The $\Sigma^*$ approach to the fine structure of $L$

by

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Abstract. We present a reformulation of the fine structure theory from Jensen [72] based on his $\Sigma^*$ theory for $K$ and introduce the Fine Structure Principle, which captures its essential content. We use this theory to prove the Square and Fine Scale Principles, and to construct Morasses.

1. The $J$-hierarchy. The most elegant hierarchy for Gödel’s $L$ is obtained through iterated first-order definability. For any set $x$ let $\text{Def}(x)$ denote $\{y \mid y \subseteq x, \text{ $y$ is definable over } \langle x, \epsilon \rangle \text{ by a first-order formula with parameters}\}$. Then $L$ is obtained as the union of all $L_\alpha$, where $L_0 = \emptyset$, $L_\lambda = \bigcup\{L_\alpha \mid \alpha < \lambda\}$ for limit $\lambda$, and

$$L_{\alpha+1} = \text{Def}(L_\alpha).$$

Unfortunately, $L_{\alpha+1}$ is not closed under pairing and for this reason, Jensen [72] defined a modified hierarchy $\langle J_\alpha \mid \alpha \in \text{ORD} \rangle$ for $L$ to get around this problem. We now present a description of the $J$-hierarchy which, as above, is based on the idea of iterated definability.

Recall the Lévy hierarchy of formulas: A formula is $\Sigma_0$ ($= \Delta_0 = \Pi_0$) if it is built from atomic formulas through the use of logical connectives and bounded quantifiers $\forall x \in y, \exists x \in y$. A formula is $\Sigma_{n+1}$ if it is of the form $\exists \vec{x} \varphi$ where $\varphi$ is $\Pi_n$. Dually, a formula is $\Pi_{n+1}$ if it is of the form $\forall \vec{x} \varphi$ where $\varphi$ is $\Sigma_n$. Every formula is logically equivalent to a $\Sigma_n$ formula for some $n$, as it can be put into prenex normal form.

We want to define the $J$-hierarchy so that $J_{\alpha+1} \cap P(J_\alpha) = \text{Def}(J_\alpha)$. $J_{\alpha+1}$ is closed under pairing and in addition, $J_{\alpha+1}$ satisfies $\Sigma_0$-Comprehension. The latter is the statement that for any $x$ we can form $\{y \in x \mid \varphi(y)\}$, where $\varphi$ is a $\Sigma_0$ formula with arbitrary parameters. This is important for the construction of universal $\Sigma_n$ predicates, a notion that we define next.

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A binary relation $W_n(e, x)$ on a transitive set $S$ is a universal $\Sigma_n$ predicate for $S$ if it is $\Sigma_n$-definable over $\langle S, \in \rangle$ without parameters and wherever $Y \subseteq S$ is $\Sigma_n$-definable over $\langle S, \in \rangle$ with parameters, there exists $e \in S$ such that

$$Y = \{x \in S \mid W_n(e, x)\}.$$ 

Thus the sets $\{x \in S \mid W_n(e, x)\}$ are exactly the sets $\Sigma_n$-definable over $\langle x, \in \rangle$ with parameters, as $e$ varies over $S$.

**Lemma 1.** Suppose that $S$ is a transitive set closed under pairing, satisfying $\Sigma_0$-Comprehension $+$ “Every set has a transitive closure.” Then there exists a universal $\Sigma_n$ predicate for $S$.

**Proof.** It is enough to treat the case $n = 1$, as for example to get $W_2$ from $W_1$ we can just define $W_2(e, x) \iff \exists y \sim W_1(e, \langle x, y \rangle)$.

Let $\langle \varphi_i \mid i \in \omega \rangle$ be a standard list of formulas with one free variable with subformulas enumerated earlier, and define $\text{Sat}(z, i, x)$ to mean: $z$ is transitive, $x \in z$ and $\langle z, \in \rangle \models \varphi_i(x)$. Sat can be expressed by a $\Sigma_1$-formula:

$$\text{Sat}(z, i, x) \iff z \text{ is transitive, } x \in z \text{ and } \exists Y \subseteq (i + 1) \times z \text{ such that } \{\forall j \leq i \mid \text{if } \varphi_j(x) \text{ is atomic then } \langle j, x \rangle \in Y \iff \varphi_j \text{ true}; \text{if } \varphi_j(x) \text{ is } \exists y \varphi_j'((x, y)) \text{ then } \langle j, x \rangle \in Y \iff \exists y \in z (\langle j', (x, y) \rangle \in Y) \text{; if } \varphi_j(x) \text{ is } \sim \varphi_j'(x) \text{ then } \langle j, x \rangle \in Y \iff \langle j', x \rangle \notin Y \text{; if } \varphi_j(x) \text{ is } \varphi_{j_1}(x) \land \varphi_{j_2}(x) \text{ then } \langle j, x \rangle \in Y \iff (\langle j_1, x \rangle \in Y \text{ and } \langle j_2, x \rangle \in Y) \text{ and } \langle i, x \rangle \in Y\}.$$ 

The fact that $S$ satisfies pairing and $\Sigma_0$-Comprehension implies that when restricted to $S$, Sat is $\Sigma_1$-definable over $\langle S, \in \rangle$, via the above definition. Finally, we set:

$$W_1(e, x) \iff e = \langle i, p \rangle \text{ and for some transitive } z, \text{ Sat}(z, i, (x, p)).$$

$W_1$ is universal, using pairing, the existence of transitive closures and the persistence of $\Sigma_1$ formulas over transitive sets.

We are ready to define the $J$-hierarchy. By induction on $\alpha$ we define $J_\alpha$ to satisfy the hypotheses of Lemma 1. Let $W_\alpha^n(e, x)$ denote the canonical universal $\Sigma_n$ predicate for $J_\alpha$ coming from the proof of Lemma 1. For $\alpha = 0$ we have $J_0 = \emptyset$ and for $\alpha = 1$ we have $J_1 = L_\omega$. For $\alpha$ limit, $J_\alpha = \bigcup \{J_\beta \mid \beta < \alpha\}$. Note that the hypotheses of Lemma 1 are met by $J_\alpha$, given that they are met by each $J_\beta$, $\beta < \alpha$.

Suppose that $J_\alpha$ and $W_\alpha^n(e, x)$ are defined for some $\alpha > 0$ and we wish to define $J_{\alpha + 1}$. An $n$-code is a pair $(n, e)$ where $e \in J_\alpha$. By induction on $n$ define

$$X(0, e) = e,$$

$$X(n + 1, e) = \{X(n, f) \mid W_\alpha^n(e, f)\}.$$ 

Then $J_{\alpha, n} = \{X(n, e) \mid e \in J_\alpha\}$ and $J_{\alpha + 1} = \bigcup \{J_{\alpha, n} \mid n \in \omega\}$. 
LEMMA 2. (a) $n \leq m \rightarrow J_{\alpha,n} \subseteq J_{\alpha,m}$.
(b) $J_{\alpha,n}$ is transitive.
(c) $\text{ORD}(J_{\alpha,n}) = \omega \alpha + n$.
(d) $J_{\alpha+1} \models \text{Pairing} + \Sigma_0\text{-Comprehension}.$
(e) $J_{\alpha+1} \cap P(J_{\alpha}) = \text{Def}(J_{\alpha}).$

Proof. (a) By induction on $n$, we define a $\Sigma_1(J_{\alpha})$ function $F(n,e)$, $e \in J_{\alpha}$, that produces $f \in J_{\alpha}$ such that $X(n,e) = X(n+1,f)$. For $n = 0$, let $F(0,e) = f$ where \(g \mid W_1^\alpha(f,g)\) = $e$; then $X(0,e) = e = \{g \mid W_1^\alpha(f,g)\} = X(1,f)$. Suppose that $F(n,e)$ has been defined for all $e$. Then let $F(n+1,e) = f$ where \(g \mid W_{n+2}^\alpha(f,g)\) = $\{F(n,h) \mid W_{n+1}^\alpha(e,h)\}$; clearly $f$ exists as $F$ restricted to pairs $(n,h)$, $h \in J_{\alpha}$, is $\Sigma_1(J_{\alpha})$ and therefore the latter set is $\Sigma_{n+1}(J_{\alpha})$ with parameter $e$. Finally, we get $X(n+2,f) = \{X(n+1,F(n,h)) \mid W_{n+1}^\alpha(e,h)\} = \{X(n,h) \mid W_{n+1}^\alpha(e,h)\}$ by induction, and the latter set is $X(n+1,e)$.

(b) $J_{\alpha,0} = J_{\alpha}$ is transitive by induction on $\alpha$, and if $x \in J_{\alpha,n+1}$ then $x \subseteq J_{\alpha,n}$ and hence $x \subseteq J_{\alpha,n+1}$ by (a).

(c) Clearly $\text{ORD}(J_{\alpha,n}) \leq \omega \alpha + n$ since $x \in J_{\alpha,n+1} \rightarrow x \subseteq J_{\alpha,n}$. By induction on $n$, define $e_{n+1}$ such that $X(n+1,e_{n+1}) = \omega \alpha + n$; for $n = 0$ we can take $e_1$ so that $\omega \alpha = \{f \mid W_1^\alpha(e_1,f)\}$. If $e_{n+1}$ is defined take $e_{n+2}$ so that \(\{f \mid W_{n+2}^\alpha(e_{n+2},f)\} = \{F(n,g) \mid W_{n+1}^\alpha(e_{n+1},g)\}\} \cup \{e_{n+1}\}$, where $F$ is from the proof of (a). Then $X(n+2,e_{n+2}) = X(n+1,e_{n+1}) \cup \{X(n+1,e_{n+1})\} = \omega \alpha + n + 1$.

(d) $J_{\alpha+1}$ is closed under pairing because all 2-element subsets of $J_{\alpha,n}$ belong to $J_{\alpha,n+1}$. For $\Sigma_0\text{-Comprehension}$ note that $J_{\alpha,n} = \{X(n,e) \mid e \in J_{\alpha}\} = X(n+1,f)$ for some $f$ so $J_{\alpha,n} \in J_{\alpha,n+1}$ and it suffices to show that if $X \subseteq J_{\alpha,n}$ is definable over $(J_{\alpha,n},e)$ then $X$ belongs to $J_{\alpha,m}$ for some $m$. But $\{e \mid X(n,e) \in X\}$ is a definable subset of $J_{\alpha}$ as $(J_{\alpha,n},e)$ with the additional function $(n,e) \rightarrow X(n,e)$ is isomorphic to a structure definable over $J_{\alpha}$. Choose $m$ so that this set is $\Sigma_m$-definable over $J_{\alpha}$ and using $F$ from the proof of (a), produce a $\Sigma_1(J_{\alpha})$ $G$ such that for each $e$, $X(n,e) = X(m,G(e))$. Then $\{G(e) \mid X(n,e) \in X\}$ is $\Sigma_m$-definable over $J_{\alpha}$ and $X = \{X(m,G(e)) \mid X(n,e) \in X\}$ belongs to $J_{\alpha,m+1}$.

(e) We get $\text{Def}(J_{\alpha}) \subseteq J_{\alpha+1}$ by (d). Conversely, if $X(n,e) \subseteq J_{\alpha}$ then $\{f \mid f \in X(n,e)\} = X(n,e)$ is a definable subset of $J_{\alpha}$, using the definition of $X(n,e)$. ■

Of course now we may define $W_{\alpha+1}^\alpha(e,x)$, using (d) of Lemma 2, thereby completing the definition of the $J$-hierarchy. It is occasionally convenient to refer to the refined hierarchy $(\tilde{J}_\alpha \mid \alpha \in \text{ORD})$ defined by $\tilde{J}_{\omega \alpha + n} = J_{\alpha,n}$ and conveniently: $\text{ORD}(\tilde{J}_\alpha) = \alpha$. 
Lemma 3. (a) $\langle \tilde{J}_\alpha \mid \alpha < \lambda \rangle$ is $\Sigma_1(\tilde{J}_\lambda)$ for limit $\lambda$, via a definition independent of $\lambda$.

(b) There is a $\Sigma_1(J_\alpha)$ well-ordering $<_\alpha$ of $J_\alpha$, via a definition independent of $\alpha$.

(c) (Condensation) If $\langle X, \varepsilon \rangle$ is $\Sigma_1$-elementary in $\langle J_\alpha, \varepsilon \rangle$ then $\langle X, \varepsilon \rangle \simeq \langle J_\alpha, \varepsilon \rangle$ for some $\tilde{\alpha}$.

Proof. (a) We have $x = \tilde{J}_\alpha \iff \exists x_\beta (\beta < \gamma) \text{ such that } x = x_\alpha$ where $x_\alpha = L_\alpha$ for finite $n < \gamma$, $x_\lambda = \bigcup \{x_\beta \mid \beta < \lambda\}$ for limit $\lambda < \gamma$ and for $\lambda + n < \gamma$, $\lambda$ limit, $x_{\lambda+n}$ is obtained from $x_\lambda$ as in the definition of $J_{\alpha,n}$ from $J_\alpha$, $\alpha > 0$. This definition works inside any $\tilde{J}_\lambda$, $\lambda$ limit.

(b) Define well-orderings $<_\alpha$ of $J_\alpha$ as follows: $<_0 = \emptyset$, $<_1$ is some $L_\omega$-definable well-ordering of $L_\omega$; $x <_\lambda y \iff x <_\alpha y$ for some $\alpha < \lambda$ for limit $\lambda$; $x <_{\alpha+1} y \iff x <_\alpha y$ or for some $n$, $y \in J_{\alpha,\alpha+1} - J_{\alpha,n}$ and either $x \in J_{\alpha,n}$ or ($<_\alpha$-least $e$ such that $X(n+1, e) = x$). Then $<_\alpha$ is $\Sigma_1(J_\alpha)$, via a definition independent of $\alpha$.

(c) Let $\langle X, \varepsilon \rangle \simeq \langle \tilde{X}, \tilde{\varepsilon} \rangle$ be the transitive collapse of $X$. Then $\langle \tilde{X}, \tilde{\varepsilon} \rangle \models \Sigma_0$-Comprehension + $\forall x \exists \beta (x \in \tilde{J}_\beta)$ + $\forall \beta \exists y (y = \tilde{J}_\beta)$. But $\Sigma_0$-Comprehension gives $(\tilde{J}_\beta)^\tilde{X} = \tilde{J}_\beta$ for $\beta \in X$ so that $\tilde{X} = \tilde{J}_{\omega\tilde{\alpha}} = J_\tilde{\alpha}$ where $\omega\tilde{\alpha} = \text{ORD}(\tilde{X})$. ■

$\Sigma_1$-Skolem functions. Condensation, as stated in Lemma 3(c), is a powerful tool for proving things about $L$. To unleash its power, we must first provide a method for generating $\Sigma_1$-elementary submodels. Fix an ordinal $\alpha > 0$.

Definition 1. Suppose $X \subseteq J_\alpha$. The $\Sigma_1$-hull of $X$ is the smallest $\Sigma_1$-elementary submodel of $J_\alpha$ containing $X$ as a subset. A $\Sigma_1$-Skolem function is a partial function $h : \omega \times J_\alpha \to J_\alpha$ with $\Sigma_1$ graph such that for any $X \subseteq J_\alpha$, $\Sigma_1$-hull of $X = \{h(n, x) \mid n \in \omega, x \text{ a finite sequence from } X\}$.

Lemma 4. For any $X \subseteq J_\alpha$, the $\Sigma_1$-hull of $X$ exists. Moreover, there is a $\Sigma_1$-Skolem function for $J_\alpha$, with a $\Sigma_1$-definition independent of $\alpha$.

Proof. Let $\varphi_0, \varphi_1, \ldots$ be a standard list of formulas of 2 free variables and define $h^*(n, x) = <_{\alpha}-\text{least pair } (y, t)$ such that $x, y \in t$, $t$ transitive, $(t, \varepsilon) \models \varphi_0(x, y)$; if no such pair $(y, t)$ exists then $h^*(n, x)$ is undefined. Then $h(n, x) = y$ when $h^*(n, x) = (y, t)$. Any $\Sigma_1$-elementary submodel of $J_\alpha$ must be closed under $h$, and clearly for any $X \subseteq J_\alpha$, $\{h(n, x) \mid x \text{ a finite sequence from } X\}$ is a $\Sigma_1$-elementary submodel of $J_\alpha$. ■

The key to Fine Structure Theory is to find a suitable generalization of Lemma 4 to higher levels of definability. We will take this up in the next section.
We close this section with an illustration of how $\Sigma_1$-hulls can be used to prove a version of Jensen’s $\diamond$-principle in $L$ (see Jensen [72]). Our version will include some technical conditions which are of use in our proof of Jensen’s Coding Theorem (see Friedman [94]). Assume $V = L$ and let $\alpha$ be an infinite cardinal.

