FUNDAMENTA
MATHEMATICAE
154 (1997)

On matrix rapid filters

by

Winfried Just (Athens, Ohio) and **Peter Vojtáš** (Košice)

Abstract. Galois–Tukey equivalence between matrix summability and absolute convergence of series is shown and an alternative characterization of rapid ultrafilters on ω is derived.

Introduction. Filters (ultrafilters) on the set ω of natural numbers play an important role in different applications of set theory in real analysis, functional analysis, topology, dynamical systems and ergodic theory, infinite combinatorics and complexity theory etc. In other words, filters on ω are useful whenever some objects are enumerated by natural numbers and we are interested in their asymptotical behaviour at infinity.

A special example of this sort with applications in analysis was introduced by G. Mokobodzki ([M]): a filter j on ω is said to be rapid if for every sequence a_n of positive reals tending to zero, there is an $X \in j$ such that the series $\sum a_n$ converges when restricted to the terms with indices in X.

Motivated by this and work of R. Atalla [A] we define a filter j on ω to be m-rapid if for every positive Toeplitz matrix A with suprema of rows tending to zero, there is an $X \in j$ such that the characteristic function of X seen as an infinite 0-1 sequence has the matrix limit calculated according to the matrix A equal to zero (is summed by A to zero).

In [V1] the notion of Galois–Tukey connection between binary relations was introduced in order to express the combinatorial content of inequalities between cardinal characteristics of real analysis in a "durable way", i.e. interesting also under CH. Note that most of the cardinal characteristics of real analysis lie between $\aleph_1$ and the continuum, and inequalities between them are no longer interesting if CH is assumed.

1991 *Mathematics Subject Classification*: 06A15, 40C05, 04A20.

This research was done in part while the second author was visiting Ohio University, and also partly while he was supported by the Alexander von Humboldt Stiftung, Bonn and by the grant 1/1224/94 of the Slovak grant agency VEGA.

We extract from the notions of rapid and m-rapid filters two relations (one between series and subsets of ω and a second one between Toeplitz matrices and subsets of ω). We show that these two relations are Galois–Tukey equivalent in a special way, namely in both conections the second mapping is the identity on subsets of ω. This enables us to show that rapid filters (ultrafilters) are exactly the m-rapid ones. Moreover, this gives "durable" combinatorial connections between matrix summation and absolute summability of series.

Connections between matrix summation and divergence of series. Let R, S be binary relations. Following [V1], we call an ordered pair of functions (E, F) a (*generalized*) *Galois–Tukey connection* (abbreviated as GT-connection) *from R to S* if the following holds:

(a) $E : \mathrm{dom}(R) \to \mathrm{dom}(S)$;
(b) $F : \mathrm{rng}(S) \to \mathrm{rng}(R)$;
(c) $\forall x \in \mathrm{dom}(R)\ \forall v \in \mathrm{rng}(S)\ (E(x), v) \in S \Rightarrow (x, F(v)) \in R$.

The existence of a GT-connection from R to S means that, in a certain sense, R is simpler than S. Moreover, if we define the *bounding number* of R by

$$\mathbf{b}(R) = \min\{|B| : B \subseteq \mathrm{dom}(R)\ \&\ \forall y \in \mathrm{rng}(R)\ \exists x \in B\ ((x, y) \notin R)\},$$

and the *dominating number* of R by

$$\mathbf{d}(R) = \min\{|D| : D \subseteq \mathrm{rng}(R)\ \&\ \forall x \in \mathrm{dom}(R)\ \exists y \in D\ ((x, y) \in R)\},$$

then the existence of a GT-connection from R to S implies that $\mathbf{b}(R) \geq \mathbf{b}(S)$ and $\mathbf{d}(R) \leq \mathbf{d}(S)$.

Two relations R and S are said to be *GT-equivalent* if there exist both a GT-connection from R to S and a GT-connection from S to R.

Let $\mathbf{M}$ denote the set of all regular matrices (also called Toeplitz matrices) $A = \{a_{i,j}\}_{i,j\in\omega}$ (i ranging over rows and j over columns) with nonnegative entries and such that the suprema of rows tend to 0. Recall that a matrix $A = \{a_{i,j}\}_{i,j\in\omega}$ with nonnegative entries is *regular* if and only if $\lim_{i\to\infty} a_{i,j} = 0$ for every $j \in \omega$ and $\lim_{i\to\infty} \sum_{j=0}^{\infty} a_{i,j} = 1$. If $b : \omega \to \mathbb{R}$ and $A \in \mathbf{M}$, then A-$\lim b$ is defined as A-$\lim b = \lim_{i\to\infty} \sum_{j=0}^{\infty} a_{i,j} b(j)$. Regular matrices have the property that for all $b : \omega \to \mathbb{R}$, if $\lim_{j\to\infty} b_j = L$, then A-$\lim b$ exists and is equal to L.

