FUNDAMENTA
MATHEMATICAE
154 (1997)

The Σ^* approach to the fine structure of L

by

Sy D. Friedman (Cambridge, Mass.)

Abstract. We present a reformulation of the fine structure theory from Jensen [72] based on his Σ^* theory for K and introduce the Fine Structure Principle, which captures its essential content. We use this theory to prove the Square and Fine Scale Principles, and to construct Morasses.

1. The J-hierarchy. The most elegant hierarchy for Gödel's L is obtained through iterated first-order definability. For any set x let $\mathbf{Def}(x)$ denote $\{y \mid y \subseteq x,\ y$ is definable over $\langle x, \in\rangle$ by a first-order formula with parameters$\}$. Then L is obtained as the union of all L_α, where $L_0 = \emptyset$, $L_\lambda = \bigcup\{L_\alpha \mid \alpha < \lambda\}$ for limit λ, and

$$L_{\alpha+1} = \mathbf{Def}(L_\alpha).$$

Unfortunately, $L_{\alpha+1}$ is not closed under pairing and for this reason, Jensen [72] defined a modified hierarchy $\langle J_\alpha \mid \alpha \in \mathrm{ORD}\rangle$ for L to get around this problem. We now present a description of the J-hierarchy which, as above, is based on the idea of iterated definability.

Recall the Lévy hierarchy of formulas: A formula is Σ_0 $(= \Delta_0 = \Pi_0)$ if it is built from atomic formulas through the use of logical connectives and bounded quantifiers $\forall x \in y, \exists x \in y$. A formula is Σ_{n+1} if it is of the form $\exists \vec{x} \varphi$ where φ is Π_n. Dually, a formula is Π_{n+1} if it is of the form $\forall \vec{x} \varphi$ where φ is Σ_n. Every formula is logically equivalent to a Σ_n formula for some n, as it can be put into prenex normal form.

We want to define the J-hierarchy so that $J_{\alpha+1} \cap P(J_\alpha) = \mathbf{Def}(J_\alpha)$, $J_{\alpha+1}$ is closed under pairing and in addition, $J_{\alpha+1}$ satisfies Σ_0-Comprehension. The latter is the statement that for any x we can form $\{y \in x \mid \varphi(y)\}$, where φ is a Σ_0 formula with arbitrary parameters. This is important for the construction of universal Σ_n predicates, a notion that we define next.

1991 *Mathematics Subject Classification*: 03E05, 03E45.

Research supported by NSF contract number 9205530.

A binary relation $W_n(e,x)$ on a transitive set S is a *universal Σ_n predicate* for S if it is Σ_n-definable over $\langle S,\in\rangle$ without parameters and wherever $Y\subseteq S$ is Σ_n-definable over $\langle S,\in\rangle$ with parameters, there exists $e\in S$ such that

$$Y=\{x\in S\mid W_n(e,x)\}.$$

Thus the sets $\{x\in S\mid W_n(e,x)\}$ are exactly the sets Σ_n-definable over $\langle x,\in\rangle$ with parameters, as e varies over S.

LEMMA 1. *Suppose that S is a transitive set closed under pairing, satisfying Σ_0-Comprehension + "Every set has a transitive closure." Then there exists a universal Σ_n predicate for S.*

Proof. It is enough to treat the case $n=1$, as for example to get W_2 from W_1 we can just define $W_2(e,x)\leftrightarrow\exists y\sim W_1(e,\langle x,y\rangle)$.

Let $\langle\varphi_i\mid i\in\omega\rangle$ be a standard list of formulas with one free variable with subformulas enumerated earlier, and define $\mathrm{Sat}(z,i,x)$ to mean: z is transitive, $x\in z$ and $\langle z,\in\rangle\models\varphi_i(x)$. Sat can be expressed by a Σ_1 formula:

> $\mathrm{Sat}(z,i,x)\leftrightarrow z$ is transitive, $x\in z$ and $\exists Y\subseteq(i+1)\times z$ such that $\{\forall j\le i$ [If $\varphi_j(x)$ is atomic then $\langle j,x\rangle\in Y\leftrightarrow\varphi_j$ true; if $\varphi_j(x)$ is $\exists y\ \varphi_{j'}(\langle x,y\rangle)$ then $\langle j,x\rangle\in Y\leftrightarrow\exists y\in z\ (\langle j',\langle x,y\rangle\rangle\in Y)$; if $\varphi_j(x)$ is $\sim\varphi_{j'}(x)$ then $\langle j,x\rangle\in Y\leftrightarrow\langle j',x\rangle\notin Y$; if $\varphi_j(x)$ is $\varphi_{j_1}(x)\wedge\varphi_{j_2}(x)$ then $\langle j,x\rangle\in Y\leftrightarrow(\langle j_1,x\rangle\in Y$ and $\langle j_2,x\rangle\in Y)$] and $\langle i,x\rangle\in Y\}$.

The fact that S satisfies pairing and Σ_0-Comprehension implies that when restricted to S, Sat is Σ_1-definable over $\langle S,\in\rangle$, via the above definition. Finally, we set:

$$W_1(e,x)\leftrightarrow e=\langle i,p\rangle \text{ and for some transitive } z,\ \mathrm{Sat}(z,i,\langle x,p\rangle).$$

W_1 is universal, using pairing, the existence of transitive closures and the persistence of Σ_1 formulas over transitive sets. ∎

We are ready to define the J-hierarchy. By induction on α we define J_α to satisfy the hypotheses of Lemma 1. Let $W_n^\alpha(e,x)$ denote the canonical universal Σ_n predicate for J_α coming from the proof of Lemma 1. For $\alpha=0$ we have $J_0=\emptyset$ and for $\alpha=1$ we have $J_1=L_\omega$. For α limit, $J_\alpha=\bigcup\{J_\beta\mid\beta<\alpha\}$. Note that the hypotheses of Lemma 1 are met by J_α, given that they are met by each $J_\beta,\ \beta<\alpha$.

Suppose that J_α and $W_n^\alpha(e,x)$ are defined for some $\alpha>0$ and we wish to define $J_{\alpha+1}$. An *n-code* is a pair (n,e) where $e\in J_\alpha$. By induction on n define

$$X(0,e)=e,$$
$$X(n+1,e)=\{X(n,f)\mid W_{n+1}^\alpha(e,f)\}.$$

Then $J_{\alpha,n}=\{X(n,e)\mid e\in J_\alpha\}$ and $J_{\alpha+1}=\bigcup\{J_{\alpha,n}\mid n\in\omega\}$.

LEMMA 2. (a) $n \le m \to J_{\alpha,n} \subseteq J_{\alpha,m}$.

(b) $J_{\alpha,n}$ *is transitive.*

(c) $\mathrm{ORD}(J_{\alpha,n}) = \omega\alpha + n$.

(d) $J_{\alpha+1} \models$ *Pairing* $+\,\Sigma_0$*-Comprehension.*

(e) $J_{\alpha+1} \cap P(J_\alpha) = \mathbf{Def}(J_\alpha)$.

Proof. (a) By induction on n, we define a $\Sigma_1(J_\alpha)$ function $F(n,e)$, $e \in J_\alpha$, that produces $f \in J_\alpha$ such that $X(n,e) = X(n+1,f)$. For $n = 0$, let $F(0,e) = f$ where $\{g \mid W_1^\alpha(f,g)\} = e$; then $X(0,e) = e = \{g \mid W_1^\alpha(f,g)\} = X(1,f)$. Suppose that $F(n,e)$ has been defined for all e. Then let $F(n+1,e) = f$ where $\{g \mid W_{n+2}^\alpha(f,g)\} = \{F(n,h) \mid W_{n+1}^\alpha(e,h)\}$; clearly f exists as F restricted to pairs (n,h), $h \in J_\alpha$, is $\Sigma_1(J_\alpha)$ and therefore the latter set is $\Sigma_{n+1}(J_\alpha)$ with parameter e. Finally, we get $X(n+2,f) = \{X(n+1,F(n,h)) \mid W_{n+1}^\alpha(e,h)\} = \{X(n,h) \mid W_{n+1}^\alpha(e,h)\}$ by induction, and the latter set is $X(n+1,e)$.

(b) $J_{\alpha,0} = J_\alpha$ is transitive by induction on α, and if $x \in J_{\alpha,n+1}$ then $x \subseteq J_{\alpha,n}$ and hence $x \subseteq J_{\alpha,n+1}$ by (a).

(c) Clearly $\mathrm{ORD}(J_{\alpha,n}) \le \omega\alpha + n$ since $x \in J_{\alpha,n+1} \to x \subseteq J_{\alpha,n}$. By induction on n, define e_{n+1} such that $X(n+1,e_{n+1}) = \omega\alpha+n$: For $n = 0$ we can take e_1 so that $\omega\alpha = \{f \mid W_1^\alpha(e_1,f)\}$. If e_{n+1} is defined take e_{n+2} so that $\{f \mid W_{n+2}^\alpha(e_{n+2},f)\} = \{F(n,g) \mid W_{n+1}^\alpha(e_{n+1},g)\} \cup \{e_{n+1}\}$, where F is from the proof of (a). Then $X(n+2,e_{n+2}) = X(n+1,e_{n+1}) \cup \{X(n+1,e_{n+1})\} = \omega\alpha + n + 1$.

(d) $J_{\alpha+1}$ is closed under pairing because all 2-element subsets of $J_{\alpha,n}$ belong to $J_{\alpha,n+1}$. For Σ_0-Comprehension note that $J_{\alpha,n} = \{X(n,e) \mid e \in J_\alpha\} = X(n+1,f)$ for some f so $J_{\alpha,n} \in J_{\alpha,n+1}$ and it suffices to show that if $X \subseteq J_{\alpha,n}$ is definable over $\langle J_{\alpha,n}, \in\rangle$ then X belongs to $J_{\alpha,m}$ for some m. But $\{e \mid X(n,e) \in X\}$ is a definable subset of J_α as $\langle J_{\alpha,n}, \in\rangle$ with the additional function $(n,e) \to X(n,e)$ is isomorphic to a structure definable over J_α. Choose m so that this set is Σ_m-definable over J_α and using F from the proof of (a), produce a $\Sigma_1(J_\alpha)$ G such that for each e, $X(n,e) = X(m,G(e))$. Then $\{G(e) \mid X(n,e) \in X\}$ is Σ_m-definable over J_α and $X = \{X(m,G(e)) \mid X(n,e) \in X\}$ belongs to $J_{\alpha,m+1}$.

(e) We get $\mathbf{Def}(J_\alpha) \subseteq J_{\alpha+1}$ by (d). Conversely, if $X(n,e) \subseteq J_\alpha$ then $\{f \mid f \in X(n,e)\} = X(n,e)$ is a definable subset of J_α, using the definition of $X(n,e)$. ∎

Of course now we may define $W_n^{\alpha+1}(e,x)$, using (d) of Lemma 2, thereby completing the definition of the J-hierarchy. It is occasionally convenient to refer to the refined hierarchy $\langle \widetilde{J}_\alpha \mid \alpha \in \mathrm{ORD}\rangle$ defined by $\widetilde{J}_{\omega\alpha+n} = J_{\alpha,n}$ and conveniently: $\mathrm{ORD}(\widetilde{J}_\alpha) = \alpha$.

LEMMA 3. (a) $\langle \widetilde{J}_\alpha \mid \alpha < \lambda \rangle$ *is* $\Sigma_1(\widetilde{J}_\lambda)$ *for limit* λ, *via a definition independent of* λ.

(b) *There is a* $\Sigma_1(J_\alpha)$ *well-ordering* $<_\alpha$ *of* J_α, *via a definition independent of* α.

(c) (Condensation) *If* $\langle X, \in \rangle$ *is* Σ_1*-elementary in* $\langle J_\alpha, \in \rangle$ *then* $\langle X, \in \rangle \simeq \langle J_{\bar{\alpha}}, \in \rangle$ *for some* $\overline{\alpha}$.

Proof. (a) We have $x = \widetilde{J}_\alpha \leftrightarrow \exists \langle x_\beta \mid \beta < \gamma \rangle$ such that $x = x_\alpha$ where $x_n = L_n$ for finite $n < \gamma$, $x_\lambda = \bigcup \{x_\beta \mid \beta < \lambda\}$ for limit $\lambda < \gamma$ and for $\lambda + n < \gamma$, λ limit, $x_{\lambda+n}$ is obtained from x_λ as in the definition of $J_{\alpha,n}$ from J_α, $\alpha > 0$. This definition works inside any $\widetilde{J}_\lambda$, λ limit.

(b) Define well-orderings $<_\alpha$ of J_α as follows: $<_0 = \emptyset$, $<_1 =$ some L_ω-definable well-ordering of L_ω; $x <_\lambda y \leftrightarrow x <_\alpha y$ for some $\alpha < \lambda$ for limit λ; $x <_{\alpha+1} y \leftrightarrow x <_\alpha y$ or for some $n, y \in J_{\alpha,n+1} - J_{\alpha,n}$ and either $x \in J_{\alpha,n}$ or ($<_\alpha$-least e such that $X(n+1, e) = x$) $<_\alpha$ ($<_\alpha$-least e such that $X(n+1, e) = y$). Then $<_\alpha$ is $\Sigma_1(J_\alpha)$, via a definition independent of α.

(c) Let $\langle X, \in \rangle \simeq \langle \overline{X}, \in \rangle$ be the transitive collapse of X. Then $\langle \overline{X}, \in \rangle \models \Sigma_0$-Comprehension $+ \forall x\ \exists \beta\ (x \in \widetilde{J}_\beta) + \forall \beta\ \exists y\ (y = \widetilde{J}_\beta)$. But Σ_0-Comprehension gives $(\widetilde{J}_\beta)^{\overline{X}} = \widetilde{J}_\beta$ for $\beta \in \overline{X}$ so that $\overline{X} = \widetilde{J}_{\omega\bar{\alpha}} = J_{\bar{\alpha}}$ where $\omega\overline{\alpha} = \mathrm{ORD}(\overline{X})$. ■

Σ_1-Skolem functions. Condensation, as stated in Lemma 3(c), is a powerful tool for proving things about L. To unleash its power, we must first provide a method for generating Σ_1-elementary submodels. Fix an ordinal $\alpha > 0$.

DEFINITION 1. Suppose $X \subseteq J_\alpha$. The Σ_1*-hull* of X is the smallest Σ_1-elementary submodel of J_α containing X as a subset. A Σ_1*-Skolem function* is a partial function $h : \omega \times J_\alpha \to J_\alpha$ with Σ_1 graph such that for any $X \subseteq J_\alpha$, Σ_1-hull of $X = \{h(n, x) \mid n \in \omega,\ x$ a finite sequence from $X\}$.

LEMMA 4. *For any* $X \subseteq J_\alpha$, *the* Σ_1*-hull of* X *exists. Moreover*, *there is a* Σ_1*-Skolem function for* J_α, *with a* Σ_1*-definition independent of* α.

Proof. Let $\varphi_0, \varphi_1, \ldots$ be a standard list of formulas of 2 free variables and define $h^*(n, x) = <_\alpha$-least pair (y, t) such that $x, y \in t$, t transitive, $\langle t, \in \rangle \models \varphi_n(x, y)$; if no such pair (y, t) exists then $h^*(n, x)$ is undefined. Then $h(n, x) = y$ when $h^*(n, x) = (y, t)$. Any Σ_1-elementary submodel of J_α must be closed under h, and clearly for any $X \subseteq J_\alpha$, $\{h(n, x) \mid x$ a finite sequence from $X\}$ is a Σ_1-elementary submodel of J_α. ■

The key to Fine Structure Theory is to find a suitable generalization of Lemma 4 to higher levels of definability. We will take this up in the next section.

We close this section with an illustration of how Σ_1-hulls can be used to prove a version of Jensen's $\diamond$-principle in L (see Jensen [72]). Our version will include some technical conditions which are of use in our proof of Jensen's Coding Theorem (see Friedman [94]). Assume $V = L$ and let α be an infinite cardinal.