**Definition 2.** $C \subseteq \alpha^+$ is closed unbounded (CUB) if $\bigcup C = \alpha^+$ and $\bigcup(C \cap \beta) \in C$ for each $\beta < \alpha^+$. $S \subseteq \alpha^+$ is stationary if $S \cap C \neq \emptyset$ for each CUB $C \subseteq \alpha^+$.

For $\mu < \alpha^+, \beta(\mu)$ denotes the largest $\beta$ such that either $\beta = \mu$ or $\mu < \beta, J_\beta \models \mu$ is a cardinal greater than $\alpha$.

**Lemma 5.** There exists $\langle D_\mu \mid \mu < \alpha^+ \rangle$ such that $D_\mu \subseteq J_\mu$ and:

(a) If $D \subseteq J_{\alpha^+}$ then $\{ \mu < \alpha^+ \mid D \cap J_\mu = D_\mu \}$ is stationary.
(b) $D_\mu$ is uniformly definable as an element of $J_{\beta'(\mu)}$, for $\mu < \alpha^+$.
(c) If $J_{\beta'(\mu)} \models \alpha^{++}$ exists or $\mu = \beta'(\mu)$ then $D_\mu = \emptyset$.

**Proof.** Let $D_\mu = \emptyset$ if $J_{\beta'(\mu)} \models \alpha^{++}$ exists or $\mu = \beta'(\mu)$ and otherwise let $\langle D_\mu, C_\mu \rangle$ be least in $J_{\beta'(\mu)}$ such that $C_\mu$ is CUB in $\mu$, $D_\mu \subseteq J_\mu$ and $\mu \in C_\mu \rightarrow D_\mu \cap J_\mu \neq D_\mu$; if $\langle D_\mu, C_\mu \rangle$ does not exist then let $D_\mu = \emptyset$. We need only prove (a).

Suppose (a) fails and let $\langle D, C \rangle$ be least in $J_{\alpha^+}$ such that $D \subseteq J_{\alpha^+}$, $C$ is CUB in $\alpha^+$ and $\mu \in C \rightarrow D \cap J_\mu \neq D_\mu$. Let $\sigma < \alpha^{++}$ be least such that $\omega \sigma = \sigma, J_\sigma \models \alpha^+$ is the largest cardinal and $\langle D, C \rangle \in J_\sigma$. Let $H = \Sigma_1$-hull of $\alpha \cup \{ \alpha^+ \}$ in $J_\sigma$ and $\mu = H \cap \alpha^+$. Then $\langle H, \in \rangle \simeq \langle J_{\beta'}, \in \rangle$ for some $\beta'$ and since $J_{\beta'} \models \mu = \alpha^+$ we have $\beta' \leq \beta'(\mu)$. But now we have $\langle D_\mu, C_\mu \rangle = \langle D \cap J_\mu, C \cap \mu \rangle$ and since $\mu = \bigcup(C \cap \mu) \in C$, this is a contradiction. ■

2. **Fine Structure Theory.** Our main goal is to develop a version of Lemma 4 for higher levels of definability. Specifically, we want to define the notion of $\Sigma_n^*$ formula so as to obtain:

(a) There is a universal $\Sigma_n^*$ predicate for $J_\alpha$ for each $n$.
(b) For any $X \subseteq J_\alpha$, the $\Sigma_n^*$-hull of $X$ in $J_\alpha$ exists for each $n$.
(c) There is a $\Sigma_n^*$-Skolem function for $J_\alpha$ for each $n$.
(d) Every formula is $\Sigma_n^*$ for some $n$.

What happens if we just take $\Sigma_n^* = \Sigma_n$? Then (a) holds by Lemma 1 and (d) is clear.

**Proposition 1.** For any $X \subseteq J_\alpha$ and $n \in \omega$ there is a least $\Sigma_n$-elementary submodel of $J_\alpha$ containing $X$ as a subset.

**Proof.** Let $M = \{ y \in J_\alpha \mid$ For some $\Sigma_n$ formula $\varphi$ with parameters from $X$, $y$ is the $<_\alpha$-least solution to $\varphi$ in $J_\alpha \}$. Then $M$ is $\Sigma_n$-elementary in
\( J_\alpha \) since if \( y_i \) is the \(<_\alpha\)-least solution to \( \varphi_i \), \( 1 \leq i \leq n \), then \( (y_1 \ldots y_n) \) is the \(<_\alpha\)-least solution to \( \varphi_1((z)_1) \land \ldots \land \varphi_n((z)_n) \), where \( (z)_i \) = \( i \)th component of \( z \). Suppose \( X \subseteq N \), \( N \) is \( \Sigma_n \)-elementary in \( J_\alpha \) and \( \varphi \) is a \( \Sigma_n \) formula with parameters from \( X \) with a solution in \( J_\alpha \). Then \( \varphi \) has a solution \( y_0 \) in \( N \) and if \( y_0 \) is not the least solution then \( N \) also has a solution \( y_1 <_\alpha y_0 \). Continuing in this way we see that in fact \( N \) does contain the \(<_\alpha\)-least solution to \( \varphi \) and hence we get \( M \subseteq N \).

So (b) holds. What fails is property (c):

**Proposition 2.** For some \( \alpha \) there is no \( \Sigma_2 \)-Skolem function for \( J_\alpha \).

**Proof.** Let \( \kappa \) denote \( \omega_1 \). For each limit \( \alpha < \omega_1 \), \( \alpha \) is the least \( \beta \) such that 
\[ \tilde{J}_{\kappa+\alpha} \models \kappa + \beta \] 

does not exist. If \( \tilde{J}_{\kappa+\alpha} \) has a \( \Sigma_2 \)-Skolem function then \( \alpha \) must be the unique solution in \( \tilde{J}_{\kappa+\alpha} \) to a \( \Sigma_2 \) formula \( \exists x \forall y \varphi_\alpha \) where \( \varphi_\alpha \) is \( \Sigma_0 \) with parameter \( \kappa \). Suppose that each \( J_{\kappa+\alpha} \) has a \( \Sigma_2 \)-Skolem function and by Fodor’s Theorem choose \( \varphi \) and \( \alpha_0 < \kappa \) such that for stationary-many \( \alpha \), \( \varphi_\alpha = \varphi \) and 
\[ \tilde{J}_{\kappa+\alpha} \models \exists x \in \tilde{J}_{\kappa+\alpha} \forall y \varphi \] 

holds at \( \alpha \). But then choose any \( \alpha < \beta \) in this stationary set, \( \alpha_0 < \alpha \) and we have \( \tilde{J}_{\kappa+\alpha} \models \exists x \forall y \varphi \) holds at both \( \alpha \) and \( \beta \). Contradiction.

**Remark.** A result similar to the previous appears in Devlin [84], pages 106–107.

It is shown in Jensen [72] that for any \( \alpha \) and any \( n \) there is a partial \( \Sigma_n \) function with parameters that can serve as a \( \Sigma_n \)-Skolem function for \( \Sigma_n \)-hulls without parameters. However, this does not achieve our goal as the definition of the necessary parameters does not reflect to arbitrary \( \Sigma_n \)-elementary submodels that contain them.

Instead we take an approach based on the idea that in a certain sense \( \Sigma_{n+1} \) can be viewed as \( \Sigma_1 \) relativized to \( \Sigma_n \), for an arbitrary \( J_\alpha \). Though this is only true for the usual Lévy hierarchy when awkward parameters are introduced, we define \( \Sigma^*_n \) in such a way that this is true using only “standard” parameters, whose definitions relativize without difficulty to \( \Sigma^*_n \)-hulls. Our approach is derived from Jensen’s \( \Sigma^* \) Theory in Jensen [7]. \( \Sigma^*_n \) in our sense corresponds to \( \Sigma^*_1 \) in Jensen’s terminology.

**The \( \Sigma^*_n \)-hierarchy.** In order to define the notion of \( \Sigma^*_n \) formula we must also define the auxiliary notions of \( n \)th reduct and \( n \)th standard parameter, all by induction on \( n \).

Let \( M \) denote some fixed \( J_\alpha \), \( \alpha > 0 \). We order finite sets of ordinals by the maximum difference order: \( x < y \) iff \( \beta \in y \), where \( \beta \) is the largest element of \( (y-x) \cup (x-y) \).

A \( \Sigma^*_1 \) formula is just a \( \Sigma_1 \) formula. The \( \Sigma^*_1 \) projectum of \( M \), denoted by \( \varrho^M \), is the least \( \varrho \) such that there is a subset of \( \omega \varrho \) which is \( \Sigma^*_1 \) with
parameters but not an element of $M$. The 1st standard parameter of $M$, denoted by $p_1^M$, is the least finite set of ordinals $p$ such that $A \cap \omega_{\mathcal{P}}^M \not\subseteq M$ for some $A$ which is $\Sigma_1^\omega$ with parameter $p$. We use $H_1^M$ to denote $J_{\omega_1^M}$ and for any $x \in M$, $A_1(x) = \{(y, m) \mid \text{the } m\text{th } \Sigma_1^\omega \text{ formula is true at } \langle y, x, p_1^M \rangle, y \in H_1^M\}$. The 1st reduct of $M$ relative to $x$, denoted by $M_1(x)$, is the structure $\langle H_{1}^M, A_1(x) \rangle$.

For $n \geq 1$: a $\Sigma_{n+1}^*$ formula is one of the form $\varphi(x) \iff M_n(x) \models \psi$, where $\psi$ is $\Sigma_1$. The $\Sigma_{n+1}^*$ projectum of $M$, denoted by $g_{n+1}^M$, is the least $\varphi$ such that there is a subset of $\omega g$ which is $\Sigma_{n+1}^*$ with parameters but not an element of $M$. The $(n+1)$st standard parameter of $M$, denoted by $p_{n+1}^M$, is $p_n^M \cup p$ where $p$ is the least finite set of ordinals such that $A \cap \omega g_{n+1}^M \not\subseteq M$ for some $A$ which is $\Sigma_{n+1}^*$ with parameter $p_n^M \cup p$. We use $H_{n+1}^M$ to denote $H_{\omega_{n+1}^M}$ and for any $x \in M$, $A_{n+1}(x) = \{(y, m) \mid \text{the } n\text{th } \Sigma_{n+1}^* \text{ formula is true at } \langle y, x, p_{n+1}^M \rangle, y \in H_{n+1}^M\}$. The $(n+1)$st reduct of $M$ relative to $x$, denoted by $M_{n+1}(x)$, is the structure $\langle H_{n+1}^M, A_{n+1}(x) \rangle$.

This completes the definition of the $\Sigma_n^*$-hierarchy. Thus a $\Sigma_{n+1}^*$ formula is a formula expressing a $\Sigma_1$ property on $n$th reducts, uniformly. In order to achieve amenability when relativizing to a $\Sigma_n^*$ predicate, we take our $n$th reduct to have ordinal height $\omega g_{n}^M$.

**Lemma 6.** (a) If $\varphi$ and $\psi$ are $\Sigma_n^*$ formulas then $\varphi \lor \psi$ and $\varphi \land \psi$ are equivalent to $\Sigma_n^*$ formulas.

(b) If $\varphi$ is a $\Sigma_n^*$ formula then both $\varphi$ and $\neg \varphi$ are equivalent to $\Sigma_{n+1}^*$ formulas.

(c) There is a universal $\Sigma_n^*$ formula, i.e., a $\Sigma_n^*$ formula $\varphi(e, x)$ such that $\psi(x)$ is $\Sigma_n^*$ then for some $e \in \omega$, $\varphi(e, x) \iff \varphi(e, x)$ for all $x$.

(d) The reduct $M_n(x) = \langle H_{n}^M, A_n(x) \rangle$ is amenable, i.e., if $y \in H_{n+1}^M$ then $y \cap A_n(x) \in H_{n+1}^M$.

**Proof.** (a) is clear because a $\Sigma_{n+1}^*$ formula is of the form $\varphi(x) \iff M_n(x) \models \psi$, $\psi$ $\Sigma_1$ and $\Sigma_1$ is closed under $\lor$ and $\land$.

(b) If $\varphi(x)$ is $\Sigma_n^*$ then so is $\varphi'(y, x, z) \iff \varphi(x)$ and choose $k$ so that $\varphi'$ is the $k$th $\Sigma_n^*$ formula. Then $\varphi(x) \iff \langle \emptyset, k \rangle \in A_n(x)$ so $\varphi$ is equivalent to a $\Sigma_{n+1}^*$ formula. Similarly for $\neg \varphi$ since $\neg \varphi(x) \iff \langle \emptyset, k \rangle \not\in A_n(x)$.

(c) If $\psi$ is a universal $\Sigma_1$ formula then $\varphi(k, x) \iff \langle H_n^M, A_n(x) \rangle \models \psi(k, \emptyset) \iff \langle H_n^M, A_n(x) \rangle \models \psi$ is a universal $\Sigma_{n+1}^*$ formula (where $\psi$ is $\Sigma_1$ and chosen to satisfy the last $\iff$).

(d) By (c) we see that $A_n(x)$ is $\Sigma_n^*$ (with parameter $p_n^M$) and hence $A_n(x) \cap y \in M$ for each $y \in H_n^M$. But either $H_n^M = M$ or $\omega g_n^M$ is a cardinal of $M$. Using Proposition 1 and condensation, we find that if $\kappa$ is an $M$-cardinal then every bounded subset of $\kappa$ in $M$ actually belongs to $J_{\kappa}^\beta$: if $x \subseteq \gamma < \kappa$ and $x$ is $\Sigma_n$-definable with parameter $p$ over $M'$, a proper initial segment of $M$, then let $H = \Sigma_n$-Skolem hull of $\gamma \cup \{p\}$ in $M'$. Then $H \simeq J_{\beta}$.
where $\beta$ is less than $\kappa$, since in $M$ the cardinality of $H$ is at most $\gamma$ (we may assume $\omega \leq \gamma < \kappa$). But $x$ is definable over $J_\beta$, so $x \in J_{\beta+1} \subseteq J_\kappa$. ■

As promised, we have the following analogue of Lemma 4, in the $\Sigma^*$ context.

**Lemma 7.** For any $X \subseteq J_\alpha$, the $\Sigma^*_n$-hull of $X$ exists. Moreover, there is a $\Sigma^*_n$-Skolem function for $J_\alpha$, via a $\Sigma^*_n$-definition independent of $\alpha$.

**Proof.** By induction on $n$. The base case $n = 1$ is Lemma 4. Suppose the result holds for $n \geq 1$ and we establish it for $n + 1$. Let $h_n(k, x)$ be a $\Sigma^*_n$-Skolem function for $J_\alpha$.

Lemma 1 holds uniformly for amenable structures so we may define a partial $\Sigma^*_{n+1}$ function $h(k, x)$ such that for each $x$, $H(x) = \{h(k, x) \mid k \in \omega, h(k, x) \text{ defined}\}$ is a $\Sigma_1$-elementary submodel of $M_n(x) = (H^M_n, A_n(x))$. Define

$$h_{n+1}(k, x) = h_n((k)_0, \langle h((k)_1, x), p^M_n \rangle)$$

where $\langle (k)_0, (k)_1 \rangle$ is a pairing function on $\omega$. Now $\text{graph}(h_{n+1})$ is a $\Sigma^*_{n+1}$ relation because $h_{n+1}(k, x) = y \leftrightarrow \exists z \in H^M_n (y = h_n((k)_0, \langle z, p^M_n \rangle) \land z = h((k)_1, x))$ and as $\text{graph}(h_n)$ is $\Sigma^*_n$, $\text{graph}(h)$ is $\Sigma^*_{n+1}$ this yields a $\Sigma^*_n$-definition of $\text{graph}(h_{n+1})$. If $H$ is a $\Sigma^*_n$-elementary submodel of $M$ then $H$ is closed under $h_{n+1}$, since it is closed under $h_n$ by induction, is closed under $h$ by $\Sigma^*_{n+1}$-elementarity and must contain $p^M_n$ since "$x = p^M_n$" is a $\Sigma^*_{n+1}$ formula.

