda = \mu^{++} \le 2^\mu$). As we can replace $\overline{\lambda}$ by any subsequence of length κ, without loss of generality $(\forall i < \kappa)(\lambda_i > 2^{\mu_i^+})$. Lastly, let $\chi = \beth_8(\mu)^+$ and $<_\chi^*$ be a well ordering of $\mathcal{H}(\chi)(:= \{x :$ the transitive closure of x is of cardinality $< \chi\})$.

Now we choose by induction on $\alpha < \mu^+$ sequences $\overline{M}_\alpha = \langle M_{\alpha,i} : i < \kappa \rangle$ such that:

(i) $M_{\alpha,i} \prec (\mathcal{H}(\chi), \in, <_\chi^*)$,

(ii) $\|M_{\alpha,i}\| = 2^{\mu_i^+}$ and ${}^{\mu_i^+}(M_{\alpha,i}) \subseteq M_{\alpha,i}$ and $2^{\mu_i^+} + 1 \subseteq M_{\alpha,i}$,

(iii) $d, c, \overline{C}, \overline{\lambda}, \overline{\mu}, \alpha \in M_{\alpha,i}$, $\langle M_{\beta,j} : \beta < \alpha, j < \kappa \rangle \in M_{\alpha,i}$, $\bigcup_{\beta \in a_i^\alpha} M_{\beta,i} \subseteq M_{\alpha,i}$ and $\langle M_{\alpha,j} : j < i \rangle \in M_{\alpha,i}$, $\bigcup_{j<i} M_{\alpha,j} \subseteq M_{\alpha,i}$,

(iv) $\langle M_{\beta,i} : \beta \in a_i^\alpha \rangle$ belongs to $M_{\alpha,i}$.

There is no problem to carry out the construction. Note that actually clause (iv) follows from (i)–(iii), as a_i^α is defined from c, α, i. Our demands imply that

$$[\beta \in a_i^\alpha \Rightarrow M_{\beta,i} \prec M_{\alpha,i}] \quad \text{and} \quad [j < i \Rightarrow M_{\alpha,j} \prec M_{\alpha,i}]$$

and $a_i^\alpha \subseteq M_{\alpha,i}$, hence $\alpha \subseteq \bigcup_{i<\kappa} M_{\alpha,i}$.

For $\alpha < \mu^+$ let $f_\alpha \in \prod_{i<\kappa} \lambda_i$ be defined by $f_\alpha(i) = \sup(\lambda_i \cap M_{\alpha,i})$. Note that $f_\alpha(i) < \lambda_i$ as $\lambda_i = \mathrm{cf}(\lambda_i) > 2^{\mu_i^+} = \|M_{\alpha,i}\|$. Also, if $\beta < \alpha$ then for every $i \in [c(\beta, \alpha), \kappa)$ we have $\beta \in M_{\alpha,i}$ and hence $\overline{M}_\beta \in M_{\alpha,i}$. Therefore, as also $\overline{\lambda} \in M_{\alpha,i}$, we have $f_\beta \in M_{\alpha,i}$ and $f_\beta(i) \in M_{\alpha,i} \cap \lambda_i$. Consequently,

$$(\forall i \in [c(\beta, \alpha), \kappa))(f_\beta(i) < f_\alpha(i)) \quad \text{and thus} \quad f_\beta <_{J_\kappa^{\mathrm{bd}}} f_\alpha.$$

Since $\{f_\alpha : \alpha < \mu^+\} \subseteq \prod_{i<\kappa} \lambda_i$ has cardinality μ^+ and $\prod_{i<\kappa} \lambda_i / J_\kappa^{\mathrm{bd}}$ is μ^{++}-directed, there is $f^* \in \prod_{i<\kappa} \lambda_i$ such that

$(*)_1$ $\quad (\forall \alpha < \mu^+)(f_\alpha <_{J_\kappa^{\mathrm{bd}}} f^*)$.

Let, for $\alpha < \mu^+$, $g_\alpha \in {}^\kappa\theta$ be defined by $g_\alpha(i) = d(\alpha, f^*(i))$. Since $|{}^\kappa\theta| < \mu < \mu^+ = \mathrm{cf}(\mu^+)$, there is a function $g^* \in {}^\kappa\theta$ such that

$(*)_2$ the set $A^* = \{\alpha < \mu^+ : g_\alpha = g^*\}$ is unbounded in μ^+.

