

On a certain map of a triangle

by

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Abstract. The paper answers some questions asked by Sharkovski concerning the map $F : (u, v) \mapsto (u(4 - u - v), uv)$ of the triangle $\Delta = \{u, v \geq 0 : u + v \leq 4\}$. We construct an absolutely continuous σ -finite invariant measure for F . We also prove the following strange phenomenon. The preimages of side $I = \Delta \cap \{v = 0\}$ form a dense subset $\bigcup F^{-n}(I)$ of Δ and there is another dense set Λ consisting of points whose orbits approach the interval I but are not attracted by I .

1. Introduction. The theory of dynamical systems deals with maps (or flows) on some spaces and investigates the structure of their trajectories from the topological, analytic or statistical point of view. There is one large class of systems where this aim is achieved: hyperbolic systems. Now the main problem is to develop a qualitative and ergodic theory for nonhyperbolic systems. The standard examples of such systems are: unimodal maps of the interval ([1]), the Hénon map ([3]), the Lozi attractor ([3]), iterations of rational maps of the Riemann sphere ([2]).

Here the formulas defining the dynamics are extremely simple (e.g. given by quadratic polynomials) whereas the analysis of the orbit structure is very difficult.

Recently Sharkovski proposed a simple map of a triangle Δ generalizing the unimodal map of the interval (similar to the Hénon map, see below). He asked whether there is an attractor inside the triangle with some good properties: transitivity, expansion, existence of an ergodic invariant measure etc. In this paper we show that if there is an attractor then it must be very strange. There exist two subsets Ω and Λ of the triangle with the following properties:

- (i) Ω and Λ are disjoint,
- (ii) Ω and Λ are each dense in Δ ,

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- (iii) Ω is the union of the preimages of a side I of the triangle Δ ,
- (iv) the trajectories of points of Λ approach the side I but they are not attracted by I .

Therefore there cannot exist a closed, transitive and topologically expanding attractor.

Moreover, we give an explicit formula for the invariant σ -finite measure. The next task is to study its ergodic properties. We plan to do it in the future.

2. Statement of the results. We consider the following dynamical system:

$$(1) \quad F(u, v) = (u(4 - u - v), uv).$$

The mapping F maps the triangle Δ with vertices $(0, 0)$, $(0, 4)$, $(4, 0)$ onto itself. Let I denote the interval $[(4, 0); (0, 4)]$. It is worth noticing that f_0 restricted to I is the map $x \mapsto x(4 - x)$ (a full parabola).

In the present paper we prove the following theorems:

THEOREM 1. *The map f_0 has an absolutely-continuous σ -finite invariant measure.*

THEOREM 2. *The preimages of I form a dense subset Ω of the triangle.*

THEOREM 3. *There exists a dense subset Λ of the triangle disjoint from Ω with the following property: for each $p \in \Lambda$ we have $\omega(p) \cap I \neq \emptyset$ and $\omega(p)$ is not included in I .*

3. Changes of variables and notation. We shall also use another chart

$$x = (u - 2)\sqrt{u(4 - u - v)}, \quad y = u(4 - u - v) - 2.$$

Here the map takes the form

$$(2) \quad f(x, y) = (y|x|, x^2 - 2).$$

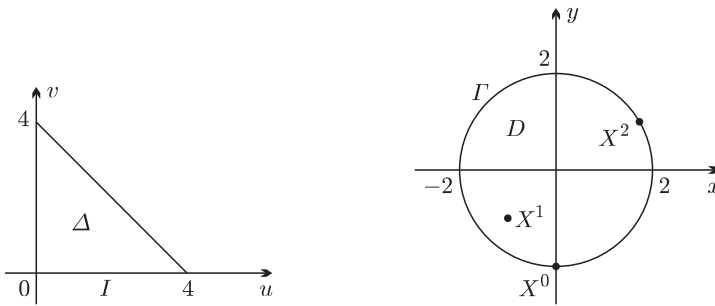


Fig. 1

Now f maps the circle $D = \{x^2 + y^2 \leq 4\}$ onto itself and $f(\partial D) = \partial D$. The fixed points of f are $X^0 = (0, -2)$, $X^1 = (-1, -1)$, $X^2 = (\sqrt{3}, 1)$.

We shall use the following notation:

- $X = (x, y)$,
- $X_k = (x_k, y_k) = f^k(X) = f \circ \dots \circ f(X)$,
- $\chi_n(X) = (x_0 x_1 x_2 \dots x_{n-1})^2$,
- $\mu(C)$ — the Lebesgue measure of a set C ,
- $Jf(X)$ — the jacobian of f at the point X ,
- $\varrho(X) = 4 - x^2 - y^2$,
- $\Gamma = \partial D = \{\varrho(X) = 0\}$,
- $A = \bigcup_{n=1}^{\infty} f^{-n}(\Gamma)$.