DEFINITION 2. $C \subseteq \alpha^+$ is *closed unbounded* (CUB) if $\bigcup C = \alpha^+$ and $\bigcup(C \cap \beta) \in C$ for each $\beta < \alpha^+$. $S \subseteq \alpha^+$ is *stationary* if $S \cap C \neq \emptyset$ for each CUB $C \subseteq \alpha^+$.

For $\mu < \alpha^+, \beta'(\mu)$ denotes the largest β such that either $\beta = \mu$ or $\mu < \beta, J_\beta \models \mu$ is a cardinal greater than α.

LEMMA 5. *There exists* $\langle D_\mu \mid \mu < \alpha^+\rangle$ *such that* $D_\mu \subseteq J_\mu$ *and*:

(a) *If* $D \subseteq J_{\alpha^+}$ *then* $\{\mu < \alpha^+ \mid D \cap J_\mu = D_\mu\}$ *is stationary.*
(b) D_μ *is uniformly definable as an element of* $J_{\beta'(\mu)}$, *for* $\mu < \alpha^+$.
(c) *If* $J_{\beta'(\mu)} \models \alpha^{++}$ *exists or* $\mu = \beta'(\mu)$ *then* $D_\mu = \emptyset$.

Proof. Let $D_\mu = \emptyset$ if $J_{\beta'(\mu)} \models \alpha^{++}$ exists or $\mu = \beta'(\mu)$ and otherwise let $\langle D_\mu, C_\mu\rangle$ be least in $J_{\beta'(\mu)}$ such that C_μ is CUB in μ, $D_\mu \subseteq J_\mu$ and $\overline{\mu} \in C_\mu \to D_\mu \cap J_{\bar\mu} \neq D_{\bar\mu}$; if $\langle D_\mu, C_\mu\rangle$ does not exist then let $D_\mu = \emptyset$. We need only prove (a).

Suppose (a) fails and let $\langle D, C\rangle$ be least in $J_{\alpha^{++}}$ such that $D \subseteq J_{\alpha^+}$, C is CUB in α^+ and $\mu \in C \to D \cap J_\mu \neq D_\mu$. Let $\sigma < \alpha^{++}$ be least such that $\omega\sigma = \sigma$, $J_\sigma \models \alpha^+$ is the largest cardinal and $\langle D, C\rangle \in J_\sigma$. Let $H = \Sigma_1$-hull of $\alpha \cup \{\alpha^+\}$ in J_σ and $\mu = H \cap \alpha^+$. Then $\langle H, \in\rangle \simeq \langle J_{\beta'}, \in\rangle$ for some β' and since $J_{\beta'} \models \mu = \alpha^+$ we have $\beta' \leq \beta'(\mu)$. But now we have $\langle D_\mu, C_\mu\rangle = \langle D \cap J_\mu, C \cap \mu\rangle$ and since $\mu = \bigcup(C \cap \mu) \in C$, this is a contradiction. ∎

2. Fine Structure Theory. Our main goal is to develop a version of Lemma 4 for higher levels of definability. Specifically, we want to define the notion of Σ_n^* formula so as to obtain:

(a) There is a universal Σ_n^* predicate for J_α for each n.
(b) For any $X \subseteq J_\alpha$, the Σ_n^*-hull of X in J_α exists for each n.
(c) There is a Σ_n^*-Skolem function for J_α for each n.
(d) Every formula is Σ_n^* for some n.

What happens if we just take $\Sigma_n^* = \Sigma_n$? Then (a) holds by Lemma 1 and (d) is clear.

PROPOSITION 1. *For any* $X \subseteq J_\alpha$ *and* $n \in \omega$ *there is a least* Σ_n-*elementary submodel of* J_α *containing* X *as a subset.*

Proof. Let $M = \{y \in J_\alpha \mid$ For some Σ_n formula φ with parameters from X, y is the $<_\alpha$-least solution to φ in $J_\alpha\}$. Then M is Σ_n-elementary in

J_α since if y_i is the $<_\alpha$-least solution to φ_i, $1 \le i \le n$, then $\langle y_1 \dots y_n\rangle$ is the $<_\alpha$-least solution to $\varphi_1((z)_1) \wedge \dots \wedge \varphi_n((z)_n)$, where $(z)_i = i$th component of z. Suppose $X \subseteq N$, N is Σ_n-elementary in J_α and φ is a Σ_n formula with parameters from X with a solution in J_α. Then φ has a solution y_0 in N and if y_0 is not the least solution then N also has a solution $y_1 <_\alpha y_0$. Continuing in this way we see that in fact N does contain the $<_\alpha$-least solution to φ and hence we get $M \subseteq N$. ∎

So (b) holds. What fails is property (c):

PROPOSITION 2. *For some α there is no Σ_2-Skolem function for J_α.*

Proof. Let κ denote ω_1. For each limit $\alpha < \omega_1, \alpha$ is the least β such that $\widetilde{J}_{\kappa+\alpha} \models \kappa+\beta$ does not exist. If $\widetilde{J}_{\kappa+\alpha}$ has a Σ_2-Skolem function then α must be the unique solution in $\widetilde{J}_{\kappa+\alpha}$ to a Σ_2 formula $\exists x\ \forall y\ \varphi_\alpha$ where φ_α is Σ_0 with parameter κ. Suppose that each $\widetilde{J}_{\kappa+\alpha}$ has a Σ_2-Skolem function and by Fodor's Theorem choose φ and $\alpha_0 < \kappa$ such that for stationary-many α, $\varphi_\alpha = \varphi$ and $\widetilde{J}_{\kappa+\alpha} \models \exists x \in \widetilde{J}_{\kappa+\alpha_0} \forall y\ \varphi$ holds at α. But then choose any $\alpha < \beta$ in this stationary set, $\alpha_0 < \alpha$ and we have $\widetilde{J}_{\kappa+\alpha} \models \exists x\ \forall y\ \varphi$ holds at both α and β. Contradiction. ∎

Remark. A result similar to the previous appears in Devlin [84], pages 106–107.

It is shown in Jensen [72] that for any α and any n there is a partial Σ_n function with parameters that can serve as a Σ_n-Skolem function for Σ_n-hulls without parameters. However, this does not achieve our goal as the definition of the necessary parameters does not reflect to arbitrary Σ_n-elementary submodels that contain them.

Instead we take an approach based on the idea that in a certain sense Σ_{n+1} can be viewed as Σ_1 relativized to Σ_n, for an arbitrary J_α. Though this is only true for the usual Lévy hierarchy when awkward parameters are introduced, we define Σ^*_n in such a way that this is true using only "standard" parameters, whose definitions relativize without difficulty to Σ^*_n-hulls. Our approach is derived from Jensen's Σ^* Theory in Jensen [?]. Σ^*_n in our sense corresponds to $\Sigma_1^{(n-1)}$ in Jensen's terminology.

The Σ^*_n-hierarchy. In order to define the notion of Σ^*_n formula we must also define the auxiliary notions of nth reduct and nth standard parameter, all by induction on n.

Let M denote some fixed J_α, $\alpha > 0$. We order finite sets of ordinals by the maximum difference order: $x < y$ iff $\beta \in y$, where β is the largest element of $(y - x) \cup (x - y)$.

A Σ^*_1 *formula* is just a Σ_1 formula. The Σ^*_1 *projectum* of M, denoted by ϱ^M_1, is the least ϱ such that there is a subset of $\omega\varrho$ which is Σ^*_1 with parameters but not an element of M. The 1st *standard parameter* of M, denoted by p_1^M, is the least finite set of ordinals p such that $A \cap \omega\varrho_1^M \notin M$ for some A which is Σ_1^* with parameter p. We use H_1^M to denote $J_{\varrho_1^M}$ and for any $x \in M$, $A_1(x) = \{\langle y, m\rangle \mid \text{the } m\text{th } \Sigma_1^* \text{ formula is true at } \langle y, x, p_1^M\rangle, y \in H_1^M\}$. The 1st *reduct* of M relative to x, denoted by $M_1(x)$, is the structure $\langle H_1^M, A_1(x)\rangle$.

For $n \geq 1$: a Σ_{n+1}^* *formula* is one of the form $\varphi(x) \leftrightarrow M_n(x) \models \psi$, where ψ is Σ_1. The Σ_{n+1}^* *projectum* of M, denoted by ϱ_{n+1}^M, is the least ϱ such that there is a subset of $\omega\varrho$ which is Σ_{n+1}^* with parameters but not an element of M. The $(n+1)$st *standard parameter* of M, denoted by p_{n+1}^M, is $p_n^M \cup p$ where p is the least finite set of ordinals such that $A \cap \omega\varrho_{n+1}^M \notin M$ for some A which is Σ_{n+1}^* with parameter $p_n^M \cup p$. We use H_{n+1}^M to denote $J_{\varrho_{n+1}^M}$ and for any $x \in M$, $A_{n+1}(x) = \{\langle y, m\rangle \mid \text{the } m\text{th } \Sigma_{n+1}^* \text{ formula is true at } \langle y, x, p_{n+1}^M\rangle, y \in H_{n+1}^M\}$. The $(n+1)$st *reduct* of M relative to x, denoted by $M_{n+1}(x)$, is the structure $\langle H_{n+1}^M, A_{n+1}(x)\rangle$.

This completes the definition of the Σ_n^*-hierarchy. Thus a Σ_{n+1}^* formula is a formula expressing a Σ_1 property on nth reducts, uniformly. In order to achieve amenability when relativizing to a Σ_n^* predicate, we take our nth reduct to have ordinal height $\omega\varrho_n^M$.

LEMMA 6. (a) *If φ and ψ are Σ_n^* formulas then $\varphi \vee \psi$ and $\varphi \wedge \psi$ are equivalent to Σ_n^* formulas.*

(b) *If φ is a Σ_n^* formula then both φ and $\sim\varphi$ are equivalent to Σ_{n+1}^* formulas.*

(c) *There is a universal Σ_n^* formula, i.e., a Σ_n^* formula $\varphi(e, x)$ such that if $\psi(x)$ is Σ_n^* then for some $e \in \omega$, $\psi(x) \leftrightarrow \varphi(e, x)$ for all x.*

(d) *The reduct $M_n(x) = \langle H_n^M, A_n(x)\rangle$ is amenable, i.e., if $y \in H_n^M$ then $y \cap A_n(x) \in H_n^M$.*

Proof. (a) is clear because a Σ_{n+1}^* formula is of the form $\varphi(x) \leftrightarrow M_n(x) \models \psi, \psi\ \Sigma_1$ and Σ_1 is closed under $\vee$ and $\wedge$.

(b) If $\varphi(x)$ is Σ_n^* then so is $\varphi'(y, x, z) \leftrightarrow \varphi(x)$ and choose k so that φ' is the kth Σ_n^* formula. Then $\varphi(x) \leftrightarrow \langle \emptyset, k\rangle \in A_n(x)$ so φ is equivalent to a Σ_{n+1}^* formula. Similarly for $\sim\varphi$ since $\sim\varphi(x) \leftrightarrow \langle \emptyset, k\rangle \notin A_n(x)$.

(c) If ψ is a universal Σ_1 formula then $\varphi(k, x) \leftrightarrow \langle H_n^M, A_n(x)\rangle \models \psi(k, \emptyset) \leftrightarrow \big\langle H_n^M, A_n(\langle k, x\rangle)\big\rangle \models \psi^*$ is a universal Σ_{n+1}^* formula (where ψ^* is Σ_1 and chosen to satisfy the last $\leftrightarrow$).

(d) By (c) we see that $A_n(x)$ is Σ_n^* (with parameter p_n^M) and hence $A_n(x) \cap y \in M$ for each $y \in H_n^M$. But either $H_n^M = M$ or $\omega\varrho_n^M$ is a cardinal of M. Using Proposition 1 and condensation, we find that if κ is an M-cardinal then every bounded subset of κ in M actually belongs to J_κ: if $x \subseteq \gamma < \kappa$ and x is Σ_n-definable with parameter p over M', a proper initial segment of M, then let $H = \Sigma_n$-Skolem hull of $\gamma \cup \{p\}$ in M'. Then $H \simeq J_\beta$

where β is less than κ, since in M the cardinality of H is at most γ (we may assume $\omega \leq \gamma < \kappa$). But x is definable over J_β, so $x \in J_{\beta+1} \subseteq J_\kappa$. ■

As promised, we have the following analogue of Lemma 4, in the Σ^* context.

LEMMA 7. *For any $X \subseteq J_\alpha$, the Σ^*_n-hull of X exists. Moreover, there is a Σ^*_n-Skolem function for J_α, via a Σ^*_n-definition independent of α.*

Proof. By induction on n. The base case $n = 1$ is Lemma 4. Suppose the result holds for $n \geq 1$ and we establish it for $n+1$. Let $h_n(k,x)$ be a Σ^*_n-Skolem function for J_α.

Lemma 1 holds uniformly for amenable structures so we may define a partial Σ^*_{n+1} function $h(k,x)$ such that for each x, $H(x) = \{h(k,x) \mid k \in \omega,\ h(k,x) \text{ defined}\}$ is a Σ_1-elementary submodel of $M_n(x) = \langle H^M_n, A_n(x)\rangle$. Define

$$h_{n+1}(k,x) = h_n((k)_0, \langle h((k)_1,x), p^M_n\rangle)$$

where $k = \langle (k)_0, (k)_1\rangle$ is a pairing function on ω. Now graph(h_{n+1}) is a Σ^*_{n+1} relation because $h_{n+1}(k,x) = y \leftrightarrow \exists z \in H^M_n\ (y = h_n((k)_0, \langle z, p^M_n\rangle) \wedge z = h((k)_1,x))$ and as graph(h_n) is Σ^*_n, graph(h) is Σ^*_{n+1} this yields a Σ^*_{n+1} definition of graph(h_{n+1}). If H is a Σ^*_{n+1}-elementary submodel of M then H is closed under h_{n+1}, since it is closed under h_n by induction, is closed under h by Σ^*_{n+1}-elementarity and must contain p^M_n since "$x = p^M_n$" is a Σ^*_{n+1} formula.

It remains to show that $H = \{h_{n+1}(k,x) \mid k \in \omega\}$ is Σ^*_{n+1}-elementary in M. (It then follows that for any $X \subseteq M$, $\{h_{n+1}(k,x) \mid x$ a finite sequence from $X\}$ is Σ^*_{n+1}-elementary in M.) As H is Σ_1-elementary in M we know that H satisfies extensionality so we may take the transitive collapse $\pi : \overline{M} \simeq H \subseteq M$. It will suffice to show that $\pi^{-1}[H \cap M_n(\pi(\overline{x}))] = \overline{M}_n(\overline{x})$ for each $\overline{x} \in \overline{M}$, for then the closure of H under h guarantees Σ^*_{n+1}-elementarity. Now $M_n(\pi(\overline{x})) = \langle H^M_n, A_n(\pi(\overline{x}))\rangle$ and $H^M_n = J_{\omega\varrho^M_n}$, $A_n(\pi(\overline{x})) = \{\langle y, m\rangle \mid$ the mth Σ^*_n formula is true at $\langle y, \pi(\overline{x}), p^M_n\rangle,\ y \in H^M_n\}$ so since by induction we have Σ^*_n-elementarity, it is enough to show

$$\pi^{-1}[\varrho^M_n] = \varrho^{\overline{M}}_n, \qquad \pi^{-1}(p^M_n) = p^{\overline{M}}_n.$$

Let $\bar{\varrho} = \pi^{-1}[\varrho^M_n]$. Suppose $\bar{A} \subseteq J_{\bar{\varrho}}$ is Σ^*_n-definable in $\overline{M}$ with parameter $\overline{q}$. For $\overline{\gamma} < \bar{\varrho}$ we have $\bar{A} \cap J_{\bar{\gamma}} \in \overline{M}$ by Σ_1-elementarity of π from $\overline{M}_n(\overline{q})$ to $M_n(\pi(\overline{q}))$. Note that if $\overline{p} = \pi^{-1}(p^M_n)$ then every $\bar{A} \in \overline{M}$ is of the form $h_n(k, \langle \overline{x}, \overline{p}\rangle)$, $\overline{x} \in J_{\bar{\varrho}}$, so the set $\{\langle k, \overline{x}\rangle \mid k \in \omega,\ \overline{x} \in J_{\bar{\varrho}},\ h_n(k, \langle \overline{x}, \overline{p}\rangle)$ defined, $\langle k, \overline{x}\rangle \notin h_n(k, \langle \overline{x}, \overline{p}\rangle)\}$ is Σ^*_n-definable in $\overline{M}$ with parameters and does not belong to $\overline{M}$. So $\bar{\varrho} = \varrho^{\overline{M}}_n$ and $\overline{p} \geq p^{\overline{M}}_n$.

