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The sequential topology on complete Boolean algebras

by

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Abstract. We investigate the sequential topology τ_s on a complete Boolean algebra B determined by algebraically convergent sequences in B. We show the role of weak distributivity of B in separation axioms for the sequential topology. The main result is that a necessary and sufficient condition for B to carry a strictly positive Maharam submeasure is that B is ccc and that the space (B, τ_s) is Hausdorff. We also characterize sequential cardinals.

1. Introduction. We deal with sequential topologies on complete Boolean algebras from the point of view of separation axioms.

Our motivation comes from the still open Control Measure Problem of D. Maharam (1947, [Ma]). Maharam asked whether every σ-complete Boolean algebra that carries a strictly positive continuous submeasure admits a σ-additive measure.

Let us review basic notions and facts concerning Maharam's problem. More details and further information can be found in Fremlin's work [Fr1].

Let B be a Boolean algebra. A *submeasure* on B is a function $\mu : B \to \mathbb{R}^+$ with the properties

(i) $\mu(\mathbf{0}) = 0$,
(ii) $\mu(a) \le \mu(b)$ whenever $a \le b$ (monotonicity),
(iii) $\mu(a \vee b) \le \mu(a) + \mu(b)$ (subadditivity).

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A submeasure μ on B is

(iv) *exhaustive* if $\lim \mu(a_n) = 0$ for every sequence $\{a_n : n \in \omega\}$ of disjoint elements,

(v) *strictly positive* if $\mu(a) = 0$ only if $a = \mathbf{0}$,

(vi) a (finitely additive) *measure* if $\mu(a \vee b) = \mu(a) + \mu(b)$ for any disjoint a and b.

If B is a σ-complete algebra, a submeasure μ on B is called a *Maharam submeasure* if it is *continuous*, i.e. $\lim \mu(a_n) = 0$ for every decreasing sequence $\{a_n : n \in \omega\}$ such that $\bigwedge\{a_n : n \in \omega\} = \mathbf{0}$. It is easy to see that a measure on a σ-complete algebra is continuous if and only if it is σ-additive.

We consider the following four classes of Boolean algebras.

- MBA: the class of all Boolean algebras that carry a strictly positive finitely additive measure.
- McBA: the class of all *measure* algebras, i.e. complete Boolean algebras that carry a strictly positive σ-additive measure.
- EBA: the class of all Boolean algebras that carry a strictly positive exhaustive submeasure.
- CcBA: the class of all complete algebras that carry a strictly positive continuous submeasure.

The diagram below shows the obvious relations between these classes:

$$\begin{array}{ccc} \boxed{\mathsf{CcBA}} & \subset & \boxed{\mathsf{EBA}} \\ \cup & & \cup \\ \boxed{\mathsf{McBA}} & \subset & \boxed{\mathsf{MBA}} \end{array}$$

The following theorem, whose proof is scattered throughout Fremlin's work [Fr1], gives additional information. Note that the relations between the classes with measure are the same as between the classes with submeasure.

1.1. Theorem. (i) *The class* MBA *consists exactly of all subalgebras of algebras in* McBA.

(ii) *The class* EBA *consists exactly of all subalgebras of algebras in* CcBA.

(iii) *The class* McBA *consists of all algebras in* MBA *that are complete and weakly distributive.*

(iv) *The class* CcBA *consists of all algebras in* EBA *that are complete and weakly distributive.*

The problem whether $\mathsf{CcBA} = \mathsf{McBA}$ is the problem of Maharam mentioned above. It follows from Theorem 1.1 that it is equivalent to the problem whether $\mathsf{EBA} = \mathsf{MBA}$.

The class MBA is closed under regular completions: Let B be a Boolean algebra and let μ be a finitely additive strictly positive measure. It follows from [Ke] that μ can be extended to a strictly positive measure on the completion $\overline{B}$.

Similarly, the class EBA is closed under regular completions (this was kindly pointed to us by S. Koppelberg): Let B be a Boolean algebra and let μ be a strictly positive exhaustive submeasure. By [Fr1], B can be embedded into a complete Boolean algebra A such that μ can be extended to a strictly positive exhaustive submeasure on A. By Sikorski's Extension Theorem ([Ko], p. 70), the completion $\overline{B}$ embeds in A, and so $\overline{B}$ also carries a strictly positive exhaustive submeasure.

Consider an algebra $B \in \mathsf{CcBA}$ and let μ be a strictly positive Maharam submeasure on B. The submeasure μ determines a topology on B: (B, ϱ_μ) is a metric space with the distance defined by $\varrho_\mu(a,b) = \mu(a \vartriangle b)$ for any $a, b \in B$. If ν is another such submeasure then ϱ_μ and ϱ_ν are equivalent; they determine the same topology on B. In [Ma], Maharam studied a sequential topology on complete Boolean algebras from the point of view of metrizability.

We study sequential topologies on complete Boolean algebras in a more general setting. Our goal is to show that the sequential topology τ_{s} on a ccc complete Boolean algebra B is Hausdorff if and only if B carries a strictly positive Maharam submeasure. Following [AnCh] and [Pl] we say that a cardinal κ is a *sequential cardinal* if there exists a continuous real-valued function on the space $(\mathcal{P}(\kappa), \tau_{\mathrm{s}})$ which is not continuous with respect to the product topology. We prove that κ is a sequential cardinal if and only if κ is uncountable and there is a nontrivial Maharam submeasure on the algebra $\mathcal{P}(\kappa)$.

2. Sequential topology.

We review some notions from topology.

2.1. Definition. Let (X, τ) be a topological space. The space X is

(i) *sequential* if a subset $A \subseteq X$ is closed whenever it contains all limits of τ-convergent sequences of elements of A;

(ii) *Fréchet* if for every $A \subseteq X$,

$$\mathrm{cl}_\tau(A) = \{x \in X : (\exists \langle x_n : n \in \omega\rangle \subseteq A)\ x_n \underset{\tau}{\to} x\}.$$

It is clear that every Fréchet space is sequential.

Now, consider a complete Boolean algebra B; σ-completeness is sufficient for the following definition. For a sequence $\langle b_n : n \in \omega\rangle$ of elements of B we define

$$\overline{\lim}\, b_n = \bigwedge_{k\in\omega} \bigvee_{n\geq k} b_n \quad \text{and} \quad \underline{\lim}\, b_n = \bigvee_{k\in\omega} \bigwedge_{n\geq k} b_n.$$

We say that a sequence $\langle b_n\rangle$ *algebraically converges* to an element $b \in B$ in symbols, $b_n \to b$, if $\overline{\lim}\, b_n = \underline{\lim}\, b_n = b$.

A sequence $\langle b_n\rangle$ algebraically converges if and only if there exist an increasing sequence $\langle a_n\rangle$ and a decreasing sequence $\langle c_n\rangle$ such that $a_n \le b_n \le c_n$ for all $n\in\omega$, and $\bigvee_{n\in\omega} a_n = \bigwedge_{n\in\omega} c_n$.

2.2. We summarize basic properties of $\to$:

(i) every sequence has at most one limit;
(ii) for a constant sequence $\langle x : n\in\omega\rangle$, we have $\langle x : n\in\omega\rangle \to x$;
(iii) $x_n \to \mathbf{0}$ iff $\overline{\lim}\, x_n = \mathbf{0}$;
(iv) if the x_n's are pairwise disjoint then $x_n \to \mathbf{0}$;
(v) $\overline{\lim}(x_n \vee y_n) = \overline{\lim}\, x_n \vee \overline{\lim}\, y_n$;
(vi) if $x_n \to x$ and $y_n \to y$ then $x_n \vee y_n \to x \vee y$ and $-x_n \to -x$;
(vii) if $\langle x_n\rangle$ is increasing then $x_n \to \bigvee_{n\in\omega} x_n$.

