Distinguishing two partition properties of $\omega_1$

by

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Abstract. It is consistent that $\omega_1 \rightarrow (\omega_1, (\omega_1 : 2))^2$ but $\omega_1 \not\rightarrow (\omega_1, \omega + 2)^2$.

One of the classic results in combinatorial set theory is the Dushnik–Miller theorem [3] which states that $\omega_1 \rightarrow (\omega_1, \omega)^2$ holds and so gives the first transfinite variant of Ramsey’s theorem. Later Erdős and Rado [4] extended this to $\omega_1 \rightarrow (\omega_1, \omega + 1)^2$ and for a long period it was open if the even stronger $\omega_1 \rightarrow (\omega_1, \omega + 2)^2$ holds. This was finally answered by A. Hajnal, who in [5] showed that if the continuum hypothesis is true then $\omega_1 \not\rightarrow (\omega_1, \omega + 2)^2$ holds. Actually, Hajnal gave a stronger example, he produced a graph witnessing $\omega_1 \not\rightarrow (\omega_1, (\omega : 2))^2$. (See [2] for applications of his method to topology.)

The consistency of the positive partition relation $\omega_1 \rightarrow (\omega_1, (\omega : 2))^2$ was then given by J. Baumgartner and A. Hajnal in [1], in fact they deduced this from MA$_{\aleph_1}$. Only much later did Todorčević prove the consistency of the relation $\omega_1 \rightarrow (\omega_1, \omega + 2)^2$ and even that of $\omega_1 \rightarrow (\omega_1, \alpha)^2$ for any countable ordinal $\alpha$ (see [6]). In an unpublished work he also showed that MA$_{\aleph_1}$ alone implies $\omega_1 \rightarrow (\omega_1, \omega^2)^2$ but at present it seems unsolved if the full positive result follows from Martin’s axiom. Here we show that the two variants of the Hajnal partition theorem are indeed different; it is consistent that $\omega_1 \rightarrow (\omega_1, (\omega : 2))^2$ holds yet $\omega_1 \not\rightarrow (\omega_1, \omega + 2)^2$.

Notation. Definitions. If $(A, <)$ is an ordered set and $A, B \subseteq V$ then $A < B$ denotes that $x < y$ holds whenever $x \in A, y \in B$. $A < \{a\}$ is denoted by $A < a$, etc. If $S$ is a set and $\kappa$ is a cardinal, then $[S]^\kappa = \{X \subseteq S : |X| = \kappa\}$ and $[S]^{<\kappa} = \{X \subseteq S : |X| < \kappa\}$. A graph is an ordered pair $(V, X)$ where $V$ is some set (the set of vertices) and $X \subseteq [V]^2$ (the set of edges). In some cases we identify the graph and $X$. If $(V, X)$ is a graph, a set $A \subseteq V$ is a...
complete subgraph if $[A]^{2} \subseteq X$, and it is an independent set if $[A]^{2} \cap X = \emptyset$. If $X$ is a graph on some ordered set $(V, <)$ and $\beta, \gamma$ are ordinals, then a subgraph of type $(\beta : \gamma)$ is a subset $B \times C \subseteq X$ where the types of $B, C$ are $\beta, \gamma$, respectively, and $B < C$.

If $\alpha, \beta, \gamma$ are ordinals, then the partition relation $\alpha \rightarrow (\beta, \gamma)^{2}$ denotes that the following statement is true: every graph on a vertex set of type $\alpha$ has either an independent set of type $\beta$ or a complete subgraph of type $\gamma$. The negation of this statement is denoted, of course, by $\alpha \not\rightarrow (\beta, \gamma)^{2}$. Similarly, $\alpha \rightarrow (\beta, (\gamma : \delta))^{2}$ denotes that in a graph on $\alpha$ if there is no independent set of type $\beta$ then there is a complete bipartite graph of type $(\gamma : \delta)$. Again, the negation is obtained by crossing the arrow.

**Theorem.** It is consistent that $\omega_1 \not\rightarrow (\omega_1, \omega + 2)^{2}$ yet $\omega_1 \rightarrow (\omega_1, (\omega : 2))^{2}$.

**Proof.** Let $V$ be a model of ZFC+GCH. We are going to construct a finite support iteration of length $\omega_2$. $(P_\alpha, Q_\alpha : \alpha < \omega_2)$. $Q_0$ will give a counterexample to $\omega_1 \rightarrow (\omega_1, \omega + 2)^{2}$, for $0 < \alpha < \omega_2$ we select a graph $Y_\alpha$ on $\omega_1$ with no subgraph of type $(\omega : 2)$ and $Q_\alpha$ will be a forcing which adds an uncountable independent set.

