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Topological realization of a family of pseudoreflection groups

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Abstract. We are interested in a topological realization of a family of pseudoreflection groups $G \subset \mathrm{GL}(n, \mathbb{F}_p)$; i.e. we are looking for topological spaces whose mod-p cohomology is isomorphic to the ring of invariants $\mathbb{F}_p[x_1, \ldots, x_n]^G$. Spaces of this type give partial answers to a problem of Steenrod, namely which polynomial algebras over $\mathbb{F}_p$ can appear as the mod-p cohomology of a space. The family under consideration is given by pseudoreflection groups which are subgroups of the wreath product $\mathbb{Z}/q \wr \Sigma_n$ where q divides $p-1$ and where p is odd. Let G be such a subgroup acting on the polynomial algebra $A := \mathbb{F}_p[x_1, \ldots, x_n]$. We show that there exists a space X such that $H^*(X; \mathbb{F}_p) \cong A^G$ which is again a polynomial algebra. Examples of polynomial algebras of this form are given by the mod-p cohomology of the classifying spaces of special orthogonal groups or of symplectic groups.

The construction uses products of classifying spaces of unitary groups as building blocks which are glued together via information encoded in a full subcategory of the orbit category of the group G. Using this construction we also show that the homotopy type of the p-adic completion of these spaces is completely determined by the mod-p cohomology considered as an algebra over the Steenrod algebra. Moreover, we calculate the set of homotopy classes of self maps of the completed spaces.

1. Introduction. In 1970, Steenrod [33] posed the question of which polynomial algebras over the field $\mathbb{F}_p$ of p elements can occur as the mod-p cohomology of a topological space. Later, work of Adams and Wilkerson [2] and Dwyer, Miller, and Wilkerson [10] shows that, at least at odd primes, such a polynomial algebra is always the ring of invariants of a pseudoreflection group acting on a polynomial algebra with generators in degree 2. More precisely, for every space X for which $H^*(X; \mathbb{F}_p)$ is a polynomial algebra on n generators, there exists a pseudoreflection group $W \subseteq \mathrm{GL}(n, \mathbb{F}_p)$ such that $H^*(X; \mathbb{F}_p) \cong H^*(BT^n; \mathbb{F}_p)^W$, where T^n denotes the n-dimensional

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torus. Actually, this isomorphism is an isomorphism of algebras over the Steenrod algebra. Here, a pseudoreflection group means a finite subgroup $W \subseteq \mathrm{GL}(n, \mathbb{F}_p)$ which is generated by pseudoreflections, i.e. by elements of finite order fixing a hyperplane of codimension one. Moreover, Dwyer, Miller, and Wilkerson also showed (still for odd p) that any such W lifts to a subgroup of $\mathrm{GL}(n, \widehat{\mathbb{Z}}_p)$, that this lift depends only on the geometric realization of the $\mathbb{F}_p$-algebra, and that W as a subgroup of $\mathrm{GL}(n, \widehat{\mathbb{Q}}_p)$ is again a pseudoreflection group [10].

Conversely, we can ask for a realization of a given pseudoreflection group $W \to \mathrm{GL}(n, \widehat{\mathbb{Z}}_p)$; e.g. for a p-complete space X with polynomial mod-p cohomology such that $H^*(X; \mathbb{F}_p) \cong H^*(BT^n; \mathbb{F}_p)^W$ (as algebras over the Steenrod algebra) and such that the lift to $\mathrm{GL}(n, \widehat{\mathbb{Z}}_p)$, determined by X, is given by the representation $W \to \mathrm{GL}(n, \mathbb{F}_p)$ we started with. Actually, for odd primes the work of Dwyer, Miller and Wilkerson shows that such a space X even satisfies some stronger conditions. These conditions will be used for a definition.

Definition 1.1. A *realization* of a pseudoreflection group $W \to \mathrm{GL}(n, \widehat{\mathbb{Z}}_p)$ consists of a p-complete space X with polynomial mod-p cohomology and a map $f : BT^\wedge_p \to X$ ($T := T^n$) such that f is equivariant up to homotopy with respect to the induced W-action on $BT^\wedge_p$ and the trivial action on X and such that f induces an isomorphism $H^*(X; \mathbb{F}_p) \cong H^*(BT; \mathbb{F}_p)^W$. Here, $\mathrm{GL}(n, \widehat{\mathbb{Z}}_p)$ acts on $BT^\wedge_p \simeq K((\widehat{\mathbb{Z}}_p)^n, 2)$ in the obvious way. A realization of a p-adic rational pseudoreflection group is given by a realization of a p-adic integral lift.

The classifying space BG of a compact connected Lie group G has polynomial mod-p cohomology if the integral cohomology of G is p-torsion free, which is true for almost all primes. In this case, the completion of BG together with the map $BT_G{}^\wedge_p \to BG^\wedge_p$ ($T_G \subset G$ a maximal torus) realizes the Weyl group W_G when regarded as a subgroup of $\mathrm{GL}(n, \widehat{\mathbb{Z}}_p)$ ($n = \mathrm{rk}(G)$). For a slightly more general definition of the realization of a pseudoreflection group, see Remark 4.10.

If X is a realization of the pseudoreflection group $W \to \mathrm{GL}(n, \widehat{\mathbb{Z}}_p)$, one can easily show that $H^*(X; \widehat{\mathbb{Z}}_p) \cong H^*(BT^\wedge_p; \widehat{\mathbb{Z}}_p)^W$ and that $H^*(X; \widehat{\mathbb{Z}}_p) \otimes \mathbb{Q} \cong (H^*(BT^\wedge_p; \widehat{\mathbb{Z}}_p) \otimes \mathbb{Q})^W$ (see Proposition 4.9).

A complete classification of all irreducible pseudoreflection groups over $\widehat{\mathbb{Q}}_p$ has been made by Clark and Ewing [8], based on the classification by Shephard and Todd of irreducible pseudoreflection groups over $\mathbb{C}$ [32]. Most of the groups on this list have already been shown to be realizable by spaces with polynomial cohomology (see [8], [36], [3], [12]); i.e. there exists a p-adic integral lift which has a realization in the above sense. In fact, there is only one case on the Clark–Ewing list for which the problem is not completely solved. Partial results for this case are achieved by Quillen [31, §10] and by Xu [35], who constructed the spaces in the cases where p^2 does not divide the order of the group. Our results here fill this last gap, in that we realize all of the groups in this class. In addition, we show that the realization is unique up to homotopy (in a sense made more precise below), and describe the monoids of self maps of the spaces.

We now set up the notation needed to describe our results explicitly. For any $q > 1$, let $\mu_q \subseteq \mathbb{C}$ be the group of qth roots of unity. We fix an identification $\mu_q \cong \mathbb{Z}/q$.

DEFINITION 1.2. For any $q > 1$, any $r \mid q$, and any $n > 1$, we set

$$A(q, r; n) = \{(z_1, \ldots, z_n) \in (\mu_q)^n \mid z_1, \ldots, z_n \in \mu_r\}$$

and define $G(q, r; n) \subseteq U(n)$ to be the subgroup generated by $A(q, r; n)$ (regarded as a group of diagonal matrices) and the group Σ_n of permutation matrices.

Thus $G(q, r; n)$ is a semidirect product of $A(q, r; n)$ and Σ_n, and $|G(q, r; n)| = q^{n-1}r \cdot n!$. Also, $G(q, q; n) \cong \mathbb{Z}/q \wr \Sigma_n$ (the wreath product).

Remark 1.3. Clearly, $G(q, r; n)$ can be considered as a subgroup of $\mathrm{GL}(n, R)$ for any commutative ring R which contains the group of qth roots of unity. Note that $G(q, r; n)$ is always a pseudoreflection group in this situation: it is generated by the pseudoreflections

$$\sigma \begin{pmatrix} u & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} \sigma^{-1} \quad \text{and} \quad \sigma \begin{pmatrix} 0 & v & 0 & \cdots & 0 \\ v^{-1} & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \sigma^{-1},$$

for all $u \in \mu_r$, all $v \in \mu_q$, and all $\sigma \in \Sigma_n$ (regarded as a permutation matrix). In particular, this applies when $R = \mathbb{F}_p$, $\widehat{\mathbb{Z}}_p$, or $\widehat{\mathbb{Q}}_p$, for any prime $p \equiv 1 \pmod q$. In this situation (and when p is understood), we denote by $V(q, r; n)$ this associated representation on $(\widehat{\mathbb{Z}}_p)^n$.

If $G(q, r; n)$ is any of the groups $G(2, 1; n)$, $G(2, 2; n)$, $G(3, 1; 2)$, $G(4, 1; 2)$, or $G(6, 1; 2)$, then it is in fact conjugate to a subgroup of $\mathrm{GL}(n, \mathbb{Z})$. Actually, they are the only pseudoreflection groups of the form $G(q, r; n)$ which can be realized over $\widehat{\mathbb{Q}}_2$ [8]. Moreover, in these cases, $G(q, r; n)$ is the Weyl group of the compact connected Lie group $SO(2n)$, $SO(2n+1)$, $SU(3)$, $Sp(2)$, or G_2, respectively, with representation given by its action on the integral lattice of the maximal torus. Hence, these groups all have topological realizations.

For $p = 2$ and $G(2, 1; n)$, $G(2, 2; n)$ and $G(6, 1; 2)$ this is not true in the sense of Definition 1.1 but in the extended sense of Remark 4.10.

We therefore can focus our attention on the groups $G = G(q, r; n)$ realized at odd primes p such that $q \mid (p - 1)$. These are exactly the pseudoreflection groups described in case no. 2a in the list of Clark and Ewing [8].

For convenience, throughout the rest of this section, we fix $r \mid q > 1$, $n \geq 2$, and let p be an odd prime such that $q \mid (p - 1)$. We first consider the ring of invariants for $G(q, r; n)$ acting on the appropriate polynomial algebras.

Proposition 1.4. *Set $R = \mathbb{F}_p$, $\widehat{\mathbb{Z}}_p$, or $\widehat{\mathbb{Q}}_p$, and fix an identification of μ_q with the group of qth roots of unity in R. Then*

$$R[x_1, \ldots, x_n]^{G(q,r;n)} \cong R[y_1, \ldots, y_{n-1}, e],$$

where for each i, $y_i = \sigma_i(x_1^q, \ldots, x_n^q)$ (the ith elementary symmetric polynomial), and where $e = (x_1, \ldots, x_n)^r$. In particular, the ring of invariants is a polynomial algebra.

Proof. Set $G = G(q, r; n)$ and $A = A(q, r; n)$. Any element $\alpha \in R[x_1, \ldots, x_n]^A$ can be written uniquely in the form $\alpha = a_0 + a_1 e + \ldots + a_{k-1} e^{k-1}$, where $k = q/r$ and $a_i \in R[(x_1)^q, \ldots, (x_n)^q]$ for each i. Then α is G-invariant if and only if each a_i lies in

$$R[(x_1)^q, \ldots, (x_n)^q]^{\Sigma_n} \cong R[y_1, \ldots, y_{n-1}, y_n]$$

(cf. [22]), where each y_i is the ith elementary symmetric polynomial on the elements $(x_1)^q, \ldots, (x_n)^q$ (and $y_n = e^{q/r}$). ■

The polynomial ring $\mathbb{F}_p[x_1, \ldots, x_n]$ can be identified with the mod-p cohomology $H^*(BT; \mathbb{F}_p)$ of the classifying space BT of an n-dimensional torus T. Thus the last proposition shows that $H^*(BT; \mathbb{F}_p)^G$ is a polynomial algebra. We can state our main results: the existence and homotopy uniqueness of spaces which realize the pseudoreflection groups $G(q, r; n)$.

Theorem 1.5. *Fix $q > 1$, $r \mid q$, and $n > 1$, and let p be an odd prime such that $q \mid (p - 1)$. Then $G = G(q, r; n) \subset \mathrm{GL}(n, \widehat{\mathbb{Z}}_p)$ can be realized by a p-complete space $BX = BX(q, r; n)$.*

The space $BX(q, r; n)$ satisfies a strong homotopy uniqueness property as the next theorem shows.

Theorem 1.6. *Fix $q > 1$, $r \mid q$, and $n > 1$, and let p be an odd prime such that $q \mid (p - 1)$. Let $BX = BX(q, r; n)$ be the space constructed in Theorem 1.5. Then for any p-complete space Y such that $H^*(BX; \mathbb{F}_p) \cong H^*(Y; \mathbb{F}_p)$ as algebras over the Steenrod algebra, the spaces Y and BX are homotopy equivalent.*

It turns out that the space $BX(q,r;n)$ behaves in many ways like the classifying space of a compact Lie group with Weyl group $G(q,r;n)$ acting on a "p-complete n-torus" via the representation $V(q,r;n)$ (Proposition 4.4). That is the reason why we switched to the notation BX (instead of X). We think of BX as the classifying space of the loop space $X := \Omega BX$, which is a p-compact group (a p-complete generalization of a compact Lie group) in the sense of Dwyer and Wilkerson. This interpretation is not necessary for the construction of the spaces BX, but is very helpful in the proof of the homotopy uniqueness property (Theorem 1.6), and in the description of the monoid of self maps $[BX, BX]$.

