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Topological invariance of the Collet–Eckmann property for S-unimodal maps

by

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Abstract. We prove that if f, g are smooth unimodal maps of the interval with negative Schwarzian derivative, conjugated by a homeomorphism of the interval, and f is Collet–Eckmann, then so is g.

Introduction

DEFINITIONS. We say that c is a *nonflat critical point* of f, a map of the interval, if $f'(c) = 0$ but for some $l_c > 1$ the limit $\lim_{x\to c} |f'(x)|/|x - c|^{l_c - 1}$ exists and is nonzero.

A C^2 map f of the interval is called *S-multimodal* if:

(i) f has a finite number of nonflat critical points,
(ii) $|f'|^{-1/2}$ is convex between the critical points.

If f has precisely one critical point c and $f''(c) \neq 0$ we call the map *S-unimodal*.

If f is C^3 then condition (ii) is equivalent to f having *nonpositive Schwarzian derivative*, namely $f'''(x)/f'(x) - 3(f''/f')^2/2 \le 0$ outside the critical points or that f expands the cross-ratio between the critical points. These properties are invariant under composition, hence hereditary for iterations (see [MS, IV.1]). In particular, they give some bounds for distortion.

Write Crit or Crit(f) for the set of all f-critical points, i.e. Crit $= \{x \in I : f'(x) = 0\}$. Write Crit$'$ for the set of those f-critical points whose forward trajectories do not hit critical points. We call an S-multimodal map f *Collet–Eckmann* if there exist $\lambda > 1$ and $C > 0$ such that for every

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$c \in \mathrm{Crit}'$ and every positive integer n,

(CE1) $$|(f^n)'(f(c))| \geq C\lambda^n.$$

The aim of this paper is to prove

THEOREM A. *If f and g are S-unimodal maps of the interval conjugated by a homeomorphism h of the interval, i.e. $h \circ f = g \circ h$, and f is Collet–Eckmann, then so is g.*

In fact, this paper provides only a concluding part of the proof. Important parts have been proved earlier in [NS] and [PR1].

Notice that we do not assume that f and g have the same order l at the critical point.

We assume that no map of the interval considered in this paper has a basin of attraction to an attracting or a parabolic periodic orbit. This property is obviously preserved under homeomorphic conjugacies.

The Collet–Eckmann condition (CE1) was introduced in [CE] in the context of the existence of an absolutely continuous invariant measure; for a general reference see [MS, V.4]. In [NP] we considered the problem of the regularity of a conjugacy between two Collet–Eckmann maps and a question arised whether (CE1) is a topological condition. According to [JS] the question was also raised by J. Guckenheimer and M. Misiurewicz. Here we give an affirmative answer.

A topological condition for S-unimodal maps which, in conjunction with (CE1), is also topological and which for a quadratic family holds for a positive measure set of parameters was given by Jakobson and Świątek in [JS, Sec. 5.3]. Later Duncan Sands in his Ph.D. thesis [S] gave a topological condition for S-unimodal maps which implies (CE1) and another one which excludes (CE1), but some cases were still left undecided. A result weaker than Theorem A, saying that quasi-symmetric conjugacy leaves (CE1) invariant, was proved in [SN].

Let us introduce the following conditions on an S-multimodal mapping $f : I \to I$:

(1) The Collet–Eckmann condition (CE1);

(2) (*exponential shrinking of components*) There exist $0 < \xi < 1$ and $\delta_2 > 0$ such that for every interval $J \subset I$ with length $|J| \leq \delta_2$, every positive integer n and every component K of $f^{-n}(J)$ one has $|K| \leq \xi^n$;

(3) (*exponential shrinking of components at critical points*) There exist $0 < \xi < 1$ and $\delta_3 > 0$ such that for every $c \in \mathrm{Crit}$ and for every positive integer n, for

$$B = B(f^n(c), \delta_3) := \{x \in I : |x - f^n(c)| \leq \delta_3\}$$

and the component K of $f^{-n}(B)$ which contains c one has $|K| \leq \xi^n$;