It is easy to see that

$$(3) \quad \varrho(f(X)) = 4 - [x^2 y^2 + (x^2 - 2)^2] = x^2(4 - x^2 - y^2) = x^2 \varrho(X)$$

so that

$$(4) \quad \varrho(X_k) = \chi_k(X) \varrho(X).$$

From the above property it immediately follows that $f(x, y) \in \Gamma$ iff either $x = 0$ and $f(x, y) = X^0$ or $(x, y) \in \Gamma$. Therefore

$$A = \bigcup_{n=1}^{\infty} f^{-n}(X^0) \cup \Gamma.$$

Of course f is not differentiable in D , but it is differentiable in $D \setminus \{(x, y) : x = 0\}$ and we have the formulas

$$(5) \quad Df(X) = \begin{bmatrix} \frac{x}{|x|} y & |x| \\ 2x & 0 \end{bmatrix}, \quad Jf(X) = 2x^2, \quad Jf^k(X) = 2^k \chi_k(X).$$

4. Proof of Theorem 1. We define $\nu(B)$ for any subset B of D as

$$\nu(B) = \int_B (4 - x^2 - y^2)^{-1} d^2 X.$$

The measure ν has a density with respect to Lebesgue measure

$$\frac{d\nu}{d\mu} = (4 - x^2 - y^2)^{-1} = \varrho(x, y)^{-1}$$

and is invariant with respect to the map f . Indeed, as f is two-to-one,

$$\nu(B) = \int_B \varrho(x, y)^{-1} d^2 X = \frac{1}{2} \int_{f^{-1}(B)} \varrho(f(x, y))^{-1} J(f(x, y)) d^2 X.$$

Now from (3) and (5) we obtain

$$\nu(B) = \frac{1}{2} \int_{f^{-1}(B)} \frac{2x^2}{x^2 \varrho(X)} d^2X = \nu(f^{-1}(B)). \blacksquare$$

5. Proof of Theorem 2. We work in the coordinate system (u, v) but we preserve some notations: Δ is the triangle, $I = \{(u, 0) \in \Delta\}$, $A = \bigcup_{n=1}^{\infty} F^{-n}(I)$.

The map F is not invertible, but F restricted to each of the sets $\Delta_- = \{0 \leq u \leq 2\}$, $\Delta_+ = \{2 \leq u \leq 4\}$ is invertible and the inverse functions are

$$g_{\pm}(u, v) = \left(2 \pm \sqrt{4 - u - v}, \frac{v}{2 \pm \sqrt{4 - u - v}} \right).$$

Wherever it does not lead to misunderstanding we shall identify points $(u, 0)$ from I with their first coordinate u .

Let us denote by $\gamma = \Delta \cap \{u = 2\}$ the segment separating Δ_+ and Δ_- . One can see that $\gamma = F^{-3}[(0, 0)]$. For any finite sequence $a = (a_1, \dots, a_k)$, $a_i \in \mathbb{Z}$, we define

$$\gamma^a = g_{\sigma_k}^{|a_k|} \circ \dots \circ g_{\sigma_1}^{|a_1|}(\gamma)$$

where σ_i denotes the sign of a_i . We have $\bigcup_a \gamma^a \subset A$. We denote the endpoints of γ^a by $\alpha^{(a)} = \gamma^a \cap I$. As $F|_I$ is a full parabola, the points $\alpha^{(a)}$ form a dense subset of I (see [1]). We consider eight curves δ_0^j , $j = 1, \dots, 8$, parts of preimages of I , such that:

- (i) $\delta_0^j = \gamma^a$ for some a ,
- (ii) $\delta_0^j \subseteq F^{-k}(\gamma)$, $k = 2, 3, 4, 5$,
- (iii) $\alpha_0^{(1)} < \alpha_0^{(2)} < \dots < \alpha_0^{(8)}$ where $\alpha_0^{(j)} = \delta_0^j \cap I$,
- (iv) $\alpha_0^{(1)} = g_-^2(2, 0)$, $\alpha_0^{(8)} = g_-^2 \circ g_+ \circ g_+^2(2, 0)$,
- (v) $\alpha_0^{(j)} = 4 \sin^2 \left[\frac{\pi}{16} \left(1 + \frac{j-1}{8} \right) \right]$.

Remark 1. The property (v) follows from the first four ones and the fact that $g_-|_I$ is conjugate to the tent map t (see the proof of Lemma 4).

