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Distinguishing two partition properties of ω_1

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Abstract. It is consistent that $\omega_1 \to (\omega_1, (\omega : 2))^2$ but $\omega_1 \not\to (\omega_1, \omega + 2)^2$.

One of the classic results in combinatorial set theory is the Dushnik–Miller theorem [3] which states that $\omega_1 \to (\omega_1, \omega)^2$ holds and so gives the first transfinite variant of Ramsey's theorem. Later Erdős and Rado [4] extended this to $\omega_1 \to (\omega_1, \omega+1)^2$ and for a long period it was open if the even stronger $\omega_1 \to (\omega_1, \omega+2)^2$ holds. This was finally answered by A. Hajnal, who in [5] showed that if the continuum hypothesis is true then $\omega_1 \not\to (\omega_1, \omega+2)^2$ holds. Actually, Hajnal gave a stronger example, he produced a graph witnessing $\omega_1 \not\to (\omega_1, (\omega : 2))^2$. (See [2] for applications of his method to topology.) The consistency of the positive partition relation $\omega_1 \to (\omega_1, (\omega : 2))^2$ was then given by J. Baumgartner and A. Hajnal in [1], in fact they deduced this from $\mathrm{MA}_{\aleph_1}$. Only much later did Todorčević prove the consistency of the relation $\omega_1 \to (\omega_1, \omega + 2)^2$ and even that of $\omega_1 \to (\omega_1, \alpha)^2$ for any countable ordinal α (see [6]). In an unpublished work he also showed that $\mathrm{MA}_{\aleph_1}$ alone implies $\omega_1 \to (\omega_1, \omega^2)^2$ but at present it seems unsolved if the full positive result follows from Martin's axiom. Here we show that the two variants of the Hajnal partition theorem are indeed different; it is consistent that $\omega_1 \to (\omega_1, (\omega : 2))^2$ holds yet $\omega_1 \not\to (\omega_1, \omega + 2)^2$.

Notation. Definitions. If $(A, <)$ is an ordered set and $A, B \subseteq V$ then $A < B$ denotes that $x < y$ holds whenever $x \in A$, $y \in B$. $A < \{a\}$ is denoted by $A < a$, etc. If S is a set and κ is a cardinal, then $[S]^\kappa = \{X \subseteq S : |X| = \kappa\}$ and $[S]^{<\kappa} = \{X \subseteq S : |X| < \kappa\}$. A *graph* is an ordered pair (V, X) where V is some set (the set of *vertices*) and $X \subseteq [V]^2$ (the set of *edges*). In some cases we identify the graph and X. If (V, X) is a graph, a set $A \subseteq V$ is a

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complete subgraph if $[A]^2 \subseteq X$, and it is an *independent set* if $[A]^2 \cap X = \emptyset$. If X is a graph on some ordered set $(V,<)$ and β, γ are ordinals, then a *subgraph of type* $(\beta : \gamma)$ is a subset $B \times C \subseteq X$ where the types of B, C are β, γ, respectively, and $B < C$.

If α, β, γ are ordinals, then the partition relation $\alpha \to (\beta,\gamma)^2$ denotes that the following statement is true: every graph on a vertex set of type α has either an independent set of type β or a complete subgraph of type γ. The negation of this statement is denoted, of course, by $\alpha \not\to (\beta,\gamma)^2$. Similarly, $\alpha \to (\beta,(\gamma : \delta))^2$ denotes that in a graph on α if there is no independent set of type β then there is a complete bipartite graph of type $(\gamma : \delta)$. Again, the negation is obtained by crossing the arrow.

THEOREM. *It is consistent that* $\omega_1 \not\to (\omega_1, \omega+2)^2$ *yet* $\omega_1 \to (\omega_1,(\omega : 2))^2$.

Proof. Let V be a model of ZFC+GCH. We are going to construct a finite support iteration of length ω_2, $(P_\alpha, Q_\alpha : \alpha < \omega_2)$. Q_0 will give a counterexample to $\omega_1 \to (\omega_1, \omega+2)^2$, for $0 < \alpha < \omega_2$ we select a graph Y_α on ω_1 with no subgraph of type $(\omega : 2)$ and Q_α will be a forcing which adds an uncountable independent set.

We define Q_0 as follows. $q = (s,g,f) \in Q_0$ iff $s \in [\omega_1]^{<\omega}$, $g \subseteq [s]^2$, $f : g \to \omega$ with the property that if $a \cup \{x,y\}$ is a complete subgraph of (s,g), i.e., $[a \cup \{x,y\}]^2 \subseteq g$, and $a < x < y$ then $|a| \le f(x,y)$. $(s',g',f') \le (s,g,f)$ iff $s' \supseteq s$, $f = f' \cap [s]^2$, $f' \supseteq f$. It is clear that Q_0 adds a graph X on ω_1 with no complete subgraph of type $\omega+2$.