(4) (*finite criticality*) There exist $M > 0$, $P_4 > 0$ and $\delta_4 > 0$ such that for every $x \in I$ there exists an increasing sequence of positive integers n_j, $j = 1, 2, \ldots$, such that $n_j \le P_4 j$ and for each j,

$$\sharp\{i : 0 \le i < n_j,\ \mathrm{Comp}_{f^i(x)} f^{-(n_j - i)}(B(f^{n_j}(x), \delta_4)) \cap \mathrm{Crit} \ne \emptyset\} \le M$$

(the subscript y at Comp, here $y = f^i(x)$, means that the component Comp_y contains y; later on, y can also be a set contained in the component);

(5) (*mean exponential shrinking of components*) There exist $P_5 > 0$, $0 < \xi < 1$ and $\delta_5 > 0$ such that for every $x \in I$ there exists an increasing sequence of positive integers n_j, $j = 1, 2, \ldots$, such that $n_j \le P_5 j$ and $|\mathrm{Comp}_x f^{-n_j}(B(f^{n_j}(x), \delta_5))| \le \xi^{n_j}$;

(6) (*uniform hyperbolicity on periodic trajectories*) There exists $\lambda > 1$ such that for every integer n and $x \in I$ of period n one has $|(f^n)'(x)| \ge \lambda^n$.

We shall prove that for every $k = 1, \ldots, 5$ the property (k) implies $(k+1)$. The implication (6)⇒(1) is a recent theorem by the first author and Duncan Sands [NS], in the unimodal case.

Notice that (4) is a topological property. We thus get Theorem A.

We do not know whether (6)⇒(1) holds in the multimodal case ([1]); this is the reason why we restricted Theorem A to the unimodal case. Negative Schwarzian is used only in (1)⇒(2), (4)⇒(5) and (6)⇒(1).

One can rewrite all the above properties for f a rational mapping on the Riemann sphere without parabolic periodic orbits. One then only considers critical points in the Julia set. One considers conjugacies on neighbourhoods of Julia sets; in this sense (4) is a topological invariant. We call this setting the *holomorphic case*.

The implication (1)⇒(2) has been proved in [PR1, Proposition 3.1] in the holomorphic case. In the interval case the proof is similar. In the unimodal case, order 2 at the critical point, this implication has been proved earlier in [NP, Main Lemma].

(2)⇒(3) is trivial.

The proof of (3)⇒(4) goes similarly to the proof of (1)⇒(4) in [PR1]; it is even simpler, one does not need to consider pre-images according to the "shrinking neighbourhoods" procedure (see [P1], [GS]), because one need not control any distortion. We shall give this proof in Section 1.

(4)⇒(5) goes by the "telescope" construction; it has been done in the rational case in [PR1, Proof of Proposition 3.1]. We adapt the proof to the interval case in Section 2.

(5)⇒(6) will also be done in Section 2. This is very easy.

([1]) *Added in revision*: It does not hold (for an idea how to construct a counterexample see [CJY, Remark 1, p. 9], [P4, Introduction] and [PR2]).

Added in revision: 1. A theorem similar to Theorem A holds in the holomorphic case provided there is at most one critical point in the Julia set (see the forthcoming paper by the second author and S. Rohde [PR2] and [P4]).

2. (5)⇒(2) is straightforward, see [P4].

1. Proof of (3)⇒(4). For every $x \in I$ and positive integer n write

$$\phi(x,n) = -\log \operatorname{dist}(f^n(x), \operatorname{Crit}(f)).$$

As $|I| = 1$, $\phi(x,n) \geq 0$. We write $\phi(n)$ if x is fixed.

The main ingredient of the proof of (3)⇒(4) is the following:

Lemma 1.1. *Let f be a differentiable mapping of the interval with a finite number of critical points and derivative Hölder continuous at these points. Then there exists a constant C_f such that for each $n \geq 1$ and $x \in I$,*

$$\sum_{j=0}^{n}{}' \phi(x,j) \leq nC_f, \tag{1.1}$$

where $\sum'$ denotes summation over all but at most $\#\operatorname{Crit}$ *indices.*

This lemma was proved in [DPU, (3.3)] in the holomorphic case. In the interval case the proof is almost the same:

The point in [DPU] is that if the sum in (1.1) is larger than Cn for C large enough, then one arrives at a disc $B = B(c,r)$ with $c \in \operatorname{Crit}(f)$ such that $f^n(B) \subset B$, which contradicts the fact that c is in the Julia set.