We define $\delta_i^j = g_-^i(\delta_0^j)$, $\alpha_i^j = \delta_i^j \cap I$, $i = 1, 2, \dots$, $j = 1, \dots, 8$. Of course $\delta_i^j \subseteq A$ for every i, j . The curves δ_i^j divide the set $P_\varepsilon = \{(u, v) : v < \varepsilon\}$ into an infinite sequence of domains ω_i , $i = 1, 2, \dots$ (see Figure 2). We define additionally ω_0 as the part of P_ε lying to the right of the curve δ_0^8 .

Assume that $C \subset \Delta$ is an open ball with $C \cap A = \emptyset$. We strive to show that there would have to exist an iteration F^n such that

- (i) $F^n(C) \cap P_\varepsilon = C_{n,\varepsilon} \neq \emptyset$,
- (ii) $C_{n,\varepsilon}$ is not contained in any ω_k .

These two properties will lead us to a contradiction with the assumption $C \cap A = \emptyset$.

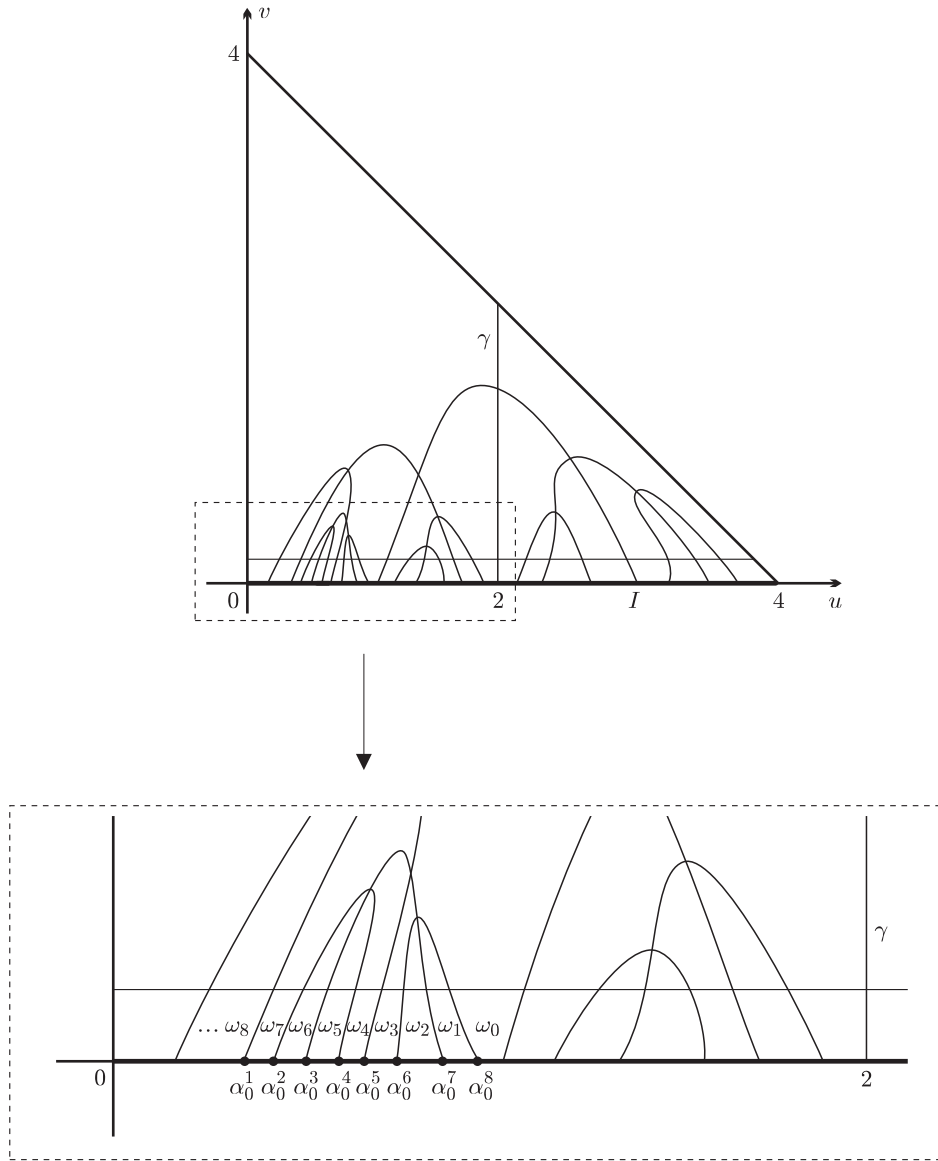


Fig. 2

The property (i) is achieved by means of the following lemma.