Now choose, by induction on $\zeta < \mu^+$, models N_ζ such that:

(a) $N_\zeta \prec (\mathcal{H}(\chi), \in, <^*_\chi)$,
(b) the sequence $\langle N_\zeta : \zeta < \mu^+\rangle$ is increasing continuous,
(c) $\|N_\zeta\| = \mu$ and ${}^{\kappa>}(N_\zeta) \subseteq N_\zeta$ if ζ is not a limit ordinal,
(d) $\langle N_\xi : \xi \le \zeta\rangle \in N_{\zeta+1}$,
(e) $\mu + 1 \subseteq N_\zeta$, $\bigcup_{\alpha<\zeta,\, i<\kappa} M_{\alpha,i} \subseteq N_\zeta$ and $\langle M_{\alpha,i} : \alpha < \mu^+,\ i < \kappa\rangle$, $\langle f_\alpha : \alpha < \mu^+\rangle$, g^*, A^* and d belong to the first model N_0.

Let $E := \{\zeta < \mu^+ : N_\zeta \cap \mu^+ = \zeta\}$. Clearly, E is a club of μ^+, and thus we can find an increasing sequence $\langle \delta_i : i < \kappa\rangle$ such that

$(*)_3$ $\delta_i \in S_{\mu_i^+} \cap \mathrm{acc}(E)$ $(\subseteq \mu^+)$ (see clause (D) at the beginning of the proof).

For each $i < \kappa$ choose a successor ordinal $\alpha_i^* \in \mathrm{nacc}(C_{\delta_i}) \setminus \bigcup\{\delta_j + 1 : j < i\}$. Take any $\alpha^* \in A^* \setminus \bigcup_{i<\kappa} \delta_i$.

We choose by induction on $i < \kappa$ an ordinal j_i and sets A_i, B_i such that:

(α) $j_i < \kappa$ and $\mu_{j_i} > \lambda_i$ (so $j_i > i$) and j_i strictly increasing in i,
(β) $f_{\delta_i}\restriction[j_i,\kappa) < f_{\alpha^*_{i+1}}\restriction[j_i,\kappa) < f_{\alpha^*}\restriction[j_i,\kappa) < f^*\restriction[j_i,\kappa)$,
(γ) for each $i_0 < i_1$ we have $c(\delta_{i_0}, \alpha^*_{i_1}) < j_{i_1}$, $c(\alpha^*_{i_0}, \alpha^*_{i_1}) < j_{i_1}$, $c(\alpha^*_{i_1}, \alpha^*) < j_{i_1}$ and $c(\delta_{i_1}, \alpha^*) < j_{i_1}$,
(δ) $A_i \subseteq A^* \cap (\alpha_i^*, \delta_i)$,
(ε) $\mathrm{otp}(A_i) = \mu_i^+$,
(ζ) $A_i \in M_{\delta_i, j_i}$,
(η) $B_i \subseteq \lambda_{j_i}$,
(θ) $\mathrm{otp}(B_i) = \lambda_{j_i}$,
(ι) $B_\varepsilon \in M_{\alpha_i^*, j_i}$ for $\varepsilon < i$,
(κ) for every $\alpha \in \bigcup_{\varepsilon\le i} A_\varepsilon \cup \{\alpha^*\}$ and $\zeta \le i$ and $\beta \in B_\zeta \cup \{f^*(j_\zeta)\}$ we have $d(\alpha,\beta) = g^*(j_\zeta)$.

If we succeed then $A = \bigcup_{\varepsilon<\kappa} A_\varepsilon \cup \{\alpha^*\}$ and $B = \bigcup_{\zeta<\kappa} B_\zeta$ are as required. During the induction at stage i concerning (ι), if $\varepsilon + 1 = i$ then for some $j < \kappa$, $B_\varepsilon \cap M_{\alpha_i^*, j}$ has cardinality λ_{j_ε}, hence we can replace B_ε by a subset of the same cardinality which belongs to the model $M_{\alpha_i^*, j}$ if j is large enough such that $\mu_j > \lambda_i$; if $\varepsilon + 1 < i$ then by the demand for $\varepsilon + 1$, we have $\bigvee_{j<\kappa} B_\varepsilon \in M_{\alpha_i^*, j}$. So assume that the sequence $\langle (j_\varepsilon, A_\varepsilon, B_\varepsilon) : \varepsilon < i\rangle$ has already been defined.