We define $Q_0$ as follows. $q = (s, g, f) \in Q_0$ iff $s \in [\omega_1]^{<\omega}$, $g \subseteq [s]^{2}$, $f : g \rightarrow \omega$ with the property that if $a \cup \{x, y\}$ is a complete subgraph of $(s, g)$, i.e., $[a \cup \{x, y\}]^{2} \subseteq g$, and $a < x < y$ then $|a| \leq f(x, y)$. $(s', g', f') \leq (s, g, f)$ iff $s' \supseteq s$, $f = f' \cap [s]^{2}$, $f' \supseteq f$. It is clear that $Q_0$ adds a graph $X$ on $\omega_1$ with no complete subgraph of type $\omega + 2$.

If $0 < \alpha < \omega_2$ and the iteration $P_\alpha$ is given assume that $Y_\alpha \in V^{P_\alpha}$ is a graph on $\omega_1$ with no subgraph of type $(\omega : 2)$. We set $q \in Q_\alpha$ iff $q \in [\omega_1]^{<\omega}$ is an independent set of $Y_\alpha$. $q' \leq q$ iff $q' \supseteq q$. It is well known that $Q_\alpha$ is ccc. This implies that there is a $\delta < \omega_1$ such that if $q \in Q_\alpha$ has $q \cap \delta = \emptyset$ then $q$ has extensions to arbitrarily large ordinals. We assume that every $q$ is as described, or, better, by removing the part of $Y_\alpha$ below $\delta$ we can make $\delta = 0$. With this, $Q_\alpha$ will really add an uncountable independent subset of $Y_\alpha$.

The results of [4] show that $Q_0$ is ccc and as all the other factors are ccc this way we get a ccc forcing $P_{\omega_2}$. (Indeed, we will prove stronger statements soon.) This makes it possible that with a bookkeeping every appropriate graph on $\omega_1$ can be some $Y_\alpha$ and so we prove the result if we show that $X$ remains a graph in $V^{P_\alpha}$ which contains no uncountable independent sets.

For $p \in P_\alpha (1 \leq \alpha \leq \omega_2)$ we denote by $\text{supp}(p)$ the support of $p$, which is a finite subset of $\alpha$. If $\beta < \alpha$, then $p|\beta$ is the restriction of $p$ to $\beta$. A condition $p \in P_\alpha$ is nice if for every $0 < \beta < \alpha$ the condition $p|\beta$ determines the finite set $p(\beta)$, that is, it is not only a name of it, but an actual set.

**Lemma 1.** For $\alpha \leq \omega_2$ the nice conditions form a dense subset of $P_\alpha$. 
Proof (by induction on $\alpha$). The statement is obvious for $\alpha = 1$. As every support is finite, there is nothing to prove for $\alpha$ limit. If $p \in P_{\alpha+1}$ pick a $p' \in P_\alpha$, $p' \leq p$ determining $p(\alpha)$. Extend $p'$ to a nice $p'' \leq p'$. Now $(p'', p(\alpha))$ is as required. 

From now on we will mostly work with nice conditions.

Assume that $0 < \alpha \leq \omega_2$, $p_0, p_1 \in P_\alpha$, $p_i(0) = (s \cup s_i, g_i, f_i)$ for $i < 2$ with $s, s_0, s_1$ disjoint. We call an extension $q \leq p_0, p_1$ edgeless if for $q(0) = (s^*, g^*, f^*)$ the graph $g^*$ contains no edge between $s_0$ and $s_1$. We will frequently use the obvious fact that if $p_i' \leq p_i$ for $i < 2$ then every edgeless extension of $p_0', p_1'$ is an edgeless extension of $p_0, p_1$.

Lemma 2. If $\alpha \leq \omega_2$, $k < \omega$, and $R_1$ conditions are given in $P_\alpha$ then some $k$ of them have an edgeless common extension.

Proof (by induction on $\alpha$). Let $p_\xi \in P_\alpha$ be given. We can assume outright that $p_\xi(0) = (s \cup s_\xi, g_\xi, f_\xi)$ with $\{s, s_\xi : \xi < \omega_1\}$ disjoint, and these conditions are compatible. We can also suppose that the supports of the conditions form a $\Delta$-system.

The statement is obvious if $\alpha = 1$.

Assume now that $\alpha$ is limit. If $\text{cf}(\alpha) \neq \omega_1$ then there is a $\beta < \alpha$ such that $P_\beta$ contains an uncountable subfamily of $\{p_\xi : \xi < \omega_1\}$ and we are done by the inductive hypothesis. If $\text{cf}(\alpha) = \omega_1$ then there is a $\beta < \alpha$ such that the supports are pairwise disjoint beyond $\beta$. This follows from the fact that they form a $\Delta$-system. These arguments give the result for limit $\alpha$.