If $0 < \alpha < \omega_2$ and the iteration P_α is given assume that $Y_\alpha \in V^{P_\alpha}$ is a graph on ω_1 with no subgraph of type $(\omega : 2)$. We set $q \in Q_\alpha$ iff $q \in [\omega_1]^{<\omega}$ is an indepedent set of Y_α. $q' \le q$ iff $q' \supseteq q$. It is well known that Q_α is ccc. This implies that there is a $\delta < \omega_1$ such that if $q \in Q_\alpha$ has $q \cap \delta = \emptyset$ then q has extensions to arbitrarily large ordinals. We assume that every q is as described, or, better, by removing the part of Y_α below δ we can make $\delta = 0$. With this, Q_α will really add an uncountable independent subset of Y_α.

The results of [4] show that Q_0 is ccc and as all the other factors are ccc this way we get a ccc forcing P_{ω_2}. (Indeed, we will prove stronger statements soon.) This makes it possible that with a bookkeeping every appropriate graph on ω_1 can be some Y_α and so we prove the result if we show that X remains a graph in V^{P_α} which contains no uncountable independent sets.

For $p \in P_\alpha$ $(1 \le \alpha \le \omega_2)$ we denote by $\mathrm{supp}(p)$ the support of p, which is a finite subset of α. If $\beta < \alpha$, then $p|\beta$ is the restriction of p to β. A condition $p \in P_\alpha$ is *nice* if for every $0 < \beta < \alpha$ the condition $p|\beta$ determines the finite set $p(\beta)$, that is, it is not only a name of it, but an actual set.

LEMMA 1. *For* $\alpha \le \omega_2$ *the nice conditions form a dense subset of* P_α.

Proof (by induction on α). The statement is obvious for $\alpha = 1$. As every support is finite, there is nothing to prove for α limit. If $p \in P_{\alpha+1}$ pick a $p' \in P_\alpha$, $p' \le p|\alpha$ determining $p(\alpha)$. Extend p' to a nice $p'' \le p'$. Now $(p'', p(\alpha))$ is as required. ■

From now on we will mostly work with nice conditions.

Assume that $0 < \alpha \le \omega_2$, $p_0, p_1 \in P_\alpha$, $p_i(0) = (s \cup s_i, g_i, f_i)$ for $i < 2$ with s, s_0, s_1 disjoint. We call an extension $q \le p_0, p_1$ *edgeless* if for $q(0) = (s^*, g^*, f^*)$ the graph g^* contains no edge between s_0 and s_1. We will frequently use the obvious fact that if $p_i' \le p_i$ for $i < 2$ then every edgeless extension of p_0', p_1' is an edgeless extension of p_0, p_1.

Lemma 2. *If $\alpha \le \omega_2$, $k < \omega$, and $\aleph_1$ conditions are given in P_α then some k of them have an edgeless common extension.*

Proof (by induction on α). Let $p_\xi \in P_\alpha$ be given. We can assume outright that $p_\xi(0) = (s \cup s_\xi, g_\xi, f_\xi)$ with $\{s, s_\xi : \xi < \omega_1\}$ disjoint, and these conditions are compatible. We can also suppose that the supports of the conditions form a Δ-system.

The statement is obvious if $\alpha = 1$.

Assume now that α is limit. If $\mathrm{cf}(\alpha) \ne \omega_1$ then there is a $\beta < \alpha$ such that P_β contains an uncountable subfamily of $\{p_\xi : \xi < \omega_1\}$ and we are done by the inductive hypothesis. If $\mathrm{cf}(\alpha) = \omega_1$ then there is a $\beta < \alpha$ such that the supports are pairwise disjoint beyond β. This follows from the fact that they form a Δ-system. These arguments give the result for limit α.

It suffices, therefore, to show the lemma for $\alpha+1$, assuming that it holds for α. Next we argue that it is enough to show it for $k = 2$. This will be done by remarking that if it is true for some $k \ge 2$ then it is true for $2k$. Indeed, if the conditions $\{p_\xi : \xi < \omega_1\}$ are given and we know the lemma for k then we can inductively choose $\{q_\xi : \xi < \omega_1\}$ such that q_ξ is an edgeless extension of $\{p_\xi : \xi \in s_\tau\}$ where the s_τ's are disjoint k-element subsets of ω_1. If now q_{τ_0} and q_{τ_1} admit an edgeless extension r then r is an edgeless extension of $\{p_\xi : \xi \in s_{\tau_0} \cup s_{\tau_1}\}$ and so our claim is proved.