For $X \subseteq \omega$ let c_X denote the characteristic function of X. Given $A \in \mathbf{M}$ we define $F_A = \{X \subseteq \omega : A\text{-}\lim c_X = 1\}$. It is not hard to see that F_A is a proper filter on ω that contains all cofinite sets. Following [A], we define the *support set* of A as the set of all ultrafilters in ω^* that extend F_A. Our requirement that the suprema of rows tend to 0 ensures that the support of this matrix is a nowhere dense subset of ω^* (see [R]).

We define a binary relation $TOEP \subseteq \mathbf{M} \times [\omega]^\omega$ as follows:

$$(A, X) \in TOEP \quad \text{if and only if} \quad A\text{-}\lim_{n\to\infty} c_X(n) = 0.$$

The relation $TOEP$ is the inverse of the relation $RLIM$ restricted to characteristic functions of subsets of ω (see [V1] for more information about $RLIM$).

As usual, let $\ell^1_+ = \{a \in {}^\omega[0,\infty) : \sum_{n=0}^\infty a(n) < \infty\}$ and $c_0^+ = \{a \in {}^\omega[0,\infty) : \lim_{n\to\infty} a(n) = 0\}$. We shall consider the binary relation $CONV \subseteq (c_0^+ \setminus \ell^1_+) \times [\omega]^\omega$ defined in [V1] by

$$(a, X) \in CONV \quad \text{if and only if} \quad \sum_{n\in X} a_n < \infty.$$

In [V1] restrictions of $RLIM$ to various domains were studied, and, roughly speaking, all **b**- and **d**-numbers of them were variants of the splitting number $\mathbf{s}$ and the refining (reaping) number $\mathbf{r}$. In [V1] it was also shown that $CONV$ and $({}^\omega\omega, \le^*)$ are also Galois–Tukey equivalent. Our first theorem will yield estimates for the bounding and dominating numbers of $TOEP = RLIM^{-1}$.

1. THEOREM. *The relations CONV and TOEP are Galois–Tukey equivalent to each other* (*and hence also to* $({}^\omega\omega, \le^*)$).

P r o o f. In both directions, the mappings F of the GT-conections will be the identity on $[\omega]^\omega$.

So to show that there exists a GT-connection from $TOEP$ to $CONV$, we have to construct a mapping $E : \mathbf{M} \to c_0^+ \setminus \ell^1_+$ such that for every matrix A and set X of natural numbers

$$\sum_{n\in X} E(A)(n) < \infty \quad \text{implies} \quad A\text{-}\lim c_X = 0.$$

In order to define E, first we define by simultaneous induction two increasing sequences of natural numbers: $(m_k)_{k\in\omega}$ (denoting rows) and $(n_k)_{k\in\omega}$ (denoting columns). Put $m_0 = 0$, $n_0 = 1$ and having defined m_k, n_k find the first row $m_{k+1} > m_k$ for which

$$\forall i \ge m_{k+1} \left(\sup_{j\in\omega} a_{i,j} < \frac{1}{n_k 2^{k+1}} \right).$$

Having defined m_{k+1} find the first column $n_{k+1} > n_k$ such that

$$\forall i \in [m_k, m_{k+1}) \Big(\sum_{j=n_{k+1}}^{\infty} a_{i,j} < 2^{-k} \Big).$$

Notice that for $k > 1$ and $i \in [m_k, m_{k+1})$ we have by induction

$$\sum_{j=0}^{n_{k-1}-1} a_{i,j} < n_{k-1}\frac{1}{n_{k-1}2^k} = \frac{1}{2^k}.$$

Having this we define

$$E(A)(j) = \frac{\log(k+1)}{k+1} \quad \text{if } j \in [n_k, n_{k+1}).$$

Now assume X is such that $\sum_{j\in X} E(A)(j) < \infty$. We claim that for all but finitely many k's, $|X \cap [n_k, n_{k+1})| \le k$. If not, then there are infinitely many k's such that

$$\sum_{j\in X} E(A)(j) \ge \sum_{j\in[n_k,n_{k+1})\cap X} E(A)(j) \ge (k+1)\frac{\log(k+1)}{k+1} = \log(k+1) \to \infty.$$

Having estimated the size of $X \cap [n_k, n_{k+1}]$ we can estimate the A-limit of c_X. For $k > 0$ and $i \in [m_k, m_{k+1})$ we have

$$\sum_{j=0}^{\infty} a_{i,j}c_X(j) = \sum_{j=0}^{n_{k-1}-1} a_{i,j}c_X(j) + \sum_{j=n_{k-1}}^{n_{k+1}-1} a_{i,j}c_X(j) + \sum_{j=n_{k+1}}^{\infty} a_{i,j}c_X(j) \le 2^{-k} + 2\cdot k\cdot 2^{-k} + 2^{-k} \le 2(k+1)2^{-k}$$

which as expected tends to zero. Thus, the first half of Theorem 1 is proved.