Finally, we show that $\overline{p} \leq p^{\overline{M}}_n$. Let $\overline{H} = \Sigma^*_n$-hull of $\{\overline{q} \mid \overline{q} < \overline{p}\}$. We may assume that $\overline{p} \neq \emptyset$ and therefore $\bar{\varrho} \subseteq \overline{H}$. Now if $\overline{H} \simeq \overline{M}$ then we get

$\overline{H} = \overline{M}$ and hence $\overline{p} \in \overline{H}$. But then by Σ_n^*-elementarity, $p_n^M \in \Sigma_n^*$-hull of $\{q \mid q < p_n^M\}$, which contradicts the definition of p_n^M. So $\overline{H} \simeq$ proper initial segment of $\overline{M}$ and therefore $\overline{A} \cap J_{\overline{\varrho}} \in \overline{M}$ whenever $\overline{A}$ is Σ_n^*-definable in $\overline{M}$ from a parameter $\overline{q} < \overline{p}$. So $\overline{p} \leq p_n^{\overline{M}}$. ∎

Our next lemma helps to clarify the meaning of the standard parameters, as well as the relationship between Σ_n^* and Σ_n.

LEMMA 8. *Let $H = \Sigma_n^*$-Skolem hull of $\varrho_n^M \cup \{p_n^M\}$ in M. Then $H = M$.*

Proof. Let $\pi : H \simeq \overline{M}$. Then $\overline{M} = M$ as $A \cap H_n^M$ is definable over $\overline{M}$ whenever A is Σ_n^*-definable in M with parameter p_n^M. So $M = \Sigma_n^*$-Skolem hull of $\varrho_n^M \cup \{\pi(p_n^M)\}$. But we must have $\pi(p_n^M) = p_n^M$, else $\pi(p_n^M) < p_n^M$ contradicts the definition of p_n^M. ∎

COROLLARY 1. *For each n, $\Sigma_n \subseteq \Sigma_n^*$ and for $m < n$, Σ_n^* is closed under existential quantification over H_m^M.*

Proof. We can assume $m = 0$ as "$x \in H_m^M$" is a Σ_{m+1}^* formula. By induction on n: Assume that we have an effective translation of Σ_n formulas into Σ_n^* formulas; then if φ is $\exists x\ \psi(x)$ where ψ is Π_n we can write $\exists x\ \psi(x) \leftrightarrow \exists \overline{x} \in H_n^M\ \exists k\ \psi(h_n(k, \langle \overline{x}, p_n^M \rangle))$ and after translating ψ into a Π_n^* formula, this gives a Σ_{n+1}^* translation of φ. ∎

Remark. With some effort, it can be shown that conversely, each Σ_n^* formula is equivalent to a Σ_n formula with parameters. But we will have no use for this fact.

It will be useful to have approximations to the Σ_n^*-hulls and Σ_n^*-Skolem functions. For $n = 1$ and limit $\sigma < \omega\alpha = \mathrm{ORD}(M)$ we let $h_1^\sigma(k, x)$ be defined by restricting the Σ_1 definition of h_1 to $\widetilde{J}_\sigma$: if $h_1(k, x) = y \leftrightarrow \exists z\ \varphi(x, y, z)$ where φ is Σ_0 then $h_1^\sigma(k, x) = y \leftrightarrow \exists z \in \widetilde{J}_\sigma\ \varphi(x, y, z)$. For any $n \geq 1$ and $\sigma < \omega\varrho_n^M$ we define $h_{n+1}^\sigma(k, x) = h_n((k)_0, \langle h^\sigma((k)_1, x), p_n^M \rangle)$, where h^σ is defined by restricting the Σ_{n+1}^* definition of h (from the proof of Lemma 7) to $\widetilde{J}_\sigma$: if $h(k, x) = y \leftrightarrow M_n(k, x, y) \models \exists z\ \varphi$ where φ is Σ_0 then $h^\sigma(k, x) = y \leftrightarrow M_n(k, x, y) \models \exists z \in \widetilde{J}_\sigma\ \varphi$. Also let $\Sigma_n^* {\restriction} \sigma$-hull$(X)$ denote $\{h_n^\sigma(k, x) \mid x \text{ is a finite sequence from } X\}$.

LEMMA 9. *For any $X \subseteq J_\alpha$, $1 \leq n \in \omega$, and every limit $\sigma < \omega\varrho_{n+1}^M$, $\Sigma_{n+1}^* {\restriction} \sigma$-hull$(X)$ is Σ_n^*-elementary in M.*

Proof. It suffices to show that the hull in question is closed under h_n. This follows from the facts that

$$\{h^\sigma(k, x) \mid x \text{ a finite sequence from } X\}$$

is closed under pairing and that $\{h_n(k, \langle y, p \rangle) \mid y \text{ a finite sequence from } Y\}$ is closed under h_n for any $Y \subseteq M,\ p \in M$. ∎

The following fact about hull approximation is very useful.

Lemma 10. *Suppose $X \subseteq J_\alpha$, $\omega < \beta \le \varrho^M_{n-1}$, β is a regular M-cardinal and $\beta \in \Sigma^*_n$-hull(X) in M. Let*

$$\bar\beta = \bigcup(\Sigma^*_n\text{-hull}(X) \cap \beta) \quad \textit{and} \quad \sigma = \bigcup(\Sigma^*_n\text{-hull}(X) \cap \varrho^M_{n-1}).$$

Then $\bar\beta = \beta \cap \Sigma^*_n{\restriction}\sigma$-hull$(X \cup \bar\beta)$ and if $n \ge 2$ then for any $\bar{\bar\beta} < \bar\beta$, and x a finite sequence from X, $\beta \cap \Sigma^*_{n-1}$-hull$(\{x\} \cup \bar{\bar\beta})$ is bounded strictly below $\bar\beta$.

Proof. Suppose that $\gamma \in \beta \cap \Sigma^*_n{\restriction}\sigma$-hull$(X \cup \bar\beta)$. Then there exists a finite sequence x from X such that $\gamma \in \Sigma^*_n{\restriction}\bar\sigma$-hull$(\{x\} \cup \bar{\bar\beta})$ where $\bar\sigma, \bar{\bar\beta} \in \Sigma^*_n$-hull$(\{x\}), \bar\sigma < \sigma$, $\bar{\bar\beta} < \bar\beta$. But $\Sigma^*_n{\restriction}\bar\sigma$-hull$(\{x\} \cup \bar{\bar\beta}) \cap \beta$ belongs to Σ^*_n-hull(X) and hence so does its supremum δ. As β is regular in M, we have $\delta < \beta$ and therefore $\delta < \bar\beta$. Since $\gamma < \delta$ we get $\gamma < \bar\beta$, as desired. The second conclusion of the lemma also follows, by Lemma 9. ∎

The Square Principle. An important application of fine structure theory is to Jensen's Square Principle, which we now establish using the Σ^* approach.

Square. Assume $V = L$. Then there is $\langle C_\mu \mid \mu \text{ a singular limit ordinal}\rangle$ such that

(a) C_μ is closed unbounded in μ.
(b) ordertype$(C_\mu) < \mu$.
(c) $\bar\mu \in \operatorname{Lim} C_\mu \to \bar\mu$ is singular and $C_{\bar\mu} = C_\mu \cap \bar\mu$.
(d) $\langle \widetilde J_\mu, C_\mu\rangle$ is amenable and if $\langle \widetilde J_{\bar\mu}, \overline C\rangle \to \langle \widetilde J_\mu, C_\mu\rangle$ is Σ_1-elementary then $\overline C = C_{\bar\mu}$.

We refer the reader to Jensen [72] for background on and applications of the Square Principle.

Let μ be a singular limit ordinal. We wish to define C_μ. Let $\beta(\mu) \ge \mu$ be the least limit ordinal β such that μ is not regular with respect to $\widetilde J_\beta$-definable functions and let $n(\mu)$ be least such that there is a $\Sigma^*_{n(\mu)}(\widetilde J_{\beta(\mu)})$ partial function (with parameters) from an ordinal less than μ cofinally into μ. Note that $\omega\varrho^{\beta(\mu)}_{n(\mu)} \le \mu$ (where ϱ^β_n denotes ϱ^N_n, $N = \widetilde J_\beta$) as otherwise such a partial function would belong to $\widetilde J_{\beta(\mu)}$, contradicting the leastness of $\beta(\mu)$. Also $\mu \le \omega\varrho^{\beta(\mu)}_{n(\mu)-1}$, else by Lemma 8 we have contradicted the leastness of $n(\mu)$.

For $X \subseteq \widetilde J_{\beta(\mu)}$ let $H(X)$ denote $\Sigma^*_{n(\mu)}$-hull(X) in $\widetilde J_{\beta(\mu)}$. For some least parameter $q(\mu) \in \widetilde J_{\beta(\mu)}$, $H(\mu \cup \{q(\mu)\}) = \widetilde J_{\beta(\mu)}$. (Actually, $q(\mu) = p^{\beta(\mu)}_{n(\mu)} - \mu - p^{\beta(\mu)}_{n(\mu)-1}$.) Also let $\alpha(\mu) = \bigcup\{\alpha < \mu \mid \alpha = H(\alpha \cup \{q(\mu)\}) \cap \mu\}$. Then $\alpha(\mu) < \mu$ and (unless $\alpha(\mu) = \bigcup\emptyset = 0$) $\alpha(\mu) = H(\alpha(\mu) \cup \{q(\mu)\}) \cap \mu$. The former is because for large enough $\alpha < \mu$, $H(\alpha \cup \{q(\mu)\})$ contains both the domain and defining parameter for a $\Sigma^*_{n(\mu)}$ partial function from an ordinal less than μ cofinally into μ.

If $\mu < \beta(\mu)$ let $p(\mu) = \langle q(\mu), \mu\rangle$ and if $\mu = \beta(\mu)$ let $p(\mu) = \emptyset$.

We are ready to define C_μ. Let $C^0_\mu = \{\overline{\mu} < \mu \mid \text{For some } \alpha \geq \alpha(\mu), \overline{\mu} = \bigcup(H(\alpha \cup \{p(\mu)\}) \cap \mu)\}$. Then C^0_μ is a closed subset of μ. If C^0_μ is unbounded in μ then let $C_\mu = C^0_\mu$. If C^0_μ is bounded but nonempty then let $\mu_0 = \bigcup C^0_\mu$ and define $C^1_\mu = \{\overline{\mu} < \mu \mid \text{For some } \alpha, \overline{\mu} = \bigcup(H(\alpha \cup \{p(\mu), \mu_0\}) \cap \mu)\}$. If C^1_μ is unbounded then let $C_\mu = C^1_\mu$. If C^1_μ is bounded but nonempty then let $\mu_1 = \bigcup C^1_\mu$ and define $C^2_\mu = \{\overline{\mu} < \mu \mid \text{For some } \alpha, \overline{\mu} = \bigcup(H(\alpha \cup \{p(\mu), \mu_0, \mu_1\}) \cap \mu)\}$. Continue in this way, defining C^k_μ for $k \in \omega$ until C^k_μ is unbounded or empty for some least $k = k(\mu)$. To see that $k(\mu)$ exists, note that $\alpha_0 > \alpha_1 > \ldots$ where α_k is greatest such that $\bigcup(H(\alpha_k \cup \{p(\mu), \mu_0 \ldots \mu_{k-1}\}) \cap \mu$ contains no ordinal $\geq \mu_k$: we get $\alpha_k \in H(\{p(\mu), \mu_0 \ldots \mu_k\})$; so $H(\alpha_k \cup \{p(\mu), \mu_0 \ldots \mu_k\}) \cap \mu \supseteq H(\alpha_k + 1 \cup \{p(\mu), \mu_0 \ldots \mu_{k-1}\}) \cap \mu$, which by definition of μ_k is unbounded in μ; hence $\alpha_{k+1} < \alpha_k$.

If $C^{k(\mu)}_\mu$ is unbounded in μ then let $C_\mu = C^{k(\mu)}_\mu$. If $C^{k(\mu)}_\mu = \emptyset$ then

$$H(\{p(\mu), \mu_0 \ldots \mu_{k(\mu)-1}\}) \cap \mu$$

is unbounded in μ. And $H(\{p(\mu), \mu_0 \ldots \mu_{k(\mu)-1}\}) \cap \omega\varrho^{\beta(\mu)}_{n(\mu)-1}$ is unbounded in $\omega\varrho^{\beta(\mu)}_{n(\mu)-1}$ else this set belongs to $\widetilde{J}_{\beta(\mu)}$ and μ is singular inside $\widetilde{J}_{\beta(\mu)}$, contradicting leastness of $\beta(\mu)$. Let $\varrho(\mu) = \omega\varrho^{\beta(\mu)}_{n(\mu)-1}$, $p = \{p(\mu), \mu_0 \ldots \mu_{k(\mu)-1}\}$ and $h_n(k, x)$ the Σ^*_n-Skolem function for $\widetilde{J}_{\beta(\mu)}$. Also let $\overline{\sigma}_m = \max(\{h_n(k,p) \mid k < m\} \cap \mu)$ and $\sigma_m = \max(\{h_n(k,p) \mid k < m\} \cap \varrho(\mu))$. We define $C_\mu = \{\delta_0, \delta_1, \ldots\}$ where δ_m is the ordertype of the transitive collapse of $\Sigma^*_{n(\mu)}{\restriction}\sigma_m$-hull$(\overline{\sigma}_m \cup \{p\})$. Note that $\delta_m < \mu$ as μ is regular inside $\widetilde{J}_{\beta(\mu)}$ and $\mathrm{card}(\delta_m) \leq \overline{\sigma}_m$ in $\widetilde{J}_{\beta(\mu)}$.

This completes the definition of C_μ. Clearly C_μ is closed unbounded in μ. The argument that $\alpha(\mu) < \mu$ also implies that ordertype$(C_\mu) < \mu$. So we need only show (c), (d) from the statement of Square.

LEMMA 11. $\overline{\mu} \in C^k_\mu \to C^k_{\overline{\mu}} = C^k_\mu \cap \overline{\mu}$.

Proof. First suppose that $k = 0$. Given $\overline{\mu} \in C^0_\mu$ choose $\alpha < \overline{\mu}$ such that $\overline{\mu} = \bigcup(H(\alpha \cup \{p(\mu)\}) \cap \mu)$, where $H(X) = \Sigma^*_{n(\mu)}$-hull(X) in $\widetilde{J}_{\beta(\mu)}$. Also let $\varrho = \bigcup(H(\alpha \cup \{p(\mu)\}) \cap \omega\varrho^{\beta(\mu)}_{n(\mu)-1})$. Let $H = \Sigma^*_{n(\mu)}{\restriction}\varrho$-hull$(\overline{\mu} \cup \{p(\mu)\})$ and $\pi : \widetilde{J}_{\overline{\beta}} \simeq H \subseteq \widetilde{J}_{\beta(\mu)}$. By Lemma 10, $H \cap \mu = \overline{\mu}$ and therefore when $\mu < \beta(\mu)$, $\pi(\overline{\mu}) = \mu$. By Lemma 9, $\pi : \widetilde{J}_{\overline{\beta}} \to \widetilde{J}_{\beta(\mu)}$ is $\Sigma^*_{n(\mu)-1}$-elementary (when $n(\mu) > 1$), so we get $\beta(\overline{\mu}) = \overline{\beta}$ and $n(\overline{\mu}) \leq n(\mu)$. By the second conclusion of Lemma 10 we get $n(\overline{\mu}) > n(\mu) - 1$, so $n(\overline{\mu}) = n(\mu)$. Thus to conclude that $C^0_{\overline{\mu}} = C^0_\mu \cap \overline{\mu}$ we need only check that $\pi(q(\overline{\mu})) = q(\mu)$ when $\mu < \beta(\mu)$, and $\alpha(\overline{\mu}) = \alpha(\mu)$.