Assume therefore that (p_ξ, q_ξ) are nice conditions in $P_{\alpha+1}$. We can as well assume that the sets $\{q_\xi : \xi < \omega_1\}$ form a Δ-system and $q_\xi = W \cup U_\xi$ holds for $\xi < \omega_1$ where $|U_\xi| = n$ for some $n < \omega$. We will ignore W as it will play no role in finding an appropriate extension. As the sets $\{U_\xi : \xi < \omega_1\}$ are disjoint, $\min(U_\xi) \ge \xi$ for almost every (closed unboundedly many) ξ.

Using the lemma itself for α we can find (by some re-indexing) a stationary set $S \subseteq \omega_1$ and conditions which are edgeless extensions

$$\overline{p}_\xi \le p_{\omega\xi}, p_{\omega\xi+1}, \ldots, p_{\omega\xi+n} \quad (\xi \in S)$$

with $\xi \le U_{\omega\xi} < U_{\omega\xi+1} < \ldots < U_{\omega\xi+n}$ and we can even assume that $\overline{p}_\xi$ determines a bound $\tau(\xi) < \omega\xi$ for those points $\gamma < \omega\xi$ which are joined to two or more points in $U_{\omega\xi} \cup \ldots \cup U_{\omega\xi+n}$. This bound exists as there are only finitely many ordinals γ as described above (by the condition that Y_α has no subgraph of type $(\omega : 2)$). By the pressing-down lemma there is a stationary subset $S' \subseteq S$ on which the function $\tau(\xi)$ is constant, $\tau(\xi) = \tau$. Using the lemma for α there are $\tau < \xi_0 < \xi_1$ with an edgeless extension $r \le \overline{p}_{\xi_0}, \overline{p}_{\xi_1}$. Now observe that r forces that any of the n points in $U_{\omega\xi_0}$ is joined to at most one point in $U_{\omega\xi_1} \cup \ldots \cup U_{\omega\xi_1+n}$. Again, we can assume that r determines these points. As there are only n elements in $U_{\omega\xi_0}$ and $n+1$ sets $U_{\omega\xi_1}, \ldots, U_{\omega\xi_1+n}$ there is some $0 \le i \le n$ with no edge between $U_{\omega\xi_0}$ and $U_{\omega\xi_1+i}$. This means that $(r, q_{\omega\xi_0} \cup q_{\omega\xi_1+i})$ is an edgeless extension of $(p_{\omega\xi_0}, q_{\omega\xi_0})$ and $(p_{\omega\xi_1+i}, q_{\omega\xi_1+i})$. ∎

LEMMA 3. *If $1 \le \alpha \le \omega_2$, $p_\xi \in P_\alpha$ for $\xi < \omega_1$, $p_\xi(0) = (s \cup s_\xi, g_\xi, f_\xi)$ with the sets $\{s, s_\xi : \xi < \omega_1\}$ disjoint, $x_\xi \in s_\xi$ and $t_\xi \subseteq s_\xi$ is independent in g_ξ, then there are $\xi < \xi'$ with a common extension r with $r(0) = (s^*, g^*, f^*)$ such that $\{x_\xi\} \times t_{\xi'} \subseteq g^*$.*

Proof (by induction on α). Assume first that $\alpha = 1$. We can assume that we are given $p_0 = (s \cup s_0, g_0, f_0)$, $p_1 = (s \cup s_1, g_1, f_1)$, $s_0 < s_1$, $x_0 \in s_0$, $t_1 \subseteq s_1$ with $g_0 \cap [s]^2 = g_1 \cap [s]^2$, $f_0|(g_0 \cap g_1) = f_1|(g_0 \cap g_1)$, t_1 independent in g_1. We try to extend p_0, p_1 to $r = (s^*, g^*, f^*)$ where $s^* = s \cup s_0 \cup s_1$, $g^* = g_0 \cup g_1 \cup (\{x_0\} \times t_1)$, $f^* \supseteq f_0, f_1$ satisfying $f^*(x_0, y) = |s|$ for $y \in t_1$. We only have to show that r is a condition. Assume that $a < y < z$ form a complete subgraph of g^* yet $|a| > f^*(y, z)$. A moment's reflection shows that the only problematic case is if $y, z \in s_1$. A "new" point joined to them can only be x_0 but this is excluded by our assumption that t_1 be independent. We therefore proved the case $\alpha = 1$.