In the interval case $f^n(B) \subset B$ can happen for arbitrarily small r for an infinitely renormalizable S-unimodal map.

Recall, however, that in [DPU] one concludes that if (1.1) is not fulfilled then $f^n(B) \subset B$ and $|(f^n)'|$ is small ($< 1/2$) on B. By the inclusion there is an f^n-fixed point $p \in B$. As $|(f^n)'(p)|$ is small, p is attracting, which contradicts the assumptions [2].

In the S-unimodal interval case Lemma 1.1 follows also immediately from the following

Lemma 1.2 [NS]. *For every $0 < \eta < 1$ there exists C such that for every $x \in I$ and every positive integer n there exists $0 \leq \widehat{n} < n$ such that $|(f^n)'(x)|/|f'(f^{\hat{n}}(x))| \geq C\eta^n$.*

[Notice that though η can be arbitrarily close to 1, this does not imply automatically that C_f in (1.1) can be arbitrarily close to 0, even if in (1.1) we replace ϕ by $\max(0, \phi - \text{Const})$ for an arbitrary Const. If C_f in (1.1) is sufficiently small then (4) holds with criticality 0, see [P2].]

[2] An appendix containing a complete proof has been added on the request of the Editorial Board of Fund. Math.

Let us continue the proof of (3)⇒(4). Fix an arbitrary $x \in I$ and write $\phi(i) := \phi(x,i)$.

Write $S_i = (i, i + \phi(i)K_f] \subset \mathbb{R}$, where we set $K_f = 1/\log(1/\xi)$.

(One could view the "graph" of $i \mapsto \phi(i)$ as the union of all vertical line segments $\{i\} \times (0, \phi(i)]$ in $\mathbb{R}^2$. Then each segment throws a *shadow* S_i on the real axis.)

The shadows of the exceptional indices in (1.1) could be infinitely long, but nevertheless (1.1) implies that many of the times n belong to boundedly many shadows: Indeed, set $N_f = 2(\#\,\mathrm{Crit} + C_f K_f)$ and

$$A = \{n \in \mathbb{N} : n \text{ belongs to at most } N_f \text{ shadows}\}.$$

For each $0 \le i \le m$ denote by χ_i the indicator function of $S_i \cap [0,m]$. By (1.1),

$$mC_f K_f \ge K_f \sum_{i=0}^{m-1}{}' \phi(x,i) = \sum_{i=0}^{m-1}{}' |S_i| \ge \sum_{i=0}^{m-1}{}' \int \chi_i = \int \sum_{i=0}^{m-1}{}' \chi_i.$$

Together with the exceptional indices we obtain

$$m(\#\,\mathrm{Crit} + C_f K_f) \ge \int \sum_{i=0}^{m-1} \chi_i \ge \#([1,m] \setminus A) \cdot N_f$$

by the definition of A. We conclude from the definition of N_f that

$$\frac{\#(A \cap [1,m])}{m} \ge \frac{1}{2}. \tag{1.2}$$

So if we order all the integers in A according to their growth we obtain $n_j \le nj$. We set $P_4 = 2$ in (4).

(Notice that if in the definition of N_f the factor 2 is replaced by an arbitrary Q then $1 - 1/Q$ stands on the right hand side of (1.2), which can therefore be arbitrarily close to 1. We can then set $P_4 = 1/(1 - 1/Q)$.)

Finally, we claim that for every $n = n_j \in A$ and $0 \le i < n$, if the set

$$B_{n,i} := \mathrm{Comp}_{f^i(x)}\, f^{-n+i}(B(f^n(x), \delta_3))$$

contains an f-critical point then n is in the shadow S_i.