To prove the opposite GT-connection we have to construct a mapping $H : c_0^+ \setminus \ell_+^1 \to \mathbf{M}$ with $H(a)\text{-}\lim c_X = 0$ implying $\sum_{n\in X} a(n) < \infty$. Again we construct two sequences $(m_k)_{k\in\omega}$ and $(n_k)_{n\subset,n\omega}$ of natural numbers, but this time we arrange that $n_k = \min\{n : \forall j \ge n\ (a_j < 2^{-k})\}$ and $m_{k+1} = m_k +$ the number of all k-element subsets of $n_{k+1} \setminus n_k$.

Now we are ready to define the matrix $H(a)$. The rows between the m_kth and the m_{k+1}th will be enumerated by k-element subsets of $n_{k+1} \setminus n_k$. For $s \in [n_{k+1} \setminus n_k]^k$ we let $a_{s,j} = 1/k$ if $j \in s$ and 0 otherwise.

2. Claim. *Suppose $X \in [\omega]^\omega$ is such that $H(a)\text{-}\lim c_X = 0$. Then*

$$\lim_{k\to\infty} \frac{|X \cap [n_k, n_{k+1})|}{k} = 0.$$

Proof. Assume towards a contradiction that there are infinitely many k's such that $|X \cap [n_k, n_{k+1})| > k\varepsilon$, for some positive $\varepsilon < 1$. For such k, there is an $i \in [m_k, m_{k+1})$ that codes a k-element subset $s \subset [n_k, n_{k+1}]$ such that $|X \cap s| \ge \varepsilon k$. For such i we obtain the following estimate:

$$\sum_{j=0}^{\infty} a_{i,j}c_X(j) > \varepsilon k\frac{1}{k} = \varepsilon > 0;$$

but this contradicts the assumption that the $H(a)$-limit of c_X is 0. We have proved Claim 2.

It follows from Claim 2 that for all but finitely many k,

$$\sum_{n\in X\cap[n_k,n_{k+1})} a_n < k\frac{1}{2^k},$$

and the latter implies that $\sum_{n\in X} a_n < \infty$.

3. Corollary. $\mathbf{b}(\mathit{TOEP}) = \mathbf{b}(\mathit{CONV}) = \mathbf{b}({}^{\omega}\omega, \le^*) = \mathbf{b}$ *and* $\mathbf{d}(\mathit{TOEP}) = \mathbf{d}(\mathit{CONV}) = \mathbf{d}({}^{\omega}\omega, \le^*) = \mathbf{d}$.

Filters on the set of natural numbers. Now let us present an application of Theorem 1. A nonprincipal ultrafilter on ω is called *rapid* if the family of functions enumerating elements of F is a dominating family in $({}^{\omega}\omega, \le^*)$. It is known (implicitly in [M]) that an ultrafilter $F \in \omega^*$ is rapid if and only if

$$\forall a \in c_0^+ \ \exists X \in F \ \Big(\sum_{n\in X} a(n) < \infty\Big). \tag{1}$$

Let us recall some notions introduced in [V2]. For $a \in c_0^+$ let $I_a = \{X \subseteq \omega : \sum_{n\in X} a_n < \infty\}$ and G_a be the corresponding dual filter. Let $\delta(G_a) = \bigcap\{A^* : A \in G_a\}$ be the closed nowhere dense subset of ω^* corresponding to G_a. Note that the set $\omega^* \setminus \bigcup\{\delta(G_a) : a \in c_0^+\}$ is exactly the set of rapid ultrafilters. In a similar vein, one might investigate those ultrafilters that omit all $\delta(F_A)$ in ω^*. Let us define *m-rapid filters* or *m-points* as those ultrafilters in

$$\omega^* \setminus \bigcup\{\delta(F_A) : A \in \mathbf{M}\},$$

i.e., all $F \in \omega^*$ such that

$$\forall A \in \mathbf{M} \ \exists X \in F \ (A\text{-}\lim c_X = 0).$$

4. Theorem. *The m-points are exactly the rapid ultrafilters.*

Proof. Suppose $F \in \omega^*$ is rapid and let $A \in \mathbf{M}$. Then $E(A) \in c_0^+ \setminus \ell_+^1$ is chosen as in the first half of the proof of Theorem 1 and by (1) there exists an $X \in F$ such that $\sum_{n\in X} E(A)(n) < \infty$. Since $(E, \mathrm{id}_{[\omega]^\omega})$ is a GT-connection, $(A, X) \in \mathit{TOEP}$, i.e., A-$\lim c_X = 0$. This shows that F is an m-point.