The description of $[BX, BX]$ is given in Theorem 7.2 below, in terms of the normalizer of G in $\mathrm{GL}(n, \widehat{\mathbb{Q}}_p)$. In particular, we prove that the homotopy classes of self maps of BX are detected by its rational p-adic cohomology:

THEOREM 1.7. *Let p be an odd prime, $q > 2$ and $r \mid q \mid (p-1)$, and set $BX = BX(q,r;n)$. Then the obvious map*

$$[BX, BX] \to \mathrm{Hom}(H^*(BX; \widehat{\mathbb{Z}}_p) \otimes \mathbb{Q}, H^*(BX; \widehat{\mathbb{Z}}_p) \otimes \mathbb{Q})$$

is a monomorphism.

For $q = 2$, the group $G(q,r;n)$ is the Weyl group of $SO(2n)$ or $SO(2n+1)$, and Theorem 1.6 shows that the space $BX(q,r;n)$ is homotopy equivalent to $BSO(2n)^\wedge_p$ or $BSO(2n+1)^\wedge_p$. For these spaces, the monoid of self maps has already been calculated in [20]. In that paper, Theorem 1.7 is also shown for $q = 2$.

Theorem 1.5 also gives new decompositions of the classifying spaces of $SO(n)$, $Sp(n)$, $SU(3)$ and G_2 at odd primes; or alternatively, reconstructs the classifying spaces of these groups at odd primes. For these compact connected Lie groups, Theorem 1.6 has already been proven with different methods in [10] (for p coprime to the order of G) and in [29] (for all odd primes).

We sketch here the construction of $BX(q,r;n)$. Let $\mathcal{C}$ be the full subcategory of the orbit category of $G = G(q,r;n)$ with objects G/H for all subgroups H conjugate to a product of symmetric groups contained in Σ_n (considered as a subgroup of G). By sending each orbit G/H of this form to the ring of invariants $R[x_1, \ldots, x_n]^H$, we get a functor from $\mathcal{C}$ into the category of algebras over the Steenrod algebra. This is then realized as the cohomology of a functor on $\mathcal{C}$, first to the homotopy category, and then to the category of spaces: a functor which sends G/H to an appropriate product of classifying spaces of unitary groups. Finally, BX is defined to be the homotopy direct limit of this topological functor. The mod-p cohomology of BX is calculated using the Bousfield–Kan spectral sequence, and shown to be isomorphic to the ring of invariants $R[x_1, \ldots, x_n]^G$. This particular construction is also used in the proof of uniqueness (Theorem 1.6), and when determining the monoid $[BX, BX]$.

The paper is organized as follows. In Section 2, we prove a vanishing result for higher derived functors of inverse limits in certain very specialized situations. This is then used in Section 3, when constructing the spaces BX. The definitions and basic results about the p-compact groups of Dwyer and Wilkerson [11] are recalled in Section 4, where we also study particular maps between p-compact groups, and show that the spaces BX constructed in Section 3 are p-compact groups. These results, in this generality, are necessary for the proof of the homotopy uniqueness of the space BX (Theorem 1.6) in Section 5, and for the calculation of the monoid $[BX, BX]$ in Section 7. The calculation of the Weyl group of $G \subseteq \mathrm{GL}(n, \widehat{\mathbb{Q}}_p)$, needed to determine $[BX, BX]$, is done in Section 6.

This paper appears only under my name, but actually, this is joint work with R. Oliver. He found this very simple construction of the spaces realizing the pseudoreflection groups in question. My only contribution consists of the idea how one can use his construction for a proof of the homotopy uniqueness property and for the calculation of the set of homotopy classes of the self maps of these spaces. It proved impossible for us to agree on a way of presenting the results, and Oliver eventually suggested that I publish a version of the work under my own name. It is a pleasure for me to thank him here for all his contributions.

I also would like to thank the referee for some clarification in the statements and the proofs of Propositions 2.1 and 2.2, and the Centre de Recerca Matemàtica in Barcelona for their hospitality when this joint work was started.

2. Inverse limits of functors on subcategories. Let $\mathcal{C}$ be a small category and $\mathrm{Fun}(\mathcal{C}, \mathsf{Ab})$ the category of (covariant) functors from $\mathcal{C}$ to Ab. Then there exist higher limits

$$\varprojlim_{\mathcal{C}}{}^{i} : \mathrm{Fun}(\mathcal{C}, \mathsf{Ab}) \to \mathsf{Ab}$$

defined as right derived functors of the inverse limit functor

$$\varprojlim_{\mathcal{C}} : \mathrm{Fun}(\mathcal{C}, \mathsf{Ab}) \to \mathsf{Ab}$$

(cf. [5, XI.6] or [30, Lemma 2]).

For any pair of categories $\mathcal{C} \supseteq \mathcal{D}$ and any object x in $\mathcal{C}$, we let $(x{\to}\mathcal{D})$ denote the category of objects in $\mathcal{D}$ "under" x: the objects are the morphisms $\varphi : x \to y$ in $\mathcal{C}$ such that y is an object in $\mathcal{D}$, and a morphism $(y_1, \varphi_1) \to (y_2, \varphi_2)$ is an element in $\mathrm{Mor}_{\mathcal{D}}(y_1, y_2)$ which makes the obvious triangle commute in $\mathcal{C}$. The category $(\mathcal{D} \to x)$ of objects "over" x is defined analogously.

Proposition 2.1. *Let $\mathcal{C}$ be a small category, and let $\mathcal{D} \subseteq \mathcal{C}$ be a full subcategory. Let $F : \mathcal{C} \to \mathsf{Ab}$ be a (covariant) functor such that for each object $y \in \mathrm{Ob}(\mathcal{C}) \setminus \mathrm{Ob}(\mathcal{D})$,*

$$\varprojlim_{y \to \mathcal{D}}{}^{*}(F|(y \to \mathcal{D})) = \begin{cases} F(y) & \text{if } * = 0, \\ 0 & \text{if } * > 0. \end{cases}$$

Then

$$\varprojlim_{\mathcal{C}}{}^{*}(F) \cong \varprojlim_{\mathcal{D}}{}^{*}(F).$$

Proof. Let $I : \mathcal{D} \to \mathcal{C}$ be the inclusion functor. The right Kan extension

$$I^{*} : \mathrm{Fun}(\mathcal{D}, \mathsf{Ab}) \to \mathrm{Fun}(\mathcal{C}, \mathsf{Ab})$$

is defined by

$$I^{*}(F)(y) = \varprojlim_{y \to \mathcal{D}} (F \,|\, (y \to \mathcal{D})).$$

There exists a composition of functors spectral sequence

$$\varprojlim_{\mathcal{C}}{}^{i}(R^{j} I^{*}(FI)) \Rightarrow \varprojlim_{\mathcal{D}}{}^{i+j}(FI)$$

converging to the higher limits of F restricted to $\mathcal{D}$ (cf. [7, XVI, 3] or [16, Appendix II, 3.6]). Here,

$$R^{j} I^{*}(FI) \cong \varprojlim_{y \to \mathcal{D}}{}^{j}(F \,|\, (y \to \mathcal{D}))$$

denotes the right derived functor of I^{*} evaluated at FI. For $y \in \mathcal{C} \setminus \mathcal{D}$ all higher right derived functors vanish and $R^{0} I^{*}(F)(y) = F(y)$ by assumption. For $y \in \mathcal{D}$ the identity $\mathrm{id} : y \to y$ is an initial object of the over category $y \to \mathcal{D}$ and the same formulas hold. Consequently, we have

$$\varprojlim_{\mathcal{C}}{}^{*}(F) \cong \varprojlim_{\mathcal{D}}{}^{*}(FI) \cong \varprojlim_{\mathcal{D}}{}^{*}(F). \ \blacksquare$$

Proposition 2.1 dealt with limits over a "terminal" subcategory. In contrast, Proposition 2.2 deals with limits over an "initial" subcategory.

Proposition 2.2. *Let $\mathcal{C}$ be a small category, and let $\mathcal{E} \subseteq \mathcal{C}$ be a full subcategory with the following property: For any $y \in \mathrm{Ob}(\mathcal{C}) \setminus \mathrm{Ob}(\mathcal{E})$ there exist $R(y) \in \mathrm{Ob}(\mathcal{E})$ and $\varphi_y \in \mathrm{Mor}_{\mathcal{C}}(R(y), y)$, such that any morphism $\varphi : x \to y$, where $x \in \mathrm{Ob}(\mathcal{E})$, factors uniquely through φ_y. (In other words, the category $(\mathcal{E} \to y)$ has a final object.) Then for any (covariant) functor $F : \mathcal{C} \to \mathsf{Ab}$,*

$$\varprojlim_{\mathcal{C}}{}^{*}(F) \cong \varprojlim_{\mathcal{E}}{}^{*}(F).$$

Proof. If $y \in \mathcal{E}$ then the over category $\mathcal{E} \to y$ has the identity $\mathrm{id} : y \to y$ as final object. Therefore by assumption all over categories are contractible. That is the inclusion $\mathcal{E} \to \mathcal{C}$ is left cofinal in the sense of Bousfield and Kan [5, XI.9.1]. The argument of the proof of [5, XI.9.2] shows that, if you have a left cofinal functor $I : \mathcal{E} \to \mathcal{C}$, the higher limits of FI are canonically isomorphic to the higher limits of F, which is the statement. ∎

Now, for any finite group G, we let $\mathcal{O}(G)$ denote the orbit category for G. Thus, $\mathrm{Ob}(\mathcal{O}(G))$ is the set of orbits G/H for all subgroups $H \subseteq G$, and $\mathrm{Mor}_{\mathcal{O}(G)}(G/H, G/K)$ is the set of all G-maps between the orbits. If $\mathcal{H}$ is any set of subgroups of G, then $\mathcal{O}_{\mathcal{H}}(G)$ will denote the full subcategory of $\mathcal{O}(G)$ whose objects are those orbits G/H for $H \in \mathcal{H}$. Also, $\mathcal{O}_p(G)$ will denote the full subcategory whose objects are the orbits G/P for p-subgroups P.

PROPOSITION 2.3. *Fix a group G and a prime p. Let $\mathcal{H}$ be a family of subgroups of G, closed under conjugation, and with the property that each p-subgroup of G is contained in a unique minimal element of $\mathcal{H}$ (minimal with respect to inclusions). Then for any $\mathbb{Z}_{(p)}[G]$-module M,*

$$\varprojlim_{G/H \in \mathcal{O}_{\mathcal{H}}(G)}{}^{*} M^H \cong \begin{cases} M^G & \text{if } * = 0, \\ 0 & \text{if } * > 0. \end{cases}$$

Proof. Let $\mathcal{P}$ be the family of p-subgroups of G, and set $\overline{\mathcal{H}} = \mathcal{H} \cup \mathcal{P}$. Let $F_M : \mathcal{O}_{\overline{\mathcal{H}}}(G)^{\mathrm{op}} \to \mathsf{Ab}$ defined by $F_M(G/H) = M^H$. We first check that the hypotheses of Propositions 2.1 and 2.2 hold, when applied to the categories $\mathcal{C} = (\mathcal{O}_{\overline{\mathcal{H}}}(G))^{\mathrm{op}}$, $\mathcal{D} = (\mathcal{O}_{\mathcal{P}}(G))^{\mathrm{op}}$, and $\mathcal{E} = (\mathcal{O}_{\mathcal{H}}(G))^{\mathrm{op}}$, and the functor F.

For each $H \in \mathcal{H}$, we can identify the categories

$$(G/H \to \mathcal{O}_{\mathcal{P}}(G)^{\mathrm{op}}) = ((G/H) \leftarrow \mathcal{O}_{\mathcal{P}}(G))^{\mathrm{op}} \cong \mathcal{O}_{\mathcal{P}}(H)^{\mathrm{op}}$$

by identifying a G-map $G/P \xrightarrow{\varphi} G/H$ (for $P \in \mathcal{P}$) with the H-orbit $\varphi^{-1}(eH)$. Also, for any $H \in \mathcal{H}$,

$$\varprojlim_{\mathcal{O}_{\mathcal{P}}(H)}{}^{*} (F|\mathcal{O}_{\mathcal{P}}(H)) = \varprojlim_{H/P \in \mathcal{O}_{\mathcal{P}}(H)}{}^{*} M^P \cong \begin{cases} M^H = F(G/H) & \text{if } * = 0, \\ 0 & \text{if } * > 0. \end{cases}$$

The hypotheses of Proposition 2.1 are thus satisfied and so

$$\varprojlim_{\mathcal{O}_{\overline{\mathcal{H}}}(G)}{}^{*} (F) \cong \varprojlim_{\mathcal{O}_{\mathcal{P}}(G)}{}^{*} (F) = \varprojlim_{G/P \in \mathcal{O}_{\mathcal{P}}(G)}{}^{*} (M^P) \cong \begin{cases} M^G & \text{if } * = 0, \\ 0 & \text{if } * > 0. \end{cases} \tag{1}$$

Here, the last step follows from [18, Theorem 5.14].