2.3. *Sequential topology on B.* Consider all topologies τ on B with the following property:

$$\text{if } x_n \to x \text{ then } x_n \underset{\tau}{\to} x.$$

There is a largest topology with respect to inclusion among all such topologies. We denote it by τ_{s} and call it the *sequential topology* on B.

The topology τ_{s} can be described as follows, by definining the closure operation: For any subset A of the algebra B let

$$u(A) = \{x : x \text{ is the limit of a sequence } \{x_n\} \text{ of elements of } A\}.$$

The closure of a set A in the topology τ_{s} is obtained by iteration of u:

$$\mathrm{cl}_{\tau_{\mathrm{s}}}(A) = \bigcup_{\alpha<\omega_1} u^{(\alpha)}(A),$$

where $u^{(\alpha+1)}(A) = u(u^{(\alpha)}(A))$, and $u^{(\alpha)}(A) = \bigcup_{\beta<\alpha} u^{(\beta)}$ for a limit α.

It is clear that the topology τ_{s} is T_1, i.e. every singleton is a closed set. Moreover, (B,τ_{s}) is a Fréchet space if and only if $\mathrm{cl}(A) = u(A)$ for every $A\subseteq B$.

We remark that a sequence $\{x_n\}$ converges to x topologically if and only if every subsequence of $\{x_n\}$ has a subsequence that converges to x algebraically.

Example (Measure algebras). Let B be a complete Boolean algebra carrying a strictly positive σ-additive measure μ. For any $a,b\in B$, let

$$\varrho(a,b) = \mu(a \vartriangle b);$$

ϱ is a metric on B and the topology given by ϱ coincides with the sequential topology. Hence (B,τ_{s}) is metrizable.

Maharam's Control Measure Problem is equivalent to the question of whether there exist complete Boolean algebras other than the algebras in the class McBA for which the sequential topology is metrizable.

Properties of the topology τ_s

2.4. Proposition. (i) *The operation of taking complement is continuous* (*and hence a homeomorphism*).

(ii) *For a fixed a, the function $a \vee x$ is a continuous function of x.*

(iii) *For a fixed a, the function $a \vartriangle x$ is continuous.*

The operation $\vee$ is generally not a continuous function of two variables. As a consequence of (iii), the space (B, τ_s) is homogeneous: given $a, b \in B$, there is a homeomorphism f such that $f(a) = b$, namely $f(x) = (x \vartriangle b) \vartriangle a$. The topology τ_s is determined by the family $\mathcal{N}_0$ of all neighborhoods of $\mathbf{0}$ as for every $a \in B$ and every set W, W is a neighborhood of a if and only if $a \vartriangle W \in \mathcal{N}_0$.

As a consequence of homogeneity of (B, τ_s), B does not have isolated points unless B is finite.

2.5. Lemma. *Let B be a σ-complete algebra. Let $\{u_n\}_{n=0}^{\infty}$ be an antichain in B, and let U be a neighborhood of $\mathbf{0}$. Then there exists a k such that $B \restriction \bigvee_{n \geq k} u_n \subset U$.*

Proof. If not, then for every k there exists an element x_k below $\bigvee_{n \geq k} u_n$ such that $x_k \notin U$. But then the sequence $\{x_k\}_k$ converges to $\mathbf{0}$ and so, because $U \in \mathcal{N}_0$, there exists some k_0 such that $x_k \in U$ for all $k > k_0$; a contradiction. ■

A subset D of a Boolean algebra B is *dense* if for every $b \in B$, $b \neq \mathbf{0}$, there is some $d \in D$, $d \neq \mathbf{0}$, such that $d \leq b$. We call D *downward closed* if $a < d \in D$ implies $a \in D$.

If H is a downward closed subset of B then $H \vartriangle H = H \vee H$, and hence if H is also an open set then so is $H \vee H$.

A downward closed dense set is called *open dense*. Since we consider a topology on B we shall call dense and open dense sets *algebraically dense* and *algebraically open dense* to avoid confusion with the corresponding topological terms.

2.6. Corollary. (i) *Every neighborhood of $\mathbf{0}$ contains all but finitely many atoms.*

(ii) *If B is atomless then every neighborhood of $\mathbf{0}$ contains an algebraically open dense subset of B.*

(iii) *If B is atomless and ccc, then for every $U \in \mathcal{N}_0$ there exists a k such that $\mathbf{1} \in U \vartriangle \ldots \vartriangle U$ (k times).*

Proof. (i) is clear.

(ii) Let V be a neighborhood of $\mathbf{0}$. If V does not contain an algebraically open dense set then $B - V$ is algebraically dense below some $u \neq \mathbf{0}$ and hence contains a pairwise disjoint set $\{x_n\}_n$. But then $\lim x_n = \mathbf{0}$ and so there is some n such that $x_n \in V$; a contradiction.

(iii) As U is algebraically dense in B, there exists a maximal antichain of B included in U, and by ccc the antichain is countable: $\{u_n\}_n \subset U$. There exists a k so that $u = \bigvee_{n>k} u_n \in U$, and then $u_0 \vee u_1 \vee \ldots \vee u_k \vee u = \mathbf{1}$. ■

2.7. Proposition. *If B is atomless and ccc, then (B, τ_{s}) is connected.*

Proof. Assume that there are two disjoint nonempty clopen sets X and Y with $X \cup Y = B$ and $\mathbf{0} \in X$, and let $a \in Y$. Let C be a maximal chain in B such that $\inf C = \mathbf{0}$ and $\sup C = a$. Let $x = \sup(C \cap X)$; by ccc, x is the limit of a sequence in $C \cap X$ and therefore $x \in X$. Let $y = \inf(Y \cap \{c \in C : c \geq x\})$. Using the ccc again we have $y \in Y$, and clearly $x < y$. By maximality, both x and y are in C. Since B is atomless, there exists some z with $x < z < y$. This contradicts the maximality of C. ■

2.8. Lemma. (i) *An ideal I on a σ-complete Boolean algebra B is a closed set in the sequential topology if and only if it is a σ-complete ideal.*

(ii) *If I is a σ-ideal on B then the sequential topology on the quotient algebra B/I is the quotient topology of τ_{s} given by the canonical projection.*

(iii) *If τ_{s} is Fréchet then so is the quotient topology.*

3. Fréchet spaces. We shall now consider those complete Boolean algebras for which the sequential topology is Fréchet. We will show that this is equivalent to an algebraic property. First we make the following observation:

3.1. Proposition. *If (B, τ_{s}) is a Fréchet space then for every $V \in \mathcal{N}_0$ there is some $U \subseteq V$ in $\mathcal{N}_0$ such that U is downward closed.*

Proof. If $V \in \mathcal{N}_0$, consider the set

$$X = \{a \in B : \text{there exists some } b \leq a \text{ such that } b \notin V\},$$

and let $u(X)$ be the set of all limits of sequences in X. As τ_{s} is Fréchet, $u(X)$ is the closure of X. We shall prove that the set $U = B - u(X)$ is downward closed and contains $\mathbf{0}$.

For the first claim it suffices to show that $a \in u(X)$ and $a < b$ implies $b \in u(X)$. Thus let $a = \lim a_n$ with $a_n \in X$. It follows that $b = \lim(a_n \vee b)$, and since $a_n \vee b \in X$, we have $b \in u(X)$.