We can find $j_i(0) < \kappa$ satisfying requirements (α), (β), (γ) and (ι) and such that $\bigwedge_{\varepsilon<i} \lambda_{j_\varepsilon} < \mu_{j_i(0)}$. Then for each $\varepsilon < i$ we have $\delta_\varepsilon \in a^{\alpha_i^*}_{j_i(0)}$ and hence $M_{\delta_\varepsilon,j_\varepsilon} \prec M_{\alpha_i^*,j_i(0)}$ (for $\varepsilon < i$). But $A_\varepsilon \in M_{\delta_\varepsilon,j_\varepsilon}$ (by clause (ζ)) and $B_\varepsilon \in M_{\alpha_i^*,j_i(0)}$ (for $\varepsilon < i$), so $\{A_\varepsilon, B_\varepsilon : \varepsilon < i\} \subseteq M_{\alpha_i^*,j_i(0)}$. Since ${}^{\kappa>}(M_{\alpha_i^*,j_i(0)}) \subseteq M_{\alpha_i^*,j_i(0)}$ (see (ii)), the sequence $\langle (A_\varepsilon, B_\varepsilon) : \varepsilon < i\rangle$ belongs to $M_{\alpha_i^*,j_i(0)}$. We know that for $\gamma_1 < \gamma_2$ in $\mathrm{nacc}(C_{\delta_i})$ we have $c(\gamma_1,\gamma_2) \le i$ (remember clause (B) and the choice of c). As $j_i(0) > i$ and so $\mu_{j_i(0)} \ge \mu_i^+$, the sequence

$$\overline{M}^* := \langle M_{\alpha,j_i(0)} : \alpha \in \mathrm{nacc}(C_{\delta_i})\rangle$$

is $\prec$-increasing and $\overline{M}^*\restriction\alpha \in M_{\alpha,j_i(0)}$ for $\alpha \in \mathrm{nacc}(C_{\delta_i})$ and $M_{\alpha_i^*,j_i(0)}$ appears in it. Also, as $\delta_i \in \mathrm{acc}(E)$, there is an increasing sequence $\langle \gamma_\xi : \xi < \mu_i^+\rangle$ of members of $\mathrm{nacc}(C_{\delta_i})$ such that $\gamma_0 = \alpha_i^*$ and $(\gamma_\xi, \gamma_{\xi+1}) \cap E \neq \emptyset$, say $\beta_\xi \in (\gamma_\xi, \gamma_{\xi+1}) \cap E$. Each element of $\mathrm{nacc}(C_\delta)$ is a successor ordinal, so every γ_ξ is a successor ordinal. Each model $M_{\gamma_\xi,j_i(0)}$ is closed under sequences of length $\le \mu_i^+$, and hence $\langle \gamma_\zeta : \zeta < \xi\rangle \in M_{\gamma_\xi,j_i(0)}$ (by choosing the right $\overline{C}$ and δ_i's we could have managed to have $\alpha_i^* = \min(C_{\delta_i})$, $\{\gamma_\xi : \xi < \mu_i^+\} = \mathrm{nacc}(C_\delta)$, without using this amount of closure).