LEMMA 1 (Exponential contraction). *There exists a constant K such that if C is as above ($C \cap A = \emptyset$) then the set E of points in C such that*

$$v_n < K \cdot 2^{-n/2}$$

has positive Lebesgue measure.

We shall prove this lemma later.

The map F restricted to the interval $I = \Delta \cap \{v = 0\}$ is a quadratic map (full parabola) and is conjugate to the tent map $u \mapsto 2u$ for $0 \leq u \leq 2$, $u \mapsto 4 - 2u$ for $2 \leq u \leq 4$. It is a strongly expanding map.

It turns out that this expanding property is not lost in the small neighbourhood P_ε of I . So, whenever some distinct points of $E \subset C$ approach I then they should diverge along I . Their v -coordinates tend to 0 very fast, but their u -coordinates should behave randomly.

The property (ii) follows from the following two lemmas.

LEMMA 2 (Strong distortion). *For every $\lambda > 0$ and every integer $m > 0$ there exists $n > m$ such that*

$$\frac{u^{(1)}}{u^{(2)}} > 2 - \lambda$$

for some $U^{(i)} = (u^{(i)}, v^{(i)}) \in F^n(E)$, $i = 1, 2$, where E is the set from Lemma 1.

LEMMA 3. *There exists $\varepsilon > 0$ such that if $U^{(1)}, U^{(2)}$ belong to one component ω_k , $k = 1, 2, \dots$, in $P_\varepsilon = \{v < \varepsilon\}$ then*

$$(1.99)^{-1} < \frac{u^{(2)}}{u^{(1)}} < 1.99.$$

Now let $U^{(1)} = (u^{(1)}, v^{(1)})$, $U^{(2)} = (u^{(2)}, v^{(2)})$ be points in $C_{n,\varepsilon}$ such that there does not exist any $i > 0$ for which $U^{(1)}, U^{(2)} \in \omega_i$. Therefore either $U^{(1)} \in \omega_i$, $U^{(2)} \in \omega_j$, $i \neq j$, or $U^{(1)}, U^{(2)} \in \omega_0$. Since $F^n(C)$ is connected, in the first case it would intersect some δ_i^j . As $\delta_i^j \subset A$ it would mean that $F^n(C) \cap A \neq \emptyset \Rightarrow C \cap A \neq \emptyset$, a contradiction.

Assume then that $U^{(1)}, U^{(2)} \in \omega_0$. This means that $u^{(1)}, u^{(2)} \geq 2 - \sqrt{2}$ and therefore $|u^{(1)} - u^{(2)}| > 0.99 \cdot (2 - \sqrt{2}) > 0.4$. This is a quite large number and the first four preimages of γ (whose equations can of course be written explicitly) divide ω_0 into subsets of diameter smaller than 0.4. Therefore also in this case $F^n(C)$ intersects A , which ends the proof of Theorem 2. ■

Proof of Lemma 3. Let $U^{(1)}, U^{(2)}$ belong to one component ω_k of $P_\varepsilon \setminus \{\delta_j^i\}$ and let ω_k be bounded by two curves $\delta', \delta'' \in \{\delta_j^i\}$, δ'' on the right of δ' .

It is enough to show the estimate of Lemma 3 for the case when the points lie on these curves: $U^{(1)} \in \delta'$, $U^{(2)} \in \delta''$.

If $U^{(1)}, U^{(2)}$ are on the side I then $v_1 = v_2 = 0$ and $u^{(1)} = \alpha_i^j$, $u^{(2)} = \alpha_{i'}^{j'}$, which can be calculated. Indeed, the map F restricted to I is conjugate to the tent map

$$p \mapsto \begin{cases} 2p, & p \in [0, \pi/4], \\ \pi/2 - 2p, & [\pi/4, \pi/2], \end{cases}$$

by means of the conjugation $u = 4 \sin^2 p$. So, we get

$$\alpha_0^{(j)} = 4 \sin^2 \frac{(j+7)\pi}{128}, \quad p_{i+1}^j = \frac{1}{2} p_i^j, \quad \alpha_{i+1}^{(j)} = 4 \sin^2(j+7)\pi 2^{-7+i}.$$

Therefore

$$\frac{\alpha_i^{(k)}}{\alpha_i^{(l)}} \leq \Theta \left[\frac{k+7}{l+7} \right]^2$$

where $\Theta \approx 1.053$. Indeed, $\sin^2 \alpha / \sin^2 \beta < \Theta(\alpha/\beta)^2$ for $\alpha, \beta \leq \pi/8$ (see the proof of Lemma 4).