Indeed, suppose that $B_{n,i}$ contains $c \in \mathrm{Crit}(f)$. Then by (3) used for $n - i$,

$$|c - f^i(x)| \le \xi^{n-i}. \tag{1.3}$$

This shows that $\phi(i) \ge -(n-i)\log\xi$ hence $n - i \le \phi(i)/\log(1/\xi)$. Hence n is in the shadow S_i.

(Inequality (1.3) also shows that each $B_{n,i}$ contains at most one f-critical point provided $\delta_4 \le \delta_3$ is small enough.)

This proves (4) with $M = N_f$. ∎

2. The implications (4)⇒(5)⇒(6). We start with the easier:

Proof of (5)⇒(6). Fix $m > 0$ and $x \in I$ so that $f^m(x) = x$. As x is a source (i.e. $|(f^m)'(x)| > 1$) there exists $a > 0$ such that $f^m(B(x,a)) \supset \operatorname{cl} B(x,a)$ and f^m has no critical points in $B(x,a)$.

Denote the periodic orbit of x by $O(x)$. For every $n > 0$ denote by g_n the branch of f^{-n} which maps x into $O(x)$. These branches are well defined on $B(x,a)$ by the definition of a.

By the finiteness of $O(x)$ and (5) there exist $y \in O(X)$ and an increasing sequence of positive integers n_j, $j = 0, 1, \ldots$, such that

$$|\operatorname{Comp}_x f^{-n_j}(B(y,\delta_5))| \le \xi^{n_j}$$

and for $K := \operatorname{Comp}_x f^{-n_0}(B(x,\delta_5))$ one has $|K| < a$.

Then $|g_{n_j-n_0}(K)|/|K| \le \xi^{n_j}/|K|$. As we are in a neighbourhood of a periodic source and the derivative of f is Hölder, all g_n's have uniformly bounded distortion on K. We conclude that $|(g_{n_j-n_0})'(x)| \le \mathrm{Const}\,\xi^{n_j-n_0}$. Letting j grow to ∞ and noticing that each $n_j - n_0$ is a multiple of m we obtain $|(f^m)'(x)| \ge \xi^{-m}$, which proves (6) with $\lambda = \xi^{-1}$. ∎

To prove (4)⇒(5) we need the following

LEMMA 2.1. *For every $N, \varepsilon > 0$ there exists k such that for every $n \ge k$ and for every interval $K \subset I$ if $f^n|_K$ has at most N critical points, then $|K| < \varepsilon$.*

Remark. In the holomorphic case this is a variant of the Mañé Lemma [M], [P1, Lemma 1.1], where one asserts $\operatorname{diam} \operatorname{Comp}_x f^{-n}(B(f^n(x), \lambda r)) < \varepsilon$, $\lambda < 1$ provided f^n has at most N critical points in $\operatorname{Comp}_x f^{-n}(B(f^n(x), r))$. In the interval case one does not need λ. (An adaptation to the interval case, silmilar to that in Lemmas 2.1 and 2.2, appeared in [P3, Sec. 3].)

Proof (of Lemma 2.1). If Lemma 2.1 were not true there would exist a sequence of intervals $J_j \subset I$ such that $|J_j| \ge \varepsilon/N$ and integers n_j, $j = 1, 2, \ldots$, such that $n_j \to \infty$ as $j \to \infty$ and f_{n_j} is monotone on J_j for each j. This leads to the existence of a *homterval*. Namely there exists an interval $J \subset I$ of length $\varepsilon/(2N)$ such that $J \subset J_{j_k}$ for a sequence $j_k \to \infty$, $k = 1, 2, \ldots$, and each $f^{n_{j_k}}$ is monotone on J, hence f^n is monotone on J for each positive integer n. However, homtervals do not exist [MS, Thm. II.6.2], so we arrived at a contradiction. ∎