Now, for each p-subgroup $P \subseteq G$, let $H_P \in \mathcal{H}$ be the unique minimal subgroup in $\mathcal{H}$ which contains P. Let $\varphi_P : G/P \to G/H_P$ be the map $\varphi_P(gP) = gH_P$. Then for any $H \in \mathcal{H}$, any $\varphi : G/P \to G/H$ factors uniquely through φ_P. Also, by hypothesis, no element of $\mathcal{H}$ is contained in any subgroup of $\mathcal{P} \setminus \mathcal{H}$. In other words, the hypotheses of Proposition

2.2 are satisfied for the pair of categories $(\mathcal{O}_{\mathcal{H}}(G))^{\mathrm{op}} \subseteq (\mathcal{O}_{\overline{\mathcal{H}}}(G))^{\mathrm{op}}$. Thus, we have

$$\varprojlim_{G/H\in\mathcal{O}_{\mathcal{H}}(G)}{}^{*} M^H = \varprojlim_{\mathcal{O}_{\mathcal{H}}(G)}{}^{*}(F) \cong \varprojlim_{\mathcal{O}_{\overline{\mathcal{H}}}(G)}{}^{*}(F)$$

and the result now follows from (1). ∎

3. The construction of the spaces $BX(q,r;n)$. By a *partition* of a set S we mean a collection of nonempty subsets $S_1,\ldots,S_k \subseteq S$ such that $S = \coprod_{i=1}^k S_i$. Let $\mathcal{P}(n)$ denote the set of partitions of $\{1,\ldots,n\}$. For any $\pi = \{S_1,\ldots,S_k\} \in \mathcal{P}(n)$, set

$$\Sigma_\pi = \Sigma(S_1) \times \ldots \times \Sigma(S_k) \subseteq \Sigma_n,$$

the product of the corresponding symmetric groups. For any prime p, let $\mathcal{P}_p(n)$ be the set of those partitions all of whose elements have p-power order.

For any $r \mid q$ and any n, let $\mathcal{H}(q,r;n)$ denote the set of all subgroups of $G = G(q,r;n)$ conjugate to subgroups of the form Σ_π for $\pi \in \mathcal{P}(n)$. And for any prime p such that $(p,q) = 1$, let $\mathcal{H}_p(q,r;n) \subseteq \mathcal{H}(q,r;n)$ denote the set of those subgroups conjugate to Σ_π for some $\pi \in \mathcal{P}_p(n)$.

LEMMA 3.1. *If $r \mid q$ and $(p,q) = 1$, then any p-subgroup of $G = G(q,r;n)$ is contained in a unique minimal element of $\mathcal{H}_p(q,r;n)$. For any $\mathbb{Z}_{(p)}[G]$-module M we have*

$$\varprojlim_{G/H\in\mathcal{O}_{\mathcal{H}}(G)}{}^{*} M^H \cong \begin{cases} M^G & \textit{if } * = 0, \\ 0 & \textit{if } * > 0. \end{cases}$$

P r o o f. Since $(q,p) = 1$, any p-subgroup is conjugate to a subgroup of Σ_n. So it suffices to consider a p-subgroup $P \subseteq \Sigma_n \subseteq G(q,r;n)$. Regard P as a group of permutations of the set $\{1,\ldots,n\}$, let $\pi_P \in \mathcal{P}_p(n)$ be the set of orbits of P, and set $H_P = \Sigma_{\pi_P} \in \mathcal{H}_p(q,r;n)$. Note in particular that $P \subseteq H_P$. It remains to show that any subgroup in $\mathcal{H}_p(q,r;n)$ which contains P also contains H_P.

Fix any $H \in \mathcal{H}_p(q,r;n)$ such that $H \supseteq P$. Let $\mathrm{pr} : G(q,r;n) \to \Sigma_n$ be the projection. Then, by construction, $\mathrm{pr}(H) \supseteq \mathrm{pr}(H_P)$. Also (by definition of $\mathcal{H}_p(q,r;n)$) there is some $g \in G(q,r;n)$ such that $gHg^{-1} \subseteq \Sigma_n$, and we can clearly take $g \in A(q,r;n)$. Then $\mathrm{pr}(gHg^{-1}) = \mathrm{pr}(H) \supseteq \mathrm{pr}(H_P)$ and both are contained in Σ_n, and it follows that $gHg^{-1} \supseteq H_P$. Also, $gPg^{-1} \subseteq gHg^{-1} \subseteq \Sigma_n$; and since $P \subseteq \Sigma_n$ it follows that $[g,P] \subseteq A(q,r;n) \cap \Sigma_n = 1$. Thus $[g,P] = 1$, and if we write $g = (z_1,\ldots,z_n)$, then this means that the z_i are constant on orbits of P. This in turn implies that $[g,H_P] = 1$, and hence that $H_P = g^{-1}(H_P)g \subseteq H$. This proves the first part. The second part follows from Proposition 2.3. ∎

For any partition $\pi = \{S_1, \ldots, S_k\} \in \mathcal{P}(n)$, let $U(\pi)$ be the corresponding product of unitary groups:

$$U(\pi) = U(S_1) \times \ldots \times U(S_k) \subseteq U(n).$$

If $\pi_1, \pi_2 \in \mathcal{P}(n)$ are two partitions, we write $\pi_1 \le \pi_2$ if each element of π_1 is contained in an element of π_2. In particular, if $\pi_1 \le \pi_2$, then $\Sigma_{\pi_1} \subseteq \Sigma_{\pi_2}$ and $U(\pi_1) \subseteq U(\pi_2)$.

THEOREM 3.2. *Fix any prime p, any $r \mid q \mid (p-1)$, and any $n > 1$. Set $G = G(q, r; n)$ and $\mathcal{H} = \mathcal{H}_p(q, r; n)$. Then there exists a functor*

$$\Psi : \mathcal{O}_{\mathcal{H}}(G) \to \mathsf{Top}$$

with the following properties:

(a) *For any partition $\pi \in \mathcal{P}_p(n)$,*

$$\Psi(G/\Sigma_\pi) \simeq BU(\pi)^\wedge_p.$$

(b) *The composite $H^*(-; \widehat{\mathbb{Z}}_p) \circ \Psi$ is isomorphic to the fixed point functor*

$$(G/H) \mapsto \widehat{\mathbb{Z}}_p[x_1, \ldots, x_n]^H.$$

(c) *If we set $BX(q, r; n) := (\underrightarrow{\mathrm{hocolim}}_{\mathcal{O}_{\mathcal{H}}(G)}(\Psi))^\wedge_p$, then*

$$H^*(BX; \mathbb{F}_p) \cong \mathbb{F}_p[x_1, \ldots, x_n]^G.$$

For the proof we need the following result about maps between classifying spaces of unitary groups.

PROPOSITION 3.3. *Let G be a product of unitary groups. Let $f : BG \to BG$ be a homotopy equivalence. Let $T \subseteq G$ be a maximal torus and $Z \subseteq T \subseteq G$ the center of G. Then the following hold*:

(a) *Fix any abelian p-toral subgroup $S \subseteq T$ and set $H = C_G(S) \subseteq G$. Then the maps*

$$BZ(H)^\wedge_p \xrightarrow{\beta_H} \mathrm{map}(BH, BG^\wedge_p)_{\mathrm{incl}} \xrightarrow{f\circ -} \mathrm{map}(BH, BG^\wedge_p)_{f|BH},$$

and

$$\beta'_H : BH^\wedge_p \to \mathrm{map}(BS, BG^\wedge_p)_{f|BS}$$

(*where β_H and β'_H are adjoint to the maps induced by multiplication*) *are homotopy equivalences.*

(b) *Let S and H be as in* (a). *Then for any map $g : BH \to BG$, $g \simeq f|BH$ iff $H^*(g; \widehat{\mathbb{Z}}_p) \otimes \mathbb{Q} = H^*(f|BH; \widehat{\mathbb{Z}}_p) \otimes \mathbb{Q}$ iff $g|BT \simeq f|BT$.*

This theorem is a special case of Proposition 4.6, where we prove a more general statement about p-compact groups. This also will show that the theorem holds for every compact connected Lie group G. In this more general case, for part (b), we only have to assume in addition that H is connected. If G is a product of unitary groups, this extra condition always holds because the centralizer of a subgroup of a unitary group is always a product of unitary groups and therefore connected.

Actually, to prove Theorem 4.6 only for compact connected Lie groups the theory of p-compact groups is not necessary as Remark 4.7 explains.

Proof of Theorem 3.2. It will be convenient to consider the subset

$$\mathcal{H}_0 = \{\Sigma_\pi \mid \pi \in \mathcal{P}_p(n)\} \subseteq \mathcal{H}.$$

Then $\mathcal{H}_0$ is the set of elements of $\mathcal{H}$ contained in Σ_n, and every element of $\mathcal{H}$ is conjugate to an element of $\mathcal{H}_0$. Hence $\mathcal{O}_{\mathcal{H}_0}(G)$ contains objects from all isomorphism classes of $\mathcal{O}_{\mathcal{H}}(G)$ and so limits over $\mathcal{O}_{\mathcal{H}_0}(G)$ and over $\mathcal{O}_{\mathcal{H}}(G)$ are the same.

Any morphism in $\mathcal{O}_{\mathcal{H}_0}(G)$ is a composite

$$(G/\Sigma_{\pi_1}) \xrightarrow{\sigma\cdot a} (G/\Sigma_{\sigma(\pi_1)}) \xrightarrow{\text{proj}} (G/\Sigma_{\pi_2}), \tag{1}$$

where

$$\sigma \in \Sigma_n, \quad a = (z_1, \ldots, z_n) \in A(q, r; n), \quad [a, \Sigma_{\pi_1}] = 1, \quad \text{and} \quad \sigma(\pi_1) \le \pi_2.$$

Note that the condition $[a, \Sigma_{\pi_1}] = 1$ is equivalent to the condition that the z_i are constant on elements in π_1.

We first define a functor

$$\Psi_0' : \mathcal{O}_{\mathcal{H}_0}(G) \to \mathsf{hTop}$$

taking values in the homotopy category of Top which satisfies conditions (a) and (b). On objects, we set

$$\Psi_0'(G/\Sigma_\pi) = BU(\pi)^\wedge_p.$$

And a morphism of type (1) above is sent to a composite of the form

$$BU(\pi_1)^\wedge_p \xrightarrow{\psi^\alpha_{\pi_1}} BU(\pi_1)^\wedge_p \xrightarrow{\sigma_*} BU(\sigma(\pi_1))^\wedge_p \xrightarrow{\text{incl}} BU(\pi_2)^\wedge_p.$$

Here, $\psi^\alpha_{\pi_1}$ means the product of the corresponding unstable Adams operations on the unitary group factors (recall that the coordinates of α are constant on each element of the partition π_1). The map σ_* is induced by the homomorphism which sends each factor of $U(\pi_1)$ to the corresponding factor of $U(\sigma(\pi_1))$. When showing that these maps compose correctly (i.e., that Ψ_0' is a functor), the only difficult parts are to show that $\psi^r \circ \psi^s \simeq \psi^{rs}$ (this follows from the homotopy uniqueness of the unstable Adams operations, shown in [19, Theorem 4.3]) and that the unstable Adams operations commute with inclusions. This last point holds since $(\psi^\alpha_\pi)^{-1} \circ \text{incl} \circ \psi^\alpha_\pi$ is homotopic to the inclusion by Proposition 3.3(b). (Note that for $U(\pi_1) \subseteq U(\pi_2)$ there exists a subtorus $S = Z(U(\pi_1)) \subseteq U(\pi_1)$ such that $U(\pi_1) = C_{U(\pi_2)}(S)$.)

Thus, Ψ_0' is a well defined functor to the homotopy category, and can be extended to a functor

$$\Psi' : \mathcal{O}_{\mathcal{H}}(G) \to \mathsf{hTop}.$$

Conditions (a) and (b) hold for Ψ' by construction.