Now we have to extend this estimate to points above I .

If ε is small then in the first eight domains $\omega_1, \dots, \omega_8$ the required estimate holds (with a very wide margin). The problem is with the domains ω_k for large k .

We shall estimate the relative positions of the points $U^{(1)}, U^{(2)}$ by the relative positions of their iterations $U_n^{(1)}, U_n^{(2)}$. Because $F(u, v) = (u(4-u-v), uv)$ the trajectories $\{U_n^{(1)}\}, \{U_n^{(2)}\}$ remain in the domain P_ε , i.e. $v_n^{(1)} < \varepsilon$ and $v_n^{(2)} < \varepsilon$ if $u_n^{(1)}, u_n^{(2)} < 2 + \sqrt{2}$ (i.e. in the domains $\bigcup_{j=1}^\infty \omega_j$).

Let n be such that $U_n^{(1)}, U_n^{(2)} \in \omega_j$, $j = 1, \dots, 8$. We shall estimate $u^{(2)}/u^{(1)}$ from above by means of $u_n^{(2)}/u_n^{(1)}$ which is good. We introduce additional two points on the side I :

$$U^{(3)} = (u^{(1)}, 0), \quad U^{(4)} = (u^{(2)}, 0)$$

(the projections of $U^{(1)}, U^{(2)}$ onto I). Let $U_n^{(3)} = (u_n^{(3)}, 0)$, $U_n^{(4)} = (u_n^{(4)}, 0)$ ($u^{(3)} = u^{(1)}$, $u^{(4)} = u^{(2)}$) be their trajectories. We write

$$\frac{u^{(2)}}{u^{(1)}} = \frac{u^{(3)}}{u^{(4)}} \cdot \frac{u^{(4)}}{u^{(1)}} \cdot \frac{u^{(2)}}{u^{(3)}}.$$

LEMMA 4. *There exists a continuous function $\xi(v)$, $v > 0$, $\xi(0) = 1$, such that if $u^{(j)} \leq 2 - \sqrt{2}$ and $u_n^{(j)} < 2$ then*

$$\frac{u^{(3)}}{u^{(4)}} \leq \Theta^2 \frac{u_n^{(3)}}{u_n^{(4)}}, \quad \frac{u^{(4)}}{u^{(1)}} \leq \Theta^2 \frac{u_n^{(4)}}{u_n^{(1)}}, \quad \frac{u^{(2)}}{u^{(3)}} \leq \Theta^3 \xi(v^{(2)}) \cdot \frac{u_n^{(2)}}{u_n^{(3)}}.$$

From this lemma we get

$$\frac{u^{(2)}}{u^{(1)}} \leq \Theta^7 \xi(v^{(2)}) \cdot \frac{u_n^{(2)}}{u_n^{(1)}}$$

where $u_n^{(2)}/u_n^{(1)}$ has a good estimate (because we are in the first eight domains):

$$\frac{u_n^{(2)}}{u_n^{(1)}} \approx [1 + O(\varepsilon)] \cdot \frac{\alpha_0^{(j+1)}}{\alpha_0^{(j)}} \leq [1 + O(\varepsilon)] \cdot \Theta \cdot \left(\frac{8}{7}\right)^2.$$

One can check that if $\Theta = 1.053$ then $\Theta^8 \cdot (8/7)^2 \approx 1.97$. From this, Lemma 3 follows. ■

Proof of Lemma 4. The map F is conjugate to the map

$$T(p, q) = (\arcsin(2\sqrt{\cos^2 p - q/4} \sin p), 4q \sin^2 p) = (p_1, q_1)$$

by means of the conjugacy $h(p, q) = (4 \sin^2 p, q)$. T restricted to the interval $[0, \pi/2] \times \{0\}$ is the tent map

$$t(p) = \begin{cases} 2p, & p \in [0, \pi/4], \\ \pi/2 - 2p, & [\pi/4, \pi/2]. \end{cases}$$

Let $P^{(i)} = (p^{(i)}, q^{(i)}) = h^{-1}(U^{(i)})$, $i = 1, 2, 3, 4$. As $(\sin x)/x$ is decreasing in $[0, \pi/2]$, we have

$$(6) \quad 1 \geq \frac{\sin x}{x} \geq \frac{8}{\pi} \sin \frac{\pi}{8} > 0.974495 \quad \text{for } x \in [0, \pi/8].$$