Now suppose F is an m-point and $a \in c_0^+$. We want to show that $\exists X \in F$ $(\sum_{n\in X} a(n) < \infty)$. If $a \in \ell_+^1$, there is nothing to prove; so assume that $a \in c_0^+ \setminus \ell_+^1$. Let $H : c_0^+ \setminus \ell_+^1 \to \mathbf{M}$ be as in the second half of the proof of Theorem 1. Since F is an m-point, we can pick $X \in F$ such that $H(a)$-$\lim c_X = 0$. Since $(H, \mathrm{id}_{[\omega]^\omega})$ is a GT-connection, the latter implies $\sum_{n\in X} a(n) < \infty$. Since a was arbitrary, we have shown that (1) holds, and thus F is rapid.

Conclusions and a problem. Let us define $\mathbf{F}_r = \{\delta(G_a) : a \in c_0^+ \setminus \ell_+^1\}$. In [V2] it is proved that $\mathbf{d}(\mathbf{F}_r, \subseteq) = \mathbf{d}$, i.e. the portion of ω^* which is covered by all nowhere dense sets from $\mathbf{F}_r$ is already covered by a subset of size $\mathbf{d}$. Recall that the *Novák number* $n(\omega^*) = \mathbf{n}$ is the minimal size of a family of nowhere dense subsets of ω^* which covers the whole ω^*. Notice that $\omega_2 \le \mathbf{n} \le 2^{\mathbf{c}}$. This was used in [V2] to prove existence theorems like: If $\mathbf{n} > \mathbf{d}$ then there are rapid ultrafilters. Moreover, the structure $(\mathbf{F}_r, \subseteq)$ considered as a forcing notion (the smaller $\delta(G_a)$, the stronger the condition) is a separative factorization of $(c_0^+ \setminus \ell_+^1, \le^*)$. Finally, under MA the partial order $RO(\mathbf{F}_r, \subseteq)$ is isomorphic to $RO(P(\omega)/\mathrm{fin})$. Now it is natural to ask similar questions about $\mathbf{F}_M = \{\delta(F_A) : A \in \mathbf{M}\}$. By Theorem 1, for every $A \in \mathbf{M}$ there is an a with $\delta(F_A) \subseteq \delta(G_a)$ and for every $a \in c_0^+ \setminus \ell_+^1$ there is an $A \in \mathbf{M}$ with $\delta(G_a) \subseteq \delta(F_A)$; hence $\mathbf{d}(\mathbf{F}_M, \subseteq) = \mathbf{d}$. On the other hand, $(\mathbf{F}_M, \subseteq)$ considered as a forcing notion is nontrivial. Note the analogy: The complexity of $(c_0^+ \setminus \ell_+^1, \le^*)$ measures the complexity of absolute divergence in the sense that a dense subset of $(c_0^+ \setminus \ell_+^1, \le^*)$ yields a set of comparison tests that allows us to detect the divergence of *every* divergent series of positive numbers, whereas a dense subset $D \subseteq (\mathbf{F}_M, \subseteq)$ has the property: every sequence of 0's and 1's containing infinitely many 1's has A-lim $= 1$ for some $A \in \mathbf{M}$ with $\delta(F_A) \in D$. This leads to the following open problem.

5. Question. *Is* $RO(\mathbf{F}_M, \subseteq)$ *isomorphic to* $RO(P(\omega)/\mathrm{fin}) = RO(\omega^*)$ (*at least consistently*)?

References

[A] R. Atalla, *Regular matrices and P-sets in $\beta N \setminus N$*, Proc. Amer. Math. Soc. 37 (1973), 157–162.

[M] G. Mokobodzki, *Ultrafiltres rapides sur N. Construction d'une densité relative de deux potentiels comparables*, in: Séminaire Brelot–Choquet–Deny (Théorie du Potentiel), 12e année, 1967/68, no. 12, pp. 12–01 to 12–22.

[R] J. Rainwater, *Regular matrices with nowhere dense supports*, Proc. Amer. Math. Soc. 29 (1971), 361.

[V1] P. Vojtáš, *Generalized Galois–Tukey-connections between explicit relations on classical objects of real analysis*, in: Israel Math. Conf. Proc. 6, 1993, 619–643.

[V2] —, *ω^* and absolutely divergent series*, Topology Proc. 19 (1994), 335–348.

Department of Mathematics
Ohio University
Athens, Ohio 45701
E-mail: just@ace.cs.ohiou.edu

Slovak Academy of Sciences
Košice, Slovakia
E-mail: vojtas@kosice.upjs.sk

Received 15 April 1996;
in revised form 4 May 1996