It remains to show that $H = \{h_{n+1}(k, x) \mid k \in \omega\}$ is $\Sigma^*_n$-elementary in $M$. (It then follows that for any $X \subseteq M$, $\{h_{n+1}(k, x) \mid x \text{ a finite sequence from } X\}$ is $\Sigma^*_n$-elementary in $M$.) As $H$ is $\Sigma_1$-elementary in $M$ we know that $H$ satisfies extensionality so we may take the transitive collapse $\pi : \overline{M} \simeq H \subseteq M$. It will suffice to show that $\pi^{-1}[H \cap M_n(\pi(x))] = \overline{M}_n(\pi)$ for each $\pi \in \overline{M}$, for then the closure of $H$ under $h$ guarantees $\Sigma^*_n$-elementarity.

Now $M_n(\pi(x)) = \langle H^M_n, A_n(\pi(x)) \rangle$ and $H^n_n = J_{\omega^\alpha^n}, A_n(\pi(x)) = \{\langle y, m \rangle \mid y \in H^M_n\}$ so since by induction we have $\Sigma^*_n$-elementarity, it is enough to show

$$\pi^{-1}[\overline{M}^n_n] = \overline{\emptyset}_n^n, \quad \pi^{-1}(p^M_n) = p^n_n.$$ 

Let $\overline{\emptyset} = \pi^{-1}[\emptyset^n_n]$. Suppose $\overline{\emptyset} \subseteq J_{\overline{\emptyset}}$ is $\Sigma^*_n$-definable in $\overline{M}$ with parameter $\overline{\varphi}$. For $\overline{\varphi} < \overline{\emptyset}$ we have $\overline{A} \cap J_{\overline{\varphi}} \subseteq \overline{M}$ by $\Sigma_1$-elementarity of $\pi$ from $\overline{M}_n(\overline{\varphi})$ to $M_n(\pi(\overline{\varphi}))$. Note that if $\overline{\varphi} = \pi^{-1}(p^M_n)$ then every $\overline{\tau} \in \overline{M}$ is of the form $h_n(k, \langle \tau, p \rangle), \tau \in J_{\overline{\varphi}}$, so the set $\{\langle k, \tau \rangle \mid k \in \omega, \tau \in J_{\overline{\varphi}}, h_n(k, \langle \tau, p \rangle) \text{ defined,} (k, \tau) \notin h_n(k, \langle \tau, p \rangle)\}$ is $\Sigma^*_n$-definable in $\overline{M}$ with parameters and does not belong to $\overline{M}$. So $\overline{\varphi} = \overline{\emptyset}^n_n$ and $\overline{\emptyset} \geq p^n_n$.

Finally, we show that $\overline{\varphi} \leq p^n_n$. Let $\overline{H} = \Sigma^*_n$-hull of $\{\overline{\varphi} \mid \overline{\varphi} < \overline{\emptyset}\}$. We may assume that $\overline{\varphi} \neq \emptyset$ and therefore $\overline{\varphi} \subseteq \overline{H}$. Now if $\overline{H} \simeq \overline{M}$ then we get
Our next lemma helps to clarify the meaning of the standard parameters, as well as the relationship between $\Sigma^*_n$ and $\Sigma_n$.

**Lemma 8.** Let $H = \Sigma^*_n$-Skolem hull of $p_n^M \cup \{p_n^M\}$ in $M$. Then $H = M$.

**Proof.** Let $\pi : H \simeq \bar{M}$. Then $\bar{M} = M$ as $A \cap H_n^M$ is definable over $\bar{M}$ whenever $A$ is $\Sigma^*_n$-definable in $M$ with parameter $p_n^M$. So $M = \Sigma^*_n$-Skolem hull of $g_n^M \cup \{\pi(p_n^M)\}$. But we must have $\pi(p_n^M) = p_n^M$, else $\pi(p_n^M) < p_n^M$ contradicts the definition of $p_n^M$. ■

**Corollary 1.** For each $n$, $\Sigma_n \subseteq \Sigma^*_n$ and for $m < n$, $\Sigma^*_n$ is closed under existential quantification over $H_n^M$.

**Proof.** We can assume $m = 0$ as $\exists x \in H_n^M$ is a $\Sigma^*_{m+1}$ formula. By induction on $n$: Assume that we have an effective translation of $\Sigma_n$ formulas into $\Sigma^*_n$ formulas; then if $\varphi$ is $\exists x \psi(x)$ where $\psi$ is $\Pi_n$ we can write $\exists x \psi(x) \leftrightarrow \exists \pi \in H_n^M \exists k \psi(h_n(k,<\pi,p_n^M)))$ and after translating $\psi$ into a $\Pi^*_n$ formula, this gives a $\Sigma^*_n$-translation of $\varphi$. ■

**Remark.** With some effort, it can be shown that conversely, each $\Sigma^*_n$ formula is equivalent to a $\Sigma_n$ formula with parameters. But we will have no use for this fact.

It will be useful to have approximations to the $\Sigma^*_n$-hulls and $\Sigma^*_n$-Skolem functions. For $n = 1$ and limit $\sigma < \omega \alpha = \text{ORD}(M)$ we let $h^*_\sigma(k,x)$ be defined by restricting the $\Sigma_1$ definition of $h_1$ to $\tilde{J}_\sigma$: if $h_1(k,x) = y \leftrightarrow \exists z \varphi(x,y,z)$ where $\varphi$ is $\Sigma_0$ then $h^*_\sigma(k,x) = y \leftrightarrow \exists z \in \tilde{J}_\sigma \varphi(x,y,z)$. For any $n \geq 1$ and $\sigma < \omega g^M_n$ we define $h^*_\sigma(k,x) = h_n((k)_0,\langle h^*(k)_1,x \rangle, p_n^M)$, where $h^*$ is defined by restricting the $\Sigma^*_n$-definition of $h$ (from the proof of Lemma 7) to $\tilde{J}_\sigma$: if $h(k,x) = y \leftrightarrow M_n(k,x,y) \models \exists z \varphi$ where $\varphi$ is $\Sigma_0$ then $h^*(k,x) = y \leftrightarrow M_n(k,x,y) \models \exists z \in \tilde{J}_\sigma \varphi$. Also let $\Sigma^*_n|_\sigma$-hull($X$) denote $\{h^*_\sigma(k,x) \mid x$ a finite sequence from $X\}$.

**Lemma 9.** For any $X \subseteq J_\alpha$, $1 \leq n \in \omega$, and every limit $\sigma < \omega g^M_{n+1}$, $\Sigma_n|_\sigma$-hull($X$) is $\Sigma^*_n$-elementary in $M$.

**Proof.** It suffices to show that the hull in question is closed under $h_n$. This follows from the facts that

\(\{h^*_\sigma(k,x) \mid x\text{ a finite sequence from } X\}\)

is closed under pairing and that $\{h_n(k,\langle y,p \rangle) \mid y\text{ a finite sequence from } Y\}$ is closed under $h_n$ for any $Y \subseteq M$, $p \in M$. ■
The following fact about hull approximation is very useful.

Lemma 10. Suppose \( X \subseteq J_\alpha, \omega < \beta \leq \mathcal{g}_{n-1}^M \), \( \beta \) is a regular \( M \)-cardinal and \( \beta \in \Sigma_n^*\text{-hull}(X) \) in \( M \). Let

\[
\tilde{\beta} = \bigcup(\Sigma_n^*\text{-hull}(X) \cap \beta) \quad \text{and} \quad \sigma = \bigcup(\Sigma_n^*\text{-hull}(X) \cap \mathcal{g}_{n-1}^M).
\]

Then \( \tilde{\beta} = \beta \cap \Sigma_n^*\sigma\text{-hull}(X \cup \beta) \) and if \( n \geq 2 \) then for any \( \tilde{\beta} < \beta \), and \( x \) a finite sequence from \( X \), \( \beta \cap \Sigma_n^*-\text{hull}(\{x\} \cup \tilde{\beta}) \) is bounded strictly below \( \tilde{\beta} \).

Proof. Suppose that \( \gamma \in \beta \cap \Sigma_n^*\sigma\text{-hull}(X \cup \beta) \). Then there exists a finite sequence \( x \) from \( X \) such that \( \gamma \in \Sigma_n^*\sigma\text{-hull}(\{x\} \cup \tilde{\beta}) \) where \( \sigma, \tilde{\beta} \in \Sigma_n^*\text{-hull}(\{x\}) \), \( \sigma < \sigma, \tilde{\beta} < \tilde{\beta} \). But \( \Sigma_n^*\sigma\text{-hull}(\{x\} \cup \tilde{\beta}) \cap \beta \) belongs to \( \Sigma_n^*\text{-hull}(X) \) and hence so does its supremum \( \delta \). As \( \beta \) is regular in \( M \), we have \( \delta < \beta \) and therefore \( \delta < \tilde{\beta} \). Since \( \gamma < \delta \) we get \( \gamma < \tilde{\beta} \), as desired. The second conclusion of the lemma also follows, by Lemma 9.

The Square Principle. An important application of fine structure theory is to Jensen’s Square Principle, which we now establish using the \( \Sigma^* \) approach.

Square. Assume \( V = L \). Then there is \( \langle C_\mu \mid \mu \text{ a singular limit ordinal} \rangle \) such that

(a) \( C_\mu \) is closed unbounded in \( \mu \).
(b) ordtype(\( C_\mu \)) < \( \mu \).
(c) \( \bar{\mu} \in \text{Lim} C_\mu \rightarrow \bar{\mu} \) is singular and \( C_{\bar{\mu}} = C_\mu \cap \bar{\mu} \).
(d) \( \langle \bar{J}_\mu, C_\mu \rangle \) is amenable and if \( \langle \bar{J}_{\bar{\mu}}, \bar{C} \rangle \rightarrow \langle \bar{J}_\mu, C_\mu \rangle \) is \( \Sigma_1 \)-elementary then \( \bar{C} = C_{\bar{\mu}} \).

We refer the reader to Jensen [72] for background on and applications of the Square Principle.

Let \( \mu \) be a singular limit ordinal. We wish to define \( C_\mu \). Let \( \beta(\mu) \geq \mu \) be the least limit ordinal \( \beta \) such that \( \mu \) is not regular with respect to \( \bar{J}_\beta \) - definable functions and let \( n(\mu) \) be least such that there is a \( \Sigma^*_n(\bar{J}_\beta(\mu)) \) partial function (with parameters) from an ordinal less than \( \mu \) cofinally into \( \mu \). Note that \( \omega \mathcal{g}_{\alpha(\mu)}^\beta \leq \mu \) (where \( \mathcal{g}_\alpha^\beta_n \) denotes \( \mathcal{g}_\alpha^\beta \), \( N = \bar{J}_\beta \)) as otherwise such a partial function would belong to \( \bar{J}_\beta(\mu) \), contradicting the leastness of \( \beta(\mu) \). Also \( \mu \leq \omega \mathcal{g}_{\beta(\mu)}^\beta \), else by Lemma 8 we have contradicted the leastness of \( n(\mu) \).

For \( X \subseteq \bar{J}_\beta(\mu) \) let \( H(X) \) denote \( \Sigma^*_n(\mu)\text{-hull}(X) \) in \( \bar{J}_\beta(\mu) \). For some least parameter \( q(\mu) \in \bar{J}_\beta(\mu), H(\mu \cup \{q(\mu)\}) = \bar{J}_\beta(\mu) \). (Actually, \( q(\mu) = \mathcal{g}_{n(\mu)}^\beta - \mu - \mathcal{g}_{n(\mu)-1}^\beta \).) Also let \( \alpha(\mu) = \bigcup \{ \alpha < \mu \mid \alpha = H(\alpha \cup \{q(\mu)\}) \cap \mu \} \). Then \( \alpha(\mu) < \mu \) and (unless \( \alpha(\mu) = \bigcup \emptyset = \emptyset \) \( \alpha(\mu) = H(\alpha(\mu) \cup \{q(\mu)\}) \cap \mu \). The
former is because for large enough \( \alpha < \mu \), \( H(\alpha \cup \{ q(\mu) \}) \) contains both the domain and defining parameter for a \( \Sigma^n_{\alpha} \) partial function from an ordinal less than \( \mu \) cofinally into \( \mu \).

If \( \mu < \beta(\mu) \) let \( p(\mu) = \{ q(\mu), \mu \} \) and if \( \mu = \beta(\mu) \) let \( p(\mu) = \emptyset \).

We are ready to define \( C_\mu \). Let \( C^0_\mu = \{ \overline{\mu} < \mu \mid \text{For some } \alpha \geq \alpha(\mu), \overline{\mu} = \bigcup(H(\alpha \cup \{ p(\mu) \}) \cap \mu) \} \). Then \( C^0_\mu \) is unbounded in \( \mu \) then let \( C_\mu = C^0_\mu \). If \( C^0_\mu \) is bounded but nonempty then let \( \mu_0 = \bigcup C^0_\mu \) and define \( C^1_\mu = \{ \overline{\mu} < \mu \mid \text{For some } \alpha, \overline{\mu} = \bigcup(H(\alpha \cup \{ p(\mu) \}) \cap \mu) \} \). If \( C^1_\mu \) is unbounded then let \( C_\mu = C^1_\mu \). If \( C_\mu \) is bounded but nonempty then let \( \mu_1 = \bigcup C^1_\mu \) and define \( C^2_\mu = \{ \overline{\mu} < \mu \mid \text{For some } \alpha, \overline{\mu} = \bigcup(H(\alpha \cup \{ p(\mu) \}) \cap \mu) \} \). Continue in this way, defining \( C^k_\mu \) for \( k \in \omega \) until \( C^k_\mu \) is unbounded or empty for some least \( k = k(\mu) \). To see that \( k(\mu) \) exists, note that \( \alpha_0 > \alpha_1 > \ldots \) where \( \alpha_k \) is greatest such that \( \bigcup(H(\alpha_k \cup \{ p(\mu), \mu_0, \ldots, \mu_{k-1} \}) \cap \mu \) contains no ordinal \( \geq \mu_k \) we get \( \alpha_k \in H(\{ p(\mu), \mu_0, \ldots, \mu_k \}) \cap \mu \); so \( H(\alpha_k \cup \{ p(\mu), \mu_0, \ldots, \mu_k \}) \cap \mu \supseteq H(\alpha_k + 1 \cup \{ p(\mu), \mu_0, \ldots, \mu_{k-1} \}) \cap \mu \), which by definition of \( \mu_k \) is unbounded in \( \mu \); hence \( \alpha_{k+1} < \alpha_k \).

If \( C^{k(\mu)}_\mu \) is unbounded in \( \mu \) then let \( C_\mu = C^{k(\mu)}_\mu \). If \( C^{k(\mu)}_\mu = \emptyset \) then

\[
H(\{ p(\mu), \mu_0, \ldots, \mu_{k(\mu)-1} \}) \cap \mu
\]

is unbounded in \( \mu \). And \( H(\{ p(\mu), \mu_0, \ldots, \mu_{k(\mu)-1} \}) \cap \mu \omega_{\omega_{\omega_{\alpha(\mu)}}}^{\beta(\mu)} \) is unbounded in \( \omega_{\omega_{\omega_{\alpha(\mu)}}}^{\beta(\mu)} \) else this set belongs to \( J_{\beta(\mu)} \) and \( \mu \) is singular inside \( J_{\beta(\mu)} \), contradicting leastness of \( \beta(\mu) \). Let \( g(\mu) = \omega_{\omega_{\omega_{\alpha(\mu)}}}^{\beta(\mu)} \cdot \bigcup, \alpha(\mu) = \{ p(\mu), \mu_0, \ldots, \mu_{k(\mu)-1} \} \) and \( h_n(k, \sigma) \) the \( \Sigma^* \)-Skolem function for \( J_{\beta(\mu)} \). Also let \( \sigma_m = max\{ \{ h_n(k, \sigma) \mid k < m \} \cap \mu \} \) and \( \sigma_m = max\{ \{ h_n(k, \sigma) \mid k < m \} \cap \mu \cap \mu \} \). We define \( C_\mu = \{ \delta_0, \delta_1, \ldots \} \) where \( \delta_0 \) is the ordertype of the transitive collapse of \( \Sigma^* \)-ordered collapse \( (\sigma_m, \text{ordtype}(\sigma_m, \{ p \})) \). Note that \( \delta_0 < \mu \) as \( \mu \) is regular inside \( J_{\beta(\mu)} \) and \( \text{ordtype}(\delta_0) \leq \sigma_m \) in \( J_{\beta(\mu)} \).