Let

$$\Theta = \left(\frac{8}{\pi} \sin \frac{\pi}{8} \right)^{-2} \approx 1.053.$$

Because

$$(7) \quad \frac{u^{(i)}}{u^{(j)}} = \frac{\sin^2 p^{(i)}}{\sin^2 p^{(j)}}, \quad i, j = 1, 2, 3, 4,$$

we have

$$(8) \quad \frac{1}{\Theta} \left[\frac{p^{(i)}}{p^{(j)}} \right]^2 \leq \frac{u^{(i)}}{u^{(j)}} \leq \Theta \left[\frac{p^{(i)}}{p^{(j)}} \right]^2, \quad i, j = 1, 2, 3, 4.$$

Now because $p_n^{(i)} = 2^n p^{(i)}$ for $i = 3, 4$ we get $p_n^{(3)}/p_n^{(4)} = p^{(3)}/p^{(4)}$ and then

$$\frac{u^{(3)}}{u^{(4)}} \leq \Theta \left[\frac{p^{(3)}}{p^{(4)}} \right]^2 = \Theta \left[\frac{p_n^{(3)}}{p_n^{(4)}} \right]^2 \leq \Theta^2 \frac{u_n^{(3)}}{u_n^{(4)}}.$$

Note that this holds because $u_n^{(i)} < 2$ by assumption. This ends the proof of the first inequality of Lemma 4. Similarly we get the inequality

$$(9) \quad \frac{u^{(2)}}{u^{(3)}} = \frac{u^{(4)}}{u^{(2)}} = \frac{u^{(4)}}{u^{(3)}} \leq \Theta^2 \frac{u_n^{(4)}}{u_n^{(3)}}$$

as $u^{(3)} = u^{(1)}$, $u^{(2)} = u^{(4)}$. As $F(u, v) = (u(4 - u - v), uv)$, we immediately get $u_n^{(3)} \geq u_n^{(1)}$ (the points on I “move faster” than the ones above). From this the second inequality of Lemma 4 follows.

If $(p_1, q_1) = T(p, q)$ and $q = 0$ then $p_1 = 2p$ and we have

$$(10) \quad p_1(p, 0) - p_1(p, q) < 2pq,$$

$$(11) \quad q_1(p, q) \leq \frac{3}{5}q.$$

Indeed, $|\arcsin a - \arcsin b| \leq 2|a - b|$ for $a, b \leq \sqrt{2}/2$. Thus

$$\begin{aligned} & \arcsin(2\sqrt{\cos^2 p} \sin p) - \arcsin(2\sqrt{\cos^2 p - q/4} \sin p) \\ & \leq 4 \sin p (\sqrt{\cos^2 p} - \sqrt{\cos^2 p - q/4}) \\ & \leq \frac{q \sin p}{\sqrt{\cos^2 p} + \sqrt{\cos^2 p - q/4}} \leq q \tan p \leq 2pq \end{aligned}$$

as $\tan p \leq 2p$ for $p \leq \pi/4$. The inequality (11) is a consequence of the fact that $4 \sin \pi/8 \leq 3/5$.

From (11) we obtain

$$q_n^{(2)} \leq \left(\frac{3}{5}\right)^n v^{(2)}.$$

Now from (10) we have

$$p_{n+1}^{(2)} = p_1(p_n^{(2)}, q_n^{(2)}) \geq p_1(p_n^{(2)}, 0) - 2q_n^{(2)} p_n^{(2)} = (1 - q_n^{(2)}) 2p_n^{(2)}$$

and hence

$$p_{n+1}^{(2)} \geq 2(1 - (3/5)^n v^{(2)}) p_n^{(2)}.$$

By simple induction we obtain

$$p_{n+1}^{(2)} \geq \left[\prod_{i=1}^{\infty} \left(1 - \left(\frac{3}{5}\right)^i v^{(2)}\right) \right] 2^n p^{(2)}.$$

We put

$$\xi(v^{(2)}) = \left[\prod_{i=1}^{\infty} \left(1 - \left(\frac{3}{5}\right)^i v^{(2)}\right) \right]^{-2}.$$

The infinite product is convergent, so the definition is correct and $\xi(v^{(2)})$ has all the desired properties. Because $p_{n+1}^{(4)} = 2^n p^{(4)} = 2^n p^{(2)}$ for the fourth point on the $q = 0$ axis, we get from this and (8),

$$\frac{u_n^{(4)}}{u_n^{(2)}} \leq \Theta \left[\frac{p_n^{(4)}}{p_n^{(2)}} \right]^2 \leq \Theta \xi(v^{(2)}).$$