To see that $\mathbf{0} \notin u(X)$, assume that $\{a_n\} \subseteq X$ and $\lim a_n = \mathbf{0}$. Then there are $x_n \leq a_n$ in $B - V$, but this is impossible because $\lim x_n = \mathbf{0}$. Hence $\mathbf{0}$ is not in $u(X)$. ■

Thus if (B, τ_{s}) is Fréchet, its topology is determined by the set $\mathcal{N}_0^{\mathrm{d}}$ of all $U \in \mathcal{N}_0$ that are downward closed. $\mathcal{N}_0^{\mathrm{d}}$ is a neighborhood base of $\mathbf{0}$.

3.2. Definition. Let κ be an infinite cardinal. A Boolean algebra B is (ω, κ)-*weakly distributive* if for every sequence $\{P_n\}$ of maximal antichains, each of size at most κ, there exists a dense set Q with the property that each $q \in Q$ meets only finitely many elements of each P_n. B is *weakly distributive* if it is (ω,ω)-weakly distributive.

If B is a κ^+-complete Boolean algebra then B is (ω,κ)-weakly distributive if and only if it satisfies the following distributive law:

$$\bigwedge_n \bigvee_\alpha a_{n\alpha} = \bigvee_{f:\omega\to[\kappa]^{<\omega}} \bigwedge_n \bigvee_{\alpha\in f(n)} a_{n\alpha}.$$

We recall two frequently used cardinal characteristics.

3.3. Definition. The *splitting number* is the least cardinal $\mathbf{s}$ of a family $\mathcal{S}$ of infinite subsets of ω such that for every infinite $X \subseteq \omega$ there is some $S \in \mathcal{S}$ such that both $X \cap S$ and $X - S$ are infinite. (S "splits" X.)

The *bounding number* is the least cardinal $\mathbf{b}$ of a family $\mathcal{F}$ of functions from ω to ω such that $\mathcal{F}$ is unbounded; i.e. for every $g \in \omega^\omega$ there is some $f \in \mathcal{F}$ such that $g(n) \le f(n)$ for infinitely many n.

The following characterization of Fréchet spaces (B,τ_{s}) uses the cardinal invariant $\mathbf{b}$ and is similar to several other results using $\mathbf{b}$, such as in [BlJe]. A consequence of Theorem 3.4 is that $(P(\kappa),\tau_{\mathrm{s}})$ is a Fréchet space if and only if $\kappa < \mathbf{b}$.

3.4. Theorem. *Let B be a complete Boolean algebra. The sequential space (B,τ_{s}) is Fréchet if and only if B is weakly distributive and satisfies the $\mathbf{b}$-chain condition.*

We first reformulate the condition stated in Theorem 3.4. Let B be a complete Boolean algebra. We call a matrix $\{a_{mn}\}$ *increasing* if each row $\{a_{mn} : n \in \omega\}$ is an increasing sequence with limit $\mathbf{1}$. Note that B is weakly distributive if and only if for every increasing matrix $\{a_{mn}\}$,

$$\bigvee_{f\in\omega^\omega} \varliminf a_{m,f(m)} = \mathbf{1}.$$

3.5. Lemma. *A complete Boolean algebra B is weakly distributive and satisfies the $\mathbf{b}$-chain condition if and only if for every increasing matrix $\{a_{mn}\}$ there exists a function $f \in \omega^\omega$ such that $\lim a_{m,f(m)} = \mathbf{1}$.*

Proof. First let B be weakly distributive and satisfy $\mathbf{b}$-c.c., and let $\{a_{mn}\}$ be an increasing matrix. By the $\mathbf{b}$-chain condition there exists a set $F \subset \omega^\omega$ of size less than $\mathbf{b}$ such that $\bigvee_{f\in F} \varliminf a_{m,f(m)} = \mathbf{1}$. Let $g : \omega \to \omega$ be an upper bound of F under eventual domination. Since the matrix is increasing, we have $\varliminf a_{m,f(m)} \le \varliminf a_{m,g(m)}$ for every $f \in F$. Therefore $\lim a_{m,g(m)} = \mathbf{1}$.

Conversely, assume that the condition holds. Then B is weakly distributive, and we verify the $\mathbf{b}$-chain condition. Thus let W be a partition of $\mathbf{1}$;

we prove that $|W| < \mathbf{b}$. Let $\{f_u : u \in W\}$ be any family of functions from ω to ω indexed by elements of W. For each m and each n we let

$$a_{mn} = \bigvee\{u \in W : f_u(m) < n\}.$$

The matrix $\{a_{mn}\}$ is increasing and therefore there exists a function $g : \omega \to \omega$ such that $\lim a_{m,g(m)} = \mathbf{1}$. Since W is an antichain, it follows that for any $u \in W$ there is some m_u such that $u \le a_{m,g(m)}$ for every $m \ge m_u$. Hence $f_u(m) < g(m)$ for every $m \ge m_u$ and it follows that g is an upper bound of the family $\{f_u : u \in W\}$. Therefore every family of functions of size $|W|$ is bounded and so $|W| < \mathbf{b}$. ∎

Proof of Theorem 3.4. We wish to show that the condition in Lemma 3.5 is necessary and sufficient for the space (B, τ_{s}) to be Fréchet. To see that the condition holds if (B, τ_{s}) is Fréchet, we recall [Ma] that for (B, τ_{s}), being Fréchet is equivalent to the following statement: whenever $\{x_{mn}\}$, $\{y_m\}$ and z are such that $\lim_n x_{mn} = y_m$ for each m and $\lim_m y_m = z$, then there is an $f : \omega \to \omega$ such that $\lim_m x_{m,f(m)} = z$.

To show that the condition implies that (B, τ_{s}) is Fréchet, let $\{x_{mn}\}$, $\{y_m\}$ and z be as above. For each m and each n let $u_{mn} = x_{mn} \vartriangle (-y_m)$, and let $a_{mn} = \bigwedge_{k \ge n} u_{mk}$. For each m, $\lim_n u_{mn} = \mathbf{1}$; the matrix $\{a_{mn}\}$ is increasing, with each row converging to $\mathbf{1}$ and so there exists some $f : \omega \to \omega$ such that $\lim a_{m,f(m)} = \mathbf{1}$. It follows that $\lim_m \bigwedge_{k \ge f(m)} u_{mk} = \mathbf{1}$, and so $\lim(x_{m,f(m)} \vartriangle (-z)) = \lim(x_{m,f(m)} \vartriangle (-y_m)) = \lim u_{m,f(m))} = \mathbf{1}$. Hence $\lim x_{m,f(m)} = z$. ∎

We conclude with the following observation that we shall use in Section 5.

3.6. Lemma. (a) *For every set* $A \subseteq B$, $\mathrm{cl}(A) = \bigcap\{A \vartriangle V : V \in \mathcal{N}_0\}$.

(b) *If* (B, τ_{s}) *is Fréchet and* A *is downward closed then* $\mathrm{cl}(A) = \bigcap\{A \vee V : V \in \mathcal{N}_0^{\mathrm{d}}\}$, *and* $\mathrm{cl}(A)$ *is downward closed.*

Proof. (a) For any $x \in B$, $x \in \mathrm{cl}(A)$ iff for all $V \in \mathcal{N}_0$, $(V \vartriangle x) \cap A \ne \emptyset$, i.e. there exist $v \in V$ and $a \in A$ such that $v \vartriangle x = a$. The latter is equivalent to $x = a \vartriangle v$, or $x \in A \vartriangle V$.

(b) If both A and V are downward closed then $A \vee V = A \vartriangle V$. ∎

3.7. Corollary. *For every* $U \in \mathcal{N}_0$, $\mathrm{cl}(U) \subseteq U \vartriangle U$. *If* (B, τ_{s}) *is Fréchet, then* $\mathrm{cl}(U) \subseteq U \vee U$ *for every* $U \in \mathcal{N}_0^{\mathrm{d}}$.