For the former, first note that $\mu \in H(\{\mu'\} \cup \{q(\mu)\})$ for some $\mu' < \mu$, $\mu' \in H(\{p(\mu)\})$, since $\mu, q(\mu) \in H(\{p(\mu)\}) \cap H(\mu \cup \{q(\mu)\})$. So in fact μ and $\alpha(\mu)$ belong to $\Sigma^*_{n(\mu)} \restriction \varrho$-hull$(\overline{\mu} \cup \{q(\mu)\})$ and hence the latter is just H. Now let $\overline{q} = \pi^{-1}(q(\mu))$. We see that $\Sigma^*_{n(\overline{\mu})}$-hull$(\overline{\mu} \cup \{\overline{q}\}) = \widetilde{J}_{\beta(\overline{\mu})}$ and hence $\overline{q} \geq q(\overline{\mu})$. But $\overline{q} \in \Sigma^*_{n(\overline{\mu})}$-hull$(\overline{\mu} \cup \{\pi(q(\overline{\mu}))\})$ in $\widetilde{J}_{\beta(\mu)}$ hence $q(\overline{\mu}) \geq \overline{q}$, else we have contradicted the definition of $q(\mu)$. So $\pi(q(\overline{\mu})) = q(\mu)$. Now since $\alpha(\mu) < \overline{\mu}$ we get $\alpha(\mu) \leq \alpha(\overline{\mu})$. Conversely, $\alpha(\overline{\mu}) < \alpha$ where $\overline{\mu} = \bigcup (H(\alpha \cup \{p(\mu)\}) \cap \mu)$ so $H(\alpha(\overline{\mu}) \cup \{p(\mu)\}) \cap \mu = \alpha(\overline{\mu})$ and we get $\alpha(\overline{\mu}) \leq \alpha(\mu)$. So $\alpha(\overline{\mu}) = \alpha(\mu)$.

Now suppose $k = 1$. The above argument shows that $\overline{\mu} \in C^1_\mu \to C^0_{\overline{\mu}} = C^0_\mu \cap \overline{\mu}$ and hence, since $\mu_0 < \overline{\mu}$, we get $\overline{\mu}_0 = \mu_0$. Then the above argument shows that $C^1_{\overline{\mu}} = C^1_\mu \cap \overline{\mu}$. The general case $k \geq 0$ follows similarly. ∎

To verify (c) in the statement of Square: if $\overline{\mu} \in \operatorname{Lim} C_\mu$ then we must have $C_\mu = C^k_\mu$ for some k and so $C^k_{\overline{\mu}} = C_\mu \cap \overline{\mu}$ is unbounded in $\overline{\mu}$. Hence $C_{\overline{\mu}} = C^k_{\overline{\mu}} = C_\mu \cap \overline{\mu}$ as desired. Now we verify (d).

Lemma 12. (a) *$A \subseteq \widetilde{J}_\mu$, $A \in \widetilde{J}_{\beta(\mu)}$ implies A is $\mathbf{\Delta}_1\langle \widetilde{J}_\mu, C_\mu\rangle$.*

(b) *Suppose $\pi : \langle \widetilde{J}_{\overline{\mu}}, \overline{C}\rangle \to \langle \widetilde{J}_\mu, C_\mu\rangle$ is Σ_1-elementary. Then $\overline{C} = C_{\overline{\mu}}$ and π extends uniquely to a $\Sigma^*_{n(\mu)}$-elementary $\widetilde{\pi} : \widetilde{J}_{\beta(\overline{\mu})} \to \widetilde{J}_{\beta(\mu)}$ such that $p(\mu) \in \operatorname{Range}(\widetilde{\pi})$.*

Proof. First suppose that $C_\mu = C^k_\mu$ for some k. For $\mu' \in C_\mu$ form $H(\mu')$ as H was formed in the proof of Lemma 11 for $\overline{\mu}$. Then $\pi(\mu') : \widetilde{J}_{\beta(\mu')} \to \widetilde{J}_{\beta(\mu)}$, with range $H(\mu')$, is $\Sigma^*_{n(\mu)-1}$-elementary and $\widetilde{J}_{\beta(\mu)} = \bigcup \{H(\mu') \mid \mu' \in C_\mu\}$. Also $\pi(\mu')$ is the identity on μ' and sends $p(\mu')$ to $p(\mu)$.

(a) If $A \subseteq \widetilde{J}_\mu$ and $A \in \widetilde{J}_{\beta(\mu)}$ then $A \cap \widetilde{J}_{\mu'}$ is $\Sigma^*_{n(\mu)}$-definable as an element of $H(\mu')$ from some fixed parameter $x \in \widetilde{J}_\mu$, uniformly for sufficiently large $\mu' \in C_\mu$. So A is $\mathbf{\Delta}_1\langle \widetilde{J}_\mu, C_\mu\rangle$. This proves (a).

(b) Let $X = \operatorname{Range}(\pi)$ and $\widetilde{X} = \Sigma^*_{n(\mu)}$-hull$(X \cup \{p(\mu)\})$ in $\widetilde{J}_{\beta(\mu)}$. If $y \in \widetilde{X} \cap \widetilde{J}_\mu$ then for some $\mu' \in C_\mu$, $y \in \Sigma^*_{n(\mu)}$-hull$((X \cap \widetilde{J}_{\mu'}) \cup \{p(\mu')\})$ in $\widetilde{J}_{\beta(\mu')}$, and as this property of μ' is $\Sigma_1\langle \widetilde{J}_\mu, C_\mu\rangle$ with parameters from X, μ' can be chosen in Σ_1-hull(X) in $\langle \widetilde{J}_\mu, C_\mu\rangle$. It follows that $y \in (\Sigma_1$-hull(X) in $\langle \widetilde{J}_\mu, C_\mu\rangle) = X$. So $\widetilde{X} \cap \widetilde{J}_\mu = X$ and if $\widetilde{\pi} : \widetilde{J}_{\overline{\beta}} \simeq \widetilde{X} \subseteq \widetilde{J}_{\beta(\mu)}$ then $\widetilde{\pi}$ is a $\Sigma^*_{n(\mu)}$-elementary embedding extending π with $p(\mu)$ in its range. Let $\mu^* = \bigcup (X \cap \mu)$.

As $\widetilde{X} \simeq \Sigma^*_{n(\mu^*)}$-hull$(X \cup \{p(\mu^*)\})$ in $\widetilde{J}_{\beta(\mu^*)}$ (by the $\Sigma^*_{n(\mu)}$-elementarity of $\pi(\mu^*)$ when $\mu^* < \mu$) we see that $\overline{\mu}$ is regular with respect to partial $\Sigma^*_{n(\mu)-1}(\widetilde{J}_{\bar{\beta}})$ functions and singular with respect to $\Sigma^*_{n(\mu)}(\widetilde{J}_{\bar{\beta}})$ partial functions. So we get $\beta(\overline{\mu}) = \overline{\beta}$ and $n(\overline{\mu}) = n(\mu)$. Then the $\Sigma^*_{n(\mu)}$-elementarity of $\widetilde{\pi}$ and the fact that $p(\mu) \in \mathrm{Range}(\widetilde{\pi})$ guarantee that $\overline{C} = C_{\bar{\mu}}$. The uniqueness of $\widetilde{\pi}$ comes from the fact that $\widetilde{J}_{\beta(\bar{\mu})} = \Sigma^*_{n(\bar{\mu})}$-hull$(\overline{\mu} \cup \{p(\overline{\mu})\})$ and $\widetilde{\pi} \restriction \overline{\mu}$ is determined by π. This proves (b).

If $C^k_\mu = \emptyset$ for some k then C_μ was defined as a special ω-sequence cofinal in μ. That definition was made precisely to enable the preceding arguments to also apply in this case. (Also note that in this case $\mu^* = \mu$.) ∎

Relativization. Square and Diamond hold relative to *reshaped strings*, a fact which is useful in the proof of Jensen's Coding Theorem (Beller–Jensen–Welch [82] and Friedman [94]). We state these versions here.

Assume that $A \subseteq \mathrm{ORD}$ and $L_\alpha[A] = H_\alpha$ for each cardinal α. For each such α define S_α to consist of all $s : [\alpha, |s|) \to 2$, $\alpha \le |s| < \alpha^+$, such that for all $\eta \le |s|$, $L[A \cap \alpha, s\restriction\eta] \models \mathrm{card}(\eta) \le \alpha$. These are the "reshaped strings" at α.

We must also define *coding structures*. For $s \in S_\alpha$ define $\mu^{<s}$ and μ^s inductively by: $\mu^{<\emptyset_\alpha} = \alpha$ (where $\emptyset_\alpha \in S_\alpha$, $|\emptyset_\alpha| = \alpha$, is the empty string), $\mu^{<s} = \bigcup\{\mu^t \mid t$ a proper intial segment of $s\}$ for $s \ne \emptyset_\alpha$, and $\mu^s =$ least $\mu > \mu^{<s}$ such that $\mu'\mu = \mu$ for $\mu' < \mu$ and $L_\mu[A\cap\alpha, s] \models \mathrm{card}(|s|) \le \alpha$. Also let $\widehat{\mu}^s =$ largest $\mu > \mu^{<s}$ such that $\mu'\mu = \mu$ for $\mu' < \mu$, $L_\mu[A \cap \alpha, s] \models |s|$ is a cardinal, if exists; if there is no such μ then $\widehat{\mu}^s = \mu^{<s}$. Then $\mathcal{A}^s = L_{\mu^s}[A \cap \alpha, s]$, $\mathcal{A}^{<s} = \langle L_{\mu^{<s}}[A \cap \alpha, \widehat{s}],\ A \cap \alpha, \widehat{s}\rangle$ and $\widehat{\mathcal{A}}^s = \langle L_{\hat{\mu}^s}[A \cap \alpha, \widehat{s}], A \cap \alpha, \widehat{s}\rangle$ where $\widehat{s} = \{\mu^{<s\restriction\eta} \mid s(\eta) = 1\}$.

And we must discuss *collapsibility*. If $\langle \mathcal{A}, C\rangle$ is an amenable structure of the form $\langle \widetilde{J}_\mu[B], B, C\rangle$ we define $\mathcal{A}^+$ to be $\langle \widetilde{J}_{\mu^*}[B], B\rangle$ where $\mu^* \ge \mu$ is the least limit ordinal such that $\widetilde{J}_{\mu^*+\omega}[B] \models \mu$ is not a cardinal (if it exists), and $\langle \mathcal{A}, C\rangle$ is *collapsible* if $\mathcal{A}^+$ exists and whenever $\pi : \langle \overline{\mathcal{A}}, \overline{C}\rangle \to \langle \mathcal{A}, C\rangle$ is Σ_1-elementary then $\overline{\mathcal{A}}^+$ exists, $\overline{C}$ is definable over $\overline{\mathcal{A}}^+$ and π lifts to a Σ_1-elementary $\pi^+ : \overline{\mathcal{A}}^+ \to \mathcal{A}^+$.

Relativized Square. Suppose α is an uncountable limit cardinal. Then there exists $\langle C^s \mid s \in S_\alpha\rangle$ such that:

(a) $s \ne \emptyset_\alpha \to C^s$ is CUB in $\mu^{<s}$, ordertype$(C^s) \le \alpha$, $C^s \in \mathcal{A}^s$.
(b) $\mu \in \mathrm{Lim}\, C^s \to \mu = \mu^{<s\restriction\eta}$ for some $\eta \le |s|$ and $C^{s\restriction\eta} = C^s \cap \mu$.
(c) $\langle \mathcal{A}^{<s}, C^s\rangle$ is collapsible.
(d) $s \ne \emptyset_\alpha$, $D \subseteq \mathcal{A}^{<s}$, $D \in (\mathcal{A}^{<s})^+ \to D$ is $\Delta_1\langle \mathcal{A}^{<s}, C^s\rangle$.

Relativized Diamond. Suppose α is an uncountable limit cardinal. Then there exists $\langle D^s \mid s \in S_\alpha\rangle$ such that:

(a) $D^s \subseteq \mathcal{A}^{<s}$ and $\langle D^t \mid t$ an initial segment of $s\rangle \in \mathcal{A}^s$.

(b) If $D \subseteq \mathcal{A}^{<s}$ and $D \in \widehat{\mathcal{A}}^s \neq \mathcal{A}^{<s}$ then $\{\eta < |s| \mid D^{s\restriction\eta} = D \cap \mathcal{A}^{<s\restriction\eta}\}$ is stationary in $\widehat{\mathcal{A}}^s$.

(c) If $\mu^{<s\restriction\eta} \in \operatorname{Lim} C^s$ and $\eta < |s|$ then $D^{s\restriction\eta} = \emptyset$. If $\widehat{\mathcal{A}}^s \models |s|^{++}$ exists then $D^s = \emptyset$. And if $\pi : \langle \mathcal{A}^{<\bar{s}}, \overline{C}\rangle \to \langle \mathcal{A}^{<s}, C^s\rangle$ is Σ_1-elementary and $\pi(\overline{\alpha}) = \alpha$ where $\bar{s} \in S_{\bar{\alpha}}$ then $D^{\bar{s}} = \pi^{-1}[D^s]$.

P r o o f. For Relativized Square, define C^s using $\langle \widetilde{J}_{\beta(s)}[A\cap\alpha, \widehat{s}], A\cap\alpha, \widehat{s}\rangle$ as we defined C_μ using $\widetilde{J}_{\beta(\mu)}$, where $\beta(s) \geq \mu^{<s}$ is least so that $\mu^{<s}$ is not regular with respect to functions definable over this structure. Note that this structure belongs to $\mathcal{A}^s$. As before, we get property (a), and (b) follows from (the analogue of) Lemma 11. Properties (c), (d) follow from (the analogue of) Lemma 12.

For Relativized Diamond, define D^s using $\langle \widetilde{J}_{\beta(s)}[A\cap\alpha, \widehat{s}], A\cap\alpha, \widehat{s}\rangle$ as we defined D_μ in Lemma 5 using $J_{\beta'(\mu)}$. Property (a) follows from (the analogue to) (b) of Lemma 5 and (b) follows from the same argument used to establish (a) of Lemma 5. Also, that argument in fact shows that (a) of Lemma 5 holds in the stronger form: if $D \subseteq J_{\alpha^+}$ then $\{\mu < \alpha^+ \mid D \cap J_\mu = D_\mu$ and $C^0_\mu = \emptyset\}$ is stationary; note that $C^0_\mu = \emptyset \to \mu \in \operatorname{Lim} C_{\mu'}$ for any $\mu < \mu'$. So by (the analogue to) this proof we may assume that the first statement of Relativized Square (c) holds. The second statement of (c) follows from (the analogue to) Lemma 5(c) and the final statement follows from (the analogues to) Lemmas 12 and 5(b). ∎

The Fine Structure Principle. We summarize here those aspects of the Σ^* theory that are used when establishing combinatorial principles in L. For any set X let $\mathrm{Seq}(X)$ denote the set of all finite sequences from X and recall the ordering $<$ on finite sets of ordinals: $p < q$ iff $\alpha \in q$ where $\alpha = \max((p - q) \cup (q - p))$. Also for any limit ordinal λ let M_λ denote $\widetilde{J}_\lambda$ ($= J_\alpha$, where $\omega\alpha = \lambda$).