For each $\xi < \mu_i^+$, we know that

$$(\mathcal{H}(\chi), \in, <^*_\chi) \models \text{“}(\exists x \in A^*)[x > \gamma_\xi \ \&\ (\forall \varepsilon < i)(\forall y \in B_\varepsilon)(d(x,y) = g^*(j_\varepsilon))]\text{”}$$

because $x = \alpha^*$ satisfies it. As all the parameters, i.e. A^*, γ_ξ, d, g^* and $\langle B_\varepsilon : \varepsilon < i\rangle$, belong to N_{β_ξ} (remember clauses (e) and (c); note that $B_\varepsilon \in M_{\alpha_i^*,j_i(0)}$, $\alpha_i^* < \beta_\xi$), there is an ordinal $\beta_\xi^* \in (\gamma_\xi, \beta_\xi) \subseteq (\gamma_\xi, \gamma_{\xi+1})$ satisfying the demands on x. Now, necessarily for some $j_i(1,\xi) \in (j_i(0), \kappa)$ we have $\beta_\xi^* \in M_{\gamma_{\xi+1},j_i(1,\xi)}$. Hence for some $j_i < \kappa$ the set

$$A_i := \{\beta_\xi^* : \xi < \mu_i^+ \ \&\ j_i(1,\xi) = j_i\}$$

has cardinality μ_i^+. Clearly $A_i \subseteq A^*$ (as each $\beta_\xi^* \in A^*$). Now, the sequence $\langle M_{\gamma_\xi,j_i} : \xi < \mu_i^+\rangle^\frown\langle M_{\delta_i,j_i}\rangle$ is $\prec$-increasing, and hence $A_i \subseteq M_{\delta_i,j_i}$. Since $\mu_{j_i}^+ > \mu_i^+ = |A_i|$ we have $A_i \in M_{\delta_i,j_i}$. Note that at the moment we know that the set A_i satisfies the demands (δ)–(ζ). By the choice of $j_i(0)$, as $j_i > j_i(0)$, clearly $M_{\delta_i,j_i} \prec M_{\alpha^*,j_i}$, and hence $A_i \in M_{\alpha^*,j_i}$. Similarly, $\langle A_\varepsilon : \varepsilon \le i\rangle \in M_{\alpha^*,j_i}$, $\alpha^* \in M_{\alpha^*,j_i}$ and

$$\sup(M_{\alpha^*,j_i} \cap \lambda_{j_i}) = f_{\alpha^*}(j_i) < f^*(j_i).$$

Consequently, $\bigcup_{\varepsilon\le i} A_\varepsilon \cup \{\alpha^*\} \subseteq M_{\alpha^*,j_i}$ (by the induction hypothesis or the above) and it belongs to M_{α^*,j_i}. Since $\bigcup_{\varepsilon\le i} A_\varepsilon \cup \{\alpha^*\} \subseteq A^*$, clearly

$$(\mathcal{H}(\chi), \in, <^*_\chi) \models \text{“}\Big(\forall x \in \bigcup_{\varepsilon\le i} A_\varepsilon \cup \{\alpha^*\}\Big)(d(x, f^*(j_i)) = g^*(j_i))\text{”}.$$

Note that

$$\bigcup_{\varepsilon\le i} A_\varepsilon \cup \{\alpha^*\}, g^*(j_i), d, \lambda_{j_i} \in M_{\alpha^*,j_i} \quad \text{and} \quad f^*(j_i) \in \lambda_{j_i} \setminus \sup(M_{\alpha^*,j_i} \cap \lambda_{j_i}).$$

Hence the set

$$B_i := \Big\{ y < \lambda_{j_i} : \Big(\forall x \in \bigcup_{\varepsilon \le i} A_\varepsilon \cup \{\alpha^*\} \Big) (d(x,y) = g^*(j_i)) \Big\}$$

has to be unbounded in λ_{j_i}. It is easy to check that j_i, A_i, B_i satisfy clauses (α)–(κ).

Thus we have carried out the induction step, finishing the proof of the theorem. $\blacksquare_{2.1}$

THEOREM 2.2. *Suppose μ is a singular limit of measurable cardinals. Then*

(1) $\binom{\mu^+}{\mu} \to \binom{\mu}{\mu}_\theta$ *if $\theta = 2$ or at least $\theta < \mathrm{cf}(\mu)$.*

(2) *Moreover, if $\alpha^* < \mu^+$ and $\theta < \mathrm{cf}(\mu)$ then* $\binom{\mu^+}{\mu} \to \binom{\alpha^*}{\mu}_\theta$.