4. Separation axioms. We will now discuss separation axioms for the topology τ_{s}. We immediately see that the sequential topology on B is T_1. The space is Hausdorff if and only if every point $b \ne \mathbf{0}$ can be separated from $\mathbf{0}$, which is equivalent to the statement that for every $b \ne \mathbf{0}$ there exists some $V \in \mathcal{N}_0$ such that $b \notin V \vartriangle V$.

4.1. Theorem. *If (B,τ_{s}) is a Hausdorff space then B is (ω,ω_1)-weakly distributive.*

We first prove a weaker statement, namely that being Hausdorff implies weak distributivity:

4.2. Lemma. *If B is not weakly distributive then there exists an $a \neq \mathbf{0}$ such that $c \in \mathrm{cl}(U)$ for every $c \le a$ and every $U \in \mathcal{N}_0$. Hence (B,τ_{s}) is not Hausdorff.*

Proof. Assume that B is not (ω,ω)-weakly distributive. There is some $a \neq \mathbf{0}$ and there exists an infinite matrix $\{a_{mn}\}$ such that each row is a partition of a, and for any nonzero $x \le a$ there is some m such that $x \wedge a_{mn} \neq \mathbf{0}$ for infinitely many n.

Let $c \le a$ and let U be an arbitrary neighborhood of $\mathbf{0}$. We will show that $c \in \mathrm{cl}(U)$. For every m and every n let $y_{mn} = c \wedge \bigvee_{i \ge n} a_{mi}$. Since the sequence $\{y_{0n}\}$ converges to $\mathbf{0}$ there exists some n_0 such that $y_{0n_0} \in U$; let $x_0 = y_{0n_0}$. Next we consider the sequence $\{y_{1n} \vee x_0\}$. This sequence converges to x_0 and so there exists some n_1 such that $x_1 \in U$ where $x_1 = y_{1n_1} \vee x_0$. We proceed by induction and obtain a sequence $\{n_m\}$ and an increasing sequence $\{x_m\}$ of elements of U. This sequence converges to c because otherwise, if we let $b \neq \mathbf{0}$ be the complement of $\bigvee_n x_n$ in c, then $b \le \bigwedge_m \bigvee_{i < n_m} a_{mi}$ and so b meets only finitely many elements in each row of the matrix. Hence $c \in \mathrm{cl}(U)$. ∎

Proof of Theorem 4.1. Let (B,τ_{s}) be a Hausdorff space. To prove that B is (ω,ω_1)-weakly distributive, let

$$A = \{a_{n\alpha} : n \in \omega,\ \alpha \in \omega_1\}$$

be a matrix such that each row is a partition of $\mathbf{1}$. Denote by X the set of all those $x \in B$ that meet at most countably many elements of each row of A. As B is (ω,ω)-weakly distributive, for every nonzero $x \in X$ there is a nonzero $y \le x$ that meets only finitely many elements of each row of A. Thus we complete the proof by showing that $\bigvee X = \mathbf{1}$.

Assume otherwise; without loss of generality we may assume that every $x \neq \mathbf{0}$ meets uncountably many elements of at least one row of A. Then the matrix A represents a Boolean-valued name for a cofinal function from ω into ω_1. Thus B collapses ω_1 and therefore there exists a matrix

$$\{b_{n\alpha} : n \in \omega,\ \alpha \in \omega_1\}$$

such that each row and each column is a partition of $\mathbf{1}$ (the name for a one-to-one mapping of ω onto ω_1). We get a contradiction to Hausdorffness by showing that $\mathbf{1}$ is in the closure of every $V \in \mathcal{N}_0$.

Let $V \in \mathcal{N}_0$ be arbitrary. By Lemma 2.5 there is for every $\alpha \in \omega_1$ some $n_\alpha \in \omega$ such that $v_\alpha = \bigvee_{i \ge n_\alpha} b_{i\alpha} \in V$. Thus there exists some n and an infinite set $\{\alpha_k\}_k$ such that $n_{\alpha_k} = n$ for all k. Now, by 2.2(v),

$$\overline{\lim_k} \bigvee_{i<n} b_{i\alpha_k} = \bigvee_{i<n} \overline{\lim_k} b_{i\alpha_k} = \mathbf{0}.$$

Therefore $\lim_k v_{\alpha_k} = \mathbf{1}$ and so $\mathbf{1}$ is in the closure of V. ■

For (ω,ω_1)-weak distributivity we refer to Namba's work [Na] which shows that it may or may not be equivalent to (ω,ω)-weak distributivity. If $\mathbf{b} = \omega_1$ then (ω,ω)-weak distributivity and (ω,ω_1)-weak distributivity are equivalent, and there is a model of ZFC in which they are not equivalent. Below (in 4.5(i)) we give another example of a complete Boolean algebra that is (ω,ω)-weakly distributive but not (ω,ω_1)-weakly distributive.

Theorem 4.1 cannot be extended by replacing ω_1 with ∞: Example 4.5(ii), due to Prikry [Pr], provides a complete Boolean algebra that is Hausdorff (therefore weakly distributive) but not (ω,κ)-weakly distributive, for a measurable κ.

In view of Theorems 3.4 and 4.1 the question arises about the relative strength of being a Hausdorff space and being a Fréchet space. Example 4.3 below shows that Hausdorff does not imply Fréchet: the space $(P(\mathbf{b}), \tau_s)$ is Hausdorff but not Fréchet.

For the other direction, see Examples 4.4 and 4.5. If T is a Suslin tree then $(B(T), \tau_s)$ is Fréchet but not Hausdorff.

4.3. Example. For every infinite cardinal κ the space $(P(\kappa), \tau_s)$ is Hausdorff. This is because each principal ultrafilter on κ is a closed and open subset of $P(\kappa)$.

We identify $P(\kappa)$ with 2^κ (via characteristic functions). For each $\alpha \in \kappa$ the set $\{X \subseteq \kappa : \alpha \in X\}$ and its complement $\{X \subseteq \kappa : \alpha \notin X\}$ are closed under limits of sequences and so are both closed and open. This implies that the topology τ_s extends the product topology, and the space $(P(\kappa), \tau_s)$ is a totally disconnected Hausdorff space. If $\kappa = \aleph_0$ then τ_s is equal to the product topology. To see this, let $U \subseteq P(\omega)$ be an open set in the sequential topology and let $A \in U$. For each n let S_n denote the basic open set (in the product topology) $\{X \subseteq \omega : X \cap n = A \cap n\}$. It suffices to show that U contains some S_n as a subset. If not, there exists for each n some $X_n \in S_n - U$. But $A = \lim_n X_n$, and since the complement of U is closed, $A \notin U$; a contradiction.

When κ is an uncountable cardinal, the space $(P(\kappa), \tau_s)$ is not compact and so τ_s is strictly stronger than the product topology.

By [Tr] the space $(P(\kappa), \tau_s)$ is sequentially compact if and only if $\kappa < \mathbf{s}$, the splitting number.

By [Gł], $(P(\kappa), \tau_s)$ is regular if and only if $\kappa = \omega$. See Corollary 4.7.

4.4. Example (Aronszajn trees). We show that the Boolean algebra associated with a Suslin tree is an example of a Fréchet space that is not Hausdorff. We point out that in ZFC, the only known examples of algebras that are Fréchet spaces are measure algebras.

Let T be an Aronszajn tree and assume that each node has at least two immediate successors. Let $B(T)$ denote the complete Boolean algebra that has upside down T as a dense set. We will show that $(B(T), \tau_s)$ is not a Hausdorff space. This shows that the converse of Theorem 4.1 is not provable: if T is a Suslin tree then $B(T)$ is a ccc ω-distributive Boolean algebra.