(FSP) There exists a sequence of recursive sets of formulas $\Sigma_1 = \Sigma^*_1 \subseteq \Sigma^*_2 \subseteq \ldots$ and partial functions $h^\lambda_n : \omega \times M_\lambda \to M_\lambda$ for λ limit and $n \in \omega$ such that

1) $\bigcup\{\Sigma^*_n \mid n \in \omega\} =$ All first-order formulas, $\Pi^*_n = \{\sim\varphi \mid \varphi \in \Sigma^*_n\} \subseteq \Sigma^*_{n+1}$ and Σ^*_n is closed under $\exists$, $\wedge$, $\vee$.

2) h^λ_n is Σ^*_n-definable and if $\varphi(x)$ is Σ^*_n then for some k, $M_\lambda \models \varphi(x) \leftrightarrow h_n(k, x)$ is defined.

3) For any $X \subseteq M_\lambda$, $H^\lambda_n(X) = \{h^\lambda_n(k, x) \mid x \in \mathrm{Seq}(X),\ k \in \omega\}$ is the least Σ^*_n-elementary submodel of M_λ containing X as a subset.

4) Let $\varrho^\lambda_n =$ least ordinal ϱ such that for some $p \in \mathrm{Seq}(\lambda)$ and $A \subseteq \lambda$ where A is $\Sigma^*_n(M_\lambda)$ in parameter p, $A \cap \omega\varrho \notin M_\lambda$. And let $p^\lambda_n =$ $<$-least

such p. Then $p_n^\lambda \in H_{n+1}^\lambda(\emptyset)$ and $M_\lambda = H_n^\lambda(\omega\varrho_n^\lambda \cup \{p_n^\lambda\})$. Also the formula "$x \in M_{\omega\varrho_n^\lambda}$" is Σ_{n+1}^*.

5) If $\pi : M_{\bar\lambda} \to M_\lambda$ is Σ_{n+1}^*-elementary then $\pi^{-1}[\varrho_n^\lambda] = \varrho_n^{\bar\lambda}$ and $\pi^{-1}(p_n^\lambda) = p_n^{\bar\lambda}$.

6) Approximations: $h_n^\lambda = \bigcup\{h_n^{\lambda,\sigma} \mid \sigma < \omega\varrho_{n-1}^\lambda,\ \sigma \text{ limit}\}$ where $\sigma < \sigma' \to h_n^{\lambda,\sigma} \subseteq h_n^{\lambda,\sigma'}$, $\{\langle\sigma,k,x,y\rangle \mid h_n^{\lambda,\sigma}(k,x) = y\}$ is $\Sigma_n^* \cap \Pi_n^*$ and when $n > 1$, for each σ, $H_n^{\lambda,\sigma}(X) = \{h_n^{\lambda,\sigma}(k,x) \mid k \in \omega,\ x \in \mathrm{Seq}(X)\}$ is Σ_{n-1}^*-elementary in M_λ. (When $n = 1$ we take $\omega\varrho_{n-1}^\lambda$ to be λ.)

It is not difficult to verify that the proof of Square that we gave can be carried out directly from the FSP. In the next section we use the FSP to construct morasses.

Remark. Σ^* theory can also be applied in core models other than L, as in its original form (Jensen [?]), however Lemma 3(c)(Condensation) may fail in this more general context. For this reason the fine structure theory for core models in general presents numerous new difficulties, some of which remain unsolved.

3. Morasses. A strong form of the gap-1 morass principle is useful in the theory of strong coding. We now establish a global form of this principle, which we call Morass with Square.

In Square we found a uniform way of writing a singular ordinal as the union of a short sequence of smaller ordinals. In Morass we find a uniform way of writing an ordinal of regular cardinality as the direct limit of ordinals of smaller cardinality. These two principles interact in Morass with Square.

Rather than begin with a statement of our principle, we first use the Fine Structure Principle to describe the actual object which will interest us. In this way it is easier to see the motivation behind a list of its combinatorial properties, expressed in Morass with Square.

An ordinal α is *cardinal-correct* if whenever $J_\alpha \models \kappa$ is a cardinal, then κ really is a cardinal. Let $S^0 = \{\alpha > \omega \mid \alpha \text{ is cardinal-correct}\}$. Then S^0 is CUB in every uncountable cardinal. For $\alpha \in S^0$ let $S_\alpha = \{\nu \mid \alpha < \nu < \alpha^+,\ \nu$ is a limit ordinal, $\widetilde{J}_\nu \models \alpha$ is regular and α is the largest cardinal$\}$. Then S_α is a closed subset of (α,α^+) and α not a cardinal, $\alpha < \beta$ in $S^0 \to \bigcup S_\alpha < \beta$. We write $\nu_0 <_0 \nu_1$ iff $\nu_0 < \nu_1$ and for some $\alpha \in S^0$, ν_0 and ν_1 both belong to S_α. When $\nu \in S_\alpha$ we write $\alpha(\nu) = \alpha$. (This is a different use of the notation $\alpha(\nu)$ than was made in the proof of Square.) Let $S^1 = \bigcup\{S_\alpha \mid \alpha \in S^0\}$.

Now we come to the main definition. For $\nu \in S^1$, $\beta(\nu) =$ least limit ordinal $\beta \geq \nu$ such that $\varrho_n^\beta \leq \alpha(\nu)$ for some n and $n(\nu) =$ least such n. And $q(\nu) =$ least $q \in \mathrm{Seq}(\beta(\nu))$ such that $\widetilde{J}_{\beta(\nu)} = H_{n(\nu)}^{\beta(\nu)}(\alpha(\nu) \cup \{q\})$. (Actually, $q(\nu) = p_{n(\nu)}^{\beta(\nu)} - \alpha(\nu)$.) We write $\bar\nu <_1 \nu$ iff there is $\pi : \ \widetilde{J}_{\beta(\bar\nu)} \to \widetilde{J}_{\beta(\nu)}$ such that π is $\Sigma^*_{n(\bar\nu)}$-elementary, $n(\bar\nu) = n(\nu), \pi(\alpha(\bar\nu)) = \alpha(\nu)$, $\pi(q(\bar\nu)) = q(\nu)$ and $\pi\restriction\alpha(\bar\nu) =$ identity; in addition we impose the Q-condition:

(Q) Whenever $\varphi(x)$ is $\Sigma^*_{n(\nu)}$ with parameter $\overline{p} \in \widetilde{J}_{\beta(\bar\nu)}$ then $\{\nu_0 < \nu \mid \widetilde{J}_{\beta(\nu)} \models \varphi(\nu_0, \pi(\overline{p}))\}$ is bounded in ν iff $\{\bar\nu_0 < \bar\nu \mid \widetilde{J}_{\beta(\bar\nu)} \models \varphi(\bar\nu_0, \overline{p})\}$ is bounded in $\bar\nu$.

This condition originates in Jensen [72].

If $\bar\nu <_1 \nu$ then π as above is unique and we write $\pi_{\bar\nu\nu} = \pi\restriction\bar\nu : \bar\nu \to \nu$, $\widetilde{\pi}_{\bar\nu\nu} = \pi$.

The above structure, together with the Square sequence $\langle C_\alpha \mid \alpha$ singular limit$\rangle$ from the preceding section, constitutes our realization of Morass with Square. Before stating this principle we make a few observations regarding the relation $<_1$. Using the fact that $\widetilde{\pi}_{\bar\nu\nu}$ is $\Sigma^*_{n(\nu)}$-elementary and sends $(\alpha(\bar\nu), q(\bar\nu))$ to $(\alpha(\nu), q(\nu))$ it follows not only that $\widetilde{\pi}_{\bar\nu\nu} = \pi$ is unique but also that $<_1$ is a tree, ν $<_0$-minimal, $<_0$-limit $\to \bar\nu$ $<_0$-minimal, $<_0$-limit. Also $\pi^{-1}[S_{\alpha(\nu)}] = S_{\alpha(\bar\nu)} \cap \bar\nu$ and $\pi(\bar\nu_0^+) = \pi(\bar\nu_0)^+$ when $\bar\nu_0^+ = ($$<_0$-successor to $\bar\nu_0)$ $<_0$ $\bar\nu$. Also if $\bar\nu_0 <_0 \bar\nu$ and $\nu_0 = \pi(\bar\nu_0)$ then $\pi\restriction\widetilde{J}_{\beta(\bar\nu_0)}$ is elementary from $\widetilde{J}_{\beta(\bar\nu_0)}$ into $\widetilde{J}_{\beta(\nu_0)}$ so $\bar\nu_0 <_1 \nu_0$ and $\pi_{\bar\nu_0\nu_0} = \pi_{\bar\nu\nu}\restriction\bar\nu_0$. Finally, $\bar{\bar\nu} <_1 \bar\nu <_1 \nu \to \pi_{\bar{\bar\nu}\nu} = \pi_{\bar\nu\nu} \circ \pi_{\bar{\bar\nu}\bar\nu}$ and $\{\alpha(\bar\nu) \mid \bar\nu <_1 \nu\}$ is always closed in $\alpha(\nu)$, unbounded if ν is not $<_0$-maximal. If $\{\alpha(\bar\nu) \mid \bar\nu <_1 \nu\}$ is unbounded then $\nu = \bigcup\{\mathrm{Range}(\pi_{\bar\nu\nu}) \mid \bar\nu <_1 \nu\}$.

There are four more properties of π which take a bit of argument. First we claim that if $\nu < \beta(\nu)$ then $\nu \in \mathrm{Range}(\pi)$: If $n(\nu) > 1$ then this is clear because $\widetilde{J}_{\beta(\nu)} \models \nu = \alpha(\nu)^+$ and the property of being a cardinal is Σ^*_2. If $n(\nu) = 1$ then we claim that $q(\nu) - \nu$ is nonempty and hence if $\gamma \in q(\nu) - \nu$ we see that $\nu = \alpha(\nu)^+$ of $\widetilde{J}_\gamma$ belongs to $H_1^{\beta(\nu)}(\{\alpha(\nu), q(\nu)\}) \subseteq \mathrm{Range}(\pi)$. The reason that $q(\nu) - \nu$ is nonempty is that as in the proof of Lemma 6(d), we can show that $\widetilde{J}_\nu$ is Σ_1-elementary in $\widetilde{J}_{\beta(\nu)}$ and hence $q(\nu) \subseteq \nu$ would contradict $H_1^{\beta(\nu)}(\alpha(\nu) \cup \{q(\nu)\}) = \widetilde{J}_{\beta(\nu)}$.

Second, we claim that if $\bar\nu$ is $<_0$-limit and $\lambda = \bigcup\mathrm{Range}(\pi) < \nu$ then $\bar\nu <_1 \lambda$ and $\pi_{\bar\nu\lambda} = \pi_{\bar\nu\nu}$: As in the proof of Square we form $H = H^{\beta(\nu),\sigma}_{n(\nu)}(\lambda \cup \{q(\nu)\})$ where $\sigma = \bigcup(\mathrm{Range}(\pi) \cap \varrho^{\beta(\nu)}_{n(\nu)-1})$. Then as in the proof of Lemma 10, $H \simeq \widetilde{J}_{\beta(\lambda)}$ and $q(\nu)$ is sent to $q(\lambda)$ under this isomorphism. By composing with π, we get a $\Sigma^*_{n(\nu)}$-elementary embedding from $\widetilde{J}_{\beta(\bar\nu)}$ into $\widetilde{J}_{\beta(\lambda)}$ sending $(\alpha(\bar\nu), q(\bar\nu))$ to $(\alpha(\lambda), q(\lambda))$. As the range of this embedding contains a cofinal subset of λ, the Q-condition is satisfied and $\bar\nu <_1 \lambda$, $\pi_{\bar\nu\lambda} = \pi_{\bar\nu\nu}$.

Third, we claim that if $\bar\nu <_1 \nu$, $\pi_{\bar\nu\nu}$ is cofinal and α is such that for each $\bar\nu_0 <_0 \bar\nu$, $\alpha = \alpha(\nu_0')$ for some $\nu_0' <_1 \pi_{\bar\nu\nu}(\bar\nu_0)$ then $\alpha = \alpha(\nu')$ for some $\nu' <_1 \nu$. For, $H = H^{\beta(\nu)}_{n(\nu)}(\alpha \cup \{q(\nu)\}) = \bigcup\{H^{\beta(\nu),\sigma}_{n(\nu)}(\alpha \cup \{q(\nu)\}) \mid \sigma \in \mathrm{Range}(\widetilde{\pi}_{\bar\nu\nu}),$

$\sigma < \omega\varrho^{\beta(\nu)}_{n(\nu)-1}\}$ and hence $H \cap \alpha(\nu) = \alpha$ since for each σ as above, $\alpha = \alpha(\nu'_\sigma)$ for some $\nu'_\sigma <_1 \nu_\sigma = \bigcup(\nu \cap H^{\beta(\nu),\sigma}_{n(\nu)}(\alpha \cup \{q(\nu)\}))$. Since $H \cap \nu$ is cofinal in ν (as we can assume that $\alpha \geq \alpha(\bar{\nu})$) we get $\alpha = \alpha(\nu')$ where $\nu' = \text{ordertype}(H \cap \nu)$.

Fourth, we claim that if ν is a $<_0$-successor then so is $\bar{\nu}$: This is clear if $\nu < \beta(\nu)$ or $n(\nu) > 1$ as being the $<_0$-predecessor to ν is $\Pi_1(\widetilde{J}_\nu)$. If $(\beta(\nu), n(\nu)) = (\nu, 1)$ then we must use the Q-condition on π to guarantee that $S_{\alpha(\bar{\nu})} \cap \bar{\nu}$ is bounded in $\bar{\nu}$.

The previous is our first use of the Q-condition on π. In strong coding we will use it to argue that if $\bar{\nu} <_1 \nu$ and ν is admissible (i.e., $L_\nu \models \Sigma_1$-Replacement) then so is $\bar{\nu}$.

We now state Morass with Square. We have shown that the structure defined above satisfies (a)–(f) in the list of properties below.

Morass with Square. There exist $\langle C_\alpha \mid \alpha \text{ singular limit}\rangle$, $\langle S_\alpha \mid \alpha \in S^0\rangle$, a binary relation $<_1$ on $S^1 = \bigcup\{S_\alpha \mid \alpha \in S^0\}$ and $\langle \pi_{\bar{\nu}\nu} \mid \bar{\nu} <_1 \nu\rangle$ such that

(a) For α a singular limit, C_α is CUB in α, $\text{ordertype}(C_\alpha) < \alpha$, $\beta \in \operatorname{Lim} C_\alpha \to \beta$ singular, $C_\beta = C_\alpha \cap \beta$.

(b) $S^0 \cap \kappa$ is CUB in κ for every uncountable cardinal κ.

(c) For $\alpha \in S^0$, S_α is a closed subset of (α, α^+). And:

- (c1) α regular $\to S_\alpha = S^0 \cap (\alpha, \alpha^+)$.
- (c2) α singular cardinal $\to S_\alpha$ is a proper initial segment of $S^0 \cap (\alpha, \alpha^+)$.
- (c3) α not a cardinal, $\alpha < \beta$ in $S^0 \to \bigcup S_\alpha < \beta$.

Notation. (c) implies that $\nu \in S^1 \to$ there is a unique α with $\nu \in S_\alpha$; denote this by $\alpha(\nu)$. Write $\nu <_0 \nu'$ if $\nu < \nu'$, $\alpha(\nu) = \alpha(\nu')$. For $\alpha \in S^0$, α not regular, let $\nu(\alpha)$ denote $\max(S_\alpha) < \alpha^+$. If $\nu \in S^1$, ν not $<_0$-maximal, then ν^+ denotes its $<_0$-successor.

(d) $<_1$ is a tree and if $\bar{\nu} <_1 \nu$ then $\alpha(\bar{\nu}) < \alpha(\nu)$ and $\bar{\nu}$ is $<_0$-minimal, successor, limit iff ν is $<_0$-minimal, successor, limit.