LEMMA 2.2. *For every $M > 0$ and $0 < p < 1$ there exists $0 < q = q(M,p) < 1$ such that for every pair of intervals $J \subset K \subset I$, every positive integer n, every pair of components J', K' of $f^{-n}(J)$ and $f^{-n}(K)$ respectively such that $J' \subset K'$, for L, R the left and right components of $K \setminus J$ and L', R' the left and right components of $K' \setminus J'$ respectively, if*

$$\sharp\{i : 0 \le i < n,\ \operatorname{Comp}_{f^i(K')} f^{-(n-i)}(K) \cap \mathrm{Crit} \ne \emptyset\} \le M$$

and if

$$|L|/|K| > p \quad \textit{and} \quad |R|/|K| > p$$

then

$$|L'|/|K'| > q \quad \textit{and} \quad |R'|/|K'| > q.$$

Remarks. This lemma also has its holomorphic analogue (see [P1, Lemma 1.4] and [PR1, Lemma 2.1]). In the interval case its proof is implicitly contained in [P3, Sec. 3] and [MS, Ch. IV, Th. 3.1, "Macroscopic Koebe Principle"] for f a smooth homeomorphism. We provide a proof below for completeness.

Proof (of Lemma 2.2). In the case $M = 0$ this lemma is called the "Koebe Principle" for distortion [MS, Chapter IV]. We shall refer to this in the proof. Denote $q(0,p)$ by $a(p)$.

Consider compatible components K_j of $f^{-j}(K)$ and J_j of $f^{-j}(J)$, i.e. such that $f(K_j) \subset K_{j-1}$ and $f(J_j) \subset J_{j-1}$ for $j = 1, \ldots, n$ and such that $K_n = K'$, $J_n = J'$.

Denote the left and right components of $K_j \setminus J_j$ by L_j and R_j respectively. If $j = n_1$ is the first j for which K_j contains a critical point, say c, then $|L_{j-1}|/|K_{j-1}| > a(p)$ and $|R_{j-1}|/|K_{j-1}| > a(p)$.

Next, $|L_j|/|K_j| > \kappa a(p)$ and $|R_j|/|K_j| > \kappa a(p)$, where κ is a constant number (of order $1/l_c$ for short K_j).

If $j = n_2$ is the next (after n_1) integer such that K_j contains a critical point we obtain $|L_{j-1}|/|K_{j-1}| > a(\kappa a(p))$ and $|R_{j-1}|/|K_{j-1}| > a(\kappa a(p))$, and so on. We end up at $j = n$, with q depending only on p and M. ∎

Proof of (4)⇒(5). Fix $\varepsilon = \delta_4/4$ and k according to Lemma 2.1 (for N easily computable from M in (4)). Fix an arbitrary $x \in I$. Denote $f^{n_{jk}}(x)$ by $x(j)$ for every $j = 0, 1, \ldots$ By Lemma 2.1,

$$W(j) = \mathrm{Comp}_{x(j)}\, f^{-(n_{(j+1)k} - n_{jk})}(B(x(j+1), \delta_4)) \subset B(x(j), \delta_4/2). \tag{2.1}$$

Denote $\mathrm{Comp}_x\, f^{-n_{kj}}(B(x(j), \delta_4))$ by V_j. By Lemma 2.2 for $f^{-n_{kj}}$ and the intervals $W(j) \subset B(x(j), \delta_4) \subset I$ and by (2.1),

$$|V_{j+1}|/|V_j| \le 1 - 2q(M, 1/4) =: \xi.$$

Combining this for $j = 0, 1, \ldots, m-1$ for an arbitrary positive integer m one obtains $|V_m| = |\mathrm{Comp}_x\, f^{-n_{km}}(B(f^{n_{km}}(x), \delta_4))| \le \xi^m$. Notice that $n_{km} \le P_4 km$. Thus we obtained (4) with the sequence $n_{kj}, j = 1, 2, \ldots$, and $P_5 = kP_4$. ∎

Remark. Condition (4) is strictly stronger than the following condition:

(4′) There exist $M > 0$, $P > 0$ and $\delta > 0$ such that for every $x \in I$ there exists an increasing sequence of positive integers n_j, $j = 1, 2, \ldots$, such that $n_j \le Pj$ and the map f^{n_j} has at most M critical points in $\mathrm{Comp}_x\, f^{-n_j} B(f^{n_j}(x), \delta)$.

For example, every "long branched" S-unimodal map, i.e. such that

$$(\exists\gamma > 0)(\forall n)(\forall K \text{ maximal such that } f^n|_K \text{ is monotone}) \quad |f^n(K)| \geq \gamma,$$

satisfies (4′), with $M = P = 1$, but need not be Collet–Eckmann [B1, B2].