We prove that $\mathbf{0}$ and $\mathbf{1}$ cannot be separated by open sets: we show that $\mathbf{1} \in V \vartriangle V$ for every open neighborhood V of $\mathbf{0}$. Let $V \in \mathcal{N}_0$. For every $\alpha \in \omega_1$, the αth level T_α of the tree is a countable partition of $\mathbf{1}$ and so there exists a finite set $u_\alpha \subseteq T_\alpha$ such that $x_\alpha = \bigvee(T_\alpha - u_\alpha) \in V$. Let $y_\alpha = \bigvee u_\alpha$. We claim that there is a β such that $y_\beta \in V$; this will complete the proof as $\mathbf{1} = x_\beta \vartriangle y_\beta \in V \vartriangle V$.

Let $f : [\omega_1]^2 \to \{0, 1\}$ be the function defined as follows: $f(\alpha, \beta) = 0$ if $y_\alpha \wedge y_\beta = \mathbf{0}$ and $f(\alpha, \beta) = 1$ otherwise. By the Dushnik–Miller Theorem there exists a set $I \subseteq \omega_1$, either homogeneous in color 0 and of size $\aleph_0$, or homogeneous in color 1 and of size $\aleph_1$. The latter case is impossible because the u_αs are disjoint finite sets in an Aronszajn tree (see [Je], Lemma 24.2). Hence there is an infinite set $\{\alpha_n : n \in \omega\}$ such that the y_{α_n} are pairwise disjoint. Thus the sequence $\{y_{\alpha_n}\}$ converges to $\mathbf{0}$ and so there exists some n such that $y_{\alpha_n} \in V$.

4.5. Examples (using large cardinals).

(i) Assume that there exists a nontrivial $\aleph_2$-saturated σ-ideal I on $P(\omega_1)$, and assume that $\mathbf{b} = \aleph_2$. Both these assumptions are consequences of Martin's Maximum (MM), with $I =$ the nonstationary ideal.

Let $B = P(\omega_1)/I$. Then B is a complete Boolean algebra and satisfies the $\aleph_2$-chain condition. Since $\mathbf{b} = \aleph_2$, the space $(P(\omega_1), \tau_s)$ is Fréchet, and so by Lemma 2.8, (B, τ_s) is Fréchet. Therefore B is weakly distributive.

Since forcing with B collapses $\aleph_1$, B is not (ω, ω_1)-weakly distributive, and hence (B, τ_s) is not Hausdorff.

The space $(P(\omega_1), \tau_s)$ is separable: this follows from MM, specifically from $\mathbf{p} = \aleph_2$ (cf. [Fr0], [To] and [Ro]). Hence (B, τ_s) is separable, and so the complete Boolean algebra B is countably generated.

This example is in the spirit of [Gł] where a similar example is presented using MA and a measurable cardinal.

(ii) Let κ be a measurable cardinal, and let B be the complete Boolean algebra associated with Prikry forcing. B is not (ω, κ)-weakly distributive as it changes the cofinality of κ to ω. But the space (B, τ_s) is Hausdorff: For any $a \in B^+$ there is a κ-complete ultrafilter on B containing a (cf. [Pr]). Every such ultrafilter is a clopen set in (B, τ_s).

Thus Hausdorffness does not imply (ω, ∞)-weak distributivity of (B, τ_s). We do not know if the large cardinal assumption is necessary.

A topological space is *regular* if points can be separated from closed sets; equivalently, for every point x and its neighborhood U there exists an open set V such that $x \in V$ and $\mathrm{cl}(V) \subseteq U$. The space (B, τ_s) is regular if and only if for every $U \in \mathcal{N}_0$ there is some $V \in \mathcal{N}_0$ such that $\mathrm{cl}(V) \subseteq U$.

A result proved independently in [Tr] and [Gł] states that the atomic algebra $P(\omega_1)$ is not regular. The following lemma uses the method employed in these papers.

4.6. LEMMA. *In the space $(P(\omega_1), \tau_s)$ for every $V \in \mathcal{N}_0$ there exists a closed unbounded set $C \subset \omega_1$ such that $\omega_1 - \beta \in \mathrm{cl}(V)$ for every $\beta \in C$.*

Proof. Let V be an open neighborhood of $\emptyset$. Let $\{A_{\alpha n} : \alpha \in \omega_1,\ n \in \omega\}$ be an Ulam matrix, i.e. a double array of subsets of ω_1 with the following properties:

$$\begin{aligned} A_{\alpha n} \cap A_{\alpha m} &= \emptyset \quad (n \neq m), \\ A_{\alpha n} \cap A_{\beta n} &= \emptyset \quad (\alpha \neq \beta), \\ \bigcup_{n \in \omega} A_{\alpha n} &= \omega_1 - \alpha. \end{aligned}$$

By Lemma 2.5 there exists for each α some k_α such that $X_\alpha = \bigcup_{n \geq k_\alpha} A_{\alpha n}$ is in V. There exist some k and an uncountable set W such that $k_\alpha = k$ for every $\alpha \in W$. Let C be the set of all limits of increasing sequences of ordinals in W. We claim that $\omega_1 - \beta \in \mathrm{cl}(V)$ for every $\beta \in C$.

Let $\alpha_0 < \alpha_1 < \ldots < \alpha_n < \ldots$ be in W such that $\beta = \lim_n \alpha_n$. Note that $\overline{\lim}_n \bigcup_{i<k} A_{\alpha_n i} = \bigcup_{i<k} \overline{\lim}_n A_{\alpha_n i} = \emptyset$, and hence $X = \lim_n X_{\alpha_n} = \omega_1 - \beta$. Therefore $X \in \mathrm{cl}(V)$. ∎

4.7. COROLLARY. *The space $(P(\omega_1), \tau_s)$ is not regular.*

Proof. Let U be the set of all $x \subset \omega_1$ whose complement is uncountable. U is an open neighborhood of $\emptyset$ and, by Lemma 4.6, does not contain $\mathrm{cl}(V)$ for any $V \in \mathcal{N}_0$. ∎

4.8. COROLLARY. *If a complete Boolean algebra B does not satisfy the countable chain condition then (B, τ_s) is not regular.*

Proof. B contains $P(\omega_1)$ as a complete subalgebra, therefore as a closed subspace. Hence it is not regular. ∎

4.9. COROLLARY. *Let $B = P(\omega_1)$, or more generally, let B be a complete Boolean algebra that does not satisfy the countable chain condition. If $\{U_n\}_n$ is a countable subset of $\mathcal{N}_0$ then $\bigcap_n \mathrm{cl}(U_n)$ is uncountable.*

P r o o f. This follows easily from Lemma 4.6 when $B = P(\omega_1)$. In the general case, (B, τ_s) contains $P(\omega_1)$ as a subspace and each $U_n \cap P(\omega_1)$ is an open neighborhood of $\emptyset$. ∎

4.10. COROLLARY. *Let B be a complete Boolean algebra, and assume that in the space (B, τ_s) there exists a countable family $\{U_n\}_n$ of neighborhoods of $\mathbf{0}$ such that $\bigcap_n \mathrm{cl}(U_n) = \{\mathbf{0}\}$. Then B satisfies the countable chain condition and (B, τ_s) is Fréchet.*

P r o o f. B satisfies ccc by Corollary 4.9. Also, (B, τ_s) is clearly Hausdorff and so B is weakly distributive by Lemma 4.2. Hence, by Theorem 3.4, B is a Fréchet space. ∎

We conclude this section with some remarks:

A Fréchet space is Hausdorff if and only if

$$\bigcap\{V \vee V : V \in \mathcal{N}_0\} = \{\mathbf{0}\}.$$

Even more is true: If the space (B, τ_s) is Fréchet and Hausdorff, then for every k,

$$\bigcap\{V \vee \ldots \vee V (k \text{ times}) : V \in \mathcal{N}_0\} = \{\mathbf{0}\}.$$

This is a consequence of the following:

4.11. LEMMA. *Let B be a σ-complete Boolean algebra such that (B, τ_s) is Fréchet. Then for every $U \in \mathcal{N}_0^{\mathrm{d}}$ there exists a $V \in \mathcal{N}_0^{\mathrm{d}}$ such that $V \vee V \vee V \subseteq U \vee U$.*

In the next section we use this consequence of Lemma 4.11:

4.12. COROLLARY. *If B is as in Lemma 4.11 and $U \in \mathcal{N}_0^{\mathrm{d}}$ then there exists a $V \subseteq U$ in $\mathcal{N}_0^{\mathrm{d}}$ such that $\mathrm{cl}(V) \vee \mathrm{cl}(V) \subseteq U \vee U$.*

(To see that this follows from Lemma 4.11, use $\mathrm{cl}(V) \subseteq V \vee V$.)