This completes the definition of \( C_\mu \). Clearly \( C_\mu \) is closed unbounded in \( \mu \). The argument that \( \alpha(\mu) < \mu \) also implies that \( \text{ordtype}(C_\mu) < \mu \). So we need only show (c), (d) from the statement of Square.

**Lemma 11.** \( \overline{\mu} \in C^k_\mu \rightarrow C^k_\mu = C^k_\mu \cap \overline{\mu} \).

**Proof.** First suppose that \( k = 0 \). Given \( \overline{\mu} \in C^0_\mu \) choose \( \alpha < \overline{\mu} \) such that \( \overline{\mu} = \bigcup(H(\alpha \cup \{ p(\mu) \}) \cap \mu) \), where \( H(X) = \Sigma^* \)-hull(X) in \( J_{\beta(\mu)} \). Also let \( g = \bigcup(H(\alpha \cup \{ p(\mu) \}) \cap \mu \omega_{\omega_{\alpha(\mu)}}^{\beta(\mu)}) \). Let \( H = \Sigma^* \cdot \omega_{\omega_{\alpha(\mu)}}^{\beta(\mu)} \cdot h(\mu \cup \{ p(\mu) \}) \) and \( \overline{\mu} \models \gamma(\mu) \). By Lemma 10, \( H \cap \mu = \overline{\mu} \) and therefore when \( \mu < \beta(\mu) \), \( \gamma(\overline{\mu}) = \mu \). By Lemma 9, \( \overline{\mu} \models \gamma(\mu \rightarrow \beta(\mu) = \beta(\mu) \). By the second
conclusion of Lemma 10 we get \( n(\pi) > n(\mu) - 1 \), so \( n(\pi) = n(\mu) \). Thus to conclude that \( C^0_{\mu} = C^0_{\mu} \cap \pi \) we need only check that \( \pi(q(\pi)) = q(\mu) \) when \( \mu < \beta(\mu) \), and \( \alpha(\pi) = \alpha(\mu) \).

For the former, first note that \( \mu \in H(\{\mu'\} \cup \{q(\mu)\}) \) for some \( \mu' < \mu \), \( \mu' \in H(\{p(\mu)\}) \), since \( \mu, q(\mu) \in H(\{p(\mu)\} \cap H(\mu \cup \{q(\mu)\})) \). So in fact \( \mu \) and \( \alpha(\mu) \) belong to \( \Sigma^*_n(\mu) \) \( \varrho \)-hull(\( \pi \cup \{q(\mu)\} \)) and hence the latter is just \( H \). Now let \( \pi = \pi^{-1}(q(\mu)) \). We see that \( \Sigma_n^*(\pi) \)-hull(\( \pi \cup \{\pi(\mu)\} \)) = \( J_{\beta(\mu)} \) and hence \( \pi \geq q(\pi) \). But \( \pi \in \Sigma_n^*(\pi) \)-hull(\( \pi \cup \{\pi(q(\pi))\} \)) in \( J_{\beta(\mu)} \), hence \( q(\pi) \geq \pi \), else we have contradicted the definition of \( q(\mu) \). So \( \pi(q(\pi)) = q(\mu) \). Now since \( \alpha(\mu) < \pi \) we get \( \alpha(\mu) \leq \alpha(\pi) \). Conversely, \( \alpha(\pi) < \alpha \) where \( \pi = \bigcup(H(\alpha \cup \{p(\mu)\}) \cap \mu) \) so \( H(\alpha(\pi) \cup \{p(\mu)\}) \cap \mu = \alpha(\pi) \) and we get \( \alpha(\pi) \leq \alpha(\mu) \). So \( \alpha(\pi) = \alpha(\mu) \).

Now suppose \( k = 1 \). The above argument shows that \( \pi \in C^1_{\mu} \to C^0_{\mu} = C^0_{\mu} \cap \pi \) and hence, since \( \mu_0 = \pi \), we get \( \pi_0 = \mu_0 \). Then the above argument shows that \( C^0_{\mu} = C^1_{\mu} \cap \pi \). The general case \( k \geq 0 \) follows similarly.

To verify (c) in the statement of Square: if \( \pi \in \text{Lim} C_\mu \) then we must have \( C_\mu = C^k_\mu \) for some \( k \) and so \( C^k_{\pi} = C_{\mu} \cap \pi \) is unbounded in \( \pi \). Hence \( C_{\pi} = C^{k}_{\mu} = C_{\mu} \cap \pi \) as desired. Now we verify (d).

**Lemma 12.** (a) \( A \subseteq \tilde{J}_\mu \), \( A \in \tilde{J}_{\beta(\mu)} \) implies \( A \in \Delta_1(\tilde{J}_\mu, C_\mu) \).

(b) Suppose \( \pi : (\tilde{J}_\mu, C_\mu) \to (\tilde{J}_\beta, C_\mu) \) is \( \Sigma_1 \)-elementary. Then \( C_\mu = C_{\beta} \) and \( \pi \) extends uniquely to a \( \Sigma^*_n(\mu) \)-elementary \( \tilde{\pi} : \tilde{J}_{\beta(\mu)} \to \tilde{J}_{\beta(\mu)} \) such that \( p(\mu) \in \text{Range}(\tilde{\pi}) \).

**Proof.** First suppose that \( C_{\mu} = C^k_{\mu} \) for some \( k \). For \( \mu' \in C_\mu \) form \( H(\mu') \) as \( H \) was formed in the proof of Lemma 11 for \( \pi \). Then \( \pi(\mu') : \tilde{J}_{\beta(\mu')} \to \tilde{J}_{\beta(\mu)} \), with range \( H(\mu') \), is \( \Sigma^*_n(\mu') \)-elementary and \( \tilde{J}_{\beta(\mu)} = \bigcup(\{H(\mu) \mid \mu' \in C_\mu\}) \). Also \( \pi(\mu') \) is the identity on \( \mu' \) and sends \( p(\mu') \) to \( p(\mu) \).

(a) If \( A \subseteq \tilde{J}_\mu \) and \( A \in \tilde{J}_{\beta(\mu)} \) then \( A \cap \tilde{J}_\mu \) is \( \Sigma^*_n(\mu) \)-definable as an element of \( H(\mu') \) from some fixed parameter \( x \in \tilde{J}_\mu \), uniformly for sufficiently large \( \mu' \in C_\mu \). So \( A \in \Delta_1(\tilde{J}_\mu, C_\mu) \). This proves (a).

(b) Let \( X = \text{Range}(\pi) \) and \( \tilde{X} = \Sigma^*_n(\mu) \)-hull(\( X \cup \{p(\mu)\} \)) in \( \tilde{J}_{\beta(\mu)} \). If \( y \in \tilde{X} \cap \tilde{J}_\mu \) then for some \( \mu' \in C_\mu \), \( y \in \Sigma^*_n(\mu') \) hull(\( X \cap \tilde{J}_\mu \cup \{p(\mu')\} \)) in \( \tilde{J}_{\beta(\mu)} \), and as this property of \( \mu' \) is \( \Sigma_1(\hat{J}_\mu, C_\mu) \) with parameters from \( X \), \( \mu' \) can be chosen in \( \Sigma_1(\text{hull}(X) \cap \tilde{J}_\mu, C_\mu) \). It follows that \( y \in (\Sigma_1(\text{hull}(X) \cap \tilde{J}_\mu, C_\mu)) = X \). So \( \tilde{X} \cap \tilde{J}_\mu = X \) and if \( \tilde{\pi} : \tilde{J}_{\beta} \simeq \tilde{X} \subseteq \tilde{J}_{\beta(\mu)} \) then \( \tilde{\pi} \) is a \( \Sigma^*_n(\mu) \)-elementary embedding extending \( \pi \) with \( p(\mu) \) in its range. Let \( \mu^* = \bigcup(X \cap \mu) \).
As $\tilde{X} \simeq \Sigma^*_\alpha$-hull ($X \cup \{p(\mu^*)\}$) in $\tilde{J}_\beta$ (by the $\Sigma^*_\alpha$-elementarity of $\pi(\mu^*)$ when $\mu^* < \mu$) we see that $\pi$ is regular with respect to partial $\Sigma^*_\alpha$-1($\tilde{J}_\beta$) functions and singular with respect to $\Sigma^*_\alpha$($\tilde{J}_\beta$) partial functions. So we get $\beta(\pi) = \tilde{\beta}$ and $n(\pi) = n(\mu)$. Then the $\Sigma^*_\alpha$-elementarity of $\pi$ and the fact that $p(\mu) \in \text{Range}(\pi)$ guarantee that $C = C_\mu$. The uniqueness of $\pi$ comes from the fact that $\tilde{J}_\beta(\tilde{\mu}) = \Sigma^*_\alpha$-hull$(\tilde{\Pi} \cup \{p(\mu)\})$ and $\tilde{\Pi}|\pi$ is determined by $\pi$. This proves (b).

If $C_\mu^k = \emptyset$ for some $k$ then $C_\mu$ was defined as a special $\omega$-sequence cofinal in $\mu$. That definition was made precisely to enable the preceding arguments to also apply in this case. (Also note that in this case $\mu^* = \mu$.)

Relativization. Square and Diamond hold relative to reshaped strings, a fact which is useful in the proof of Jensen’s Coding Theorem (Beller–Jensen–Welch [82] and Friedman [94]). We state these versions here.

Assume that $A \subseteq \text{ORD}$ and $L_\alpha[A] = H_\alpha$ for each cardinal $\alpha$. For each such $\alpha$ define $S_\alpha$ to consist of all $s : [\alpha, [s]) \rightarrow 2, \alpha \leq |s| < \alpha^+$, such that for all $\eta \leq |s|, L[A \cap \alpha, s|\eta] = \text{card}(\eta) \leq \alpha$. These are the “reshaped strings” at $\alpha$.

We must also define coding structures. For $s \in S_\alpha$ define $\mu^<s$ and $\mu^s$ inductively by: $\mu^<\emptyset_\alpha = \alpha$ (where $\emptyset_\alpha \in S_\alpha$, $|\emptyset_\alpha| = \alpha$, is the empty string), $\mu^<s = \bigcup \{\mu^t \mid t \text{ a proper initial segment of } s\}$ for $s \neq \emptyset_\alpha$, and $\mu^s$ = least $\mu > \mu^<s$ such that $\mu'\mu = \mu$ for $\mu' < \mu$ and $L_\mu[A \cap \alpha, s] = \text{card}(|s|) \leq \alpha$. Also let $\hat{\mu}^s$ = largest $\mu > \mu^<s$ such that $\mu'\mu = \mu$ for $\mu' < \mu$, $L_\mu[A \cap \alpha, s] = |s|$ is a cardinal, if exists; if there is no such $\mu$ then $\hat{\mu}^s = \mu^<s$. Then $A^s = L_{\hat{\mu}^s}[A \cap \alpha, s]$, $A^<s = \langle L_{\mu^<s}[A \cap \alpha, \hat{s}], A \cap \alpha, \hat{s} \rangle$ and $\tilde{A}^s = \langle L_{\hat{\mu}^s}[A \cap \alpha, \hat{s}], A \cap \alpha, \hat{s} \rangle$ where $\hat{s} = \{\mu^<s|\eta \mid s(\eta) = 1\}$.

And we must discuss collapsibility. If $\langle A, C \rangle$ is an amenable structure of the form $\langle \tilde{J}_\mu[B], B, C \rangle$ we define $A^+$ to be $\langle \tilde{J}_\mu^+[B], B \rangle$ where $\mu^* \geq \mu$ is the least limit ordinal such that $\tilde{J}_\mu^+[B] = [\mu]$ (if it exists), and $\langle A, C \rangle$ is collapsible if $A^+$ exists and whenever $\pi : \langle \tilde{A}, \tilde{C} \rangle \rightarrow \langle A, C \rangle$ is $\Sigma_1$-elementary then $\tilde{A}^+$ exists, $\tilde{C}$ is definable over $\tilde{A}^+$ and $\pi$ lifts to a $\Sigma_1$-elementary $\pi^+ : \tilde{A}^+ \rightarrow A^+$.

Relativized Square. Suppose $\alpha$ is an uncountable limit cardinal. Then there exists $\langle C^s \mid s \in S_\alpha \rangle$ such that:

(a) $s \neq \emptyset_\alpha \rightarrow C^s$ is CUB in $\mu^<s$, ordertype($C^s) \leq \alpha$, $C^s \in A^s$.
(b) $\mu \in \text{Lim} C^s \rightarrow \mu = \mu^<s|\eta$ for some $\eta \leq |s|$ and $C^s|\eta = C^s \cap \mu$.
(c) $\langle A^<s, C^s \rangle$ is collapsible.
(d) $s \neq \emptyset_\alpha$, $D \subseteq A^<s$, $D \in \{A^<s\}^+ \rightarrow D$ is $\Delta_1(A^<s, C^s)$.

Relativized Diamond. Suppose $\alpha$ is an uncountable limit cardinal. Then there exists $\langle D^s \mid s \in S_\alpha \rangle$ such that:
(a) \( D^s \subseteq \mathcal{A}^{<s} \) and \( \langle D^t \mid t \text{ an initial segment of } s \rangle \in \mathcal{A}^s \).

(b) If \( D \subseteq \mathcal{A}^{<s} \) and \( D \in \mathcal{A}^s \neq \mathcal{A}^{<s} \) then \( \{ \eta < |s| \mid D^{s|\eta} = D \cap \mathcal{A}^{<s|\eta} \} \) is stationary in \( \mathcal{A}^s \).

(c) If \( \mu^{<s|\eta} \in \text{Lim} \mathcal{C}^s \) and \( \eta < |s| \) then \( D^{s|\eta} = \emptyset \). If \( \mathcal{A}^s \models |s|^{++} \exists x \forall \eta < \mu \) then \( D^s = \emptyset \). And if \( \pi : (\mathcal{A}^{<s}, \mathcal{C}) \to (\mathcal{A}^{<s}, \mathcal{C}^s) \) is \( \Sigma_1 \)-elementary and \( \pi(\sigma) = \alpha \) where \( s \in S_\alpha \) then \( D^s = \pi^{-1}[D^s] \).

Proof. For Relativized Square, define \( C^s \) using \( \langle \tilde{J}_{\beta(s)}[A \cap \alpha, \hat{s}], A \cap \alpha, \hat{s} \rangle \) as we defined \( C_u \) using \( \tilde{J}_{\beta(u)} \), where \( \beta(s) \geq \mu^{<s} \) is least so that \( \mu^{<s} \) is not regular with respect to functions definable over this structure. Note that this structure belongs to \( \mathcal{A}^s \). As before, we get property (a), and (b) follows from (the analogue of) Lemma 11. Properties (c), (d) follow from (the analogue of) Lemma 12.

For Relativized Diamond, define \( D^s \) using \( \langle \tilde{J}_{\beta(s)}[A \cap \alpha, \hat{s}], A \cap \alpha, \hat{s} \rangle \) as we defined \( D_\mu \) in Lemma 5 using \( J_{\beta(\mu)} \). Property (a) follows from (the analogue to) (b) of Lemma 5 and (b) follows from the same argument used to establish (a) of Lemma 5. Also, that argument in fact shows that (a) of Lemma 5 holds in the stronger form: if \( D \subseteq \tilde{J}_\alpha \) then \( \{ \mu < \alpha^+ \mid D \cap J_\mu = D_\mu \) and \( C_\mu^0 = \emptyset \} \) is stationary; note that \( C_\mu^0 = \emptyset \rightarrow \mu \in \text{Lim} C_\mu \) for any \( \mu < \mu' \). So by (the analogue to) this proof we may assume that the first statement of Relativized Square (c) holds. The second statement of (c) follows from (the analogue to) Lemma 5(c) and the final statement follows from (the analogues to) Lemmas 12 and 5(b).

The Fine Structure Principle. We summarize here those aspects of the \( \Sigma^* \) theory that are used when establishing combinatorial principles in \( L \). For any set \( X \) let \( \text{Seq}(X) \) denote the set of all finite sequences from \( X \) and recall the ordering \( < \) on finite sets of ordinals: \( p < q \) iff \( \alpha \in q \) where \( \alpha = \max((p-q) \cup (q-p)) \). Also for any limit ordinal \( \lambda \) let \( M_\lambda \) denote \( \tilde{J}_\lambda \) (= \( J_\alpha \), where \( \omega \alpha = \lambda \)).