(e) If $\bar{\nu} <_1 \nu$ then $\pi = \pi_{\bar{\nu}\nu} : \bar{\nu} \to \nu$ is order-preserving, $\pi^{-1}[S_{\alpha(\nu)}] = S_{\alpha(\bar{\nu})} \cap \bar{\nu}$, $\pi(\bar{\nu}_0^+) = \pi(\bar{\nu}_0)^+$ when $\bar{\nu}_0^+ <_0 \bar{\nu}$. If $\bar{\nu}_0 <_0 \bar{\nu}$ and $\nu_0 = \pi(\bar{\nu}_0)$ then $\bar{\nu}_0 <_1 \nu_0$ and $\pi_{\bar{\nu}_0\nu_0} = \pi \restriction \bar{\nu}_0$. If $\bar{\nu}$ is $<_0$-limit, $\lambda = \bigcup \text{Range}(\pi)$ then $\bar{\nu} <_1 \lambda$ and $\pi_{\bar{\nu}\lambda} = \pi$; and if $\nu = \bigcup \text{Range}(\pi)$, $\alpha = \alpha(\nu'_0)$ for some $\nu'_0 <_1 \pi_{\bar{\nu}\nu}(\bar{\nu}_0)$ for each $\bar{\nu}_0 < \bar{\nu}$ then $\alpha = \alpha(\nu')$ for some $\nu' <_1 \nu$.

(f) $\bar{\bar{\nu}} <_1 \bar{\nu} <_1 \nu \to \pi_{\bar{\nu}\nu}\pi_{\bar{\bar{\nu}}\bar{\nu}} = \pi_{\bar{\bar{\nu}}\nu}$. For $\nu \in S^1$, $\{\alpha(\bar{\nu}) \mid \bar{\nu} <_1 \nu\}$ is closed in $\alpha(\nu)$ and is unbounded if ν is not $<_0$-maximal. If $\{\alpha(\bar{\nu}) \mid \bar{\nu} <_1 \nu\}$ is unbounded in $\alpha(\nu)$ then $\nu = \bigcup\{\text{Range}(\pi_{\bar{\nu}\nu}) \mid \bar{\nu} <_1 \nu\}$.

Now let C'_α denote the limit points of C_α less than α, for α singular limit.

(g) Suppose ν is $<_1$-limit and $\alpha(\nu)$ singular. Then for α in a final segment of $C'_{\alpha(\nu)}$ there exists $\nu_\alpha <_1 \nu$ with $\nu_\alpha \in S_\alpha$; and for ν $<_0$-limit if $\lambda_\alpha = \bigcup \mathrm{Range}(\pi_{\nu_\alpha \nu})$ then $\lambda_\alpha \in C'_\nu \cup \{\nu\}$ and $\alpha < \beta \in C'_{\alpha(\nu)} \to \lambda_\alpha \in \mathrm{Range}(\pi_{\nu_\beta \nu}) \cup \{\nu\}$.

(h) Suppose ν is $<_1$-minimal and $<_0$-limit. Then for α in a final segment of $C'_{\alpha(\nu)}$, $\nu(\alpha)$ is $<_1$-minimal, $<_0$-limit and there is $\nu_\alpha <_0 \nu$ such that $\nu(\alpha) <_1 \nu_\alpha \in C'_\nu$, $\alpha < \beta \to \nu_\alpha \in \mathrm{Range}(\pi_{\nu(\beta)\nu_\beta})$, and $\beta \in \mathrm{Lim}\, C'_{\alpha(\nu)} \to \nu_\beta = \bigcup\{\nu_\alpha \mid \alpha \in C'_\beta\}$.

(i) Suppose ν is a $<_1$-successor and $<_0$-limit. Let $\bar\nu <^*_1 \nu$ express the property that $\bar\nu = {<_1}$-predecessor to ν. Then for a final segment of $\alpha \in C'_{\alpha(\nu)}$, $\nu(\alpha)$ is $<_1$-successor and $<_0$-limit and there exist $\nu_\alpha <_0 \nu$ as in (h) such that in addition, $\nu = \lambda = \bigcup \mathrm{Range}(\pi_{\bar\nu\nu}) \to \nu_\alpha = \pi_{\bar\nu\nu}(\bar\nu_\alpha)$ where $\bar\nu_\alpha <^*_1 \nu(\alpha)$, $\lambda < \nu \to \lambda \in \mathrm{Range}(\pi_{\nu(\alpha)\nu_\alpha})$, $\bar\nu <^*_1 \nu(\alpha)$.

Proof. (g) Suppose $\alpha \in C'_{\alpha(\nu)}$. Then for some k, $\alpha \in \mathrm{Lim}\, C^k_{\alpha(\nu)}$ and therefore for some $\gamma \geq \gamma(\alpha(\nu))$,

$$\alpha = \bigcup(\alpha(\nu) \cap H^{\beta(\alpha(\nu))}_{n(\alpha(\nu))}(\gamma \cup \{p(\alpha(\nu)), \alpha(\nu)_0 \ldots \alpha(\nu)_{k-1}\}))$$

where $\beta(\alpha(\nu))$, $n(\alpha(\nu)), p(\alpha(\nu)) = \langle q(\alpha(\nu)), \alpha(\nu)\rangle$ and $\alpha(\nu)_i$ are defined as in the proof of Square. (We have changed the notation $\alpha(\mu)$ to $\gamma(\mu)$ so as to avoid confusion.) The fact that ν is a $<_1$-limit implies that $(\beta(\nu), n(\nu)) < (\beta(\alpha(\nu)), n(\alpha(\nu))$. (I.e., either $\beta(\nu) < \beta(\alpha(\nu))$ or $\beta(\nu) = \beta(\alpha(\nu))$, $n(\nu) < n(\alpha(\nu))$. Note that as $\widetilde{J}_\nu \models$ There is a largest cardinal, $\beta(\nu)$ and $n(\nu)$ have the same meaning in this section as they did in the proof of Square.) Thus for sufficiently large $\alpha \in C'_{\alpha(\nu)}$ we see by Lemma 10 that $\alpha = \alpha(\nu) \cap H^{\beta(\nu)}_{n(\nu)}(\alpha \cup \{q(\nu)\})$, where $q(\nu)$ is defined in this section. (We need only choose α large enough so that $H^{\beta(\nu)}_{n(\nu)}(\alpha \cup \{q(\nu)\}) \subseteq H^{\beta(\alpha(\nu))}_{n(\alpha(\nu))-1}(\alpha \cup \{p(\alpha(\nu))\})$.) To verify the Q-condition we must argue as follows. Either α can be chosen large enough so that $H^{\beta(\nu)}_{n(\nu)}(\alpha \cup \{q(\nu)\}) \cap \nu$ is cofinal in ν, in which case the Q-condition is automatic, or we claim that the Q-condition implies that $H^{\beta(\nu)}_{n(\nu)}(\alpha \cup \{q(\nu)\})$ is $\Sigma^*_{n(\nu)+1}$-elementary in $\widetilde{J}_{\beta(\nu)}$. In the latter case the assumption that ν is a $<_1$-limit yields that in fact $(\beta(\nu), n(\nu)+1) < (\beta(\alpha(\nu)), n(\alpha(\nu)))$, and hence $H^{\beta(\nu)}_{n(\nu)}(\alpha \cup \{q(\nu)\}) = H^{\beta(\nu)}_{n(\nu)+1}(\alpha \cup \{q(\nu)\})$ obeys the Q-condition.

To prove the above claim suppose $\varphi(x)$ is $\Sigma^*_{n(\nu)+1}$ and note that $\varphi(x) \leftrightarrow \exists\gamma < \alpha(\nu)\ \exists k \in \omega\ (x \in y = h^{\beta(\nu)}_{n(\nu)}(k, \langle\gamma, q(\nu)\rangle), y \models \varphi(x), y\ \Sigma^*_{n(\nu)}$-elementary in $\widetilde{J}_{\beta(\nu)})$. To each $\sigma < \omega\varrho^{\beta(\nu)}_{n(\nu)-1} = \varrho$ associate the least $(\gamma(\sigma), k(\sigma))$ such that the above holds with $h^{\beta(\nu)}_{n(\nu)}$ replaced by $h^{\beta(\nu),\sigma}_{n(\nu)}$ and "y $\Sigma^*_{n(\nu)}$-elementary in $\widetilde{J}_{\beta(\nu)}$" replaced by "$y = H^{\beta(\nu),\sigma}_{n(\nu)}(y)$." Then $\varphi(x) \leftrightarrow A = \{\sigma \mid$ For some

γ, k, σ is least so that $(\gamma(\sigma), k(\sigma)) = (\gamma, k)\}$ is bounded in $\varrho \leftrightarrow B = \{H^{\beta(\nu),\sigma}_{n(\nu)}(\alpha(\nu) \cup \{q(\nu)\}) \cap \nu \mid \sigma \in A\}$ is bounded in ν. So ν is equivalent to the boundedness of a $\Sigma^*_{n(\nu)}$ subset of ν, hence the Q-condition implies $\Sigma^*_{n(\nu)+1}$-elementarity.

Thus we have $\nu_\alpha <_1 \nu$ where $\nu_\alpha = \text{ordertype}(\nu \cap H^{\beta(\nu)}_{n(\nu)}(\alpha \cup \{q(\nu)\})$. Again since $(\beta(\nu), n(\nu)) < (\beta(\alpha(\nu)), n(\alpha(\nu)))$, if $\alpha < \beta$ in $C'_{\alpha(\nu)}$ and $H^{\beta(\nu)}_{n(\nu)}(\alpha \cup \{q(\nu)\})$ is bounded in ν then its supremum below ν belongs to $H^{\beta(\nu)}_{n(\nu)}(\beta \cup \{q(\nu)\}) \supseteq H^{\beta(\nu)}_{n(\nu)+1}(\{\alpha, q(\nu)\}) \cap \nu$. So it only remains to show that $\lambda_\alpha = \bigcup(\nu \cap H^{\beta(\nu)}_{n(\nu)}(\alpha \cup \{q(\nu)\})) \in C'_\nu \cup \{\nu\}$. Note that if $p(\nu)$ and ν_0 are defined as in the proof of Square then $q(\nu)$ as defined in this section belongs to $H^{\beta(\nu)}_{n(\nu)}(\{p(\nu), \nu_0\})$: $q(\nu)$ is just $p^{\beta(\nu)}_{n(\nu)} - p^{\beta(\nu)}_{n(\nu)-1} - \alpha(\nu)$, and so by definition of $p(\nu)$ we have $q(\nu) - \nu$ in $H^{\beta(\nu)}_{n(\nu)}(\{p(\nu)\})$; but if $q(\nu) \cap \nu$ is nonempty then it consists of a single ordinal δ, and δ is largest so that $H^{\beta(\nu)}_{n(\nu)}(\delta \cup \{p^{\beta(\nu)}_n - \nu\}) \cap \nu = \delta$. This is precisely the ordinal used to provide a lower bound on C^0_ν in the proof of Square. As $C^0_\nu = \{\delta\}$ in this case we get $\nu_0 = \delta$. So if $\lambda_\alpha < \nu$ for sufficiently large α then C_ν is a final segment of $\{\bigcup(H^{\beta(\nu)}_{n(\nu)}(\alpha \cup \{q(\nu)\}) \cap \nu) \mid \alpha < \alpha(\nu)\}$. And the fact that $(\beta(\nu), n(\nu)) < (\beta(\alpha(\nu)), n(\alpha(\nu)))$ implies that $\lambda_\alpha \in C'_\nu$ for sufficiently large α. Of course the alternative is that $\lambda_\alpha = \nu$ for sufficiently large $\alpha \in C'_{\alpha(\nu)}$ and so (g) is proved.

(h) There are two cases: either $(\beta(\nu), n(\nu)) = (\beta(\alpha(\nu)), n(\alpha(\nu)))$ or $(\beta(\nu), n(\nu)+1) = (\beta(\alpha(\nu)), n(\alpha(\nu)))$. In the latter case we must have $H^{\beta(\nu)}_{n(\nu)}(\alpha \cup \{q(\nu)\}) \cap \nu$ bounded in ν for each $\alpha < \alpha(\nu)$, else we could contradict the $<_1$-minimality of ν by forming $H^{\beta(\nu)}_{n(\nu)}(\alpha_0 \cup \{q(\nu)\})$ where $\alpha_0 = \bigcup(\alpha(\nu) \cap H^{\beta(\nu)}_{n(\nu)}(\alpha \cup \{q(\nu)\}))$, $H^{\beta(\nu)}_{n(\nu)}(\alpha \cup \{q(\nu)\}) \cap \nu$ unbounded in ν, $\alpha < \alpha(\nu)$.

First we treat the former case. Suppose $\alpha \in C'_{\alpha(\nu)}$. Then for some k, $\alpha \in C^k_{\alpha(\nu)}$ and so for some γ ($\geq \gamma(\alpha(\nu))$ if $k = 0$),

$$\alpha = \bigcup(\alpha(\nu) \cap H^{\beta(\nu)}_{n(\nu)}(\gamma \cup \{p(\alpha(\nu)), \alpha(\nu)_0 \ldots \alpha(\nu)_{k-1}\}))$$

where $p(\alpha(\nu)) = \langle q(\alpha(\nu)), \alpha(\nu)\rangle$ and $\alpha(\nu)_i$ are defined as in the proof of Square. (Again to avoid confusion we now write $\gamma(\mu)$ in place of $\alpha(\mu)$.) Note that $q(\alpha(\nu))$ is precisely the $q(\nu)$ as defined in this section. Write p for $\{p(\alpha(\nu)), \alpha(\nu)_0 \ldots \alpha(\nu)_{k-1}\}$ and ϱ for $\omega\varrho^{\beta(\nu)}_{n(\nu)-1}$.

Now let $\nu_\alpha = \bigcup(\nu \cap H^{\beta(\nu)}_{n(\nu)}(\gamma \cup \{p\}))$ and $\sigma_\alpha = \bigcup(\varrho \cap H^{\beta(\nu)}_{n(\nu)}(\gamma \cup \{p\}))$. Then as $\alpha > \gamma(\alpha(\nu))$ we get $\nu_\alpha < \nu$ and $\sigma_\alpha < \varrho$, and by Lemma 10, $\alpha =$

$\alpha(\nu)\cap H^{\beta(\nu),\sigma_\alpha}_{n(\nu)}(\alpha\cup\{p\})$. We get an embedding $\pi:\widetilde{J}_\beta\simeq H^{\beta(\nu),\sigma_\alpha}_{n(\nu)}(\alpha\cup\{p\})$ and $\pi^{-1}[\nu_\alpha]=\nu(\alpha)$, $\beta=\beta(\nu(\alpha))$, $n(\nu)=n(\nu(\alpha))$. In fact, $\nu(\alpha)<_1\nu_\alpha$ as π maps $\nu(\alpha)$ cofinally into ν_α. It is not clear that $\nu(\alpha)$ is $<_1$-minimal as it is possible that there exists $\bar\nu<_1\nu(\alpha)$ with $\alpha(\bar\nu)\le\gamma(\alpha)=\gamma(\alpha(\nu))$. ($\alpha(\bar\nu)>\gamma(\alpha)$ is ruled out because of the definition of $\gamma(\alpha)$.) However, as ν is $<_1$-minimal the Q-condition must fail between $H^{\beta(\nu)}_{n(\nu)}(\gamma(\alpha)\cup\{q(\nu)\})$ and $\widetilde{J}_{\beta(\nu)}$ so we may choose α large enough in $C'_{\alpha(\nu)}$ so that this failure is captured by $H^{\beta(\nu),\sigma_\alpha}_{n(\nu)}(\alpha\cup\{p\})$, and therefore $\nu(\alpha)$ is $<_1$-minimal.