Of course, in the holomorphic case, (4) is equivalent to (4′) since f maps $\mathrm{Comp}_{f^i(x)} f^{-(n-i)} B(f^n(x), \delta)$ onto $\mathrm{Comp}_{f^{i+1}(x)} f^{-(n-i-1)} B(f^n(x), \delta)$.

We thank Henk Bruin and Gerhard Keller for calling our attention to this.

Appendix: On the distance of a trajectory from the critical set for differentiable maps of the interval. This is an adaptation to the interval case, without significant changes, of a part of the analogous theory for holomorphic maps by M. Denker, F. Przytycki and M. Urbański in [DPU]. The appendix has been added on the request of the Editorial Board, advised by the referee.

Let $T : I \to I$ be a differentiable map of the unit interval I. Let $c \in I$ be a critical point, i.e. $T'(c) = 0$.

For every $x \in I$ and $r > 0$ set $B(x, r) := \{z \in I : |x - z| < r\}$.

Define a function $k_c : I \to \{0, 1, 2, \ldots\} \cup \{\infty\}$ by setting

$$k_c(x) = \min\{n \geq 0 : x \notin B(c,\ e^{-(n+1)})\},$$

and $k_c(x) = \infty$ if $x = c$.

Write $k(x) = \sup_{c \in \mathrm{Crit}} k_c(x)$.

We call a real function φ on I *Hölder continuous at a point* c if there exist $\vartheta, \alpha > 0$ such that for every x, $|\varphi(x) - \varphi(c)| \leq e^{\vartheta}|x - c|^{\alpha}$.

Theorem. *Let $T : I \to I$ be a differentiable map of the unit interval I. Suppose it has $N < \infty$ critical points and at each of them the derivative T' is Hölder continuous. Suppose also that T has no attracting periodic orbit. Then there exists a constant $Q > 0$ not depending on N such that for every $x \in I$,*

$$\sum k(T^j(x)) \leq NQn$$

where the sum is taken over all integers j between 0 *and* n (0 *and* n *included*) *except at most N of them.*

Lemma. *Let a differentiable $T : I \to I$ have derivative Hölder continuous at a critical point c. Suppose also that T has no attracting periodic orbit. Then there exists a constant $Q > 0$ such that if $x \in I$ satisfies*

(A1) $$k_c(T^j(x)) \leq k_c(T^n(x)) \quad \textit{for every } j = 1, \ldots, n-1,$$

for an integer $n \geq 1$, then

(A2) $$\min\{k_c(x), k_c(T^n(x))\} + \sum_{j=1}^{n-1} k_c(T^j(x)) \leq Qn.$$

P r o o f. The proof is by induction on n. The procedure will be as follows: Given $x, T(x), \ldots, T^n(x)$ satisfying (A1) we shall decompose this string into two blocks: (a) $x, T(x), \ldots, T^m(x), 0 < m \le n$, for which we shall prove (A2); (b) $T^m(x), \ldots, T^n(x)$ for which we can apply the induction hypothesis. Summing these two estimates we prove (A2) for $x, T(x), \ldots, T^n(x)$.

Let $k' = \min\{k_c(x), k_c(T^n(x))\}$ and $B = B(c, e^{-(k'-1)})$.

Let $1 \le m \le n$ be the first positive integer such that either

$$\text{(i)} \qquad k_c(T^m(x)) - \inf\{k_c(T^m(z)) : z \in B\} > 1$$

or

$$\text{(ii)} \qquad k_c(T^m(x)) \ge k'.$$

In both cases, if $m < n$, the sequence $y = T^m(x), T(y), \ldots, T^{n-m}(y)$ satisfies the assumption (A1) automatically and, moreover, $k_c(y) = \min\{k_c(y), k_c(T^{n-m}(y)\}$. Hence by the induction hypothesis

$$\text{(A3)} \qquad \sum_{j=m}^{n-1} k_c(T^j(x)) \le Q(n-m).$$

By the definition of m, for every $0 < j < m$, and for every $z \in B$, we have $k_c(T^j(x)) \le k_c(T^j(z)) + 1$. Hence

$$|(T^{m-1})'(T(z))| \le e^{(m-1)\vartheta} e^{-\alpha \sum_{j=1}^{m-1} (k_c(T^j(x))-1)}.$$