(FSP) There exists a sequence of recursive sets of formulas \( \Sigma_1 = \Sigma_1^s \subseteq \Sigma_2 \subseteq \ldots \) and partial functions \( h_\lambda^s : \omega \times M_\lambda \to M_\lambda \) for \( \lambda \) limit and \( n \in \omega \) such that

1) \( \bigcup \{ \Sigma_n^s \mid n \in \omega \} \) = All first-order formulas, \( \Pi_n^s = \{ \neg \varphi \mid \varphi \in \Sigma_n^s \} \) \( \subseteq \Sigma_{n+1}^s \) and \( \Sigma_n^s \) is closed under \( \exists, \land, \lor \).

2) \( h_\lambda^s \) is \( \Sigma_n^s \)-definable and if \( \varphi(x) \in \Sigma_n^s \) then for some \( k, M_\lambda \models \varphi(x) \leftrightarrow h_\lambda^s(k, x) \) is defined.

3) For any \( X \subseteq M_\lambda, H_n^\lambda(X) = \{ h_\lambda^s(k, x) \mid x \in \text{Seq}(X), k \in \omega \} \) is the least \( \Sigma_n^s \)-elementary submodel of \( M_\lambda \) containing \( X \) as a subset.

4) Let \( q^s_\lambda = \text{least ordinal } q \) such that for some \( p \in \text{Seq}(\lambda) \) and \( A \subseteq \lambda \) where \( A \) is \( \Sigma_n^s(M_\lambda) \) in parameter \( p \), \( A \cap \omega q \notin M_\lambda \). And let \( p^s_\lambda = \text{<}-\text{least} \).
such $p$. Then $p^\lambda_n \in H_{n+1}^\lambda(\emptyset)$ and $M^\lambda_n = H_{n}^\lambda(\omega g_n^\lambda \cup \{p^\lambda_n\})$. Also the formula “$x \in M_{\omega g_n^\lambda}$” is $\Sigma^*_n$.

5) If $\pi : M^\lambda_n \to M^\lambda_n$ is $\Sigma^*_n$-elementary then $\pi^{-1}(g^\lambda_n) = g^\lambda_n$ and $\pi^{-1}(p^\lambda_n) = p^\lambda_n$.

6) Approximations: $h^\lambda_n = \bigcup\{h^\lambda_{n'} \mid \sigma < \omega g_{n'-1}, \sigma \text{ limit}\}$ where $\sigma < \sigma' \to h^\lambda_{n'} \subseteq h^\lambda_{n'}$, $\{\langle\sigma, k, x, y\rangle \mid h^\lambda_{n'}(k, x) = y\}$ is $\Sigma^*_n \cap \Pi^*_n$ and when $n > 1$, for each $\sigma$, $H_{n}^\lambda(X) = \{h^\lambda_{n'}(k, x) \mid k \in \omega, x \in \text{Seq}(X)\}$ is $\Sigma^*_n$-elementary in $M^\lambda_n$. (When $n = 1$ we take $\omega g^\lambda_{n-1}$ to be $\lambda$.)

It is not difficult to verify that the proof of Square that we gave can be carried out directly from the FSP. In the next section we use the FSP to construct morasses.

Remark. $\Sigma^*$ theory can also be applied in core models other than $L$, as in its original form (Jensen [??]), however Lemma 3(c)(Condensation) may fail in this more general context. For this reason the fine structure theory for core models in general presents numerous new difficulties, some of which remain unsolved.

3. Morasses. A strong form of the gap-1 morass principle is useful in the theory of strong coding. We now establish a global form of this principle, which we call Morass with Square.

In Square we found a uniform way of writing a singular ordinal as the union of a short sequence of smaller ordinals. In Morass we find a uniform way of writing an ordinal of regular cardinality as the direct limit of ordinals in a uniform way. In Square we found a uniform way of writing a singular ordinal as the direct limit of ordinals in a uniform way. These two principles interact in Morass with Square.

Rather than begin with a statement of our principle, we first use the Fine Structure Principle to describe the actual object which will interest us. In this way it is easier to see the motivation behind a list of its combinatorial properties, expressed in Morass with Square.

An ordinal $\alpha$ is cardinal-correct if whenever $J_\alpha \models \kappa$ is a cardinal, then $\kappa$ really is a cardinal. Let $S^0 = \{\alpha > \omega \mid \alpha \text{ is cardinal-correct}\}$. Then $S^0$ is CUB in every uncountable cardinal. For $\alpha \in S^0$ let $S_\alpha = \{\nu \mid \alpha < \nu < \alpha^+, \nu \text{ is a limit ordinal, } J_\nu \models \alpha \text{ is regular and } \alpha \text{ is the largest cardinal}\}$. Then $S_\alpha$ is a closed subset of $(\alpha, \alpha^+]$ and $\alpha$ not a cardinal, $\alpha < \beta$ in $S^0 \to \bigcup S_\alpha < \beta$.

We write $\nu_0 < \nu_1$ iff $\nu_0 < \nu_1$ and for some $\alpha \in S^0$, $\nu_0$ and $\nu_1$ both belong to $S_\alpha$. When $\nu \in S_\alpha$ we write $\alpha(\nu) = \alpha$. (This is a different use of the notation $\alpha(\nu)$ than was made in the proof of Square.) Let $S^1 = \bigcup\{S_\alpha \mid \alpha \in S^0\}$.

Now we come to the main definition. For $\nu \in S^1$, $\beta(\nu)$ is least limit ordinal $\beta \geq \nu$ such that $g^\lambda_n \leq \alpha(\nu)$ for some $n$ and $\alpha(\nu)$ is least such $n$. And $q(\nu) = \text{least } q \in \text{Seq}(\beta(\nu))$ such that $\beta(\nu) = H_{n(\nu)}^\lambda(\alpha(\nu) \cup \{q\})$. (Actually, $q(\nu) = p^\lambda_{n(\nu)}(\alpha(\nu)$.) We write $\nu <_{\text{fin}} \nu$ iff there is $\pi : J_{\beta(\nu)} \to J_{\beta(\nu)}$ such
that $\pi$ is $\Sigma^*_n(\nu)$-elementary, $n(\tilde{\nu}) = n(\nu), \pi(\alpha(\tilde{\nu})) = \alpha(\nu), \pi(q(\tilde{\nu})) = q(\nu)$ and $\pi|\alpha(\tilde{\nu}) = \text{identity}$; in addition we impose the $Q$-condition:

\[(Q)\] Whenever $\varphi(\tilde{\nu})$ is $\Sigma^*_n(\nu)$ with parameter $\rho \in \tilde{J}_\beta(\nu)$ then \(\{\nu_0 < \nu \mid \tilde{J}_\beta(\nu) = \varphi(\nu_0, \pi(\rho))\}\) is bounded in $\nu$ iff \(\{\rho_0 < \rho \mid \tilde{J}_\beta(\nu) = \varphi(\rho_0, \rho)\}\) is bounded in $\sigma$.

This condition originates in Jensen [72].

If $\sigma <_1 \nu$ then $\pi$ as above is unique and we write $\pi_{\rho\nu} = \pi|\sigma: \sigma \rightarrow \nu$, $\tilde{\pi}_{\rho\nu} = \pi$.

The above structure, together with the Square sequence $(C_\alpha \mid \alpha$ singular limit) from the preceding section, constitutes our realization of Morass with Square. Before stating this principle we make a few observations regarding the relation $<_1$. Using the fact that $\tilde{\pi}_{\rho\nu}$ is $\Sigma^*_n(\nu)$-elementary and $\pi(\alpha(\tilde{\nu}), q(\tilde{\nu}))$ to $(\alpha(\nu), q(\nu))$ it follows not only that $\tilde{\pi}_{\rho\nu} = \pi$ is unique but also that $<_1$ is a tree, $\nu <_{\to \text{-} \text{minimal}}, <_{\to \text{-} \text{limit}} \rightarrow \sigma <_{\to \text{-} \text{minimal}}, <_{\to \text{-} \text{limit}}$ limit. Also $\pi^{-1}[S(\alpha(\nu)) = S(\alpha(\tilde{\nu})) \cap \sigma$ and $\pi(\rho(\nu)) = \pi(\rho(\tilde{\nu})$ $\rightarrow \rho(\nu)$ $\rightarrow \rho(\tilde{\nu})$ $\rightarrow \rho(\nu)$ $\rightarrow \rho(\tilde{\nu})$ $\rightarrow \rho(\nu)$ $\rightarrow \rho(\tilde{\nu})$ $\rightarrow \rho(\nu)$ $\rightarrow \rho(\tilde{\nu})$. Finally, $\sigma <_1 \nu <_1 \nu \rightarrow \pi_{\rho\nu} = \pi_{\rho\nu} \circ \pi_{\rho\nu}$ and $\{\alpha(\nu) \mid \sigma <_1 \nu\}$ is always closed in $\alpha(\nu)$, unbounded if $\nu$ is not $<_0$-maximal. If $\{\alpha(\nu) \mid \sigma <_1 \nu\}$ is unbounded then $\nu = \bigcup\{\text{Range}(\pi_{\rho\nu}) \mid \sigma <_1 \nu\}$.

There are four more properties of $\pi$ which take a bit of argument. First we claim that if $\nu < \beta(\nu) \rightarrow \nu \in \text{Range}(\pi)$: If $n(\nu) > 1$ then this is clear because $\tilde{J}_\beta(\nu) = \nu = \alpha(\nu)$ and the property of being a cardinal is $\Sigma^*_n$. If $n(\nu) = 1$ then we claim that $q(\nu) = \nu$ is nonempty and hence if $\gamma \in q(\nu) - \nu$ we see that $\nu = \alpha(\nu)$ of $\tilde{J}_\gamma$ belongs to $H(\alpha(\nu), q(\nu)) \subseteq \text{Range}(\pi)$. The reason that $q(\nu) - \nu$ is nonempty is that as in the proof of Lemma 6(d), we can show that $\tilde{J}_\gamma$ is $\Sigma_1$-elementary in $\tilde{J}_\beta(\nu)$ and hence $q(\nu) \subseteq \nu$ would contradict $\tilde{J}_\gamma(\alpha(\nu)) = \tilde{J}_\beta(\nu)$.

Second, we claim that if $\rho < _0 \text{-} \text{limit}$ and $\lambda = \bigcup\text{Range}(\pi) < \nu$ then $\sigma <_1 \lambda$ and $\pi_{\sigma\lambda} = \pi_{\rho\nu}$. As in the proof of Square we form $H = H_{n(\nu), \sigma}(\lambda \cup \{q(\nu)\})$ where $\sigma = \bigcup(\text{Range}(\pi) \cap \rho_{\nu(\nu), -1})$. Then as in the proof of Lemma 10, $H \simeq \tilde{J}_\beta(\lambda)$ and $q(\nu)$ is sent to $q(\lambda)$ under this isomorphism. By composing with $\pi$, we get a $\Sigma^*_n(\nu)$-elementary embedding from $\tilde{J}_\beta(\nu)$ into $\tilde{J}_\beta(\lambda)$ sending $(\alpha(\tilde{\nu}), q(\tilde{\nu}))$ to $(\alpha(\lambda), q(\lambda))$. As the range of this embedding contains a cofinal subset of $\lambda$, the $Q$-condition is satisfied and $\sigma <_1 \lambda, \pi_{\sigma\lambda} = \pi_{\rho\nu}$.

Third, we claim that if $\sigma <_1 \nu, \pi_{\rho\nu}$ is cofinal and $\alpha$ is such that for each $\rho_0 <_\nu \sigma, \alpha = \alpha(\rho_0)$ for some $\rho_0 <_1 \pi_{\rho\nu}(\rho_0)$ then $\alpha = \alpha(\nu')$ for some $\nu' <_1 \nu$. For, $H = H_{n(\nu), \sigma}(\alpha \cup \{q(\nu)\}) = \bigcup\{H_{n(\nu), \sigma}(\alpha \cup \{q(\nu)\}) \mid \sigma \in \text{Range}(\pi_{\rho\nu})\}$.
$\sigma < \omega_{\beta(\nu)}$ and hence $H \cap \alpha(\nu) = \alpha$ since for each $\sigma$ as above, $\alpha = \alpha(\nu')$ for some $\nu' < \nu$ such that $\nu' \in H \cap \nu_\alpha$. Since $H \cap \nu$ is cofinal in $\nu$ (as we can assume that $\alpha \geq \alpha(\nu)$) we get $\alpha = \alpha(\nu')$ where $\nu' = \text{ordertype}(H \cap \nu)$.

Fourth, we claim that if $\nu$ is a $<_\alpha$-successor then so is $\nu$. This is clear if $\nu < \beta(\nu)$ or $n(\nu) > 1$ as being the $<_\alpha$-predecessor to $\nu$ is $\Pi_1(\bar{\nu})$. If $\beta(\nu), n(\nu) = (\nu, 1)$ then we must use the $Q$-condition on $\pi$ to guarantee that $S_\alpha(\nu) \cap \bar{\nu}$ is bounded in $\nu$.

The previous is our first use of the $Q$-condition on $\pi$. In strong coding we will use it to argue that if $\sigma < \nu$ and $\nu$ is admissible (i.e., $L_\nu \models \Sigma_1$-Replacement) then so is $\sigma$.

We now state Morass with Square. We have shown that the structure defined above satisfies (a)–(f) in the list of properties below.

**Morass with Square.** There exist $\langle C_\alpha \mid \alpha \text{ singular limit} \rangle$, $\langle S_\alpha \mid \alpha \in S^0 \rangle$, a binary relation $<_\alpha$ on $S^1 = \bigcup \{ S_\alpha \mid \alpha \in S^0 \}$ and $\langle \pi_{\nu_\nu} \mid \sigma < \nu \rangle$ such that

(a) For $\alpha$ a singular limit, $C_\alpha$ is CUB in $\alpha$, ordertype($C_\alpha$) $< \beta, \beta \in \text{Lim}C_\alpha \to _\beta$ singular, $C_\beta = C_\alpha \cap \beta$.

(b) $S^0 \cap \kappa$ is CUB in $\kappa$ for every uncountable cardinal $\kappa$.

(c) For $\alpha \in S^0$, $S_\alpha$ is a closed subset of $\langle \alpha, \alpha^+ \rangle$. And:
   
   (c1) $\alpha$ regular $\to S_\alpha = S^0 \cap (\alpha, \alpha^+)$.  
   
   (c2) $\alpha$ singular cardinal $\to S_\alpha$ is a proper initial segment of $S^0 \cap (\alpha, \alpha^+)$.  
   
   (c3) $\alpha$ not a cardinal, $\alpha < \beta$ in $S^0 \to \bigcup S_\alpha < \beta$.

**Notation.** (c) implies that $\nu \in S^1 \to$ there is a unique $\alpha$ with $\nu \in S_\alpha$; denote this by $\alpha(\nu)$. Write $\nu <_\alpha \nu'$ if $\nu < \nu'$, $\alpha(\nu) = \alpha(\nu')$. For $\alpha \in S^0$, $\alpha$ not regular, let $\nu(\alpha)$ denote max($S_\alpha$) $< \alpha^+$. If $\nu \in S^1$, $\nu$ not $<_\alpha$-maximal, then $\nu^+$ denotes its $<_\alpha$-successor.

(d) $<_\alpha$ is a tree and if $\sigma <_\alpha \nu$ then $\alpha(\sigma) < \alpha(\nu)$ and $\sigma$ is $<_\alpha$-minimal, successor limit iff $\nu$ is $<_\alpha$-minimal, successor, limit.

(e) If $\bar{\nu} < \nu$ then $\pi = \pi_{\bar{\nu} \nu} : \bar{\nu} \to \nu$ is order-preserving, $\pi^{-1}[S_\alpha(\nu)] = S_\alpha(\nu) \cap \sigma, \pi(\nu^+) = \pi(\nu^+)$ when $\nu^+ < \nu^+$. If $\nu^+ < \nu$ and $\pi(\nu^+) = \pi(\nu^+)$ then $\sigma < \nu_0$ and $\pi(\nu_0) = \pi(\nu_0)$. If $\sigma$ is $<_\alpha$-limit, $\lambda = \bigcup \text{Range}(\pi)$ then $\sigma < \lambda$ and $\pi(\nu_0) = \pi(\nu)$; and if $\nu = \bigcup \text{Range}(\pi)$, $\alpha = \alpha(\nu)$ for some $\nu_0 < \pi(\nu_0)$ for each $\nu_0 < \nu$ then $\alpha = \alpha(\nu')$ for some $\nu' < \nu$.