We now claim that Ψ' is a *centric* functor in the sense of Dwyer and Kan [9]. This means that for any morphism $\varphi : G/H_1 \to G/H_2$ in $\mathcal{O}_{\mathcal{H}}(G)$ the map

$$\mathrm{map}(\Psi'(G/H_1), \Psi'(G/H_1))_{\mathrm{Id}} \xrightarrow{\Psi'(\varphi)\circ -} \mathrm{map}(\Psi'(G/H_1), \Psi'(G/H_2))_{\Psi'(\varphi)}$$

is a homotopy equivalence. This is automatic when φ is an isomorphism, so it suffices to check it for inclusions of the form $\Sigma_{\pi_1} \subseteq \Sigma_{\pi_2}$. It follows from Proposition 3.3(a) (applied with $G = \Psi'(G/H_2)$ and with $G = H = \psi'(G/H_1)$). Furthermore, for each $H \in \mathcal{H}$,

$$\mathrm{map}(\Psi'(G/H), \Psi'(G/H))_{\mathrm{Id}} \simeq BZ(H)^\wedge_p,$$

again by Lemma 3.3(b). Since $Z(\psi'(G/H))$ is a torus ($\psi'(G/H)$ is a product of unitary groups), the only nonvanishing homotopy group in these mapping spaces is

$$\pi_2(\mathrm{map}(\Psi'(G/H), \Psi'(G/H))_{\mathrm{Id}}) \cong \pi_2(BZ(\psi'(G/H)))^\wedge_p \cong [(\widehat{\mathbb{Z}}_p)^n]^H.$$

So by Lemma 3.1,

$$\varprojlim_{\mathcal{O}_{\mathcal{H}}(G)}{}^{i}\ \pi_j(\mathrm{map}(\Psi'(-), \Psi'(-))_{\mathrm{Id}}) \cong 0$$

for all $i, j > 0$. The obstruction groups for lifting Ψ' to a functor to topological spaces are given by some of these higher derived limits [9, Theorem 1.1]. Since all these groups vanish, we obtain a lifting

$$\Psi : \mathcal{O}_{\mathcal{H}}(G) \to \mathsf{Top}$$

which satisfies conditions (a) and (b) above.

Finally, by Lemma 3.1 again,

$$\begin{aligned}
\varprojlim_{\mathcal{O}_{\mathcal{H}}(G)}{}^{i}\ (H^*(\Psi(-); \mathbb{F}_p)) &\cong \varprojlim_{G/H\in\mathcal{O}_{\mathcal{H}}(G)}{}^{i}\ (\mathbb{F}_p[x_1, \ldots, x_n]^H) \\
&\cong \begin{cases} \mathbb{F}_p[x_1, \ldots, x_n]^G & \text{if } i = 0, \\ 0 & \text{if } i > 0. \end{cases}
\end{aligned}$$

So by the spectral sequence for the cohomology of a homotopy colimit (cf. [5, XII.4.5] or [21]),

$$H^*(\underset{\mathcal{O}_{\mathcal{H}}(G)}{\underrightarrow{\mathrm{hocolim}}}(\Psi); \mathbb{F}_p) \cong \mathbb{F}_p[x_1, \ldots, x_n]^G.$$

The homotopy colimit is thus mod-p simply connected, and hence its p-completion $BX(q, r; n)$ has the same mod-p cohomology. ∎

Proof of Theorem 1.5. Let $T \subset U(n)$ be the maximal torus given by the diagonal. Making the identification $H^*(BT^n{}^\wedge_p; \widehat{\mathbb{Z}}_p) \cong \widehat{\mathbb{Z}}_p[x_1, \ldots, x_n]$, the composition $BT^\wedge_p \to BU(n)^\wedge_p \to BX := BX(q, r; n)$ becomes equivariant with respect to the action of $G(q, r; n)$ and realizes the isomorphism $H^*(BX; \mathbb{F}_p) \cong H^*(BT; \mathbb{F}_p)^G$. All this follows directly from the construction of the space BX. ∎

4. p-compact groups. The concept of p-compact groups was introduced by Dwyer and Wilkerson in [13], where they showed that p-compact groups behave very much like compact Lie groups. In particular, a p-compact group always has a maximal torus and a Weyl group, which satisfy properties analogous to those of the maximal torus and Weyl group of a compact Lie group. We recall here the basic definitions and results from [13].

A *p-compact group* is a triple $X = (X, BX, e)$, where BX is a p-complete pointed space, where X is mod-p finite (i.e., $H^*(X; \mathbb{F}_p)$ is finite), and where $e : \Omega BX \xrightarrow{\simeq} X$ is a homotopy equivalence. The space BX is thought of as the classifying space of the p-compact group X.

The motivating examples of p-compact groups come from compact Lie groups. If G is any compact Lie group for which $\pi_0(G)$ is a finite p-group, then the triple $(G^\wedge_p, BG^\wedge_p, G^\wedge_p \simeq \Omega BG^\wedge_p)$ is a p-compact group. Particular compact Lie groups establish particular p-compact groups. A *p-compact torus* is a triple of the form $(T^\wedge_p, BT^\wedge_p, \simeq)$ where T is a torus and a finite p-compact group is of the form $(\pi, B\pi, \simeq)$ where π is a finite p-group. And p-toral groups give *p-compact toral groups* which in general are p-compact groups whose component of the unit is a p-compact torus. (The *component of the unit* is given by the universal cover of BX or by the component of X related to the constant loop in BX.) Note that for any p-compact group (X, BX, e), $\pi_1(BX) \cong \pi_0(X)$ is a finite p-group (since BX is p-complete).

A *homomorphism* $f : X \to Y$ between two p-compact groups is a pointed map $Bf : BX \to BY$. A homomorphism f is called an *isomorphism* if Bf is a homotopy equivalence; and a *monomorphism* if the homotopy fiber of Bf, denoted by Y/X, is mod-p finite. Two homomorphisms $f, g : X \to Y$ are *conjugate* if Bf and Bg are freely homotopic.

Let P be a p-toral or p-compact toral group. Dwyer and Wilkerson showed that for every homomorphism $f : P \to X$ of p-compact groups, the mapping space $BC_X(f) := \mathrm{map}(BP, BX)_{Bf}$ is the classifying space of another p-compact group; i.e. that $BC_X(f)$ is p-complete and its loop space $C_X(f)$ is mod-p finite. The triple $C_X(f) = (C_X(f), BC_X(f), \mathrm{Id})$ is called the *centralizer* of f. Evaluation at the base point induces a map $BC_X(f) \to BX$ respectively a homomorphism $C_X(f) \to X$ which is always a monomorphism [13, 5.2, 6.1]. If P is an abelian compact Lie group, or if $\mathrm{map}(BP, BP)_{\mathrm{Id}} \simeq BP$, then there is an obvious map $BP \to BC_X(f)$ [13, 8.2]. The notion of the centralizer is motivated by:

PROPOSITION 4.1 ([15] and [28]). *For any homomorphism $\varrho : P \to G$ from a p-toral group P into a compact connected Lie group G, let $C_G(\varrho)$ denote the centralizer of* $\mathrm{Im}(\varrho)$ (*in the group theoretic sense*). *Then the map*

$$BC_G(\varrho)^\wedge_p \xrightarrow{\simeq} \mathrm{map}(BP, BG^\wedge_p)_{B\varrho},$$

adjoint to $B(\varrho, \mathrm{incl}) : BC_G(\varrho) \times BP \to BG$, *is a homotopy equivalence.*

Note that this form of the statement, which will be used later, is only contained implicitly in the two mentioned references. It is derived explicitly in [19, Theorem 3.2].

Now we recall the notion of maximal tori and Weyl groups. A *maximal torus* of a p-compact group X is a monomorphism $f : T \to X$ from a p-compact torus into X such that the induced homomorphism $T \to C_X(T)$ is an equivalence to the identity component of the centralizer. The *rank* of X is defined as the dimension of T.

THEOREM 4.2 [13, 8.11, 8.13, 9.1]. *Every p-compact group X has a maximal torus $T_X \to X$, and any two maximal tori of X are conjugate.*

When defining the Weyl group W_X of a p-compact group X, we will assume here for simplicity that X is connected. For a fixed maximal torus $f : T_X \to X$ of X, W_X is defined to be the set of all homotopy classes of self maps of $w : BT_X \to BT_X$ such that $Bf \circ w$ and Bf are homotopic. It turns out that W_X has many of the well known properties of Weyl groups of a compact connected Lie group. This definition is equivalent to the one given by Dwyer and Wilkerson in [13, 9.11] (see [27, 1.8]); their definition also applies to disconnected p-compact groups.

PROPOSITION 4.3 [13, 9.5 and 9.7]. *Let $T_X \to X$ be a maximal torus of a connected p-compact group X of rank n. Then the following hold*:

(1) *W_X is a finite group.*

(2) *The action of W_X on BT_X induces a faithful representation*

$$W_X \to \mathrm{GL}(n, \widehat{\mathbb{Q}}_p)$$

which represents W_X as a pseudoreflection group.

(3) *The map $T_X \to X$ induces an isomorphism*

$$H^*(BX; \widehat{\mathbb{Z}}_p) \otimes \mathbb{Q} \to (H^*(BT_X; \widehat{\mathbb{Z}}_p) \otimes \mathbb{Q})^{W_X}.$$

The next proposition describes the structure of the spaces BX constructed in Section 3, as p-compact groups.

PROPOSITION 4.4. *Fix $r \mid q \mid (p-1)$ and $n \geq 2$ and set $G = G(q,r;n)$. Let BX be any p-complete space such that $H^*(BX;\mathbb{F}_p) \cong \mathbb{F}_p[x_1,\dots,x_n]^G$. Then the following hold*:

(1) *The triple $X = (\Omega BX, BX, \mathrm{Id})$ is a connected p-compact group.*

(2) *There exists a maximal torus $T_X \to X$ and an identification $H^*(BT_X;\mathbb{F}_p) \cong \mathbb{F}_p[x_1,\dots,x_n]$ such that the composition*

$$\mathbb{F}_p[x_1,\dots,x_n]^G \cong H^*(BX;\mathbb{F}_p) \to H^*(BT_X;\mathbb{F}_p) \cong \mathbb{F}_p[x_1,\dots,x_n]$$

is the obvious iclusion.

(3) *$W_X \cong G$.*

Proof. To prove the first part we only have to show that $X = \Omega BX$ is mod-p finite. This follows easily from an Eilenberg–Moore spectral sequence argument.

Using Lannes' T-functor, one can show that there exists a realization $BT_X \to BX$ of the given algebraic map. The following construction is similar to that in [10] (see also [29, Section 7]). Set $V \cong (\mathbb{Z}/p)^n$. By [23, Théorème 0.4] the composite

$$H^*(BX;\mathbb{F}_p) \cong \mathbb{F}_p[x_1,\dots,x_n]^G \hookrightarrow \mathbb{F}_p[x_1,\dots,x_n] \to H^*(BV;\mathbb{F}_p)$$

can be realized by a map $f_0 : BV \to BX$. A calculation of the mod-p cohomology of $\mathrm{map}(BV, BX)_{f_0}$, using Lannes' T-functor, shows that this mapping space has the same mod-p cohomology as the classifying space BT_X of a p-compact torus T_X and hence, that they can be identified. The evaluation at the basepoint

$$BT_X = \mathrm{map}(BV;BX)_{f_0} \xrightarrow{\mathrm{ev}} BX$$

induces the inclusion $\mathbb{F}_p[x_1,\dots,x_n]^G \subseteq \mathbb{F}_p[x_1,\dots,x_n]$ (in mod-p cohomology). Because $H^*(BT_X;\mathbb{F}_p)$ is finitely generated over $H^*(BX;\mathbb{F}_p)$, the map $BT_X \to BX$ is a monomorphism of p-compact groups [13, 9.11]. Moreover, the sequence of monomorphisms $T_X \hookrightarrow C_X(T_X) \hookrightarrow C_X(V) \cong T_X$ [13, 5.2, 6.1] shows that $T_X \cong C_X(T_X)$ and that $T_X \to X$ is a maximal torus. This proves the second part.

The group G acts in a basepoint preserving way on the space BV and fixes the induced map $H^*(f_0;\mathbb{F}_p)$. Hence, again by [23, Théorème 0.4], this action also fixes the component of $\mathrm{map}(BV, BX)_{f_0} \simeq BT_X$ of the mapping space $\mathrm{map}(BV, BX)$ and is therefore a subgroup of W_X.

Passing to invariants the inclusion of the maximal torus $T_X \to X$ establishes the composition

$$H^*(BX;\mathbb{F}_p) \to H^*(BT_X;\mathbb{F}_p)^{W_X} \to H^*(BT_X;\mathbb{F}_p)^G.$$

The second arrow is a monomorphism, the composition is an isomorphism and so are both arrows. The action of W_X on $H^*(BT_X;\mathbb{F}_p)$ is faithful because this is true for $\mathbb{Z}_p^\wedge$ as coefficients (Proposition 4.3) and because for odd primes the kernel of $\mathrm{GL}(n,\mathbb{Z}_p^\wedge) \to \mathrm{GL}(n,\mathbb{F}_p)$ is torsion free.