Using also $|T'(z)| \le e^{\vartheta} e^{-\alpha(k'-1)}$ we obtain, for every $z \in B$,

$$\text{(A4)} \qquad |(T^m)'(z)| \le e^{m\vartheta + m\alpha - \alpha(k' + \sum_{j=1}^{m-1} k_c(T^j(x)))}.$$

Hence

$$\text{(A5)} \qquad \frac{\operatorname{diam} T^m(B)}{\operatorname{diam} B} \le e^{m\vartheta + m\alpha - \alpha(k' + \sum_{j=1}^{m-1} k_c(T^j(x)))}.$$

In case (i) but not (ii) we have by definition

$$\begin{aligned} \operatorname{diam} T^m(B) &\ge e^{-(k_c(T^m(x))-1)} - e^{-k_c(T^m(x))} \\ &\ge e^{-k'}(e-1) = (e^{-(k'-1)} - e^{-k'}). \end{aligned}$$

This together with (A5) gives

$$\frac{e-1}{2e} \le e^{m(\vartheta+\alpha) - \alpha(k' + \sum_{j=1}^{m-1} k_c(T^j(x)))},$$

hence

$$\text{(A6)} \qquad k' + \sum_{j=1}^{m-1} k_c(T^j(x)) \le \alpha^{-1}(m(\vartheta+\alpha) + \log 2 - \log(1-1/e)).$$

In case (ii) we also obtain (A6). Otherwise using the opposite inequality and (A4) we obtain $|(T^m)'| \le (e-1)/(2e) < 1$ on B and $T^m(B) \subset B$. By the latter there is a T^m-fixed point in I, by the former it attracts, which contradicts the assumptions.

Thus, defining $Q = \alpha^{-1}(\log 2 + \vartheta + \alpha - \log(1 - 1/e))$, (A.6) and (A.3) imply

$$k' + \sum_{j=1}^{n-1} k_c(T^j(x)) \le Qn.$$

This finishes the proof. ■

Proof of the Theorem. Denote the set of critical points for T by Crit. Fix $x \in I$ and fix $c \in \mathrm{Crit}$ for the moment.

Let $q(c) = t_1$ denote the index $t \in \{0, 1, \ldots, n\}$ for which $k_c(T^t(x))$ attains its maximum (recall that even $k_c(T^t(x)) = \infty$ is possible, if $c = T^t(x)$, but there exists at most one such t, otherwise c would be a (super)attracting periodic point). Recursively, define t_l to be that index in $\{t_{l-1} + 1, \ldots, n\}$ where $k_c(T^t(x))$ attains its maximum. This procedure terminates after finitely many steps, say u steps, with $t_u = n$.

We decompose the trajectory $x, T(x), \ldots, T^n(x)$ into blocks (with overlapping ends)

$$(x, \ldots, T^{t_1}(x)),\ (T^{t_1}(x), \ldots, T^{t_2}(x)),\ \ldots,\ (T^{t_{u-1}}(x), \ldots, T^{t_u}(x)).$$

Observe that these pieces satisfy the assumptions of the Lemma and

$$k_c(T^{t_1}(x)) \ge k_c(T^{t_2}(x)) \ge \ldots \ge k_c(T^{t_{u-1}}(x)) \ge k_c(T^{t_u}(x)).$$

Applying the Lemma to all the blocks we obtain

$$\text{(A7)} \qquad \sum_{j=0}^{t_1-1} k_c(T^j(x)) + \sum_{j=t_1+1}^{n} k_c(T^j(x)) \le Qn.$$

Considering now all critical points we get, by (A7),

$$\sum k(T^j(x)) \le NQn,$$

where the sum is over all integers $j \in \{0, 1, \ldots, n\} \setminus \{q(c) : c \in \mathrm{Crit}\}$. ■

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