(f) $\bar{\nu} < \nu \to \nu - \pi_{\bar{\nu} \nu} = \pi_{\nu_0}$. For $\nu \in S^1$, $\{ \alpha(\bar{\nu}) \mid \bar{\nu} < \nu \}$ is closed in $\alpha(\nu)$ and is unbounded if $\nu$ is not $<_\alpha$-maximal. If $\{ \alpha(\sigma) \mid \sigma < \nu \}$ is unbounded in $\alpha(\nu)$ then $\nu = \bigcup \{ \text{Range}(\pi_{\nu_\nu}) \mid \sigma < \nu \}$.

Now let $C'_\alpha$ denote the limit points of $C_\alpha$ less than $\alpha$, for $\alpha$ singular limit.
(g) Suppose $\nu$ is $<_{1}$-limit and $\alpha(\nu)$ singular. Then for $\alpha$ in a final segment of $C^*_\alpha(\nu)$ there exists $\nu_\alpha <_{1} \nu$ with $\nu_\alpha \in S_\alpha$; and for $\nu <_{0}$-limit if $\lambda_\alpha = \bigcup \text{Range}(\pi_\alpha(\nu))$ then $\lambda_\alpha \in C^*_{\nu} \cup \{\nu\}$ and $\alpha < \beta \in C^*_{\alpha(\nu)} \rightarrow \lambda_\alpha \in \text{Range}(\pi_{\nu, \nu}) \cup \{\nu\}$.

(h) Suppose $\nu$ is $<_{1}$-minimal and $<_{0}$-limit. Then for $\alpha$ in a final segment of $C^*_{\alpha(\nu)}$, $\nu(\alpha)$ is $<_{1}$-minimal, $<_{0}$-limit and there is $\nu_\alpha <_{0} \nu$ such that $\nu(\alpha) <_{1} \nu_\alpha \in C^*_{\nu}$, $\alpha < \beta \in \text{Range}(\pi_{\nu(\beta), \nu(\beta)})$, and $\beta \in \text{Lim} C^*_{\alpha(\nu)} \rightarrow \nu_\beta = \bigcup \{\nu_\alpha \mid \alpha \in C^*_{\beta}\}$.

(i) Suppose $\nu$ is a $<_{1}$-successor and $<_{0}$-limit. Let $\sigma < ^* \nu$ express the property that $\sigma = <_{1}$-predecessor to $\nu$. Then for a final segment of $\alpha \in C^*_{\alpha(\nu)}$, $\nu(\alpha)$ is $<_{1}$-successor and $<_{0}$-limit and there exist $\nu_\alpha <_{0} \nu$ as in (h) such that in addition, $\nu = \lambda = \bigcup \text{Range}(\pi_{\nu, \nu}) \rightarrow \nu_\alpha = \pi_{\nu, \nu}(\rho_\alpha)$ where $\rho_\alpha < ^* \nu(\alpha)$, $\lambda < \nu \rightarrow \lambda \in \text{Range}(\pi_{\nu(\alpha)(\nu)}), \sigma < ^* \nu(\alpha)$.

Proof. (g) Suppose $\alpha \in C^*_{\alpha(\nu)}$. Then for some $k$, $\alpha \in \text{Lim} C^k_{\alpha(\nu)}$ and therefore for some $\gamma \geq \gamma(\alpha(\nu))$,

$$\alpha = \bigcup \{\alpha(\nu) \cap H^\beta(\alpha(\nu)) \cap \{p(\alpha(\nu)), (\alpha(\nu))_0 \ldots (\alpha(\nu))_{k-1}\}\}$$

where $\beta(\alpha(\nu)), n(\alpha(\nu)), p(\alpha(\nu)) = (q(\alpha(\nu)), (\alpha(\nu))$ and $\alpha(\nu)_i$ are defined as in the proof of Square. (We have changed the notation $\alpha(\mu)$ to $\gamma(\mu)$ so as to avoid confusion.) The fact that $\nu$ is a $<_{1}$-limit implies that $(\beta(\nu), n(\nu)) < (\beta(\alpha(\nu)), n(\alpha(\nu)))$. (I.e., either $\beta(\nu) < \beta(\alpha(\nu))$ or $\beta(\nu) = \beta(\alpha(\nu))$, $n(\nu) < n(\alpha(\nu))$. Note that as $\tilde{J}_\nu \models$ There is a largest cardinal, $\beta(\nu)$ and $n(\nu)$ have the same meaning in this section as they did in the proof of Square.) Thus for sufficiently large $\alpha \in C^*_{\alpha(\nu)}$ we see by Lemma 10 that $\alpha = \alpha(\nu) \cap H^\beta(\alpha(\nu)) (\alpha \cup \{q(\nu)\})$, where $q(\nu)$ is defined in this section. (We need only choose $\alpha$ large enough so that $H^\beta(\alpha(\nu))(\alpha \cup \{q(\nu)\}) \subseteq H^\beta(\alpha(\nu)) \cap (\alpha \cup \{(\alpha(\nu))\})$. To verify the $Q$-condition we must argue as follows. Either $\alpha$ can be chosen large enough so that $H^\beta(\alpha(\nu))(\alpha \cup \{q(\nu)\}) \cap \nu$ is cofinal in $\nu$, in which case the $Q$-condition is automatic, or we claim that the $Q$-condition implies that $H^\beta(\alpha(\nu)) (\alpha \cup \{q(\nu)\})$ is $\Sigma^*_{\alpha(\nu)+1}$-elementary in $\tilde{J}_\beta(\nu)$. In the latter case the assumption that $\nu$ is a $<_{1}$-limit yields that in fact $(\beta(\nu), n(\nu)+1) < (\beta(\alpha(\nu)), n(\alpha(\nu)))$, and hence $H^\beta(\alpha(\nu))(\alpha \cup \{q(\nu)\}) = H^\beta(\alpha(\nu))(\alpha \cup \{q(\nu)\})$ obeys the $Q$-condition.

To prove the above claim suppose $\varphi(x)$ is $\Sigma^*_{\alpha(\nu)+1}$ and note that $\varphi(x) \leftrightarrow \exists \gamma < \alpha(\nu) \exists k \in \omega \ (x \in y = h^\beta(\nu)(k, (\gamma, q(\nu))))$, $y \models \varphi(x)$, $y$ $\Sigma^*_{\alpha(\nu)-}\text{elementary}$ in $\tilde{J}_\beta(\nu)$. To each $\sigma < \omega \mu_\alpha^{-\beta(\nu)} = y$ associate the least $(\gamma(\sigma), k(\sigma))$ such that the above holds with $h^\beta(\nu)(\alpha)$ replaced by $h^\beta(\nu)(\sigma)$ and $\text{"y $\Sigma^*_{\alpha(\nu)-}\text{elementary}$ in $\tilde{J}_\beta(\nu)$" replaces by $y = H^\beta(\alpha(\nu))(y)$}$. Then $\varphi(x) \leftrightarrow A = \{\sigma \mid \text{For some}}$
γ, k, σ is least so that \((γ(σ), k(σ)) = (γ, k)\) is bounded in \(q \leftrightarrow B = \{H_{n(ν)}^{α(ν)}(\{q(ν)\})\} \cap ν \mid \sigma \in A\) is bounded in ν. So ν is equivalent to the boundedness of a \(Σ^*_n(ν)\) subset of ν, hence the Q-condition implies \(Σ^*_n(ν)\) boundedness.

Thus we have \(ν_α < ν\) where \(ν_α = \text{ordertype}(ν \cap H_n^{α(ν)}(\{q(ν)\}))\). Again since \((β(ν), n(ν)) < (β(α(ν)), n(α(ν)))\), if \(α < β\) in \(C^\prime_α(ν)\) and \(H_n^{α(ν)}(α \cup \{q(ν)\})\) is bounded in ν then its supremum below ν belongs to \(H_n^{β(ν)}(β \cup \{q(ν)\})\). So it only remains to show that \(ν_α ≤ \bigcup(ν \cap H_n^{α(ν)}(α \cup \{q(ν)\})) = C^\prime_ν \cup \{ν\}\). Note that if \(p(ν)\) and \(ν_0\) are defined as in the proof of Square then \(q(ν)\) as defined in this section belongs to \(H_n^{β(ν)}(p(ν), ν_0)\): \(q(ν)\) is just \(p_n(ν) - p_n(ν) - α(ν)\), and so by definition of \(p(ν)\) we have \(q(ν) \cap ν\) if nonempty then it consists of a single ordinal \(δ\), and \(δ\) is largest so that \(H_n^{β(ν)}(δ \cup \{p_n(ν) - α(ν)\}) \cap ν = δ\). This is precisely the ordinal used to provide a lower bound on \(C^\prime_ν\) in the proof of Square. As \(C^\prime_ν = \{δ\}\) in this case we get \(ν_0 = δ\). So if \(ν_α < ν\) for sufficiently large \(α\) then \(C^\prime_ν\) is a final segment of \(\{∪(ν \cap H_n^{α(ν)}(α \cup \{q(ν)\})) \cap ν \mid α < α(ν)\}\). And the fact that \((β(ν), n(ν)) < (β(α(ν)), n(α(ν)))\) implies that \(ν_α \cap C^\prime_ν\) for sufficiently large \(α\). Of course the alternative is that \(ν_α = ν\) for sufficiently large \(α\) in \(C^\prime_α(ν)\) and so (g) is proved.

(h) There are two cases: either \((β(ν), n(ν)) = (β(α(ν)), n(α(ν)))\) or \((β(ν), n(ν) + 1) = (β(α(ν)), n(α(ν)))\). In the latter case we must have \(H_n^{β(ν)}(α \cup \{q(ν)\}) \cap ν\) bounded in ν for each \(α < α(ν)\), else we could contradict the \(<\)-minimality of ν by forming \(H_n^{β(ν)}(α_0 \cup \{q(ν)\})\) where \(α_0 = \bigcup(α \cap H_n^{β(ν)}(α \cup \{q(ν)\}) \cap ν\) unbounded in ν, \(α < α(ν)\).

First we treat the former case. Suppose \(α \in C^\prime_α(ν)\). Then for some \(k\), \(α \in C^\prime_κ(ν)\) and so for some \(γ \geq γ(α(ν))\) if \(k = 0\),

\[
α = ∪(α \cap H_n^{β(ν)}(γ \cup \{p(α(ν)), α(ν) \cdots α(ν)\})
\]

where \(p(α(ν)) = q(α(ν)), α(ν))\) and \(α(ν)\) are defined as in the proof of Square. (Again to avoid confusion we now write \(γ(μ)\) in place of \(α(μ)\).) Note that \(q(α(ν))\) is precisely the \(q(ν)\) as defined in this section. Write \(p\) for \(\{p(α(ν)), α(ν) \cdots α(ν)\}\) and \(q\) for \(ω_α^{β(ν)}\).

Now let \(ν_α = ∪(ν \cap H_n^{β(ν)}(γ \cup \{p\}))\) and \(γ_α = ∪(q \cap H_n^{β(ν)}(γ \cup \{p\}))\). Then as \(α > γ\) we get \(ν_α < ν\) and \(γ_α < γ\), and by Lemma 10, \(α =

\(\alpha(\nu) \cap H^{\beta(\nu), \sigma_\alpha}(\alpha \cup \{p\})\). We get an embedding \(\tilde{J}_\beta \simeq H^{\beta(\nu), \sigma_\alpha}(\alpha \cup \{p\})\) and \(\pi^{-1}[\nu_\alpha] = \nu(\alpha), \beta(\nu(\alpha)), n(\nu) = n(\nu(\alpha))\). In fact, \(\nu(\alpha) < \nu_\alpha\) as \(\pi\) maps \(\nu(\alpha)\) cofinally into \(\nu_\alpha\). It is not clear that \(\nu(\alpha)\) is \(<_1\)-minimal as it is possible that there exists \(\varphi < \nu(\alpha)\) with \(\alpha(\varphi) \leq \gamma(\alpha) = \gamma(\alpha(\nu))\).
\((\alpha(\varphi) > \gamma(\alpha)\) is ruled out because of the definition of \(\gamma(\alpha)\).) However, as \(\nu\) is \(<_1\)-minimal the \(Q\)-condition must fail between \(H^{\beta(\nu)}(\gamma(\alpha) \cup \{q(\nu)\})\) and \(\tilde{J}_{\beta(\nu)}\) so we may choose \(\alpha\) large enough in \(C'_{\alpha(\nu)}\) so that this failure is captured by \(H^{\beta(\nu), \sigma_\alpha}(\alpha \cup \{p\})\), and therefore \(\nu(\alpha)\) is \(<_1\)-minimal.

To see that \(\nu_\alpha \in C'_{\varphi}\) for sufficiently large \(\alpha\), the same analysis as in the proof of (g) shows that if \(q(\nu) \cap \nu \neq \emptyset\) then \(C'_{\nu} = \{\nu' < \nu|\) For some \(\gamma\), \(\nu' = \bigcup(\nu \cap H^{\beta(\nu)}(\gamma \cup \{q(\nu)\})\), and if \(q(\nu) \cap \nu = \emptyset\) then \(C''_{\nu}\) is a final segment of this set, beyond an ordinal \(<_1\) \(\gamma(\alpha(\nu))\). Thus it follows that either \(C'_{\nu}\) or \(C''_{\nu}\) agrees with \(\{\nu_\alpha| \alpha \in C'_{\alpha(\nu)}\}\) for \(\alpha \geq \gamma(\alpha(\nu))\). If \(C'_{\alpha(\nu)}\) is unbounded in \(\alpha(\nu)\) then we are done because then \(C'_{\nu}\) or \(C''_{\nu}\) as above is unbounded in \(\nu\). If not then we need only note that \(\alpha(\nu) \in H^{\beta(\nu)}(\{q(\nu), \nu^\star\})\) and \(\nu^\star \in H^{\beta(\nu)}(\{q(\nu), \nu(\alpha)\})\) where \(\nu^\star = \bigcup C'_{\nu}\) or \(\bigcup C''_{\nu}\) as above. Thus \(\{\nu_\alpha| \alpha \in C'_{\alpha(\nu)}\}\) agrees with \(C'_{\nu}\) or \(C''_{\nu}\) for \(\alpha \geq \gamma(\alpha(\nu))\) and continuing in this way we get \(\nu_\alpha \in C'_{\nu}\) for sufficiently large \(\alpha\).

The last part of (h) is clear from the definition of the \(\nu_\alpha\)'s and the fact that \(\alpha < \beta \rightarrow \nu_\alpha < \nu_\beta\).

Now we consider the case \((\beta(\nu), n(\nu) + 1) = (\beta(\alpha(\nu)), n(\alpha(\nu)))\) and recall that \(H^{\beta(\nu)}(\alpha \cup \{q(\nu)\}) \cap \nu\) is bounded in \(\nu\) for each \(\alpha < \alpha(\nu)\). Thus as in the proof of (g), for \(\alpha \in C'_{\alpha(\nu)}\) we have \(\alpha = \alpha(\nu) \cap H^{\beta(\nu)}(\alpha \cup \{q(\nu)\})\) and \(\tilde{\nu}(\alpha) < \nu(\alpha)\) where \(\nu(\alpha) = \bigcup(\nu \cap H^{\beta(\nu)}(\alpha \cup \{q(\nu)\}))\) and \(\tilde{\nu}(\alpha) = \text{ordtype}(\nu \cap H^{\beta(\nu)}(\alpha \cup \{q(\nu)\}))\). Also \(\varphi < \tilde{\nu}(\alpha)\) implies as in the proof of (g) that \(\tilde{\nu}(\alpha)\) is \(\Sigma^*_\alpha(\nu)\)-elementary, hence \(\alpha(\varphi) \leq \gamma(\alpha(\nu))\); but as in the first part of the present proof, this is ruled out, for \(\alpha\) sufficiently large. So for such \(\alpha \in C'_{\alpha(\nu)}\) we have \(\tilde{\nu}(\alpha) = \nu(\alpha) < \nu(\alpha)\) and \(\nu(\alpha)\) is \(<_1\)-minimal. The proof that \(\nu_\alpha \in C'_{\nu}\) for sufficiently large \(\alpha\) is as in the proof of (g) and the last part of (h) is clear from the definition of \(\nu_\alpha\) and the fact that \((\beta(\alpha(\nu)), n(\alpha(\nu))) > (\beta(\nu), n(\nu))\).