(3) *If $\theta < \mu$, $\alpha^* < \mu^+$ and d is a function from $\mu^+ \times \mu$ to θ then for some $A \subseteq \mu^+$, $\mathrm{otp}(A) = \alpha^*$, and $B = \bigcup_{i<\mathrm{cf}(\mu)} B_i \subseteq \mu$, $d \restriction A \times B_i$ is constant for each $i < \mathrm{cf}(\mu)$.*

Proof. Clearly (3)⇒(2)⇒(1), so we shall prove part (3).

Let $d : \mu^+ \times \mu \to \theta$. Let $\kappa := \mathrm{cf}(\mu)$. Choose sequences $\langle \lambda_i : i < \kappa \rangle$ and $\langle \mu_i : i < \kappa \rangle$ such that $\langle \mu_i : i < \kappa \rangle$ is increasing continuous, $\mu = \sum_{i<\kappa} \mu_i$, $\mu_0 > \kappa + \theta$, each λ_i is measurable and $\mu_i < \lambda_i < \mu_{i+1}$ (for $i < \kappa$). Let D_i be a λ_i-complete uniform ultrafilter on λ_i. For $\alpha < \mu^+$ define $g_\alpha \in {}^\kappa\theta$ by $g_\alpha(i) = \gamma$ iff $\{\beta < \lambda_i : d(\alpha,\beta) = \gamma\} \in D$ (as $\theta < \lambda_i$ it exists). The number of such functions is $\theta^\kappa < \mu$ (as μ is necessarily strong limit), so for some $g^* \in {}^\kappa\theta$ the set $A := \{\alpha < \mu^+ : g_\alpha = g^*\}$ is unbounded in μ^+. For each $i < \kappa$ we define an equivalence relation e_i on μ^+:

$$\alpha e_i \beta \quad \text{iff} \quad (\forall \gamma < \lambda_i)[d(\alpha,\gamma) = d(\beta,\gamma)].$$

So the number of e_i-equivalence classes is $\le {}^{\lambda_i}\theta < \mu$. Hence we can find an increasing continuous sequence $\langle \alpha_\zeta : \zeta < \mu^+ \rangle$ of ordinals $< \mu^+$ such that:

(∗) for each $i < \kappa$ and e_i-equivalence class X, either $X \cap A \subseteq \alpha_0$, or for every $\zeta < \mu^+$, $(\alpha_\zeta, \alpha_{\zeta+1}) \cap X \cap A$ has cardinality μ.

Let $\alpha^* = \bigcup_{i<\kappa} a_i$, $|a_i| = \mu_i$, $\langle a_i : i < \kappa \rangle$ pairwise disjoint. Now, by induction on $i < \kappa$, we choose A_i, B_i such that:

(a) $A_i \subseteq \bigcup \{(\alpha_\zeta, \alpha_{\zeta+1}) : \zeta \in a_i\} \cap A$ and each $A_i \cap (\alpha_\zeta, \alpha_{\zeta+1})$ is a singleton,

(b) $B_i \in D_i$,

(c) if $\alpha \in A_i$, $\beta \in B_j$, $j \le i$ then $d(\alpha,\beta) = g^*(j)$.

Now, at stage i, $\langle (A_\varepsilon, B_\varepsilon) : \varepsilon < i \rangle$ are already chosen. Let us choose A_ε. For each $\zeta \in a_i$ choose $\beta_\zeta \in (\alpha_\zeta, \alpha_{\zeta+1}) \cap A$ such that if $i > 0$ then for some

$\beta' \in A_0$, $\beta_\zeta e_i \beta'$, and let $A_i = \{\beta_\zeta : \zeta \in a_i\}$. Now clause (a) is immediate, and the relevant part of clause (c), i.e. $j < i$, is O.K. Next, as $\bigcup_{j \le i} A_j \subseteq A$, the set

$$B_i := \bigcap_{j \le i} \bigcap_{\beta \in A_j} \{\gamma < \lambda_i : d(\beta, \gamma) = g^*(i)\}$$

is the intersection of $\le \mu_i < \lambda_i$ sets from D_i and hence $B_i \in D_i$. Clearly clause (b) and the remaining part of clause (c) (i.e. $j = i$) holds. So we can carry out the induction and hence finish the proof. $\blacksquare_{2.2}$

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