The ring $R := H^*(BT_X;\mathbb{F}_p)$ is an integrally closed integral domain. Let $F(R)$ denote the field of fractions. Then we have $F(R^G) \cong F(R)^G$ and $F(R)^G \subset F(R)$ is a Galois extension with Galois group G. The same formula is true for the Weyl group W_X and $F(R)^G \cong F(R)^{W_X}$. Thus, both groups have the same order and $G = W_X$. This proves the third part. ∎

Now we can start to study particular maps between classifying spaces of p-compact groups. The next lemma describes a well known trick in the theory of classifying spaces.

LEMMA 4.5. *Fix a prime p and a p-compact group X. Let G be a compact Lie group, let $P \lhd G$ be a normal p-toral subgroup, and let $\kappa : G \to G/P$ be the projection. Then for any $f : B(G/P) \to BX$,*

$$(1)\qquad (-\circ B\kappa) : \mathrm{map}(B(G/P), BX)_f \xrightarrow{\simeq} \mathrm{map}(BG, BX)_{f\circ B\kappa}$$

is a homotopy equivalence.

Proof. The map $BP \to \mathrm{map}(BP, BX)_{\mathrm{const}}$ is an equivalence. This follows from the Sullivan conjecture and is stated in [14, 9.7, 10.1]. Using this fact the statement might be found in [4, 7.3]. ∎

We are now ready to state our main proposition needed to describe certain mapping spaces between p-compact groups. We have to distinguish between centralizers in the (Lie) group theoretic sense and in the sense od p-compact groups. In the latter case we will use the notation of mapping spaces and keep the notation as centralizers for homomorphisms between compact Lie groups.

PROPOSITION 4.6. *Let G be a compact connected Lie group with maximal torus $T \subseteq G$ and with central subgroup $Z \subseteq Z(G) \subseteq G$. Let $i : T \hookrightarrow G$ and $j : Z \hookrightarrow G$ be the inclusions, and let* mult $: G \times Z \to G$ *denote multiplication (a homomorphism since Z is central). Let X be a connected p-compact group, and let $f : BG \to BX$ be a map such that*

$$(1)\qquad f' : BG_p^\wedge \xrightarrow{\simeq} \mathrm{map}(BZ, BX)_{f\circ Bj},$$

adjoint to $f \circ B\,\mathrm{mult}$, is a homotopy equivalence. Then the following hold:

(a) *The composite $BT \xrightarrow{B_i} BG \xrightarrow{f} BX$ is a maximal torus of X.*

(b) *Fix any abelian p-toral subgroup $S \subseteq T$ with $Z \subseteq S$ and assume that $H := C_G(S) \subseteq G$ is connected. Then the maps*

$$BZ(H)_p^\wedge \xrightarrow{\beta_H} \mathrm{map}(BH, BG_p^\wedge)_{\mathrm{incl}} \xrightarrow{f\circ -} \mathrm{map}(BH, BX)_{f|BH},$$

and

$$\beta'_H : BH^\wedge_p f \to \mathrm{map}(BS, BX^\wedge_p)_{f|BS}$$

are homotopy equivalences. Here, β_H *and* β'_H *are adjoint to the maps induced by* $B\,\mathrm{mult} : BZ(H) \times BH \to BG$ *and* $BH \times BS \xrightarrow{B\,\mathrm{mult}} BG \xrightarrow{f} BX$.

(c) *Let* S *and* H *be as in* (b) *and assume that* H *is connected. Then for any map* $g : BH \to BX$, $g \simeq f|BH$ *iff* $H^*(g; \widehat{\mathbb{Z}}_p) \otimes \mathbb{Q} = H^*(f|BH; \widehat{\mathbb{Z}}_p) \otimes \mathbb{Q}$ *iff* $g|BT \simeq f|BT$.

Remark. The fact corresponding to (a) for Lie groups is this: Let X be a compact connected Lie group. Because G is connected, Z is contained in a maximal torus for G and therefore in a maximal torus for X; consequently, $G \to X$ is of maximal rank.

If $S \subset T$ is a subtorus, then H is always connected [19, Proposition A.4].

Proof (of Proposition 4.6). Throughout the proof, $\mathrm{Ad} : BH^\wedge_p \to \mathrm{map}(BK, BG^\wedge_p)$ and $\mathrm{Ad}(f) : BH^\wedge_p \to \mathrm{map}(BK, BX)$ will denote the adjoint maps to $B(\mathrm{mult})$ and $f \circ B(\mathrm{mult})$, for any commuting pair of subgroups $H, K \subseteq G$ and the multiplication $\mathrm{mult} : H \times K \to G$. And similarly for any other map $f' : BG' \to BX$, for any $G' \subseteq G$. In particular, β_H is of the form Ad and β'_H of the form $\mathrm{Ad}(f)$.

For any $H \subseteq G$ containing Z, we have

$$\begin{aligned}(2)\qquad & \mathrm{map}(BH, BG^\wedge_p)_{\mathrm{incl}} \\ & \simeq \mathrm{map}(BH, \mathrm{map}(BZ, BX)_{f\circ Bj})_{\mathrm{Ad}(f)} \\ & \simeq \mathrm{map}(BZ \times BH, BX)_{f\circ B\,\mathrm{mult}} \simeq \mathrm{map}(BH, BX)_{f|BH}.\end{aligned}$$

The first equivalence holds by (1), the second by taking adjoints, and the third by Lemma 4.5 using the fibration $BZ \to BZ \times BH \xrightarrow{B\,\mathrm{mult}} BH$. Thus, the second map $(f \circ -)$ in (b) is a homotopy equivalence. This even holds if we only assume that $\pi_0(G)$ is a finite p-group.

Now we assume that H is the centralizer in G of a p-toral subgroup $S \subseteq T$. In particular, we have $H \supseteq T$ and $\pi_0(H)$ is a finite p-group (cf. [19, Proposition A.4]). Then by Proposition 4.1,

$$(3)\qquad \mathrm{Ad} : BH^\wedge_p \xrightarrow{\simeq} \mathrm{map}(BS, BG^\wedge_p)_{\mathrm{incl}}$$

is a homotopy equivalence. Hence by (2) (after replacing G by H and BX by $BG^\wedge_p$), the map

$$(4)\qquad \mathrm{map}(BH, BH^\wedge_p)_{\mathrm{Id}} \xrightarrow[\simeq]{(\mathrm{incl})\circ -} \mathrm{map}(BH, BG^\wedge_p)_{\mathrm{incl}}$$

is a homotopy equivalence. Furthermore,

$$(5)\qquad \mathrm{map}(BH, BH^\wedge_p)_{\mathrm{Id}} \simeq BZ(H)^\wedge_p.$$

For connected H, this follows from [19, Theorem 4.2] when H is simple, and from [20, Proposition 2.7] in the nonsimple case. For disconnected H, this is proved in [14], but can also be proved in the context of compact Lie groups starting from the connected case. Points (2)–(5) now combine to prove that β_H as well as the composition $(f\circ -)\beta_H$ are equivalences. The second part of (b) follows by applying (2) in the case of $H=S$ and by (3) if we assume in addition that $Z\subseteq S$. Otherwise there is no multiplication $Z\times S\to S$.

For $H=T$, point (b) takes the form of a homotopy equivalence

$$BT^\wedge_p \xrightarrow{\simeq} \mathrm{map}(BT,BX)_{f|BT}$$

(recall that $T=C_G(T)$ [6, Theorem 2.3]). This proves part (a), namely that $BT\hookrightarrow BG\xrightarrow{f} BX$ is a maximal torus (see the above definitions).

It remains to prove part (c). Assume that H is connected and let $g: BH\to BX$ be another map such that $H^*(g;\widehat{\mathbb{Z}}_p)\otimes\mathbb{Q}=H^*(f|BH;\widehat{\mathbb{Z}}_p)\otimes\mathbb{Q}$. By [13, Proposition 8.11], $g|BT$ and $f|BT$ lift to maps $g',f': BT\to BT^\wedge_p$. Since $H^*(g';\widehat{\mathbb{Z}}_p)\otimes\mathbb{Q}$ and $H^*(f';\widehat{\mathbb{Z}}_p)\otimes\mathbb{Q}$ agree on $H^*(BX;\widehat{\mathbb{Z}}_p)\otimes\mathbb{Q}\cong(H^*(BT;\widehat{\mathbb{Z}}_p)\otimes\mathbb{Q})^{W_X}$ (Proposition 4.3), there exists $w\in W_X$ such that g' and $w\circ f'$ induce the same map on $H^*(BT;\widehat{\mathbb{Z}}_p)\otimes\mathbb{Q}$ and that $g'\simeq w\circ f'$. The first conclusion follows from [28, 7.1] and the second is obvious. Hence, we can assume that $g|BT\simeq f|BT$. We consider the following diagram:

$$\begin{array}{ccccccc}
BT & \longrightarrow & \mathrm{map}(BS,BT)_{B\,\mathrm{incl}} & = & \mathrm{map}(BS,BT)_{B\,\mathrm{incl}} & & \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle (g|BT\circ -)\simeq(f|BT\circ -)} & & \\
BH & \xrightarrow{\mathrm{Ad}} & \mathrm{map}(BS,BH^\wedge_p)_{B\,\mathrm{incl}^\wedge_p} & \underset{f\circ -}{\overset{g\circ -}{\longrightarrow}} & \mathrm{map}(BS,BX)_{g|BS} & \xrightarrow{\mathrm{ev}} & BX, \\
 & & & & =\mathrm{map}(BS,BX)_{f|BS} & &
\end{array}$$

where incl always denotes the obvious inclusion. The composition in the lower row is nothing but g or f. The compositions $(g\circ -)\mathrm{Ad}$ and $(f\circ -)\mathrm{Ad}\simeq(\mathrm{id}_{BH})^\wedge_p$ induce the same map after restriction to BT and are therefore homotopic by [20, 2.5]. And so are f and g. ∎

Proof of Theorem 3.3. Theorem 3.3 is a special case of (b) and (c) of Theorem 4.6. We only have to assume that $X=G$. To satisfy the extra condition $Z\subset S$ we notice that we can replace S by the group $S'\subset T$ generated by S and Z, since $H=C_G(S)=C_G(S')$. Moreover, notice that, for G a product of unitary groups, $H=C_G(S)$ is always connected. ∎

Remark 4.7. In the case of X being the completion of a compact connected Lie group, the proof does not depend on the theory of p-compact groups. In the proof of parts (b) and (c) we only have to replace some references, namely Proposition 4.3 by the fact that $H^*(BG;\mathbb{Q})\simeq H^*(BT_G;\mathbb{Q})^{W_G}$

for any compact connected Lie group and [13, Proposition 8.11] by [20, Proposition 1.2]. In particular, this shows that Proposition 3.3, which is contained in (b) and (c), can be proved in the context of compact Lie groups.

The following proposition describes the principal situation in which Proposition 4.6 will be applied.

PROPOSITION 4.8. *Fix $r \mid q \mid (p-1)$ and $n \geq 2$, and set $G = G(q,r;n)$. Let BX be a p-complete space, together with an isomorphism*

$$\Phi : H^*(BX;\mathbb{F}_p) \xrightarrow{\cong} \mathbb{F}_p[x_1,\ldots,x_n]^G$$

of algebras over the Steenrod algebra. Then there exists a map $g : BU(n) \to BX$ such that the square

$$\begin{CD} H^*(BX;\mathbb{F}_p) @>{\Phi}>> \mathbb{F}_p[x_1,\ldots,x_n]^G \\ @V{g^*}VV @VV{\mathrm{incl}}V \\ H^*(BU(n);\mathbb{F}_p) @>{\Phi_U}>{\cong}> \mathbb{F}_p[x_1,\ldots,x_n]^{\Sigma_n} \end{CD} \tag{1}$$

commutes (where Φ_U is the standard isomorphism). Furthermore, the following hold for any such map $g : BU(n) \to BX$:

(a) *For any maximal torus $T \subseteq U(n)$, $g|BT : BT \to BX$ is a maximal torus of X.*

(b) *Let $S \subseteq U(n)$ be any subtorus with $Z(U(n)) \subseteq S$, and set $H = C_{U(n)}(S)$. Then the maps*

$$BH^{\wedge}_p \to \mathrm{map}(BS,BX)_{g|BS} \quad \textit{and} \quad BZ(H)^{\wedge}_p \to \mathrm{map}(BH,BX)_{g|BH} \tag{2}$$

(adjoint to $g \circ B\,\mathrm{mult}$) are homotopy equivalences.