The case when α is limit can be treated exactly as in Lemma 2.

Assume now that we are given the nice conditions $(p_\xi, q_\xi) \in P_{\alpha+1}$ with $p_\xi(0) = (s \cup s_\xi, g_\xi, f_\xi)$ where the sets $\{s, s_\xi : \xi < \omega_1\}$ are disjoint, and we are also given $x_\xi \in s_\xi$, and the independent $t_\xi \subseteq s_\xi$. We will call x_ξ *the distinguished element* of p_ξ and t_ξ *the distinguished subset* of p_ξ. Again, as in Lemma 2 we assume that the sets $\{q_\xi : \xi < \omega_1\}$ form a Δ-system, and $q_\xi = W \cup U_\xi$ holds for $\xi < \omega_1$ where $|U_\xi| = n$ for some $n < \omega$. Using Lemma 2 $\aleph_1$ times we can create the edgeless extensions

$$\overline{p}_\xi \le p_{\omega\xi}, p_{\omega\xi+1}, \ldots, p_{\omega\xi+n} \qquad (\xi \in S)$$

for a stationary $S \subseteq \omega_1$ with $\omega\xi \le U_{\omega\xi} < \ldots < U_{\omega\xi+n}$. We let $x_{\omega\xi}$ be the distinguished element and $t_{\omega\xi} \cup \ldots \cup t_{\omega\xi+n}$ the distinguished subset of $\overline{p}_\xi$. This is possible, as we made an edgeless extension, so the above set is independent. As in Lemma 2, we assume that $\overline{p}_\xi$ forces a bound $\tau(\xi) < \omega\xi$ for those points below $\omega\xi$ which are joined to two or more vertices in $U_{\omega\xi} \cup \ldots \cup U_{\omega\xi+n}$. On a stationary set, $\tau(\xi) = \tau$. Pick two elements of it, $\tau < \xi < \xi'$, for which the inductive hypothesis applies, that is, there is a condition $r \le \overline{p}_\xi, \overline{p}_{\xi'}$ in which $x_{\omega\xi}$ is joined to $t_{\omega\xi} \cup \ldots \cup t_{\omega\xi+n}$ and also determining the edges between $U_{\omega\xi}$ and $U_{\omega\xi'} \cup \ldots \cup U_{\omega\xi'+n}$. As every point of $U_{\omega\xi}$ is joined to at most one point in $U_{\omega\xi'} \cup \ldots \cup U_{\omega\xi'+n}$, there is a $0 \le i \le n$ such that $r \Vdash U_{\omega\xi} \cup U_{\omega\xi'+i}$ is independent. Now $(r, q_{\omega\xi} \cup q_{\omega\xi'+i})$ is an extension of $(p_{\omega\xi}, q_{\omega\xi})$, $(p_{\omega\xi'+i}, q_{\omega\xi'+i})$ as required. ∎

With Lemma 3 we can conclude the proof of the Theorem. Assume that $p \in P_{\omega_2}$ forces that A is an uncountable independent subset of X in $V^{P_{\omega_2}}$. There exist, for $\xi < \omega_1$, conditions $p_\xi \le p$, and distinct ordinals x_ξ, such that $p_\xi \Vdash x_\xi \in A$. We assume that $p_\xi(0) = (s \cup s_\xi, g_\xi, f_\xi)$ with $x_\xi \in s_\xi$. Let x_ξ be the distinguished element and $\{x_\xi\}$ the distinguished subset of p_ξ. By Lemma 3 we can find $\xi < \xi'$ with a common extension of p_ξ, $p_{\xi'}$ which adds the edge $\{x_\xi, x_{\xi'}\}$ to X, and therefore forces a contradiction. ∎

References

[1] J. E. Baumgartner and A. Hajnal, *A proof (involving Martin's axiom) of a partition relation*, Fund. Math. 78 (1973), 193–203.

[2] W. W. Comfort and S. Negrepontis, *Chain Conditions in Topology*, Cambridge Univ. Press, 1982.

[3] B. Dushnik and E. W. Miller, *Partially ordered sets*, Amer. J. Math. 63 (1941), 600–610.

[4] P. Erdős and R. Rado, *A partition calculus in set theory*, Bull. Amer. Math. Soc. 62 (1956), 427–489.

[5] A. Hajnal, *Some results and problems on set theory*, Acta Math. Acad. Sci. Hungar. 11 (1960), 277–298.

[6] S. Todorčević, *Forcing positive partition relations*, Trans. Amer. Math. Soc. 280 (1983), 703–720.

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