To see that $\nu_\alpha\in C'_\nu$ for sufficiently large α, the same analysis as in the proof of (g) shows that if $q(\nu)\cap\nu\ne\emptyset$ then $C^1_\nu=\{\nu'<\nu\mid$ For some γ, $\nu'=\bigcup(\nu\cap H^{\beta(\nu)}_{n(\nu)}(\gamma\cup\{q(\nu)\})\}$, and if $q(\nu)\cap\nu=\emptyset$ then C^0_ν is a final segment of this set, beyond an ordinal $\le\gamma(\alpha(\nu))$. Thus it follows that either C^1_ν or C^0_ν agrees with $\{\nu_\alpha\mid\alpha\in C^0_{\alpha(\nu)}\}$ for $\alpha\ge\gamma(\alpha(\nu))$. If $C^0_{\alpha(\nu)}$ is unbounded in $\alpha(\nu)$ then we are done because then C^1_ν or C^0_ν as above is unbounded in ν. If not then we need only note that $\alpha(\nu)_0\in H^{\beta(\nu)}_{n(\nu)}(\{q(\nu),\nu^*\})$ and $\nu^*\in H^{\beta(\nu)}_{n(\nu)}(\{q(\nu),\alpha(\nu)_0\})$ where $\nu^*=\bigcup C^1_\nu$ or $\bigcup C^0_\nu$ as above. Thus $\{\nu_\alpha\mid\alpha\in C^1_{\alpha(\nu)}\}$ agrees with C^2_ν or C^1_ν for $\alpha\ge\gamma(\alpha(\nu))$ and continuing in this way we get $\nu_\alpha\in C'_\nu$ for sufficiently large α.

The last part of (h) is clear from the definition of the ν_α's and the fact that $\alpha<\beta\to\nu_\alpha<\nu_\beta$.

Now we consider the case $(\beta(\nu),n(\nu)+1)=(\beta(\alpha(\nu)),n(\alpha(\nu)))$ and recall that $H^{\beta(\nu)}_{n(\nu)}(a\cup\{q(\nu)\})\cap\nu$ is bounded in ν for each $\alpha<\alpha(\nu)$. Thus as in the proof of (g), for $\alpha\in C'_{\alpha(\nu)}$ we have $\alpha=\alpha(\nu)\cap H^{\beta(\nu)}_{n(\nu)}(\alpha\cup\{q(\nu)\})$ and $\widehat{\nu}(\alpha)<_1\nu_\alpha$ where $\nu_\alpha=\bigcup(\nu\cap H^{\beta(\nu)}_{n(\nu)}(\alpha\cup\{q(\nu)\}))$ and $\widehat{\nu}(\alpha)=\mathrm{ordertype}(\nu\cap H^{\beta(\nu)}_{n(\nu)}(\alpha\cup\{q(\nu)\}))$. Also $\bar\nu<_1\widehat{\nu}(\alpha)$ implies as in the proof of (g) that $\widetilde{\pi}_{\bar\nu\hat\nu(\alpha)}$ is $\Sigma^*_{n(\nu)+1}$-elementary, hence $\alpha(\bar\nu)\le\gamma(\alpha(\nu))$; but as in the first part of the present proof, this is ruled out, for α sufficiently large. So for such $\alpha\in C'_{\alpha(\nu)}$ we have $\widehat{\nu}(\alpha)=\nu(\alpha)<_1\nu(\alpha)$ and $\nu(\alpha)$ is $<_1$-minimal. The proof that $\nu_\alpha\in C'_\nu$ for sufficiently large α is as in the proof of (g) and the last part of (h) is clear from the definition of ν_α and the fact that $(\beta(\alpha(\nu)),n(\alpha(\nu))>(\beta(\nu),n(\nu))$.

(i) As in (h) there are two cases: either $(\beta(\nu),n(\nu))=(\beta(\alpha(\nu)),n(\alpha(\nu)))$ or $(\beta(\nu),n(\nu)+1)=(\beta(\alpha(\nu)),n(\alpha(\nu)))$ and $\alpha<\alpha(\nu)\to H^{\beta(\nu)}_{n(\nu)}(\alpha\cup\{q(\nu)\})\cap\nu$ is bounded in ν.

We begin with the first case. As in (h), write $\alpha=\bigcup(\alpha(\nu)\cap H^{\beta(\nu)}_{n(\nu)}(\gamma\cup\{p\}))$ where $\gamma\ge\gamma(\alpha(\nu))$ if $C_{\alpha(\nu)}=C^0_{\alpha(\nu)}$ and $p=\{p(\alpha(\nu)),\ \alpha(\nu)_0\ldots\alpha(\nu)_{k-1}\}$.

Also let $\sigma_\alpha = \bigcup(\varrho \cap H^{\beta(\nu)}_{n(\nu)}(\gamma \cup \{p\}))$ where $\varrho = \omega\varrho^{\beta(\nu)}_{n(\nu)-1}$ and $\nu_\alpha = \bigcup(\nu \cap H^{\beta(\nu)}_{n(\nu)}(\gamma \cup \{p\}))$. Then $\nu(\alpha) <_1 \nu_\alpha$ and the ν_α's obey the property expressed in (h). And there exists $\bar\nu_\alpha \le_0 \bar\nu <^*_1 \nu$ such that $\bar\nu_\alpha <_1 \nu(\alpha)$ as we can take $\bar\nu_\alpha = \text{ordertype}(\nu \cap H^{\beta(\nu),\sigma_\alpha}_{n(\nu)}(\alpha(\bar\nu) \cup \{q(\nu)\}))$. As in (h) we can arrange that $\bar\nu_\alpha <^*_1 \nu(\alpha)$ for sufficiently large α by capturing a witness to the failure of the Q-condition between $H^{\beta(\nu)}_{n(\nu)}(\gamma(\alpha(\nu)) \cup \{q(\nu)\})$ and $\widetilde J_{\beta(\nu)}$. Note that in fact $k > 0$ and we must have $\gamma < \alpha(\bar\nu)$ for sufficiently large $\alpha = \bigcup(\alpha(\nu) \cap H^{\beta(\nu)}_{n(\nu)}(\gamma \cup \{p\}))$ so we get $\nu_\alpha \in \text{Range}(\pi_{\bar\nu\nu})$ for $\nu_\alpha < \lambda = \bigcup \text{Range}(\pi_{\bar\nu\nu})$. Similarly, $\lambda \in \text{Range}(\pi_{\nu(\alpha)\nu_\alpha})$ when $\nu_\alpha > \lambda$ and we get $\bar\nu_\alpha = \bar\nu$.

Note that in the second case, $\pi_{\bar\nu\nu}$ is not cofinal. The argument now is very similar to the second case of the proof of (h), arranging $\bar\nu <^*_1 \nu(\alpha)$ as in the first case of the present proof. ∎

Our version of Morass with Square originates in Friedman [87] and is related to the concept of Morass with Linear Limits; see Donder [85].

The next principle arises in the proof of Jensen's Coding Theorem in the general case. It is similar in some respects to the Squared Scales of Donder–Jensen–Stanley [85].

Again we first describe the object, obtained through use of the Fine Structure Principle, which satisfies this principle before stating the principle itself. Let $T = \{\nu \mid \nu$ is a limit ordinal, $\widetilde J_\nu \models$ there is a largest cardinal $\alpha(\nu)$ and the cardinality of ν equals $\alpha(\nu)\}$. We do not require that $\alpha(\nu) = \text{card}(\nu)$ is regular. Let $\beta(\nu) \ge \nu$ be the least limit ordinal such that for some n, $\varrho^{\beta(\nu)}_n = \alpha(\nu)$, let $n(\nu)$ be the least such n, and $p(\nu) = \langle p^{\beta(\nu)}_{n(\nu)}, \alpha(\nu)\rangle$. Also $\widehat\beta(\nu) = \beta(\nu) + \omega = T$-successor to ν.

Now for any $k \ge 0$ in ω and infinite cardinal $\alpha < \alpha(\nu)$ let $H(\nu,k,\alpha) = H^{\beta(\nu)}_{n(\nu)+k}(\alpha \cup \{p(\nu)\})$ and $\bar H(\nu,k,\alpha)$ its transitive collapse. Then $f(\nu,k,\alpha) = \alpha^+$ of $\bar H(\nu,k,\alpha)$. Note that $f(\nu,k,\alpha) \in T$ and $\alpha(f(\nu,k,\alpha)) = \alpha$.

For $\nu \in T$ we let $\widetilde C_\nu \subseteq \nu$ come from the Square Principle; then $\widetilde C_\nu$ is CUB in ν, $\text{ordertype}(\widetilde C_\nu) \le \alpha(\nu)$ and $\bar\nu \in \text{Lim}\, \widetilde C_\nu \to \widetilde C_{\bar\nu} = \widetilde C_\nu \cap \bar\nu$. We let $C_\nu = \widetilde C_\nu$ if $\text{ordertype}(\widetilde C_\nu) < \alpha(\nu)$ and otherwise $C_\nu = \{\bar\nu < \nu \mid$ For some $\alpha < \alpha(\nu)$, $\bar\nu = \bigcup(\nu \cap H^{\beta(\nu)}_{n(\nu)}(\alpha \cup \{p(\nu)\}))\}$.

For $\nu \in T$, $k \ge 0$ in ω, and $\alpha(\nu)$ an uncountable limit cardinal we define a CUB $D_{\nu,k} \le \alpha(\nu)$ as follows. If $D = \{\alpha < \alpha(\nu) \mid \alpha = \alpha(\nu) \cap H^{\beta(\nu)}_{n(\nu)+k+1}(\alpha \cup \{p(\nu)\})\}$ is unbounded in $\alpha(\nu)$ then set $D_{\nu,k} = D$. If $H^{\beta(\nu)}_{n(\nu)+k+1}(\alpha \cup \{p(\nu)\}) \cap \alpha(\nu)$ is unbounded in $\alpha(\nu)$ for some $\alpha < \alpha(\nu)$ then we can choose $D_{\nu,k}$ CUB in $\alpha(\nu)$ of ordertype $< \alpha(\nu)$ so that $D_{\nu,k} \cap \alpha$ is $\Sigma^*_{n(\nu)+k+1}$-definable over $H^{\beta(\nu)}_{n(\nu)+k}(\alpha \cup \{p(\nu)\})$ uniformly for $\alpha \in \text{Lim}\, D_{\nu,k}$. Otherwise define $D_{\nu,k} =$

$\{\alpha_0, \alpha_1, \ldots\}$ where $\alpha_0 = 0$ and $\alpha_{n+1} = \bigcup(\alpha(\nu) \cap H^{\beta(\nu)}_{n(\nu)+k+1}(\alpha_n \cup \{p(\nu)\}))$.

Fine Scale Principle. There exist $\langle f(\nu, k, \alpha) \mid \nu \in T,\ k \in \omega,\ \alpha$ an infinite cardinal $< \alpha(\nu)\rangle$, $\langle C_\nu \mid \nu \in T\rangle$, $\langle D_{\nu,k} \mid \nu \in T,\ \alpha(\nu)$ an uncountable limit cardinal, $k \in \omega\rangle$ such that

(a) $\nu \in T \to \nu$ is a limit ordinal and not a cardinal, and $T \cap \alpha^+$ is CUB in α^+. $\alpha(\nu)$ denotes the cardinality of ν.

(b) $f(\nu, k, \alpha) \in T \cap \alpha^+$; C_ν is CUB in ν, $\mathrm{ordertype}(C_\nu) \le \alpha(\nu)$; $D_{\nu,k}$ is CUB in $\alpha(\nu)$. For α an uncountable limit cardinal and any $f : \alpha \to \alpha$ such that $f(\alpha_0) < \alpha_0^+$ for every $\alpha_0 < \alpha$, there is $\nu \in T \cap \alpha^+$ such that $f(\alpha_0^+) < f(\nu, 0, \alpha_0^+)$ for sufficiently large $\alpha_0 < \alpha$.

(c) For any $\nu \in T$ there is $\alpha_0(\nu) < \alpha(\nu)$ such that for $\alpha_0(\nu) \le \alpha < \alpha(\nu)$, α an infinite cardinal and $\bar\nu \in \mathrm{Lim}\, C_\nu$:

(c1) $f(\bar\nu, 0, \alpha) = \bigcup\{f(\bar{\bar\nu}, 0, \alpha) \mid \bar{\bar\nu} \in C_\nu \cap \bar\nu\}$,

(c2) $\{f(\bar{\bar\nu}, 0, \alpha) \mid \bar{\bar\nu} \in C_\nu \cap \bar\nu\} \in \widetilde{J}_\beta$ where $\beta = T$-successor to $f(\bar\nu, 0, \alpha)$.

(d) For any $\nu \in T$ and $k \ge 0$ there is $\alpha_0(\nu, k) < \alpha(\nu)$ such that for $\alpha_0(\nu, k) \le \alpha_0 < \alpha(\nu)$, α_0 an infinite cardinal and $\alpha \in \mathrm{Lim}\, D_{\nu,k}$:

(d1) $f(f(\nu, k, \alpha), 1, \alpha_0) = \bigcup\{f(f(\nu, k, \bar\alpha), 1, \alpha_0) \mid \bar\alpha \in D_{\nu,k} \cap \alpha\}$,

(d2) $\{f(f(\nu, k, \bar\alpha), 1, \alpha_0) \mid \bar\alpha \in D_{\nu,k} \cap \alpha\} \in \widetilde{J}_\beta$ where $\beta = T$-successor to $f(f(\nu, k, \alpha), 1, \alpha_0)$.

Proof. (c) Choose $\alpha_0(\nu)$ larger than $\mathrm{ordertype}(C_\nu)$ if the latter is less than $\alpha(\nu)$. In this case the properties follow from the $\Sigma^*_{n(\nu)}$-elementarity of $H(\bar\nu, 0, \alpha)$ in $\widetilde{J}_{\beta(\bar\nu)}$ and the $\Pi^*_{n(\nu)}$-definability of $C_{\bar\nu} = C_\nu \cap \bar\nu$. In case $\mathrm{ordertype}(C_\nu) = \alpha(\nu)$ note that $f(\bar\nu, 0, \alpha) = f(\bar\nu_\alpha, 0, \alpha)$ where $\bar\nu_\alpha \le \bar\nu$ are in C_ν and $\bar\nu_\alpha = \alpha$th element of C_ν. So the argument also works in this case.

(d) If $\mathrm{ordertype}(D_{\nu,k}) < \alpha(\nu)$ then choose $\alpha_0(\nu, k)$ larger than this ordertype. Note that $D_{\nu,k} \cap \alpha$ is $\Sigma^*_{n(\nu)+k+1}$ or $\Pi^*_{n(\nu)+k+1}$-definable over $H(\nu, k, \alpha)$ when $\alpha < \alpha(\nu)$; so the result follows from the $\Sigma^*_{n(\nu)+k}$-elementarity of $H(\nu, k, \bar\alpha)$ in $H(\nu, k, \alpha)$ and the fact that $n(f(\nu, k, \alpha)) = n(\nu) + k$. Also in case $\mathrm{ordertype}(D_{\nu,k}) = \alpha(\nu)$ note that $f(f(\nu, k, \alpha), 1, \alpha_0)$ is constant for $\alpha \ge \alpha_0$th element of $\mathrm{Lim}\, D_{\nu,k}$. ■

The key clause in the Fine Scale Principle is (d). It says that $f(\nu, k+1, -)$ can be uniformly approximated by functions which differ from $f(\nu, k, -)$ only on a proper initial segment of $\alpha(\nu)$, in such a way that at limit stages α, the αth approximation can easily recover the α-sequence of smaller approximations. This is a powerful tool for proving a statement for each $f(\nu, k, -)$, by induction on (ν, k). In the case of Jensen coding, extendibility of conditions can be proved in this way.

We conclude with a discussion of gap 2 morasses. Again we begin with a description of the intended object.