It suffices, therefore, to show the lemma for $\alpha + 1$, assuming that it holds for $\alpha$. Next we argue that it is enough to show it for $k = 2$. This will be done by remarking that if it is true for some $k \geq 2$ then it is true for $2k$. Indeed, if the conditions $\{p_\xi : \xi < \omega_1\}$ are given and we know the lemma for $k$ then we can inductively choose $\{q_\xi : \xi < \omega_1\}$ such that $q_\xi$ is an edgeless extension of $\{p_\xi : \xi \in s_\tau\}$ where the $s_\tau$'s are disjoint $k$-element subsets of $\omega_1$. If now $q_{\tau_0}$ and $q_{\tau_1}$ admit an edgeless extension $r$ then $r$ is an edgeless extension of $\{p_\xi : \xi \in s_{\tau_0} \cup s_{\tau_1}\}$ and so our claim is proved.

Assume therefore that $(p_\xi, q_\xi)$ are nice conditions in $P_{\alpha+1}$. We can as well assume that the sets $\{q_\xi : \xi < \omega_1\}$ form a $\Delta$-system and $q_\xi = W \cup U_\xi$ holds for $\xi < \omega_1$ where $|U_\xi| = n$ for some $n < \omega$. We will ignore $W$ as it will play no role in finding an appropriate extension. As the sets $\{U_\xi : \xi < \omega_1\}$ are disjoint, $\min(U_\xi) \geq \xi$ for almost every (closed unboundedly many) $\xi$.

Using the lemma itself for $\alpha$ we can find (by some re-indexing) a stationary set $S \subseteq \omega_1$ and conditions which are edgeless extensions

$$p_\xi \leq p_{\omega \xi}, p_{\omega \xi+1}, \ldots, p_{\omega \xi+n} \quad (\xi \in S)$$

with $\xi \leq U_\omega \xi < U_{\omega \xi+1} < \ldots < U_{\omega \xi+n}$ and we can even assume that $p_\xi$ determines a bound $\tau(\xi) < \omega_1$ for those points $\gamma < \omega_1$ which are joined.
to two or more points in \( U_\omega \cup \ldots \cup U_{\omega^{\xi+n}} \). This bound exists as there are only finitely many ordinals \( \gamma \) as described above (by the condition that \( Y_\alpha \)
has no subgraph of type \((\omega : 2)\)). By the pressing-down lemma there is a stationary subset \( S' \subseteq S \) on which the function \( \tau(\xi) \) is constant, \( \tau(\xi) = \tau \).

Using the lemma for \( \alpha \) there are \( \tau < \xi_0 < \xi_1 \) with an edgeless extension \( r \leq \bar{p}_\xi, p_\xi \). Now observe that \( r \) forces that any of the \( n \) points in \( U_\omega \) is \( \tau \) for \( \alpha \) at most one point in \( U_{\omega^{\xi_1}} \cup \ldots \cup U_{\omega^{\xi+n}} \). Again, we can assume that \( r \) determines these points. As there are only \( n \) elements in \( U_\omega \) and \( n+1 \) sets \( U_{\omega^{\xi_1}} \), \( \ldots \), \( U_{\omega^{\xi+n}} \) there is some \( 0 \leq i \leq n \) with no edge between \( U_\omega \) and \( U_{\omega^{\xi_1+i}} \). This means that \( (r, \omega^{\xi_0} \cup \omega^{\xi_1+i}) \) is an edgeless extension of \( (p_\omega^{\xi_0}, q_\omega^{\xi_0}) \) and \( (p_\omega^{\xi_1+i}, q_\omega^{\xi_1+i}) \).

**Lemma 3.** If \( 1 \leq \alpha \leq \omega_2 \), \( p_\xi \in P_\alpha \) for \( \xi < \omega_1 \), \( p_\xi(0) = (s \cup s_\xi, g_\xi, f_\xi) \) with the sets \( \{s, s_\xi : \xi < \omega_1\} \) disjoint, \( x_\xi \in s_\xi \) and \( t_\xi \subseteq s_\xi \) is independent in \( g_\xi \), then there is \( \xi' < \xi \) with a common extension \( r \) with \( r(0) = (s', g', f') \) such that \( \{x_\xi\} \times t_{\xi'} \subseteq g' \).