(c) *Let S and H be as in (b). If $f : BH \to BX$ is any map such that $H^*(f;\widehat{\mathbb{Z}}_p)\otimes\mathbb{Q} = H^*(g|BH;\widehat{\mathbb{Z}}_p)\otimes\mathbb{Q}$, then $f \simeq g|BH$.*

Proof. Let $\varphi : H^*(BX;\mathbb{F}_p) \to H^*(B\mathbb{Z}/p;\mathbb{F}_p)$ be the composite

$$\varphi : H^*(BX;\mathbb{F}_p) \xrightarrow[\cong]{\Phi} \mathbb{F}_p[x_1,\ldots,x_n]^G \xrightarrow{\Phi_U^{-1}\circ\mathrm{incl}} H^*(BU(n);\mathbb{F}_p) \xrightarrow{\mathrm{incl}^*} H^*(B\mathbb{Z}/p;\mathbb{F}_p),$$

where the last map is the restriction to the central subgroup of order p. By [23, Théorème 0.4], there is a map $g_0 : B\mathbb{Z}/p \to BX$ such that $H^*(g_0;\mathbb{F}_p) = \varphi$. Let $BT_X \to BX$ be the maximal torus of X as in Proposition 4.4. Then a lift of g_0 to BT_X is given by the imbedding $k : B\mathbb{Z}/p \to BT_X$. The isotropy group of k, i.e. the group of all elements of W_X such that $w \circ k \simeq k$, is given by $\Sigma_n \subseteq G = W_X$.

Since $H^*(BX;\mathbb{F}_p) \cong \mathbb{F}_p[x_1,\ldots,x_n]^G$, an application of [29, 10.1, 10.2] calculates the mod-p cohomology of the mapping space. We get

$$H^*(\mathrm{map}(B\mathbb{Z}/p, BX)_{g_0};\mathbb{F}_p) \cong H^*(BT_X;\mathbb{F}_p)^{\Sigma_n} \cong H^*(BU(n);\mathbb{F}_p).$$

The uniqueness theorems in [29] yield that

$$\mathrm{map}(B\mathbb{Z}/p, BX)_{g_0} \simeq BU(n)^\wedge_p. \tag{3}$$

The composite

$$g : BU(n)^\wedge_p \simeq \mathrm{map}(B\mathbb{Z}/p, BX)_{g_0} \xrightarrow{\mathrm{ev}} BX$$

makes diagram (1) commutative.

Now let $g : BU(n) \to BX$ be any map satisfying (1), and let $g_0 : B\mathbb{Z}/p \to BX$ be the restriction of g to the central subgroup of order p in $U(n)$. Then using Proposition 4.1 and (3), we see that multiplication $\mathbb{Z}/p\times U(n) \to U(n)$ induces homotopy equivalences

$$BU(n)^\wedge_p \xrightarrow{\simeq} \mathrm{map}(B\mathbb{Z}/p, BU(n)^\wedge_p)_{\mathrm{incl}} \xrightarrow{\simeq} \mathrm{map}(B\mathbb{Z}/p, BX)_{g_0}.$$

Points (a), (b) and (c) now follow directly from Proposition 4.6. ∎

We end this section with some remarks we referred to in the introduction. Let $f : BT^\wedge_p \to BX$ be the realization of a pseudoreflection group $W \to \mathrm{GL}(n,\widehat{\mathbb{Z}}_p)$. We claimed that, for several coefficients, the map f induces an isomorphism between the cohomology of BX and the invariants of W acting on the cohomology of BT. This is stated in the next proposition.

PROPOSITION 4.9. *Let $W \subset \mathrm{GL}(n,\widehat{\mathbb{Z}}_p)$ be a finite subgroup, T be a torus and $BT^\wedge_p \to BX$ be a map invariant with repect to the action of W. If $H^*(BX;\mathbb{F}_p) \to H^*(BT;\mathbb{F}_p)^W$ is an isomorphism, then so are the maps $H^*(BX;\widehat{\mathbb{Z}}_p) \to H^*(BT;\widehat{\mathbb{Z}}_p)^W$ and $H^*(BX;\widehat{\mathbb{Z}}_p)\otimes\mathbb{Q} \to (H^*(BT;\widehat{\mathbb{Z}}_p)\otimes\mathbb{Q})^W$.*

Proof. By assumption the mod-p cohomology of BX is concentrated in even degrees. Hence, the p-adic integral cohomology is torsionfree, and the map $H^*(BX;\widehat{\mathbb{Z}}_p)\otimes\mathbb{F}_p \to H^*(BX;\mathbb{F}_p)$ is an isomorphism. Therefore the isomorphism for mod-p cohomology can be written as the composition

$$H^*(BX;\mathbb{F}_p) \xrightarrow{\cong} H^*(BX;\widehat{\mathbb{Z}}_p)\otimes\mathbb{F}_p \to H^*(BT;\widehat{\mathbb{Z}}_p)^W\otimes\mathbb{F}_p \to H^*(BT;\mathbb{F}_p)^W.$$

The right arrow is an epimorphism, because the composition is one. Passing to invariants is left exact. Thus this arrow is also a monomorphism and the middle arrow is an isomorphism. As a consequence of the Nakayama Lemma, this shows that $H^*(BX;\widehat{\mathbb{Z}}_p) \to H^*(BT;\widehat{\mathbb{Z}}_p)^W$ is an isomorphism. The functor $\otimes\widehat{\mathbb{Q}}_p$ commutes with taking invariants. This establishes the second isomorphism asserted. ∎

R e m a r k 4.10. Pseudoreflection groups appear as the Weyl group of connected p-compact groups (Theorem 4.3). Hence, one could say that a realization of a pseudoreflection group $W \to \mathrm{GL}(n, \widehat{\mathbb{Z}}_p)$ consists of a connected p-compact group X of rank n such that the action of the Weyl group W_X on the maximal torus T_X is described by the representation $W \to \mathrm{GL}(n, \widehat{\mathbb{Z}}_p)$. In this sense, every connected p-compact group is a realization of its Weyl group.

Because p-complete spaces with polynomial cohomology give rise to a connected p-compact group (see Proposition 4.4), this notion is a generalisation of the definition given in the introduction.

The work of several people [8], [36], [3], [14] and this paper (Theorem 1.4) show that every irreducible pseudoreflection group over $\widehat{\mathbb{Q}}_p$ (all primes) has a realization in this interpretation. The fact that *every* pseudoreflection group over $\widehat{\mathbb{Q}}_p$ has a realization in this sense follows immediately from the observation that such a group splits (canonically) as a product of irreducible pseudoreflection groups.

5. Homotopy uniqueness. In this section, we show that the spaces $BX(q,r;n)$ are determined by their mod-p cohomology: any p-complete space whose $\mathbb{F}_p$-cohomology is isomorphic to that of $BX(q,r;n)$ has the homotopy type of $BX(q,r;n)$. The next theorem is nothing but Theorem 1.6.

THEOREM 5.1. *Fix $r \mid q \mid (p-1)$ $(q>1)$ and $n>1$, set $G = G(q,r;n)$ and $BX = BX(q,r;n)$. Let BY be any p-complete space such that $H^*(BY;\mathbb{F}_p) \cong H^*(BX;\mathbb{F}_p)$ as algebras over the Steenrod algebra. Then $BY \simeq BX$.*

R e m a r k. We used the notation BY instead of Y, because a p-complete space with the above properties can be thought of as the classifying space of a p-compact group (see Proposition 4.4).

P r o o f (of Theorem 5.1). Let

$$j_Y : BU(n) \to BY$$

be the map constructed in Proposition 4.8, whose induced map in cohomology is the same as that of $j : BU(n) \to BX$. For each space $BU(\pi)$ in the decomposition of BX (Theorem 5.2), set $f_\pi = j_Y | BU(\pi) : BU(\pi) \to BY$. We want to show that the f_π all fit nicely together to define a map

$$\widehat{f} : BX = (\underset{\mathcal{O}_{\mathcal{H}}(G)}{\underrightarrow{\mathrm{hocolim}}}(BU(\pi)))^\wedge_p \to BY, \tag{1}$$

which will then be a homotopy equivalence by the assumption on the cohomology of BY. Here, we take $\mathcal{H} = \{\Sigma_\pi \mid \pi \in \mathcal{P}_p(n)\}$.

We first want to show that for any morphism $\varphi : G/\Sigma_{\pi_1} \to G/\Sigma_{\pi_2}$ in $\mathcal{O}_{\mathcal{H}}(G)$, the diagram

$$
\begin{tikzcd}
BU(\pi_1) \arrow[r, "f_{\pi_1}"] \arrow[d, "\varphi"'] & BY \arrow[d, equal] \\
BU(\pi_2) \arrow[r, "f_{\pi_2}"] & BY
\end{tikzcd}
\tag{2}
$$

commutes up to homotopy. Since the restriction of j_Y to BT ($T \subseteq U(n)$ a maximal torus) gives a maximal torus of BY (Proposition 4.8(a)), and because $j_Y^* : H^*(BY; \widehat{\mathbb{Z}}_p) \to H^*(BT; \widehat{\mathbb{Z}}_p)$ is a monomorphism, the diagrams clearly commute in cohomology. Moreover, by Proposition 4.8(b) we have $BU(\pi_1) \simeq \mathrm{map}(BZ(U(\pi_i)), BY)_{f_{\pi_i}|BZ(U(\pi_i))}$. (Note that $Z(U(\pi_i)) = T^{\Sigma_{\pi_i}}$ is a torus containing the center $Z(U(n))$ of $U(n)$.) Hence, Proposition 4.8(c) shows that the diagrams (2) commute up to homotopy.

The commutativity up to homotopy of the diagrams (2) shows that the maps f_π define a map from the 1-skeleton of the homotopy colimit into BY. The obstructions for extending this to a map on the full homotopy direct limit lie in the groups

$$
\varprojlim_{G/H \in \mathcal{O}_{\mathcal{H}}(G)}{}^{i+1} \ \pi_i(\mathrm{map}(BU(\pi), BY)_{f_\pi})
\tag{3}
$$

(for $i \geq 1$; see [34]), where

$$
\begin{aligned}
\pi_i(\mathrm{map}(BU(\pi), BY)_{f_\pi}) &\cong \pi_i(BZ(U(\pi))^\wedge_p) \qquad \text{(Prop. 4.8(b))}, \\
&\cong \begin{cases} \pi_2(BT^\wedge_p)^{\Sigma_\pi} & \text{if } i = 2, \\ 0 & \text{if } i \neq 2. \end{cases}
\end{aligned}
$$

Lemma 3.1 now implies that the obstruction groups in (3) all vanish. Hence there exists a map $\widehat{f} : BX \to BY$ as in (1), which is a homotopy equivalence since BX and BY are p-complete and $\widehat{f}$ induces an isomorphism in $\mathbb{F}_p$-cohomology. ∎

We can also offer a characterization of BX respectively X as a p-compact group in terms of the Weyl group data. We say that two p-compact groups X and Y have the *same Weyl group type* if the ranks are equal, say equal to n, and if the two representations $W_X, W_Y \to \mathrm{GL}(n, \widehat{\mathbb{Z}}_p)$, induced by the action on the maximal torus, are conjugate.

THEOREM 5.2. *Fix $r \mid q \mid (p-1)$ $(q > 1)$ and $n > 1$ and set $BX = BX(q, r; n)$. Let Y be a p-compact group which has the same Weyl group type as X. Then the p-compact groups X and Y are isomorphic.*

Proof. Without loss of generality we can assume that $T := T_X = T_Y$, that $W := W_X = W_Y$ and that the two representations $W \to \mathrm{GL}(n, \widehat{\mathbb{Z}}_p)$

are equal. Let $j_0 : \mathbb{Z}/p \hookrightarrow T$ be the diagonal inclusion. Then the centralizer $C := C_Y(S^1)$ is a p-compact group of rank n with Weyl group given by the isotropy group of the inclusion $S^1 \hookrightarrow T$. That is to say, $W_C = \Sigma_n$ and C has the same Weyl group type as $U(n)$. Hence, by [27], the two p-compact groups C and $U(n)^\wedge_p$ are isomorphic. The index of $\Sigma_n \subseteq W$ is coprime to p. This implies that $H^*(BY;\mathbb{F}_p) \to H^*(BU(n);\mathbb{F}_p)$ is a monomorphism ([14] or [26]), that $H^*(BY;\widehat{\mathbb{Z}}_p)$ is torsionfree, that $H^*(BY;\mathbb{Z}^\wedge_p) \cong H^*(BT;\mathbb{Z}^\wedge_p)^W$ [27, 4.2] and that

$$H^*(BY;\mathbb{F}_p) \cong H^*(BT;\mathbb{Z}^\wedge_p)^W \otimes \mathbb{F}_p \cong H^*(BT;\mathbb{F}_p)^W \cong H^*(BX;\mathbb{F}_p).$$

The second isomorphism follows from the proof of Proposition 4.9. Now, an application of Theorem 5.1 yields that $BX \simeq BY$, which is to say that $X \cong Y$ as p-compact groups. ∎

6. The normalizer of $G(q,r;n)$. In this section we calculate the quotient group $N_{\mathrm{GL}(n,\widehat{\mathbb{Q}}_p)}(G)/G$, when $G = G(q,r;n) \subseteq \mathrm{GL}(n,\widehat{\mathbb{Q}}_p)$. This will be needed in the next section when describing the self maps of $BX(q,r;n)$.