(i) As in (h) there are two cases: either \((\beta(\nu), n(\nu)) = (\beta(\alpha(\nu)), n(\alpha(\nu)))\) or \((\beta(\nu), n(\nu) + 1) = (\beta(\alpha(\nu)), n(\alpha(\nu)))\) and \(\alpha < \alpha(\nu) \rightarrow H^{\beta(\nu)}(\alpha \cup \{q(\nu)\}) \cap \nu\) is bounded in \(\nu\).

We begin with the first case. As in (h), write \(\alpha = \bigcup(\alpha(\nu) \cap H^{\beta(\nu)}(\gamma \cup \{p\}))\) where \(\gamma \geq \gamma(\alpha(\nu))\) if \(C'_{\alpha(\nu)} = C''_{\alpha(\nu)}\) and \(p = \{p(\alpha(\nu)), \alpha(\nu)_0 \ldots \alpha(\nu)_{k-1}\}\. \)
Also let \( \sigma_\alpha = \bigcup (\varrho \cap H^\beta(\nu) (\gamma \cup \{ p \})) \) where \( \varrho = \omega \beta^\gamma_{n(\nu)-1} \) and \( \nu_\alpha = \bigcup (\nu \cap H^\beta(\nu) (\gamma \cup \{ p \})) \). Then \( \nu(\alpha) < \nu_\alpha \) and the \( \nu_\alpha \)'s obey the property expressed in (h). And there exists \( \sigma_\alpha \leq \sigma < \nu \) such that \( \sigma_\alpha < \nu(\alpha) \) as we can take \( \tilde{\varrho}_\alpha = \text{ordertype}(\nu \cap H^\beta(\nu), \sigma_\alpha (\alpha(\nu) \cup \{ q(\nu) \})) \). As in (h) we can arrange that \( \sigma_\alpha < \nu(\alpha) \) for sufficiently large \( \alpha \) by capturing a witness to the failure of the \( Q \)-condition between \( H^\beta(\nu)(\alpha(\nu) \cup \{ q(\nu) \}) \) and \( \tilde{J}_\beta(\nu) \). Note that in fact \( k > 0 \) and we must have \( \gamma < \alpha(\bar{\sigma}) \) for sufficiently large \( \alpha = \bigcup (\alpha(\nu) \cap H^\beta(\nu)(\gamma \cup \{ p \})) \) so we get \( \nu_\alpha \in \text{Range}(\pi_{\bar{\nu}\nu}) \) for \( \nu_\alpha < \lambda = \bigcup \text{Range}(\pi_{\bar{\nu}\nu}) \). Similarly, \( \lambda \in \text{Range}(\pi_{\nu(\alpha)\nu_\alpha}) \) when \( \nu_\alpha > \lambda \) and we get \( \sigma_\alpha = \sigma \).

Note that in the second case, \( \pi_{\bar{\nu}\nu} \) is not cofinal. The argument now is very similar to the second case of the proof of (h), arranging \( \bar{\varrho} < \nu(\alpha) \) as in the first case of the present proof. 

Our version of Morass with Square originates in Friedman [87] and is related to the concept of Morass with Linear Limits; see Donder [85].

The next principle arises in the proof of Jensen’s Coding Theorem in the general case. It is similar in some respects to the Squared Scales of Donder–Jensen–Stanley [85].

Again we first describe the object, obtained through use of the Fine Structure Principle, which satisfies this principle before stating the principle itself. Let \( T = \{ \nu \mid \nu \) is a limit ordinal, \( \tilde{T}_\nu \models \) there is a largest cardinal \( \alpha(\nu) \) and the cardinality of \( \nu \) equals \( \alpha(\nu) \} \). We do not require that \( \alpha(\nu) = \text{card}(\nu) \) is regular. Let \( \beta(\nu) \geq \nu \) be the least limit ordinal such that for some \( n \), \( \bar{g}_n^{\beta(\nu)} = \alpha(\nu) \), let \( n(\nu) \) be the least such \( n \), and \( p(\nu) = \langle p_n^{\beta(\nu)} (\nu, \alpha(\nu)) \rangle \). Also \( \beta(\nu) = \beta(\nu) + \omega = T\text{-successor to } \nu \).

Now for any \( k \geq 0 \) in \( \omega \) and infinite cardinal \( \alpha < \alpha(\nu) \) let \( H(\nu, k, \alpha) = H_n(\nu, k+1) (\alpha \cup \{ p(\nu) \}) \) and \( \tilde{H}(\nu, k, \alpha) \) its transitive collapse. Then \( f(\nu, k, \alpha) = \alpha^+ \) of \( \tilde{H}(\nu, k, \alpha) \). Note that \( f(\nu, k, \alpha) \in T \) and \( \alpha(\nu, k, \alpha) = \alpha \).

For \( \nu \in T \) we let \( \tilde{C}_\nu \subseteq H \) come from the Square Principle; then \( \tilde{C}_\nu \) is CUB in \( \nu \), ordertype(\( \tilde{C}_\nu \) \( \leq \alpha(\nu) \) and \( \bar{\sigma} \in \text{Lim} \tilde{C}_\nu \rightarrow \tilde{C}_\nu \cap \tilde{\sigma} \). We let \( C_\nu = \tilde{C}_\nu \) if ordertype(\( C_\nu \) \( < \alpha(\nu) \) and otherwise \( C_\nu = \{ \sigma < \nu \mid \) For some \( \alpha < \alpha(\nu), \sigma = \bigcup (\nu \cap H_n(\nu, \alpha \cup \{ p(\nu) \})) \} \).

For \( \nu \in T, k \geq 0 \) in \( \omega \), and \( \alpha(\nu) \) an uncountable limit cardinal we define a CUB \( D_{\nu, k} \leq \alpha(\nu) \) as follows. If \( D = \{ \alpha < \alpha(\nu) \mid \alpha = \alpha(\nu) \cap H_n(\nu, k+1) (\alpha \cup \{ p(\nu) \}) \} \) is unbounded in \( \alpha(\nu) \) then set \( D_{\nu, k} = D \). If \( H_n(\nu, k+1) (\alpha \cup \{ p(\nu) \}) \cap \alpha(\nu) \) is unbounded in \( \alpha(\nu) \) for some \( \alpha < \alpha(\nu) \) then we can choose \( D_{\nu, k} \text{ CUB in } \alpha(\nu) \) of ordertype \( < \alpha(\nu) \) so that \( D_{\nu, k} \cap \alpha \) is \( \Sigma^*_n(\nu, k+1) \)-definable over \( H_n(\nu, k+1) (\alpha \cup \{ p(\nu) \}) \) uniformly for \( \alpha \in \text{Lim} D_{\nu, k} \). Otherwise define \( D_{\nu, k} = \)
\{α₀, α₁, \ldots\} \) where \( α₀ = 0 \) and \( α_{n+1} = \bigcup(α(ν) \cap H^{β(ν)}_{n(ν)+k+1}(α \cup \{p(ν)\})) \).

**Fine Scale Principle.** There exist \( (f(ν, k, α) \mid ν \in T, \ k \in ω, \ α \) an infinite cardinal \( < α(ν)) \), \( (C_ν \mid ν \in T) \), \( (D_{ν, k} \mid ν \in T, \ α(ν)) \) an uncountable limit cardinal, \( k \in ω \) such that

\[
\text{(a) } ν \in T \rightarrow ν \text{ is a limit ordinal and not a cardinal, and } T \cap α^+ \text{ is CUB in } α^+. \ α(ν) \text{ denotes the cardinality of } ν.
\]

\[
\text{(b) } f(ν, k, α) \in T \cap α^+; \ C_ν \text{ is CUB in } ν, \ \text{ordertype}(C_ν) \leq α(ν); \ D_{ν, k} \text{ is CUB in } α(ν). \ \text{For } α \text{ an uncountable limit cardinal and any } f : α \rightarrow α \text{ such that } f(α₀) < α₀^+ \ \text{for every } α₀ < α, \ \text{there is } ν \in T \cap α^+ \text{ such that } f(α₀^+) < f(ν, 0, α₀^+) \text{ for sufficiently large } α₀ < α.
\]

\[
\text{(c) For any } ν \in T \text{ there is } α₀(ν) < α(ν) \text{ such that for } α₀(ν) < α < α(ν), \ \text{an infinite cardinal and } η \in \text{Lim } C_ν:
\]

\[
\begin{align*}
\text{(c1) } f(σ, 0, α) = \bigcup\{f(\bar{σ}, 0, α) \mid \bar{σ} \in C_ν \cap η\}, \\
\text{(c2) } \{f(\bar{σ}, 0, α) \mid \bar{σ} \in C_ν \cap η\} \in \tilde{J}_β \text{ where } β = T\text{-successor to } f(σ, 0, α).
\end{align*}
\]

\[
\text{(d) For any } ν \in T \text{ and } k \geq 0 \text{ there is } α₀(ν, k) < α(ν) \text{ such that for } α₀(ν, k) \leq α₀ < α(ν), \ \text{an infinite cardinal and } α \in \text{Lim } D_{ν, k}:
\]

\[
\begin{align*}
\text{(d1) } f(f(ν, k, α), 1, α₀) = \bigcup\{f(f(ν, k, \bar{σ}), 1, α₀) \mid \bar{σ} \in D_{ν, k} \cap α\}, \\
\text{(d2) } \{f(f(ν, k, \bar{σ}), 1, α₀) \mid \bar{σ} \in D_{ν, k} \cap α\} \in \tilde{J}_β \text{ where } β = T\text{-successor to } f(f(ν, k, α), 1, α₀).
\end{align*}
\]

**Proof.** (c) Choose \( α₀(ν) \) larger than ordertype\( (C_ν) \) if the latter is lesser than \( α(ν) \). In this case the properties follow from the \( Σ^∗_{n(ν)} \)-elementarity of \( H(σ, 0, α) \) in \( \tilde{J}_{β(ν)} \) and the \( \Pi^∗_{n(ν)} \)-definability of \( C_ν = C_ν \cap η \). In case \( \text{ordertype}(C_ν) = α(ν) \) note that \( f(σ, 0, α) = f(σ_α, 0, α) \) where \( σ_α \leq σ \) are in \( C_ν \) and \( σ_α = α_0 \text{th element of } C_ν \). So the argument also works in this case.

(d) If \( \text{ordertype}(D_{ν, k}) < α(ν) \) then choose \( α₀(ν, k) \) larger than this order-type. Note that \( D_{ν, k} \cap α \) is \( Σ^∗_{n(ν)+k+1} \) or \( \Pi^∗_{n(ν)+k+1} \)-definable over \( H(ν, k, α) \) when \( α < α(ν) \); so the result follows from the \( Σ^∗_{n(ν)+k} \)-elementarity of \( H(ν, k, σ) \) in \( H(ν, k, α) \) and the fact that \( n(f(ν, k, α)) = n(ν) + k \). Also in case \( \text{ordertype}(D_{ν, k}) = α(ν) \) note that \( f(f(ν, k, α), 1, α₀) \) is constant for \( α \geq α_0 \text{th element of } \text{Lim } D_{ν, k} \).

The key clause in the Fine Scale Principle is (d). It says that \( f(ν, k+1, −) \) can be uniformly approximated by functions which differ from \( f(ν, k, −) \) only on a proper initial segment of \( α(ν) \), in such a way that at limit stages \( α \), the \( α \text{th approximation can easily recover the } α\text{-sequence of smaller approximations.} \) This is a powerful tool for proving a statement for each \( f(ν, k, −) \), by induction on \( (ν, k) \). In the case of Jensen coding, extendibility of conditions can be proved in this way.
We conclude with a discussion of gap 2 morasses. Again we begin with a description of the intended object.

Let \( S^0 = \{ \alpha > \omega \mid \alpha \) is a limit ordinal, \( \alpha \) is cardinal-correct \}, \( S^1 = \{ \nu \mid \nu \) is a limit ordinal and for some \( \alpha(\nu) \in S^0, J_\nu \models \alpha(\nu) \) is the largest cardinal and \( \alpha(\nu) \) is regular \}, \( S^2 = \{ \mu \mid \mu \) is a limit ordinal, \( \mu \) is not a cardinal and for some \( \nu(\mu) \in S^1, J_\mu \models \nu(\mu) \) is the largest cardinal \}. Thus if \( \mu \in S^2 \) then \( J_\mu \models \alpha(\nu(\mu)) \) is regular, \( \nu(\mu) = \alpha(\nu(\mu))^{+} \) is the largest cardinal. We write \( \nu_0 < \nu_1 \) if \( \nu_0 < \nu_1 \) and for some \( \alpha, \nu_0 \) and \( \nu_1 \) both belong to \( S_\alpha = \{ \nu \mid \alpha(\nu) = \alpha \} \); also we write \( \mu_0 < \mu_1 \) if \( \mu_0 < \mu_1 \) and for some \( \nu \in S^1 \), \( \mu_0 \) and \( \mu_1 \) both belong to \( S_\nu = \{ \mu \mid \nu(\mu) = \nu \} \). For \( \alpha \in S^0 \), \( \nu(\alpha) \) denotes \( \max S_\alpha \) (when \( \alpha \) is not regular) and for \( \nu \in S^1 \), \( \nu(\mu) \) denotes \( \nu \cup \max S_\nu \) (when \( \nu \) is not regular).

Now the main definition. For \( \nu \in S^1 \), \( \nu \) not regular, let \( \beta(\nu) \geq \nu \) be the least limit ordinal such that \( \bar{q}(\nu) = p(\nu) \) for some least \( n(\nu) \), and let \( q(\nu) = \bar{p}(\nu) \neq \alpha(\nu) \). (Thus \( q(\nu) \) is least so that \( H(\alpha(\nu)) \cup \{ q(\nu) \} = J_\beta(\nu) \).) The previous, as well as the definition of \( \bar{\nu} \) for \( \nu \) in gap 1 case: \( \bar{\nu} < \nu \) if there exists \( \bar{\pi}_{\bar{\nu}} = \pi : J_\beta(\nu) \rightarrow J_\beta(\nu) \) which is \( \sum^*_{\alpha(\nu)} \)-elementary, \( \bar{n}(\bar{\nu}) = n(\nu) \), \( \pi(\alpha(\bar{\nu})) = \alpha(\nu) \), \( \pi(q(\bar{\nu})) = q(\nu) \) and the \( Q \)-condition is satisfied: whenever \( \phi(x) \) is \( \sum^*_{\alpha(\nu)} \) in parameters \( \bar{\pi} \in J_\beta(\nu) \) then \( \{ \bar{\nu} \mid J_\beta(\nu) \models \phi(\bar{\nu}) \} \) is bounded in \( \bar{\nu} \) if \( \{ \nu' < \nu \mid J_\beta(\nu) \models \phi(\nu', \pi(\bar{\nu})) \} \) is bounded in \( \nu \). We write \( \bar{\pi}_{\bar{\nu}} \) for \( \pi_{\mu(n(\nu))} \). Now in addition, for \( \mu \in S^2 \), define \( \beta(\mu), n(\mu), q(\mu) \) in the same way, with \( \alpha(\nu) \) replaced by \( \nu(\mu) \). Also define \( \bar{\nu} < \mu \) in the same way, with \( \alpha(\bar{\nu}), \alpha(\nu) \) replaced by \( \nu(\bar{\nu}), \nu(\mu) \). We write \( \bar{\pi}_{\bar{\nu}} \) for \( \mu(n(\nu)) \).

Note that we defined \( \bar{\pi}_{\bar{\nu}} \) for \( \bar{\nu} < \nu \) in \( S^1 \) to be \( \pi_{\mu(\bar{\nu})} \) and not simply \( \pi_{\bar{\nu}} \). This means that \( \bar{\pi}_{\bar{\nu}} \) moves ordinals \( \bar{\pi} \in S^2 \), \( \bar{\nu} < \mu(\bar{\nu}) \), and raises interesting questions about how the relation \( \bar{\nu} < \nu \) on such ordinals is affected by applying \( \bar{\pi}_{\bar{\nu}} \). Thus our gap 2 morass properties pertain not only to the “gap 1” relationships \( \bar{\nu} < \nu \) and \( \bar{\nu} < \mu \) but also to the way in which they interact.