**Proof** (by induction on \( \alpha \)). Assume first that \( \alpha = 1 \). We can assume that we are given \( p_0 = (s \cup s_0, g_0, f_0) \), \( p_1 = (s \cup s_1, g_1, f_1) \), \( s_0 < s_1 \), \( x_0 \in s_0 \), \( t_1 \subseteq s_1 \) with \( g_0 \cap [s]^2 = g_1 \cap [s]^2 \), \( f_0(g_0 \cap g_1) = f_1(g_0 \cap g_1) \), \( t_1 \) independent in \( g_1 \). We try to extend \( p_0, p_1 \) to \( r = (s', g', f') \) where \( s' = s \cup s_0 \cup s_1 \), \( g' = g_0 \cup g_1 \cup \{t_0 \times t_1\} \), \( f' \supseteq f_0, f_1 \) satisfying \( f'(x_0, y) = |s| \) for \( y \in t_1 \).

We only have to show that \( r \) is a condition. Assume that \( a < y < z \) form a complete subgraph of \( g' \) yet \( |a| > f'(y, z) \). A moment’s reflection shows that the only problematic case is if \( y, z \in s_1 \). A “new” point joined to them can only be \( x_0 \) but this is excluded by our assumption that \( t_1 \) be independent.

We therefore proved the case \( \alpha = 1 \).

The case when \( \alpha \) is limit can be treated exactly as in Lemma 2.

Assume now that we are given the nice conditions \((p_\xi, q_\xi) \in P_{\alpha+1} \) with \( p_\xi(0) = (s \cup s_\xi, g_\xi, f_\xi) \) where the sets \( \{s, s_\xi : \xi < \omega_1\} \) are disjoint, and we are also given \( x_\xi \in s_\xi \) and the independent \( t_\xi \subseteq s_\xi \). We will call \( x_\xi \) the distinguished element of \( p_\xi \) and \( t_\xi \) the distinguished subset of \( p_\xi \). Again, as in Lemma 2 we assume that the sets \( \{q_\xi : \xi < \omega_1\} \) form a \( \Delta \)-system, and \( q_\xi = W \cup U_\xi \) holds for \( \xi < \omega_1 \) where \( |U_\xi| = n \) for some \( n < \omega \). Using Lemma 2 \( n_1 \) times we can create the edgeless extensions

\[ \bar{p}_\xi \leq p_\omega \xi, p_\omega \xi+1, \ldots, p_\omega \xi+n \quad (\xi \in S) \]

for a stationary \( S \subseteq \omega_1 \) with \( \omega_\xi \leq U_\omega \xi < \ldots < U_{\omega^{\xi+n}} \). We let \( x_\omega \xi \) be the distinguished element and \( t_\omega \xi \cup \ldots \cup t_\omega \xi+n \) the distinguished subset of \( \bar{p}_\xi \).

This is possible, as we made an edgeless extension, so the above set is independent. As in Lemma 2, we assume that \( \bar{p}_\xi \) forces a bound \( \tau(\xi) < \omega_\xi \) for those points below \( \omega_\xi \) which are joined to two or more vertices in \( U_\omega \xi \cup \ldots \cup U_{\omega^{\xi+n}} \). On a stationary set, \( \tau(\xi) = \tau \). Pick two elements of
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it, $\tau < \xi < \xi'$, for which the inductive hypothesis applies, that is, there is a condition $r \leq p_\xi, p_\xi'$ in which $x_\xi$ is joined to $t_\omega \cup \ldots \cup t_{\omega \xi + n}$ and also determining the edges between $U_\omega \xi$ and $U_\omega \xi' \cup \ldots \cup U_\omega \xi' + n$. As every point of $U_\omega \xi$ is joined to at most one point in $U_\omega \xi' \cup \ldots \cup U_\omega \xi' + n$, there is a $0 \leq i \leq n$ such that $r \upharpoonright U_\omega \xi \cup U_\omega \xi' + i$ is independent. Now $(r, q_\omega, q_\omega')$ is an extension of $(p_\omega, q_\omega, q_\omega')$ as required.

With Lemma 3 we can conclude the proof of the Theorem. Assume that $p \in P_{\omega_2}$ forces that $A$ is an uncountable independent subset of $X$ in $V^{P_{\omega_2}}$. There exist, for $\xi < \omega_1$, conditions $p_\xi \leq p$, and distinct ordinals $x_\xi$, such that $p_\xi \upharpoonright x_\xi \in A$. We assume that $p_\xi(0) = (s \cup s_\xi, g_\xi, f_\xi)$ with $x_\xi \in s_\xi$. Let $x_\xi$ be the distinguished element and $\{x_\xi\}$ the distinguished subset of $p_\xi$. By Lemma 3 we can find $\xi < \xi'$ with a common extension of $p_\xi, p_\xi'$ which adds the edge $\{x_\xi, x_\xi'\}$ to $X$, and therefore forces a contradiction. ■

References


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