Now the first inequality of Lemma 4 follows from (9). ■

6. Proof of Lemmas 1 and 2. In this section we shall work in the (x, y) chart.

Proof of Lemma 1. As C is a connected set disjoint from $A = \bigcup_{n=1}^{\infty} f^{-n}(\Gamma)$, $f^k|_C$ is a diffeomorphism onto its image for every k . We have (from (5))

$$\mu(f^k(C)) = \int_C J f^k d^2 X = 2^k \int_C \chi_k(X) d^2 X.$$

Let $K > 0$ be a constant whose value will be specified later. We define

$$C_n = \{X \in C : \varrho(X_n) \geq K \cdot 2^{-n/2}\} \quad \text{for } n = 0, 1, 2, \dots$$

We shall show that

$$(12) \quad \mu(C_n) \leq \frac{16\pi}{K} 2^{-n/2}.$$

From this, Lemma 1 follows because the set $C \setminus \bigcup C_n$ consists of points which approach the boundary Γ very fast, and has positive Lebesgue measure for large K (the explicit relation between $\varrho(X_n)$ and v_n will be given later).

To show (12) we use the formula

$$2^n \int_{C_n} \chi_n(X) d^2 X = \mu(f^n(C_n)) \leq \mu(D) = 4\pi$$

and the estimate

$$\chi_n(X) \geq \chi_n(X) \frac{\varrho(X)}{4} = \frac{1}{4} \varrho(X_n) \geq K \cdot 2^{-n/2} / 4$$

for $X \in C_n$. Therefore

$$2^n K \cdot 2^{-n/2-2} \mu(C_n) \leq 4\pi. \quad \blacksquare$$

We shall call the property $\varrho(X_n) < K \cdot 2^{-n/2}$, $X \in C \setminus \bigcup C_n$, $n = 1, 2, \dots$, *exponential contraction*. The ratio $\sqrt{2}$ of exponential contraction can be replaced by any other ratio $d > 1$.

Proof of Lemma 2. First we show that

$$\frac{\sup_{X \in E} \chi_n(X)}{\inf_{X \in E} \chi_n(X)} > K \cdot 2^n$$

for some constant $K = K(E)$ not depending on n . (Here $E = C \setminus \bigcup C_n$ is the subset defined above.)

We have the estimate

$$\mu(f^n(E)) = 2^n \int_E \chi_n(X) d^2 X \geq 2^n \mu(E) \inf_{X \in E} \chi_n(X).$$

On the other hand,

$$\varrho(X_n) = \chi_n(X)^2 \varrho(X) \leq 4 \sup_{Y \in E} \chi_n(Y)^2, \quad X \in E,$$

and

$$\mu(f^n(E)) \leq \mu(\{\varrho(X) \leq 4 \sup_E \chi_n\}) \leq 4\pi \sup_E \chi_n$$

(because $\varrho(X) = 4 - |X|^2$). From this the inequality

$$\frac{\sup_{X \in E} \chi_n(X)}{\inf_{X \in E} \chi_n(X)} \geq \frac{\mu(E)}{4\pi} \cdot 2^n$$

follows. Now because $\varrho(X_n) = \chi_n(X)\varrho(X)$ we get

$$\frac{\sup_{X \in E} \varrho(X_n)}{\inf_{X \in E} \varrho(X_n)} > K_1 \cdot 2^n$$

where K_1 is a constant depending on the initial ball C .

We pass to the (u, v) coordinates. It turns out that the function $\varrho(X)$ has a simple expression here: it is the second component v_2 of the second iteration of $U = (u, v)$. Indeed,

$$\varrho(x, y) = 4 - x^2 - y^2 = vu^2(4 - u - v) = v_2.$$

Therefore the *exponential contraction* property also holds in the coordinates (u, v) , which means that for any open set $C \subset \Delta$ there exists an open subset E of C such that $v_n < K \cdot 2^{-n/2}$ for $U \in E$, $n = 1, 2, \dots$ (compare proof of Lemma 1). Therefore

$$\frac{\sup_{X \in E} v_n}{\inf_{X \in E} v_n} > K_2 \cdot 2^n.$$

Assume now that the statement of Lemma 2 is false. This would mean that there exist λ, m such that for every $n > m$, $u_n^{(1)}/u_n^{(2)} < 2 - \lambda$ for every $U_n^{(1)}, U_n^{(2)} \in F^n(E)$, $U^{(i)} = (u_n^{(i)}, v_n^{(i)})$. One can check that $v_n = v_0 u_0 u_1 u_2 \dots u_{n-1}$. Therefore