P r o o f (of Lemma 4.11). Assume that for every $V \in \mathcal{N}_0^{\mathrm{d}}$ there exist x, y and z in V such that $x \vee y \vee z \notin U \vee U$. Note that $U \vee U = U \triangle U$ and is downward closed.

Let $V_0 = U$; by induction we define neighborhoods V_n and points x_n, y_n, z_n as follows: For each n let $x_n, y_n, z_n \in V_n$ be such that $x_n \vee y_n \vee z_n \notin U \vee U$. Then let $V_{n+1} \subseteq V_n$ be in $\mathcal{N}_0^{\mathrm{d}}$ and such that the sets $x_n \vee V_{n+1}$, $y_n \vee V_{n+1}$ and $z_n \vee V_{n+1}$ are all included in V_n; such a neighborhood exists by the one-sided continuity of $\vee$.

Let $X = \bigcap_n \mathrm{cl}(V_n)$ and $\overline{x} = \overline{\lim}_n x_n, \overline{y} = \overline{\lim}_n y_n$ and $\overline{z} = \overline{\lim}_n z_n$. The set X is topologically closed and downward closed, and $X \subseteq \mathrm{cl}(U) \subseteq U \vee U$.

We claim that $\overline{x}, \overline{y}, \overline{z} \in X$. Thus let us prove that $\overline{x} \in \mathrm{cl}(V_n)$ for each n. We have $x_n \in V_n$, and by induction on $k > 0$ we see that $x_n \vee x_{n+1} \vee \ldots \vee x_{n+k} \in V_n$. Thus $\bigvee_{i \geq n} x_i \in \mathrm{cl}(V_n)$ and $\overline{x} \in \mathrm{cl}(V_n)$.

Next we claim that $\overline{x} \vee X \subseteq X$ (and similarly for $\overline{y}, \overline{z}$). Let n be arbitrary; we show that $\overline{x} \vee X \subseteq \mathrm{cl}(V_n)$. For any k we have $x_n \vee \ldots \vee x_{n+k} \vee V_{n+k+1} \subseteq V_n$, and by the one-sided continuity of $\vee$ it follows that $x_n \vee \ldots \vee x_{n+k} \vee \mathrm{cl}(V_{n+k+1}) \subseteq \mathrm{cl}(V_n)$. Hence $x_n \vee \ldots \vee x_{n+k} \vee X \subseteq \mathrm{cl}(V_n)$, and so $\bigvee_{i \ge n} x_i \vee X \subseteq \mathrm{cl}(V_n)$. As $\overline{x} \le \bigvee_{i \ge n} x_i$ and $\mathrm{cl}(V_n)$ is downward closed, we have $\overline{x} \vee X \subseteq \mathrm{cl}(V_n)$.

Now it follows that $\overline{x} \vee \overline{y} \vee \overline{z}$ is in X and hence in $U \vee U$. But $\overline{x} \vee \overline{y} \vee \overline{z} = \overline{\lim}_n (x_n \vee y_n \vee z_n)$. As the complement of $U \vee U$ is upward closed, we have $\bigvee_{i \ge n}(x_i \vee y_i \vee z_i) \notin U \vee U$ for each n, and because $U \vee U$ is topologically open, we have $\overline{x} \vee \overline{y} \vee \overline{z} \notin U \vee U$, a contradiction. ∎

5. Metrizability. We will show that for complete ccc Boolean algebras, Hausdorffness of the sequential topology is a strong property: it implies metrizability, and equivalently, the existence of a strictly positive Maharam submeasure. We remark that the assumption of completeness is essential.

5.1. Theorem. *If B is a complete Boolean algebra, then the following are equivalent*:

(i) *B is ccc and (B, τ_s) is a Hausdorff space*,

(ii) *there exists a countable family $\{U_n\}_n$ of open neighborhoods of $\mathbf{0}$ such that $\bigcap_n \mathrm{cl}(U_n) = \{\mathbf{0}\}$*,

(iii) *the operation $\vee$ is continuous at $(\mathbf{0}, \mathbf{0})$, i.e. for every $V \in \mathcal{N}_0$ there exists a $U \in \mathcal{N}_0$ such that $U \vee U \subseteq V$*,

(iv) *(B, τ_s) is a regular space*,

(v) *(B, τ_s) is a metrizable space*,

(vi) *B carries a strictly positive Maharam submeasure.*

The equivalence of (v) and (vi) is proved in [Ma], and (v) implies (i). We shall prove in this section that properties (i)–(iv) are equivalent and imply (vi). First we claim that each of the four properties implies that B satisfies ccc, and that the space (B, τ_s) is Fréchet.

If B is ccc and Hausdorff, then by Theorems 4.1 and 3.4 it is Fréchet.

Property (ii) implies Fréchet by Corollary 4.10, and property (iv) implies (i) (and hence Fréchet) by Corollary 4.8.

To complete the claim, 5.2–5.5 below prove that (iii) implies Fréchet. Let B be a complete Boolean algebra and assume that $\vee$ is continuous at $(\mathbf{0}, \mathbf{0})$.

5.2. Lemma. *B satisfies the countable chain condition.*

If B does not satisfy ccc then (B, τ_s) contains $P(\omega_1)$ as a closed subspace. Thus the lemma is a consequence of the following lemma closely related to Corollary 4.7:

5.3. LEMMA. *In* $(P(\omega_1), \tau_s)$ *the operation* $\cup$ *is not continuous at* $(\emptyset, \emptyset)$.

Proof. Let U be the set of all $x \subset \omega_1$ whose complement is uncountable. U is an open neighborhood of $\emptyset$. We will show that for every $V \in \mathcal{N}_0$ there exist Y and Z in V such that $Y \cup Z \notin U$. Thus let $V \in \mathcal{N}_0$.

By Lemma 4.6 there exists an X such that $X \notin U$ while $X \in \mathrm{cl}(V)$. By Corollary 3.7 there exist Y and Z in V such that $X = Y \vartriangle Z$. But $Y \vartriangle Z \subseteq Y \cup Z$ and therefore $Y \cup Z \notin U$. ∎

5.4. LEMMA. *B is weakly distributive.*

Proof. Assume that B is not weakly distributive. By Lemma 4.2 there exists some $a \neq 0$ such that $a \in \mathrm{cl}(V)$ for every $V \in \mathcal{N}_0$.

Let $U = \{x \in B : x \not\geq a\}$; then U is a neighborhood of $\mathbf{0}$. We claim that $V \vee V \not\subseteq U$ for every $V \in \mathcal{N}_0$, contradicting the continuity of $\vee$. Thus let $V \in \mathcal{N}_0$ be arbitrary.