Set $A_{\max} = (\mu_q)^n = A(q,q;n)$, and set $G_{\max} = G(q,q;n)$. As usual, we write $G = G(q,r;n)$ and $A = A(q,r;n)$ for short. An easy calculation shows that the image of the homomorphism

$$H' : A_{\max} \times (\widehat{\mathbb{Q}}_p)^* \to \mathrm{GL}(n,\widehat{\mathbb{Q}}_p),$$

where $H'((a_1,\ldots,a_n),k) = \mathrm{diag}(ka_1,\ldots,ka_n)$, is contained in the normalizer $N_{\mathrm{GL}(n,\widehat{\mathbb{Q}}_p)}(G)$. Since H' sends $A \times 1$ into G, and the subgroup $\mu_q \cong \{((k,\ldots,k),k^{-1}) \mid k \in \mu_q\} \subseteq A_{\max} \times \widehat{\mathbb{Q}}_p$ into 1, it factors through the quotient homomorphism

$$H : A_{\max}/A \times_{\mu_q} (\widehat{\mathbb{Q}}_p)^* \to N_{\mathrm{GL}(n,\widehat{\mathbb{Q}}_p)}(G)/G.$$

LEMMA 6.1. *Fix $r \mid q \mid (p-1)$ and $n > 1$, and assume $q > 2$. Regard $G = G(q,r;n)$ as a subgroup of* $\mathrm{GL}(n,\widehat{\mathbb{Q}}_p)$. *Then the homomorphism*

$$H : A_{\max}/A \times_{\mu_q} (\widehat{\mathbb{Q}}_p)^* \to N_{\mathrm{GL}(n,\widehat{\mathbb{Q}}_p)}(G)/G$$

is an isomorphism except in the following two cases:

(a) $G = G(4,2;2)$, *and* $N_{\mathrm{GL}(2,\widehat{\mathbb{Q}}_p)}(G)/G \cong \Sigma_3 \times ((\widehat{\mathbb{Q}}_p)^*/\mu_4)$,

(b) $G = G(3,1;3)$, *and* $N_{\mathrm{GL}(3,\widehat{\mathbb{Q}}_p)}(G)/G \cong A_4^* \times_{(-1)} ((\widehat{\mathbb{Q}}_p)^*/\mu_3)$ (A_4^* *denotes the binary tetrahedral group*).

In both cases (a) *and* (b), *the first factor is contained in* $\mathrm{GL}(n,\widehat{\mathbb{Z}}_p)$.

P r o o f. S t e p 1: We first check that H is a monomorphism. For any $(a_1,\ldots,a_n) \in A_{\max}$ and any $k \in (\widehat{\mathbb{Q}}_p)^*$, let $[a_1,\ldots,a_n;k]$ denote the class of $((a_1,\dots,a_n),k)$ in $A_{\max}/A\times_{\mu_q}(\widehat{\mathbb{Q}}_p)^*$. If $[a_1,\dots,a_n;k]\in\mathrm{Ker}(H)$, then $\mathrm{diag}(ka_1,\dots,ka_n)\in G$, so $k\in\mu_q$ and $(ka_1,\dots,ka_n)\in A(q,r;n)$, and

$$[a_1,\dots,a_n;k]=[ka_1,\dots,ka_n;1]=1\quad\text{ in } A_{\max}/A\times_{\mu_q}(\widehat{\mathbb{Q}}_p)^*.$$

Step 2: Fix an element $M\in N_{\mathrm{GL}(n,\widehat{\mathbb{Q}}_p)}(G)$, and let $[M]$ denote its class mod G. We show that $[M]\in\mathrm{Im}(H)$ if M normalizes A.

As an A-representation, $(\widehat{\mathbb{Q}}_p)^n$ splits as a sum of n distinct 1-dimensional representations. So if $M\in N_{\mathrm{GL}(n,\widehat{\mathbb{Q}}_p)}(A)$, then M must be monomial (i.e., a product of a diagonal matrix and a permutation matrix). In particular, after multiplying M by an appropriate element in $\Sigma_n\subseteq G$, we can assume that M is diagonal.

Write $M=\mathrm{diag}(u_1,\dots,u_n)$, where $u_i\in\widehat{\mathbb{Q}}_p$. For any $\sigma\in\Sigma_n$,

$$\begin{aligned}M\sigma M^{-1}&=M\cdot(\sigma M^{-1}\sigma^{-1})\cdot\sigma\\&=(u_1u^{-1}_{\sigma^{-1}(1)},\dots,u_nu^{-1}_{\sigma^{-1}(n)})\cdot\sigma\in G.\end{aligned}$$

It follows that $u_1\equiv\ldots\equiv u_n\pmod{\mu_q}$, so

$$M=H'((1,u_2u_1^{-1},\dots,u_nu_1^{-1}),u_1)$$

and $[M]\in\mathrm{Im}(H)$.

Step 3: Again fix $M\in N_{\mathrm{GL}(n,\widehat{\mathbb{Q}}_p)}(G)$, and set $\varrho=(g\mapsto MgM^{-1})\in\mathrm{Aut}(G)$. We now show that $\varrho(A)=A$, except in the cases (a) and (b) listed above, by showing that no other subgroup of G is isomorphic to A. In fact, G contains no other abelian subgroup with the same order as that of A.

Assume first that for some prime $p'\mid q$, $O_{p'}(\Sigma_n)=1$. Here, $O_{p'}(-)$ denotes the intersection of all Sylow p'-subgroups. In this case, $O_{p'}(G)\subseteq A$ and $A=C_G(O_{p'}(G))$, and so $\varrho(A)=A$. Hence the only cases which remain to consider are those where q is a power of p', and where $(n,p')=(2,2)$, $(3,3)$, or $(4,2)$. (Recall that for $n\ge 5$, the alternating group $A_n\subseteq\Sigma_n$ is the only nontrivial normal subgroup.)

Now, if $n=3$ and q is a power of 3, and if $\varrho(A)\ne A$, then $\phi(\varrho(A))$ must contain a 3-cycle, and so $\varrho(A)\cap A$ is contained in the diagonal subgroup of order at most $\min(q,3r)$. Thus, $q^2r=|\varrho(A)|\mid\min(3q,9r)$, and hence $q=3$ and $r=1$. If $n=2$ or 4 and q is a power of 2, then a similar argument shows that $(q,r;n)=(4,1;2)$ or $(4,2;2)$ (recall that $q>2$ by assumption). Finally, the group $G(4,1;2)$ is dihedral of order 8, and A is its unique cyclic subgroup of order 4.

Step 4: If $G=G(4,2;2)$, then it contains a total of three subgroups isomorphic to A. So by Step 2, $\mathrm{Im}(H)$ has index at most 3 in $N_{\mathrm{GL}(n,\widehat{\mathbb{Q}}_p)}(G)/G$. Using this, one checks that $N_{\mathrm{GL}(n,\widehat{\mathbb{Q}}_p)}(G)/G\cong\Sigma_3\times(\widehat{\mathbb{Q}}_p)^*/\mu_4$, where the second factor consists of multiples of the identity, and where the first factor is generated by the matrices

$$S_1 = \frac{1}{1+i}\begin{pmatrix} i & 1 \\ i & -1 \end{pmatrix} \quad \text{and} \quad S_2 = \begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix}.$$

Note that $1+i$ is a unit in $\widehat{\mathbb{Z}}_p$, since 2 is invertible.

If $G = G(3,1;3)$, then it contains a total of four subgroups isomorphic to A. Hence $\mathrm{Im}(H)$ has index at most four in $N_{\mathrm{GL}(n,\widehat{\mathbb{Q}}_p)}(G)/G$. One now checks that $N_{\mathrm{GL}(n,\widehat{\mathbb{Q}}_p)}(G)/G \cong A_4^* \times_{(-1)} (\widehat{\mathbb{Q}}_p)^*/\mu_3$, where the second factor consists of multiples of the identity and the first is generated by the matrices

$$T_i = \frac{1}{\sqrt{-3}}\begin{pmatrix} 1 & 1 & 1 \\ 1 & \zeta & \zeta^2 \\ 1 & \zeta^2 & \zeta \end{pmatrix}, \quad T_j = \frac{1}{\sqrt{-3}}\begin{pmatrix} 1 & \zeta^2 & \zeta^2 \\ \zeta & \zeta & \zeta^2 \\ \zeta & \zeta^2 & \zeta \end{pmatrix},$$

$$T_k = \frac{1}{\sqrt{-3}}\begin{pmatrix} 1 & \zeta & \zeta \\ \zeta^2 & \zeta & \zeta^2 \\ \zeta^2 & \zeta^2 & \zeta \end{pmatrix}, \quad \text{and} \quad T' = \mathrm{diag}(\zeta, 1, 1).$$

Here, ζ denotes a primitive cube root of unity in $\widehat{\mathbb{Z}}_p$. ∎

7. Classification of self maps. Let $BX = BX(q,r;n)$ be a generalized Grassmannian. In this section, we want to study the monoid $[BX, BX]$ of homotopy classes of self maps of BX. As usual we assume that $q \mid (p-1)$, but throughout this section we also assume that $q > 2$. This restriction is necessary because we later use Lemma 6.1. When $q = 2$, $BX \simeq BSO(2n)^\wedge_p$ or $BSO(2n+1)^\wedge_p$, and so $[BX, BX]$ is described in [20, Theorem 2].

PROPOSITION 7.1. *Let p be an odd prime and let $r \mid q \mid (p-1)$. Let $BX = BX(q,r;n)$ be the p-complete space constructed in Theorem* 3.2. *Then*, *for any self map $f : BX \to BX$ which induces a nontrivial map in cohomology*, *the map $H^*(f; \widehat{\mathbb{Z}}_p)$ extends to a map*

$$H^*(f_T) \in \mathrm{End}(H^*(BT; \widehat{\mathbb{Z}}_p)) \cap \mathrm{Aut}(H^*(BT; \widehat{\mathbb{Z}}_p) \otimes \mathbb{Q}) \cong M_n(\widehat{\mathbb{Z}}_p) \cap \mathrm{GL}(n, \widehat{\mathbb{Q}}_p).$$

This extension $H^*(f_T)$ is contained in $N(G) = N_{\mathrm{GL}(n,\widehat{\mathbb{Q}}_p)}(G)$ and is unique modulo G. If $n \geq p$, then $H^*(f_T) \in \mathrm{GL}(n, \widehat{\mathbb{Z}}_p)$.

The proof of this statement also works for $q = 2$.

P r o o f. By [13, Proposition 8.11], for any map $f : BX \to BX$, the map $f|BT : BT \to BX$ lifts to a self map $f_T : BT^\wedge_p \to BT^\wedge_p$ of the classifying space of the maximal torus $BT \xrightarrow{i} BX$ of X. Moreover, $H^*(f; \widehat{\mathbb{Z}}_p)$ is nontrivial if and only if $H^*(f_T; \widehat{\mathbb{Z}}_p)$ is nontrivial. For any $w \in G = G(q,r;n)$, the two maps $f_T \circ Bw$ and f_T are homotopic after composition with i. So by [28, 7.1], there is $w' \in G$ such that $H^*(f_T \circ Bw) = H^*(Bw' \circ f_T)$, and hence $f_T \circ Bw \simeq Bw' \circ f_T$. In other words, there is a map of sets $\varrho : G \to G$

such that $f_T^* = H^*(f_T;\widehat{\mathbb{Z}}_p)$ is equivariant with respect to ϱ. An algebraic version dealing with rational cohomology, which is sufficient for our purpose, is already contained in [1] (see also [25, 0.2]).

In particular, this shows that the kernel of $H^2(f_T;\widehat{\mathbb{Z}}_p)$ is W_X-invariant. Also, $H^2(BT;\widehat{\mathbb{Z}}_p)\otimes\mathbb{Q}$ is an irreducible representation of $G = G(q,r;n) = W_X$, since it splits as a sum of distinct 1–dimensional $A = A(q,r;n)$-representations which are permuted transitively by $\Sigma_n = G/A$. It follows that f_T induces either an isomorphism in rational cohomology and that ϱ is an isomorphism of groups (given by conjugation with f_T^* considered as a self map of $H^2(BT;\widehat{\mathbb{Z}}_p)\otimes\mathbb{Q}$), or that $H^*(f_T;\widehat{\mathbb{Z}}_p)\otimes\mathbb{Q}$ is the trivial map. We only have to consider the first case. Now the above argument says that $H^*(f_T)\in N_{\mathrm{GL}(n,\widehat{\mathbb{Q}}_p)}(G)$ (and clearly $H^*(f_T)\in M_n(\widehat{\mathbb{Z}}_p)$). By [28, 7.1] again, $H^*(f_T)$ is uniquely defined modulo G. This proves the first part of the statement.