$$(13) \quad \frac{v_n^{(1)}}{v_n^{(2)}} = \frac{v_0^{(1)} u_0^{(1)} u_1^{(1)} u_2^{(1)} \dots u_{n-1}^{(1)}}{v_0^{(2)} u_0^{(2)} u_1^{(2)} u_2^{(2)} \dots u_{n-1}^{(2)}} < \frac{v_0^{(1)} u_0^{(1)} u_1^{(1)} u_2^{(1)} \dots u_{m-1}^{(1)}}{v_0^{(2)} u_0^{(2)} u_1^{(2)} u_2^{(2)} \dots u_{m-1}^{(2)}} (2 - \lambda)^{n-m} < L(2 - \lambda)^n$$

where

$$L = \frac{1}{(2 - \lambda)^m} \sup_{U^{(1)}, U^{(2)} \in E} \left(\frac{v_0^{(1)} u_0^{(1)} u_1^{(1)} \dots u_{m-1}^{(1)}}{v_0^{(2)} u_0^{(2)} u_1^{(2)} \dots u_{m-1}^{(2)}} \right).$$

From (13) we get the inequality

$$L(2 - \lambda)^n > K_2 \cdot 2^n, \quad n = 1, 2, \dots,$$

which cannot hold for large n . ■

7. Proof of Theorem 3. Set $B = \{U : \omega(U) \cap I \neq \emptyset, \omega(U) \cap (\Delta \setminus I) \neq \emptyset\}$. We have to prove that for any ball $C_0 \subseteq \Delta$, $C_0 \cap B \neq \emptyset$.

We use the form (II). The map F has the fixed point $(3, 0)$ on the invariant interval $I = \{0 \leq u \leq 4, v = 0\}$. It is a repelling fixed point, as the linear part of F at $(3, 0)$ is $\begin{bmatrix} -2 & -3 \\ 0 & 3 \end{bmatrix}$.

Take the set $R = \{(u, v) : 0 < v < 1/8, |u - 3| < v\}$ (triangle with one vertex at the source $(3, 0)$) and the sets $P_k = \{(u, v) \in \Delta : v < 1/(8k)\}$, $k = 1, 2, \dots$ (stripes approaching the interval I ; see Figure 3).

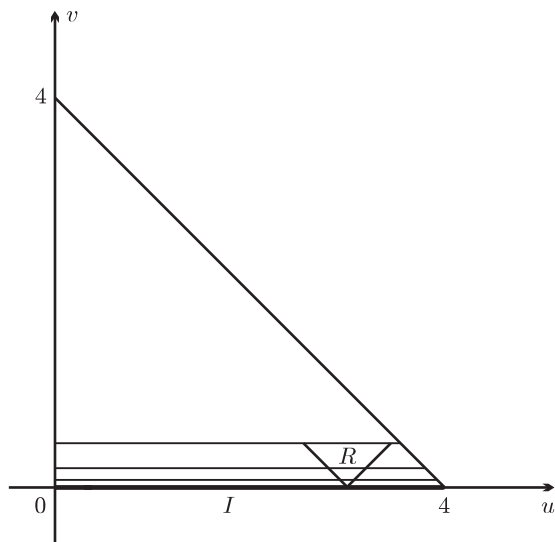


Fig. 3

Let C_0 be the ball. From Theorem 1 we know that C_0 intersects $A = \bigcup_{n=1}^{\infty} F^{-n}(\{U^{(0)}\})$, where $U^{(0)} = (0, 0)$ corresponds to the fixed point $X^{(0)}$ for the map f . So there exists a connected open subset $C'_0 \subseteq C_0$ such that $F^n(C'_0) \subseteq P_1$ for some n and $U^{(0)} \in \text{cl } F^n(C'_0)$. After several further iterations the set C'_0 starts to intersect the triangle R , $F^m(C'_0) \cap R \neq \emptyset$ ($m \geq n$). The points of $F^m(C'_0) \cap R$ leave P_1 under next iterations.

We repeat the same analysis with P_1 replaced with P_2 and C_0 replaced with $C_1 = C'_0 \cap F^{-m}(R)$. We get some C'_1 such that $F^{m_1}(C'_1 \cap R) \cap P_2 \neq \emptyset$. Next we apply the same to $C_2 = C'_1 \cap F^{-m_1}(R \cap P_2)$ etc.

By repeating this procedure we obtain a sequence of nonempty sets C_i such that $C_0 \supset C_1 \supset C_2 \supset \dots$. As Δ is a compact set the intersection $\bigcap_i C_i$ is nonempty. Let $U_0 \in \bigcap_i C_i$. Then:

- (i) for every k there exists n such that $F^n(U_0) \in P_k$,
- (ii) for every N there exists $n > N$ such that $F^n(U_0) \in P_k F^n(U_0) \in R$.

Therefore $U_0 \in B$. This ends the proof of Theorem 3. ■

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