**Gap 2 Morass.** There exist \( \langle S_\alpha \mid \alpha \in S^0 \rangle, \langle S_\nu \mid \nu \in S^1 \rangle = \bigcup \{ S_\alpha \mid \alpha \in S^0 \} \), a binary relation \( \leq \) on \( S^1 \times S^1 \cup (S^2 \times S^2) \) where \( S^2 = \bigcup \{ S_\nu \mid \nu \in S^1 \} \) and \( \langle \bar{\pi}_{\bar{\nu}} \mid \bar{\nu} < \nu \in S^1 \rangle, \langle \bar{\pi}_{\bar{\nu}} \mid \bar{\nu} < \nu \in S^2 \rangle \) such that:

(a) \( S^0 \cap \kappa \) is CUB in \( \kappa \) for each uncountable cardinal \( \kappa \).
(b) For \( \alpha \in S^0 \), \( S_\alpha \) is a closed subset of \( (\alpha, \omega^+ \) and for \( \nu \in S^1 \), \( S_\nu \) is a closed subset of \( (\nu, \nu^+) \). And:

(b1) \( \alpha \) regular \( \rightarrow S_\alpha = S^0 \cap (\alpha, \omega^+ \),
(b2) \( \alpha \) singular cardinal \( \rightarrow S_\alpha \) is a proper initial segment of \( S^0 \cap (\alpha, \omega^+) \),
(b3) \( \alpha < \alpha' \in S^0 \), \( \alpha \) not a cardinal \( \rightarrow \bigcup S_\alpha < \alpha' \),
(b4) \( \nu < \nu' \in S^1 \), \( \nu \) not a cardinal \( \rightarrow \bigcup S_\nu < \nu' \).
Notation. For \( \nu \in S^1 \), \( \alpha(\nu) \) denotes the \( \alpha \) such that \( \nu \in S_\alpha \) and for \( \mu \in S^2 \), \( \nu(\mu) \) denotes the \( \nu \) such that \( \mu \in S_\nu \). We write \( \nu <_0 \nu' \) if \( \nu < \nu' \) and \( \alpha(\nu) = \alpha(\nu') \), and \( \mu <_0 \mu' \) if \( \mu < \mu' \) and \( \nu(\mu) = \nu(\mu') \). If \( \alpha \in S^0 \) then \( \nu(\alpha) = \max S_\alpha \) and if \( \nu \in S^1 \) is not a cardinal then \( \nu(\nu) = \nu \cup \max S_\nu \). If \( \nu \) is not \( <_0 \)-maximal then \( \nu^+ \) then denotes its \( <_0 \)-successor (similarly for \( \mu \in S^2 \)).

(c) \( <_1 \) is a tree and if \( \bar{\nu} <_1 \nu \) in \( S^1 \) then \( \alpha(\bar{\nu}) < \alpha(\nu) \) and \( \nu \) is not a cardinal. If \( \bar{\nu} <_1 \mu \) in \( S^2 \) then \( \nu(\bar{\nu}) < \nu(\mu) \). If \( \bar{\nu} <_1 \nu \) then \( \bar{\nu} \) is \( <_0 \)-minimal, successor, limit iff \( \mu <_0 \bar{\nu} \), \( \mu = \pi_{\bar{\nu}\nu}(\bar{\nu}) \) or when \( (\pi, \mu) = (\mu(\nu), \nu(\nu)) \). If \( \bar{\nu} <_1 \mu \) then \( \bar{\nu} \) is \( <_0 \)-minimal, successor, limit iff \( \mu \) is \( <_0 \)-minimal, successor, limit.

(d) If \( \sigma <_1 \nu \) then \( \pi = \pi_{\bar{\nu}\nu} : \mu(\bar{\nu}) \to \mu(\nu) \) is order-preserving, \( \pi^{-1}[\alpha(\nu)] = \alpha(\nu) \cap \sigma \) and \( \pi^{-1}[\nu] = \pi(\nu)^+ \) whenever \( \pi(\nu)^+ < \mu(\nu) \). If \( \sigma_0 <_0 \sigma \) and \( \nu_0 = \pi(\sigma_0) \) then \( \sigma_0 <_1 \nu_0 \) and \( \pi_{\sigma_0\nu_0} = \pi(\sigma_0(\nu)) \). If \( \sigma \) is a \( <_0 \)-limit and \( \lambda = \bigcup \text{Range}(\mu(\nu)) \) then \( \sigma <_1 \lambda \) and if \( \mu_\lambda = \bigcup \text{Range}(\pi_{\sigma\nu}) \) and \( \mu = \bigcup \text{Range}(\pi_{\sigma\nu}) \) then \( \mu_\lambda <_1 \mu \) and \( \pi_{\sigma\nu} = \pi_{\mu_\lambda\nu} \circ \pi_{\sigma\lambda} \). If \( \bigcup \text{Range}(\mu(\nu)) = \nu(\nu) \) and \( \alpha = \alpha(\nu) \) for some \( \nu_0 <_1 \pi_{\sigma\nu}(\sigma_0) \) for each \( \sigma_0 < \sigma \) then \( \alpha = \alpha(\nu) \) for some \( \nu' <_1 \nu \). Similarly for \( \pi_{\sigma\mu} \) when \( \mu <_1 \mu \), with \( (\mu(\nu), \mu(\mu), \alpha(\nu), \alpha(\nu)) \) replaced by \( \pi, \nu(\nu), \nu(\mu), \pi_{\sigma(\mu)} \) and \( \pi_{\mu(\nu)} \) replaced by the identity.

(e) \( \bar{\nu} <_1 \sigma <_1 \nu \to \pi_{\sigma\nu} = \pi_{\nu\nu} \circ \pi_{\sigma\nu} \). For \( \nu \in S^1 \), \( \{ \alpha(\sigma) \mid \sigma <_1 \nu \} \) is closed in \( \alpha(\nu) \) and unbounded unless \( \nu \) is \( <_0 \)-maximal. If \( \{ \alpha(\sigma) \mid \sigma < \nu \} \) is unbounded in \( \alpha(\nu) \) then \( \nu(\nu) = \bigcup \{ \text{Range}(\pi_{\sigma\nu}) \mid \sigma <_1 \nu \} \). Similarly for \( \mu \in S^2 \), with \( \alpha(\nu), \mu(\nu) \) replaced by \( \nu(\mu), \mu \).

(f) Suppose \( \sigma <_1 \nu \). Then \( \sigma < \mu(\bar{\nu}) \) iff \( \nu < \mu(\nu) \). Suppose now that \( \sigma < \bar{\nu} \) and \( \nu < \mu \). Then \( \bar{\nu} \) is \( <_1 \)-minimal, successor, limit iff \( \mu \) is \( <_1 \)-minimal, successor, limit, as for \( \bar{\mu}_0 <_1 \bar{\mu}_1 < \bar{\mu}_2 \). \( \bar{\mu}_0 <_1 \bar{\mu}_1 \) if \( \pi_{\sigma\nu}(\bar{\mu}_0) <_1 \pi_{\sigma\nu}(\bar{\mu}_1) \), and in addition \( \bar{\mu}_0 = \nu(\bar{\mu}_0) \) for some \( \bar{\mu}_0 <_1 \bar{\mu} \) if \( \pi_{\sigma\nu}(\bar{\nu}) = \nu(\bar{\mu}_0) \) for some \( \mu_0 <_1 \mu \).

(g) Suppose \( \sigma <_1 \nu \), \( \sigma < \mu(\bar{\nu}) \) and \( \bar{\nu} = (\pi(\bar{\nu})) \) is a \( <_1 \)-successor. Let \( \bar{\mu}_0 <_1 \bar{\mu} \) denote that \( \bar{\mu}_0 \) is the \( <_1 \)-predecessor to \( \bar{\mu} \). Then if \( \mu_0 <_1 \mu \) we have \( \pi_{\sigma\nu}(\nu(\bar{\mu}_0)) = \nu(\mu_0) \). And \( \pi_{\sigma\mu} \) is cofinal iff \( \pi_{\sigma\mu} \) is cofinal. If \( \pi_{\sigma\mu} \) is not cofinal and \( \lambda = \bigcup \text{Range}(\pi_{\sigma\mu}) \) then \( \pi_{\nu\nu}(\lambda_0) = \mu_0 \) and \( \pi_{\nu\nu}(\lambda) = \bigcup \text{Range}(\mu_{0\lambda}) \).

(h) Suppose \( \sigma < \mu(\bar{\nu}) = \bar{\nu} \) and \( \bar{\nu} \) is not a \( <_1 \)-limit. Then \( \pi_{\sigma\nu} \) is cofinal iff \( \pi_{\sigma\nu} \) is cofinal.

(i) Suppose \( \sigma < \mu(\bar{\nu}) = \bar{\nu} \) and \( \sigma <_1 \nu \). If \( \bar{\mu}_0 <_1 \bar{\mu}_1 \leq \bar{\mu} \), \( \mu_0 <_1 \mu_1 = \pi_{\nu\nu}(\bar{\mu}_1) \) (or \( \mu_0 <_1 \mu(\nu) \) if \( \bar{\nu} = \pi(\bar{\nu}) \)) and \( \nu(\mu_0) = \pi_{\nu\nu}(\nu(\bar{\mu}_0)) \) then \( \pi_{\nu\nu}(\bar{\nu}_0) = \pi_{\nu\nu}(\bar{\mu}_0) \).

Remarks. (a) Jensen points out that we cannot have perfect tree preservation, which would say: \( \bar{\nu}_0 <_1 \bar{\nu}_1 \iff \pi_{\nu\nu}(\bar{\nu}_0) \leq \pi_{\nu\nu}(\bar{\nu}_1) \) for \( \bar{\nu}_1 \leq \mu(\bar{\nu}) \). (We take \( \pi_{\nu\nu}(\nu(\bar{\nu})) = \nu(\nu) \) rather than \( \pi_{\nu\nu}(\bar{\nu}) = \mu' \). So \( \pi_{\nu\nu} \) may send \( \bar{\nu} \) to \( \mu(\bar{\nu}) \) to \( \mu \) where
ν(μ) = ν(μ'), μ' < * μ(ν), μ ≠ μ'. If π_ρ(ν) is not cofinal, however, this will not happen and we get μ = μ'.

(b) Though π_ρ may fail to preserve the relation π_0 <_1 π_1 when π_1 = μ(σ) it does preserve the following relation ⊣: π_0 ⊨ π_1 iff there are π_0 <_1 π_1, π_1 <_1 π_1, π_0 <_1 π_1 and π_0 <_1 π_ρ(π_0). Moreover, in case π_ρ is cofinal (the troublesome case for <_1-preservation) then π_1 is the direct limit of {π_0 | π_0 ⊨ π_1} via natural maps (if σ ⊨ π_γ then f_σγ is π_σγ where σ <_1 σ, π <_1 γ = π_τ(σ)).

(c) Of course one could formulate “Gap 2 Morass with Square” but as we know of no applications of this principle, we have elected not to do so here for the sake of simplicity.

Proof. (a)–(c). This is just as in the gap 1 case, with one exception: we must show that if π is <_0-limit, π < μ(σ), π <_1 ν, π_ρ | π not cofinal, λ = ∪ Range(π_ρ | π), μ_λ = ∪ Range(π_ρ) and μ = ∪ Range(π_ρ) then μ_λ <_1 μ and π_ρ = π_ρ | π o π_ρ. Consider H_λ = H_σ(ν, λ ∪ {ν(ν)}) and H = H_σ(ν, ν ∪ {ν(ν)}) where σ = ∪(ν, λ−1) ∩ H_σ(ν, α(π) ∪ {ν(ν)}).

Then H_λ is Σ^*_n(π)-elementary in H and after transitive collapse yields π_ρ. And π_ρ = π_ρ o π_ρ follows from the fact that π_ρ, π_ρ are obtained respectively by collapsing the inclusion of H_σ(ν, α(π) ∪ {ν(ν)}) in H, H_λ.

(f) The fact that π > π if μ > ν is clear from Σ^*_n(π)-elementarity of π_ρ. For the rest, first suppose that for some μ_0 <_1 μ, π_ρ_0 is cofinal. If μ is a <_1-successor then we can take μ_0 <_1 μ and then we have ν(μ_0) ∈ q(ν) and therefore ν(μ_0) = ν(π_0) for some π_0, where π = π_ρ. Then H_σ(ν, ν | π_0) where σ = ∪ Range(π(π_0) ∩ {ν(π)}) is cofinal in π and π ⊨ π = π_0. So we get π_0 <_1 π and ν(π_0) = π_0. If μ is a <_1-limit then ν(μ) = ν(π) and π is therefore Σ^*_n(π)-elementary. Thus Range(π) ∩ {ν(μ_0) | μ_0 <_1 μ} is unbounded in Range(π) ∩ μ so π is a <_1-limit as all maps π_ρ_0, μ_0 <_1 μ sufficiently large, are cofinal.

Second, suppose that there is no cofinal π_ρ_0 with μ_0 <_1 μ. If μ (n(μ) ∩ (μ_0 <_1 μ) we must have Σ^*_n(μ)+1-elementarity for π_ρ_0 (see the proof of (g) from the gap 1 case). Thus if μ is a <_1-limit then n(μ) ≥ n(μ)+2 and we see that π is a <_1-limit as {ν(μ) | μ_0 <_1 μ} is Π^*_n(μ)+1. If μ is not a <_1-limit then max{ν(μ_0) | μ_0 <_1 μ} belongs to Range(π) as it is either in q(ν) or 0. Thus π is not a <_1-limit and if μ_0 <_1 μ then ν(μ_0) = ν(π_0) where π_0 <_1 π. If μ is <_1-minimal then so is π. Finally, if n(μ) = n(ν) then μ is not a <_1-limit; if ν_0 = ∪{ν' < ν | ν' = ν ∩ H_σ(ν', ν' ∪ (q(μ)))} then ν_0 ∈ Range(π) but now since π is Q-elementary we get H_σ(ν_0 ∪ (q(μ))) ∩ μ bounded below μ* = ∪(Range(π) ∩ μ). It follows that μ_0 <_1 μ if μ_0 <_1 μ*, so π is not a <_1-limit and if μ_0 <_1 μ then ν(μ_0) = π(π_0) where π_0 <_1 π. If μ is <_1-minimal then so is π.
Note that in the above argument we also verified the final statement of (f). The remaining claim in (f) is clear by $\Sigma_1$-elementarity.

(g) The argument in the proof of (f) showed that $\pi_{\bar{\nu}}(\nu(\bar{\mu}_0)) = \nu(\mu_0)$ and $\pi_{\bar{\mu}_0}\mu$ cofinal if $\pi_{\mu_0}\mu$ cofinal. Finally, if $\lambda = \bigcup \text{Range}(\pi_{\mu_0}\mu) < \mu$ then we get $\lambda \in \text{Range}(\pi_{\nu})$ by the argument in (f), and hence $\mu_0 \in \text{Range}(\pi_{\bar{\nu}})$. Then we must have $\pi^{-1}(\mu_0) = \bar{\mu}_0$.

(h) If $\mu$ is not a $<_1$-limit then either $n(\mu) = n(\nu)$ and the result follows easily or $H_{n(\mu)}(\nu_0 \cup \{q(\mu)\}) \cap \mu$ is bounded in $\mu$ for each $\nu_0 < \nu$, which means that $X \cap \mu$ bounded in $\mu$ iff $X \cap \nu$ bounded in $\nu$ for any $X$ which is $\Sigma_1^*\nu$-elementary in $\tilde{J}_\beta(\nu)$.

(i) This is clear if $\bar{\mu}_1 < \bar{\mu}$. Otherwise it follows immediately when $\pi_{\bar{\nu}}(\bar{\mu}_0) = \mu_0$ and otherwise by the fact that $\pi_{\bar{\mu}_0}\mu$ is given by $H_{n(\bar{\mu})}(\nu(\bar{\mu}_0) \cup \{q(\bar{\mu})\})$, $\pi_{\mu_0}\mu$ is given by $H_{n(\mu)}(\nu(\mu_0) \cup \{q(\mu)\})$ and $\pi_{\bar{\nu}}(\nu(\bar{\mu}_0)) = \nu(\mu_0)$.

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