Lemma 6.1 shows that, up to homotopy equivalences, the map f_T looks like an unstable Adams operation of degree p^sk with $(p,k)=1$. Hence, we can assume that $f_T = \Psi^{p^sk}$ is an unstable Adams operation.

Let $g_S : BS^1 = BZ(U(n)) \to BX$ be the inclusion of the center. Then we have $\Psi^{p^sk}g_S \simeq g_S\Psi^{p^sk}$ (Proposition 4.8), and Ψ^{p^sk} induces a map

$$BU(n)_p^\wedge \simeq \mathrm{map}(BZ(U(n)),BX)_{g_S} \to \mathrm{map}(BS^1,BX)_{g_S} \simeq BU(n)_p^\wedge,$$

which is again an unstable Adams operation of the same degree p^sk. By a result of Ishiguro [17], the degree p^sk is a p-adic unit, equal to 0 or $n<p$. Hence, for $n\ge p$ every non-nullhomotopic map $f : BX\to BX$ gives rise to an element in $\mathrm{GL}(n,\widehat{\mathbb{Z}}_p)$. This finishes the proof. ∎

The fact that the degree of an unstable Adams operations on BX is a p-adic unit when $n\ge p$ is also implicitly contained in a theorem of Møller [25, 4.5].

By Lemma 6.1 and Proposition 7.1, there are well defined homomorphisms of monoids

$$[BX,BX] \xrightarrow{D} \{0\} \amalg N_{\mathrm{GL}(n,\widehat{\mathbb{Q}}_p)}(G)/G \xleftarrow{H} \{0\}\amalg((A_{\max}/A)\times_{\mu_q}(\widehat{\mathbb{Z}}_p\setminus 0)),$$

where H is an isomorphism except in the cases $BX(4,2;2)$ and $BX(3,1;3)$. Using these maps, we can now formulate the main theorem of this section.

THEOREM 7.2. *Let p be an odd prime, $r\,|\,q\,|\,(p-1)$ and $q>2$. Let $BX = BX(q,r;n)$ be the generalized Grassmannian constructed in Theorem* 3.2. *Then $H^{-1}\circ D$ induces an isomorphism*

$$[BX,BX]\cong\{0\}\amalg((A_{\max}/A)\times_{\mu_q}(\widehat{\mathbb{Z}}_p)^*)$$

if $n\ge p$, and an isomorphism

$$[BX,BX]\cong\{0\}\amalg((A_{\max}/A)\times_{\mu_q}(\widehat{\mathbb{Z}}_p\setminus 0))$$

if $n < p$ *and* $(q, r; n) \neq (4,2;2)$ *or* $(3,1;3)$. *In the two exceptional cases*, D *induces isomorphisms*

$$\begin{aligned}[BX(4,2;2), BX(4,2;2)] &\cong \{0\} \amalg (M_n(\widehat{\mathbb{Z}}_p) \cap N_{\mathrm{GL}(n,\widehat{\mathbb{Q}}_p)}(G)/G) \\ &\cong \{0\} \amalg (\Sigma_3 \times (\widehat{\mathbb{Z}}_p \smallsetminus 0))\end{aligned}$$

and

$$\begin{aligned}[BX(3,1;3), BX(3,1;3)] &\cong \{0\} \amalg (M_n(\widehat{\mathbb{Z}}_p) \cap N_{\mathrm{GL}(n,\widehat{\mathbb{Q}}_p)}(G)/G) \\ &\cong \{0\} \amalg (A_4^* \times_{(-1)} (\widehat{\mathbb{Z}}_p \smallsetminus 0)).\end{aligned}$$

The following corollary contains Theorem 1.7.

COROLLARY 7.3. *Let* BX *be as in Theorem* 7.2. *Then the following conditions are equivalent for any pair of maps* $f, g : BX \to BX$.

(a) *The maps* f *and* g *are homotopic.*
(b) *The restrictions* $f|BT, g|BT : BT \to BX$ *are homotopic.*
(c) $f^* = g^* : H^*(BX; \widehat{\mathbb{Z}}_p) \to H^*(BX; \widehat{\mathbb{Z}}_p)$.
(d) $f^* \otimes \mathbb{Q} = g^* \otimes \mathbb{Q} : H^*(BX; \widehat{\mathbb{Z}}_p) \otimes \mathbb{Q} \to H^*(BX; \widehat{\mathbb{Z}}_p) \otimes \mathbb{Q}$.

Proof. Clearly, (a)⇒(b)⇒(c)⇒(d): the second implication holds since $H^*(BX; \widehat{\mathbb{Z}}_p) \otimes \mathbb{Q}$ injects into $H^*(BT; \widehat{\mathbb{Z}}_p) \otimes \mathbb{Q}$ (Proposition 4.3). Finally, (d) implies (a) by Theorem 7.2 (the injectivity of D). ∎

Proof of Theorem 7.2. Set $G = G(q, r; n)$. If $n < p$, then

$$BX \simeq (EG \times_G BT_p^{\wedge})_p^{\wedge} \simeq (\underrightarrow{\mathrm{hocolim}}_{G}(BT_p^{\wedge}))_p^{\wedge}$$

by [10, Theorem 1.2] (where T is the n-torus) and where we think of G as a category with one object and morphisms given by the elements of G. In this case, we will show that

$$\begin{aligned}[BX, BX] &\xrightarrow{\cong} \{f_T \in [BT, BT] \mid \text{equivariant w.r.t. some } \varrho : G \to G\}/G \\ &\xleftarrow{\cong} \{0\} \amalg (M_n(\widehat{\mathbb{Z}}_p) \cap N_{\mathrm{GL}(n,\widehat{\mathbb{Q}}_p)}(G))/G.\end{aligned}$$

The latter isomorphism is obvious. The action of G on the second set is induced from the action of G on the target space $BT_p^{\wedge}$. Because self maps of $BT_p^{\wedge}$ are detected by rational cohomology, the arguments in the proof of Theorem 7.1 establish the first map. Now let $f_T : BT_p^{\wedge} \to BT_p^{\wedge}$ be a self map equivariant w.r.t. some $\varrho : G \to G$. The obstructions for extending the composition $BT_p^{\wedge} \xrightarrow{fT} BT_p^{\wedge} \to BX$, also denoted by f_T, to a self map of BX lie in

$$\varprojlim_G{}^{i+1}\pi_i(\mathrm{map}(BT_p^\wedge, BX)_{f_T}) \cong \varprojlim_G{}^{i+1}\pi_i(BT_p^\wedge) \cong H^{i+1}(G;\pi_i(BT_p^\wedge)).$$

Let $f, f' : BX \to BX$ be two maps which, up to homotopy and the G-action, give rise to the same self map of $BT_p^\wedge$. The obstructions for f and f' being homotopic lie in

$$\varprojlim_G{}^{i}\pi_i(\mathrm{map}(BT_p^\wedge, BX)_{f_T}) \cong \varprojlim_G{}^{i}\pi_i(BT_p^\wedge) \cong H^{i}(G;\pi_i(BT_p^\wedge)).$$

In both cases this follows from [34]. Because the order of G is coprime to p, all the obstruction groups involved vanish, which proves the statement for $n < p$.

Now we assume that $n \geq p$. In particular, by Proposition 7.1, any map $f : BX \to BX$ which does not induce the trivial map in rational cohomology induces an isomorphism on $H^*(BX;\widehat{\mathbb{Z}}_p)$, and hence is a homotopy equivalence.

We first construct the unstable Adams operations. Fix $k \in (\widehat{\mathbb{Z}}_p)^*$. Let $g : BU(n) \to BX$ be the inclusion of $BU(n)$ into the homotopy direct limit constructed in Theorem 4.8. For each partition $\pi \in \mathcal{P}_p(n)$, set $g_\pi = g|BU(\pi)$; and let $f_\pi : BU(\pi) \to BX$ be the composite

$$f_\pi : BU(\pi)_p^\wedge \xrightarrow{\psi_\pi^k}_{\simeq} BU(\pi)_p^\wedge \xrightarrow{g_\pi} BX,$$

where ψ_π^k denotes an unstable Adams operation of degree k on $BU(\pi)$. Furthermore, by Proposition 4.8(c), the f_π commute up to homotopy with maps induced by morphisms $\varphi : G/\Sigma_{\pi_1} \to G/\Sigma_{\pi_2}$ (since they commute in cohomology). The f_π thus combine to define an element

$$\widehat{f} \in \varprojlim_{G/H \in \mathcal{O}_{\mathcal{H}}(G)} \pi_0(\mathrm{map}(BU(\pi), BY)_{g_\pi}).$$

Here, $\mathcal{H} = \{\Sigma_\pi \mid \pi \in \mathcal{P}_p(n)\}$ is the family of subgroups of G constructed in Section 5.

Since ψ_π^k is an equivalence ($p \nmid k$), Proposition 4.8(b) applies again to show that there are homotopy equivalences

$$\begin{aligned} BZ(U(\pi))_p^\wedge &\xrightarrow{\simeq} \mathrm{map}(BU(\pi), BX)_{g_\pi} \\ &\xrightarrow[\simeq]{-\circ\psi_\pi^k} \mathrm{map}(BU(\pi), BX)_{f_\pi} \xleftarrow{\simeq} BZ(U(\pi))_p^\wedge. \end{aligned}$$

These equivalences are compatible with all morphisms of the category $\mathcal{O}_{\mathcal{H}}(G)$, and

$$\pi_i(BZ(U(\pi))_p^\wedge) \cong \begin{cases} 0 & \text{if } i \neq 2, \\ \pi_2(BT_p^\wedge)^{\Sigma_\pi} & \text{if } i = 2. \end{cases}$$

This is a fixed point functor. So by Proposition 2.3,

$$\varprojlim_{G/H \in \mathcal{O}_{\mathcal{H}}(G)}{}^{j} \pi_2(\mathrm{map}(BU(\pi), BY)_{f_\pi}) = 0$$

for all $j \geq 1$. Thus, the obstructions to extending the f_π to a map

$$BX \cong \underset{\mathcal{O}_{\mathcal{H}}(G)}{\underrightarrow{\mathrm{hocolim}}}(BU(\pi)) \to BX$$

all vanish, and we get a map $f : BX \to BX$ which (since the cohomology of BX injects into the cohomology of $BU(n)$) is an unstable Adams operation of degree k.

Next we want to realize the elements of $A_{\max}/A$ as self maps of BX. Fix $\alpha \in A_{\max}$, and set

$$\Phi = \mathrm{incl} \circ B\alpha_X^* : H^*(BX; \mathbb{F}_p) \to H^*(BU(n); \mathbb{F}_p).$$

By Proposition 4.8, there exists a map $g_\alpha : BU(n) \to BX$ such that $H^*(g_\alpha) = \Phi$. If we set $f_\pi = g_\alpha | BU(\pi)$ for each $\pi \in \mathcal{P}_p(n)$, then the same arguments as those used to construct unstable Adams operations apply to show that the f_π extend to a map $g : BX \to BX$. Then $H^*(f; \widehat{\mathbb{Z}}_p) = \alpha_X^*$ (mod p), so $D(g) = (\alpha, k)$ for some $k \equiv 1 \pmod p$.

It remains to show that D is injective. If $D(f) = D(f')$ and they are nontrivial in rational cohomology, then as noted above, f and f' are both homotopy equivalences. Hence, upon replacing f by $f \circ (f')^{-1}$, we can assume that $H^*(f; \widehat{\mathbb{Z}}_p)$ is the identity, and must show that $f \simeq \mathrm{Id}_{BX}$. By Proposition 4.8(b),

$$f \circ g \simeq g : BU(n) \to BX,$$

where $g : BU(n) \to BX$ denotes the "canonical" map of Proposition 4.8. It follows that f and Id_{BX} have the same image under the restriction map

$$[BX, BX] \cong [\underset{G/\Sigma_\pi \in \mathcal{O}_{\mathcal{H}}(G)}{\underrightarrow{\mathrm{hocolim}}} (BU(\pi)), BX] \to \varprojlim_{G/\Sigma_\pi \in \mathcal{O}_{\mathcal{H}}(G)} [BU(\pi), BX].$$

By [34] again, the obstructions to f being homotopic to Id_{BX} lie in the groups

$$\varprojlim_{G/\Sigma_\pi \in \mathcal{O}_{\mathcal{H}}(G)}{}^{i} \pi_i(\mathrm{map}(BU(\pi), BX)_{g_\pi}).$$

Because $\mathrm{map}(BU(\pi), BX)_{g_\pi} \simeq BZ(U(\pi))_p^\wedge$ by Proposition 4.8(b), these higher limits vanish by Proposition 2.3.

It remains to check that $f : BX \to BX$ is nullhomotopic if it induces the trivial map in rational cohomology. This is nothing but a special case of [25, 5.7]. ∎

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