We have $a \in \mathrm{cl}(V)$. By Corollary 3.7, $a \in V \vartriangle V$ and so there exist x and y in V such that $a = x \vartriangle y$. If we let $b = x \vee y$ then $b \geq a$ and therefore $b \notin U$. But $b \in V \vee V$, completing the proof. ∎

5.5. COROLLARY. (B, τ_s) *is Fréchet.*

Proof. Use Theorem 3.4. ∎

For the rest of Section 5 we assume that B is a complete Boolean algebra that satisfies the countable chain condition, and that the space (B, τ_s) is Fréchet. In particular, $\mathcal{N}_0^{\mathrm{d}}$ is a neighborhood base, so we shall only consider those neighborhoods of $\mathbf{0}$ that are downward closed.

To prove that (i)–(iv) are equivalent, we first observe that (iii) implies (iv):

5.6. PROPOSITION. *If* $\vee$ *is continuous at* $(\mathbf{0}, \mathbf{0})$ *then* (B, τ_s) *is regular.*

Proof. Let $V \in \mathcal{N}_0$. By homogeneity, it suffices to find an open U such that $\mathrm{cl}(U) \subseteq V$. Since $\vee$ is continuous at $(\mathbf{0}, \mathbf{0})$ and since (B, τ_s) is Fréchet, by Corollary 3.7 there exists a $U \in \mathcal{N}_0^{\mathrm{d}}$ such that $\mathrm{cl}(U) \subseteq U \vee U \subseteq V$. ∎

As (iv) implies (i), it remains to show that (i) implies (ii) and that (ii) implies (iii). Lemma 5.7 proves the latter:

5.7. LEMMA. *Assume that* (B, τ_s) *satisfies* (ii). *Then the operation* $\vee$ *is continuous at* $(\mathbf{0}, \mathbf{0})$.

Proof. As (B, τ_s) is Fréchet, the set $\mathcal{N}_0^{\mathrm{d}}$ of all downward closed open neighborhoods of $\mathbf{0}$ is a neighborhood base. Thus let us assume that there exists a $U \in \mathcal{N}_0^{\mathrm{d}}$ such that for every $V \in \mathcal{N}_0$ there exist x and y in V with $x \vee y \notin U$.

Let $\{V_n\}_n$ in $\mathcal{N}_0^{\mathrm{d}}$ be such that $\bigcap_n \mathrm{cl}(V_n) = \{\mathbf{0}\}$. We construct a descending sequence of neighborhoods U_n in $\mathcal{N}_0^{\mathrm{d}}$ as follows: Let $U_0 = V_0 \cap U$. Given

U_n let $x_n, y_n \in U_n$ be such that $x_n \vee y_n \notin U$. By (separate) continuity of $\vee$ there exists a set $U_{n+1} \in \mathcal{N}_0^{\mathrm{d}}$ such that $x_n \vee U_{n+1} \subset U_n$ and $y_n \vee U_{n+1} \subset U_n$; moreover, we may assume that U_{n+1} is included in V_{n+1}.

Let $\overline{x} = \overline{\lim}\, x_n$ and $\overline{y} = \overline{\lim}\, y_n$. First we claim that $\overline{x} = \overline{y} = \mathbf{0}$ and therefore $\overline{x} \vee \overline{y} = \mathbf{0} \in U$.

We have $\overline{x} = \bigwedge_n z_n$ where $z_n = \bigvee_k x_{n+k}$. It suffices to prove that for each n, $\bigwedge_m z_m$ is in the closure of U_n, and for that it is enough to show that $z_m \in \mathrm{cl}(U_n)$ for each $m \geq n$.

Let n be arbitrary and let $m \geq n$. As for each k we have $x_{m+k} \vee U_{m+k+1} \subset U_{m+k}$, it follows (by induction on k) that $x_m \vee x_{m+1} \vee \ldots \vee x_{m+k} \in U_m \subset U_n$. Hence $z_m \in \mathrm{cl}(U_n)$.

Now we get a contradiction by showing that $\overline{x} \vee \overline{y} \notin U$. We have $\overline{x} \vee \overline{y} = \overline{\lim}(x_n \vee y_n) = \bigwedge_n z_n$ where $z_n = \bigvee_{k \geq n}(x_k \vee y_k)$. As U is a downward closed open set and $x_k \vee y_k \notin U$ for each k, we have $z_n \notin U$ for each n and therefore $\bigwedge_n z_n \notin U$. ■

We now prove that (i) implies (ii):

5.8. Lemma. *Let B be a complete ccc Boolean algebra such that (B, τ_{s}) is a Hausdorff space. Then there exists a sequence $\{U_n\}_n$ in $\mathcal{N}_0$ such that $\bigcap_n \mathrm{cl}(U_n) = \{\mathbf{0}\}$.*

Proof. For any given $b \in B^+$ we shall find a sequence $\{V_n\}_n$ in $\mathcal{N}_0^{\mathrm{d}}$ such that $c_b = b - \bigvee(\bigcap_n \mathrm{cl}(V_n)) \neq \mathbf{0}$. Then the set of all such c_b is algebraically dense and therefore there exists a partition $\{c_k\}_k$ of $\mathbf{1}$ and sequences $\{V_n^k\}_n$ with $\bigvee(\bigcap_n \mathrm{cl}(V_n^k)) \wedge c_k = \mathbf{0}$. Now if we let $U_n = V_n^0 \cap V_n^1 \cap \ldots \cap V_n^n$ for each n, we get a sequence with the desired properties.

Thus let $b \neq \mathbf{0}$. We construct the sequence $\{V_n\}_n$. For every set $S \subseteq B$ let $S^{(n)}$ denote the n-fold joint $S \vee \ldots \vee S$.

As the space is Hausdorff, there exists a $V_0 \in \mathcal{N}_0^{\mathrm{d}}$ such that $b \notin V_0 \vee V_0$. By Lemma 4.11 and Corollary 4.12 there exists for each n some $V_{n+1} \in \mathcal{N}_0^{\mathrm{d}}$ such that $\mathrm{cl}(V_{n+1}) \vee \mathrm{cl}(V_{n+1}) \subseteq V_n \vee V_n$, and $V_{n+1}^{(3)} \subseteq V_n^{(2)}$. Let $X = \bigcap_n \mathrm{cl}(V_n)$ and $a = \bigvee X$.

In order to prove that $b - a \neq \mathbf{0}$, it suffices to show that $a \in V_0 \vee V_0$, because that set is downward closed and b is outside it. By ccc, $a = \lim_n a_n$ where $a_n \in X^{(n)}$ for each n. We claim that $X^{(n)} \subseteq V_2 \vee V_2$ for each n. Then $a \in \mathrm{cl}(V_2 \vee V_2) \subseteq V_2^{(4)} \subseteq V_1^{(3)} \subseteq V_0^{(2)}$.

The claim is proved as follows (we may assume that n is even):

$$X^{(n)} \subseteq (\mathrm{cl}(V_{n+1}))^{(n)} \subseteq V_n^{(n)} \subseteq \ldots \subseteq V_2^{(2)}. \ ■$$

This completes the proof of the equivalence of properties (i)–(iv). We make the following remark:

5.9. Corollary. *Let B be a complete Boolean algebra such that (B,τ_{s}) is a regular space. Then the Boolean operations $\wedge$, $-$ and $\triangle$ are continuous, and $(B,\triangle,\mathbf{0},\tau_{\mathrm{s}})$ is a topological group. Moreover, (B,τ_{s}) is a completely regular space.*

Proof. As (B,τ_{s}) is Fréchet, $\mathbf{0}$ has a neighborhood base $\mathcal{N}_0^{\mathrm{d}}$ of sets for which $U \triangle U = U \vee U$. Because $\vee$ is continuous at $(\mathbf{0},\mathbf{0})$, $\triangle$ is also continuous at $(\mathbf{0},\mathbf{0})$. From that it easily follows that $\triangle$ is continuous (at every $(u,v)\in B\times B$) and that $(B,\triangle,\mathbf{0},\tau_{\mathrm{s}})$ is a topological group. Consequently, $\vee$ and $\wedge$ are also continuous everywhere. Finally, every regular topological group is completely regular (cf. [HeRo]). ∎

We now prove (vi), assuming that (B,τ_{s}) is regular.

5.10. Lemma. (a) *There exists a sequence $\{U_n\}_n$ of elements of $\mathcal{N}_0^{\mathrm{d}}$ such that $\mathrm{cl}(U_{n+1})\subset U_{n+1}\vee U_{n+1}\subset U_n$ for every n and such that $\bigcap_n U_n=\{\mathbf{0}\}$.*

(b) *Moreover, $\{U_n\}_n$ is a neighborhood base of $\mathbf{0}$.*

Proof. (a) By continuity of $\vee$ there is a sequence $\{U_n\}_n$ in $\mathcal{N}_0^{\mathrm{d}}$ such that $U_{n+1}\vee U_{n+1}\subset U_n$ for every n. By (ii) we may assume that $\bigcap_n U_n=\{\mathbf{0}\}$.

(b) We prove that the U_n form a neighborhood base. Assume not. Then there exists a $V\in\mathcal{N}_0$ such that $U_n\not\subseteq V$ for every n. For each n let x_n be such that $x_n\in U_n-V$.

It follows by induction on k that $x_{n+1}\vee x_{n+2}\vee\ldots\vee x_{n+k}\in U_n$ for each n and each k. Thus $\bigvee_k x_{n+k}\in\mathrm{cl}(U_n)$ and it follows that $\overline{\lim}\,x_n\in U_m$ for each m; hence $\overline{\lim}\,x_n=\mathbf{0}$ and so $\lim x_n=\mathbf{0}$. This is a contradiction because V is a neighborhood of $\mathbf{0}$. ∎

We are now ready to prove (vi). Let $\{U_n\}_n$ be a neighborhood base of $\mathbf{0}$ as in Lemma 5.10, with $U_0=B$. Let $\mathbf{D}$ be the set of all rational numbers of the form $r=\sum_{i=1}^k 2^{-n_i}$ where $\{n_1,\ldots,n_k\}$ is a finite increasing sequence of positive integers. For each $r\in\mathbf{D}$ as above, let $V_r=U_{n_1}\vee\ldots\vee U_{n_k}$, and let $V_1=U_0=B$. For each $a\in B$, we define

$$\mu(a)=\inf\{r\in\mathbf{D}\cup\{1\}: a\in V_r\}.$$

5.11. Lemma. *The function μ is a strictly positive Maharam submeasure.*

Proof. We repeatedly use the following fact that follows by induction on k: For every increasing sequence $\{n_1,\ldots,n_k\}$ of nonnegative integers, $U_{n_1+1}\vee\ldots\vee U_{n_k+1}\subseteq U_{n_1}$.

First, if $a\le b$ then $\mu(a)\le\mu(b)$; this is because for all $r,s\in\mathbf{D}$, if $r\le s$ then $V_r\subseteq V_s$.

Second, $\mu(a\vee b)\le\mu(a)+\mu(b)$ for all a and b; this is because $V_r\vee V_s\subseteq V_{r+s}$ for all r and s such that $r+s<1$.

Third, the submeasure μ is strictly positive: if $a\ne\mathbf{0}$ then there exists a positive integer n such that $a\notin U_n=V_{1/2^n}$, and so $\mu(a)\ge 1/2^n$.

Next we show that μ is continuous: if $\{a_n\}_n$ is a descending sequence converging in B to $\mathbf{0}$ then for every k eventually all a_n are in U_k, hence $\mu(a_n) \le 1/2^k$ for eventually all n, and so $\lim_n \mu(a_n) = 0$.

Finally, the topology induced by the submeasure μ coincides with τ_{s}: this is because $U_n \subseteq \{a \in B : \mu(a) \le 1/2^n\} \subseteq \bigcap_{k>n}(U_n \vee U_k) = \mathrm{cl}(U_n) \subseteq U_{n-1}$ for each $n > 0$. ∎

6. Sequential cardinals. We now turn our attention to the atomic Boolean algebra $P(\kappa)$ where κ is an infinite cardinal. We compare two topologies on $P(\kappa)$: the product topology τ_{c} (when $P(\kappa)$ is identified with the product space $\{0,1\}^\kappa$) and the sequential topology τ_{s}.

If f is a real-valued function on B we say that f is *sequentially continuous* if it is continuous in the sequential topology τ_{s} on B. Equivalently, $f(a_n)$ converges to $f(a)$ whenever a_n converges algebraically to a.

As τ_{s} is stronger than τ_{c}, every real-valued function on $P(\kappa)$ that is continuous in the product topology is sequentially continuous. Following [AnCh] we say that κ is a *sequential cardinal* if there exists a discontinuous real-valued function that is sequentially continuous.

A submeasure μ on $P(\kappa)$ is *nontrivial* if $\mu(\kappa) > 0$ and $\mu(\{\alpha\}) = 0$ for every $\alpha \in \kappa$. If μ is a Maharam submeasure on $P(\kappa)$ then it is a sequentially continuous function. If μ is nontrivial then it is discontinuous in the product topology, because it takes the value 0 on the dense set $[\kappa]^{\aleph_0}$. Thus if $P(\kappa)$ carries a nontrivial Maharam submeasure then κ is a sequential cardinal. In particular, the least real-valued measurable cardinal is sequential. Keisler and Tarski asked in [KeTa] whether the least sequential cardinal is real-valued measurable.

It follows from Theorem 6.2 below that if the Control Measure Problem has a positive answer then so does the Keisler–Tarski question.

We use the following theorem of G. Plebanek ([Pl], Theorem 6.1). A σ-complete Boolean algebra B carries a *Mazur functional* if there exists a sequentially continuous real-valued function f on B such that $f(\mathbf{0}) = 0$ and $f(b) > 0$ for all $b \ne \mathbf{0}$.

6.1. Theorem (Plebanek). *If κ is a sequential cardinal then there exists a σ-complete proper ideal H on $P(\kappa)$ containing all singletons and such that the algebra $P(\kappa)/H$ carries a Mazur functional.*

6.2. Theorem. *An infinite cardinal is sequential if and only if the algebra $P(\kappa)$ carries a nontrivial Maharam submeasure.*

Proof. Let κ be a sequential cardinal. By Theorem 6.1 the σ-complete algebra $B = P(\kappa)/H$ carries a Mazur functional f. First we claim that B satisfies the countable chain condition, and hence is a complete algebra. If not, there is an uncountable antichain, and it follows that there is some $\varepsilon > 0$

and there are infinitely many pairwise disjoint elements a_n, $n = 0, 1, 2, \ldots$, such that $|f(a_n)| \geq \varepsilon$ for all n. This contradicts the sequential continuity of f as $\lim_n a_n = \mathbf{0}$.

For each n, let U_n be the set of all $a \in B$ such that $|f(a)| < 1/n$. The U_n are neighborhoods of $\mathbf{0}$ and satisfy property (ii) of Theorem 5.1.

By Theorem 5.1, B carries a strictly positive Maharam submeasure. This submeasure induces a strictly positive Maharam submeasure on $P(\kappa)$ that vanishes H and therefore on singletons. Thus $P(\kappa)$ carries a nontrivial Maharam submeasure. ■

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