Let $S^0 = \{\alpha > \omega \mid \alpha$ is a limit ordinal, α is cardinal-correct$\}$, $S^1 = \{\nu \mid \nu$ is a limit ordinal and for some $\alpha(\nu) \in S^0$, $J_\nu \models \alpha(\nu)$ is the largest cardinal and $\alpha(\nu)$ is regular$\}$, $S^2 = \{\mu \mid \mu$ is a limit ordinal, μ is not a cardinal and for some $\nu(\mu) \in S^1, \widetilde{J}_\mu \models \nu(\mu)$ is the largest cardinal$\}$. Thus if $\mu \in S^2$ then $\widetilde{J}_\mu \models \alpha(\nu(\mu))$ is regular, $\nu(\mu) = \alpha(\nu(\mu))^+$ is the largest cardinal. We write $\nu_0 <_0 \nu_1$ if $\nu_0 < \nu_1$ and for some α, ν_0 and ν_1 both belong to $S_\alpha = \{\nu \mid \alpha(\nu) = \alpha\}$; also we write $\mu_0 <_0 \mu_1$ if $\mu_0 < \mu_1$ and for some $\nu \in S^1$, μ_0 and μ_1 both belong to $S_\nu = \{\mu \mid \nu(\mu) = \nu\}$. For $\alpha \in S^0$, $\nu(\alpha)$ denotes $\max S_\alpha$ (when α is not regular) and for $\nu \in S^1$, $\mu(\nu)$ denotes $\nu \cup \max S_\nu$ (when ν is not regular).

Now the main definition. For $\nu \in S^1$, ν not regular, let $\beta(\nu) \geq \nu$ be the least limit ordinal such that $\varrho^{\beta(\nu)}_{n(\nu)} \leq \alpha(\nu)$ for some least $n(\nu)$, and let $q(\nu) = p^{\beta(\nu)}_{n(\nu)} - \alpha(\nu)$. (Thus $q(\nu)$ is least so that $H^{\beta(\nu)}_{n(\nu)}(\alpha(\nu) \cup \{q(\nu)\}) = \widetilde{J}_{\beta(\nu)}$.) The previous, as well as the definition of $\bar{\nu} <_1 \nu$ are as in the gap 1 case: $\bar{\nu} <_1 \nu$ iff there exists $\widetilde{\pi}_{\bar{\nu}\nu} = \pi : \widetilde{J}_{\beta(\bar{\nu})} \to \widetilde{J}_{\beta(\nu)}$ which is $\Sigma^*_{n(\nu)}$-elementary, $n(\bar{\nu}) = n(\nu)$, $\pi \restriction \alpha(\bar{\nu}) =$ identity, $\pi(\alpha(\bar{\nu})) = \alpha(\nu)$, $\pi(q(\bar{\nu})) = q(\nu)$ and the Q-condition is satisfied: whenever $\varphi(x)$ is $\Sigma^*_{n(\nu)}$ in parameters $\overline{p} \in \widetilde{J}_{\beta(\bar{\nu})}$ then $\{\bar{\nu}' < \bar{\nu} \mid \widetilde{J}_{\beta(\bar{\nu})} \models \varphi(\bar{\nu}', \overline{p})\}$ is bounded in $\bar{\nu}$ iff $\{\nu' < \nu \mid \widetilde{J}_{\beta(\nu)} \models \varphi(\nu', \pi(\overline{p}))\}$ is bounded in ν. We write $\pi_{\bar{\nu}\nu}$ for $\pi \restriction \mu(\bar{\nu}) \cup \bar{\nu}$. Now in addition, for $\mu \in S^2$, define $\beta(\mu), n(\mu), q(\mu)$ in the same way, with $\alpha(\nu)$ replaced by $\nu(\mu)$. Also define $\overline{\mu} <_1 \mu$ in the same way, with $\alpha(\bar{\nu}), \alpha(\nu)$ replaced by $\nu(\overline{\mu}), \nu(\mu)$. We write $\pi_{\bar{\mu}\mu}$ for $\pi \restriction \overline{\mu}$.

Note that we defined $\pi_{\bar{\nu}\nu}$ for $\bar{\nu} <_1 \nu$ in S^1 to be $\pi \restriction \mu(\bar{\nu})$ and not simply $\pi \restriction \bar{\nu}$. This means that $\pi_{\bar{\nu}\nu}$ moves ordinals $\overline{\mu} \in S^2$, $\overline{\mu} < \mu(\bar{\nu})$, and raises interesting questions about how the relation $<_1$ on such ordinals is affected by applying $\pi_{\bar{\nu}\nu}$. Thus our gap 2 morass properties pertain not only to the "gap 1" relationships $\bar{\nu} <_1 \nu$ and $\overline{\mu} <_1 \mu$ but also to the way in which they interact.

Gap 2 Morass. There exist $\langle S_\alpha \mid \alpha \in S^0\rangle$, $\langle S_\nu \mid \nu \in S^1 = \bigcup\{S_\alpha \mid \alpha \in S^0\}\rangle$, a binary relation $<_1$ on $(S^1 \times S^1) \cup (S^2 \times S^2)$ where $S^2 = \bigcup\{S_\nu \mid \nu \in S^1\}$ and $\langle \pi_{\bar{\nu}\nu} \mid \bar{\nu} <_1 \nu$ in $S^1\rangle$, $\langle \pi_{\bar{\mu}\mu} \mid \overline{\mu} <_1 \mu$ in $S^2\rangle$ such that:

(a) $S^0 \cap \kappa$ is CUB in κ for each uncountable cardinal κ.

(b) For $\alpha \in S^0$, S_α is a closed subset of $(\alpha, \alpha^+]$ and for $\nu \in S^1$, S_ν is a closed subset of (ν, ν^+). And:

- (b1) α regular $\to S_\alpha = S^0 \cap (\alpha, \alpha^+]$,
- (b2) α singular cardinal $\to$ S_α is a proper initial segment of $S^0 \cap (\alpha, \alpha^+)$,
- (b3) $\alpha < \alpha'$ in S^0, α not a cardinal $\to \bigcup S_\alpha < \alpha'$,
- (b4) $\nu < \nu'$ in S^1, ν not a cardinal $\to \bigcup S_\nu < \nu'$.

NOTATION. For $\nu \in S^1$, $\alpha(\nu)$ denotes the α such that $\nu \in S_\alpha$ and for $\mu \in S^2$, $\nu(\mu)$ denotes the ν such that $\mu \in S_\nu$. We write $\nu <_0 \nu'$ if $\nu < \nu'$ and $\alpha(\nu) = \alpha(\nu')$, and $\mu <_0 \mu'$ if $\mu < \mu'$ and $\nu(\mu) = \nu(\mu')$. If $\alpha \in S^0$ then $\nu(\alpha) = \max S_\alpha$ and if $\nu \in S^1$ is not a cardinal then $\mu(\nu) = \nu \cup \max S_\nu$. If ν is not $<_0$-maximal then ν^+ denotes its $<_0$-successor (similarly for $\mu \in S^2$).

(c) $<_1$ is a tree and if $\bar\nu <_1 \nu$ in S^1 then $\alpha(\bar\nu) < \alpha(\nu)$ and ν is not a cardinal. If $\overline{\mu} <_1 \mu$ in S^2 then $\nu(\overline{\mu}) < \nu(\mu)$. If $\bar\nu <_1 \nu$ then $\overline{\mu}$ is $<_0$-minimal, successor, limit iff μ is $<_0$-minimal, successor, limit when $\overline{\mu} < \mu(\bar\nu)$, $\mu = \pi_{\bar\nu\nu}(\overline{\mu})$ or when $(\overline{\mu}, \mu) = (\mu(\bar\nu), \mu(\nu))$. If $\overline{\mu} <_1 \mu$ then $\overline{\mu}$ is $<_0$-minimal, successor, limit iff μ is $<_0$-minimal, successor, limit.

(d) If $\bar\nu <_1 \nu$ then $\pi = \pi_{\bar\nu\nu} : \mu(\bar\nu) \to \mu(\nu)$ is order-preserving, $\pi^{-1}[S_{\alpha(\nu)}] = S_{\alpha(\bar\nu)} \cap \bar\nu$, $\pi^{-1}[S_\nu] = S_{\bar\nu}$, $\pi(\overline{\mu}^+) = \pi(\overline{\mu})^+$ whenever $\pi(\overline{\mu})^+ < \mu(\nu)$. If $\bar\nu_0 <_0 \bar\nu$ and $\nu_0 = \pi(\bar\nu_0)$ then $\bar\nu_0 <_1 \nu_0$ and $\pi_{\bar\nu_0\nu_0} = \pi \restriction \mu(\bar\nu_0)$. If $\bar\nu$ is a $<_0$-limit and $\lambda = \bigcup \mathrm{Range}(\pi \restriction \bar\nu)$ then $\bar\nu <_1 \lambda$ and if $\mu_\lambda = \bigcup \mathrm{Range}(\pi_{\bar\nu\lambda})$ and $\mu = \bigcup \mathrm{Range}(\pi_{\bar\nu\nu})$ then $\mu_\lambda <_1 \mu$ and $\pi_{\bar\nu\nu} = \pi_{\mu_\lambda\mu} \circ \pi_{\bar\nu\lambda}$. If $\bigcup \mathrm{Range}(\pi \restriction \bar\nu) = \nu$ and $\alpha = \alpha(\nu_0')$ for some $\nu_0' <_1 \pi_{\bar\nu\nu}(\bar\nu_0)$ for each $\bar\nu_0 < \bar\nu$ then $\alpha = \alpha(\nu')$ for some $\nu' <_1 \nu$. Similarly for $\pi_{\overline{\mu}\mu}$ when $\overline{\mu} <_1 \mu$, with $\mu(\bar\nu), \mu(\nu)$, $\alpha(\bar\nu), \alpha(\nu)$ replaced by $\overline{\mu}, \mu, \nu(\overline{\mu}), \nu(\mu)$ and $\pi_{\mu_\lambda\mu}$ replaced by the identity.

(e) $\bar{\bar\nu} <_1 \bar\nu <_1 \nu \to \pi_{\bar{\bar\nu}\nu} = \pi_{\bar\nu\nu} \circ \pi_{\bar{\bar\nu}\bar\nu}$. For $\nu \in S^1$, $\{\alpha(\bar\nu) \mid \bar\nu <_1 \nu\}$ is closed in $\alpha(\nu)$ and unbounded unless ν is $<_0$-maximal. If $\{\alpha(\bar\nu) \mid \bar\nu <_1 \nu\}$ is unbounded in $\alpha(\nu)$ then $\mu(\nu) = \bigcup \{\mathrm{Range}(\pi_{\bar\nu\nu}) \mid \bar\nu <_1 \nu\}$. Similarly for $\mu \in S^2$, with $\alpha(\nu), \mu(\nu)$ replaced by $\nu(\mu)$, μ.

(f) Suppose $\bar\nu <_1 \nu$. Then $\bar\nu < \mu(\bar\nu) = \overline{\mu}$ iff $\nu < \mu(\nu) = \mu$. Suppose now that $\bar\nu < \overline{\mu}$ and $\nu < \mu$. Then $\overline{\mu}$ is $<_1$-minimal, successor, limit iff μ is $<_1$-minimal, successor, limit, as for $\overline{\mu}_0 < \overline{\mu}_1 < \overline{\mu}$, $\overline{\mu}_0 <_1 \overline{\mu}_1$ iff $\pi_{\bar\nu\nu}(\overline{\mu}_0) <_1 \pi_{\bar\nu\nu}(\overline{\mu}_1)$, and in addition $\bar\nu_0 = \nu(\overline{\mu}_0)$ for some $\overline{\mu}_0 <_1 \overline{\mu}$ iff $\pi_{\bar\nu\nu}(\bar\nu_0) = \nu(\mu_0)$ for some $\mu_0 <_1 \mu$.

(g) Suppose $\bar\nu <_1 \nu$, $\bar\nu < \mu(\bar\nu)$ and $\overline{\mu} = \mu(\bar\nu)$ is a $<_1$-successor. Let $\overline{\mu}_0 <_1^* \overline{\mu}$ denote that $\overline{\mu}_0$ is the $<_1$-predecessor to $\overline{\mu}$. Then if $\mu_0 <_1^* \mu$ we have $\pi_{\bar\nu\nu}(\nu(\overline{\mu}_0)) = \nu(\mu_0)$. And $\pi_{\overline{\mu}_0\overline{\mu}}$ is cofinal iff $\pi_{\mu_0\mu}$ is cofinal. If $\pi_{\overline{\mu}_0\overline{\mu}}$ is not cofinal and $\overline{\lambda} = \bigcup \mathrm{Range}(\pi_{\overline{\mu}_0\overline{\mu}})$ then $\pi_{\bar\nu\nu}(\overline{\mu}_0) = \mu_0$ and $\pi_{\bar\nu\nu}(\overline{\lambda}) = \bigcup \mathrm{Range}(\pi_{\mu_0\mu})$.

(h) Suppose $\bar\nu < \mu(\bar\nu) = \overline{\mu}$ and $\overline{\mu}$ is not a $<_1$-limit. Then $\pi_{\bar\nu\nu}$ is cofinal iff $\pi_{\bar\nu\nu} \restriction \bar\nu$ is cofinal.

(i) Suppose $\bar\nu < \mu(\bar\nu) = \overline{\mu}$ and $\bar\nu <_1 \nu$. If $\overline{\mu}_0 <_1 \overline{\mu}_1 \le \overline{\mu}$, $\mu_0 <_1 \mu_1 = \pi_{\bar\nu\nu}(\overline{\mu}_1)$ (or $\mu_0 <_1 \mu(\nu)$ if $\overline{\mu}_1 = \overline{\mu}$) and $\nu(\mu_0) = \pi_{\bar\nu\nu}(\nu(\overline{\mu}_0))$ then $\pi_{\mu_0\mu_1}(\pi_{\bar\nu\nu} \restriction \overline{\mu}_0) = \pi_{\bar\nu\nu}\pi_{\overline{\mu}_0\overline{\mu}_1}$.

Remarks. (a) Jensen points out that we cannot have perfect tree preservation, which would say: $\overline{\mu}_0 <_1 \overline{\mu}_1 \leftrightarrow \pi_{\bar\nu\nu}(\overline{\mu}_0) <_1 \pi_{\bar\nu\nu}(\overline{\mu}_1)$ for $\overline{\mu}_1 \le \mu(\bar\nu)$. (We take $\pi_{\bar\nu\nu}(\mu(\bar\nu))$ to be $\mu(\nu)$.) Thus in (g) we wrote only $\pi_{\bar\nu\nu}(\nu(\overline{\mu}')) = \nu(\mu')$ rather than $\pi_{\bar\nu\nu}(\overline{\mu}') = \mu'$. So $\pi_{\bar\nu\nu}$ may send $\overline{\mu} <_1^* \mu(\bar\nu)$ to μ where

$\nu(\mu) = \nu(\mu')$, $\mu' <_1^* \mu(\nu)$, $\mu \neq \mu'$. If $\pi_{\overline{\mu}\mu(\overline{\nu})}$ is not cofinal, however, this will not happen and we get $\mu = \mu'$.

(b) Though $\pi_{\overline{\nu}\nu}$ may fail to preserve the relation $\overline{\mu}_0 <_1 \overline{\mu}_1$ when $\overline{\mu}_1 = \mu(\overline{\nu})$ it does preserve the following relation $\dashv$: $\overline{\mu}_0 \dashv \overline{\mu}_1$ iff there are $\overline{\overline{\mu}}_0 <_1^* \overline{\mu}_0$, $\overline{\overline{\mu}}_1 <_1^* \overline{\mu}_1$, $\overline{\overline{\mu}}_0 <_0 \overline{\overline{\mu}}_1$ and $\overline{\mu}_0 <_1 \pi_{\overline{\overline{\mu}}_1\overline{\mu}_1}(\overline{\overline{\mu}}_0)$. Moreover, in case $\pi_{\overline{\overline{\mu}}_1\overline{\mu}_1}$ is cofinal (the troublesome case for $<_1$-preservation) then $\overline{\mu}_1$ is the direct limit of $\{\overline{\mu}_0 \mid \overline{\mu}_0 \dashv \overline{\mu}_1\}$ via natural maps (if $\sigma \dashv \tau$ then $f_{\sigma\tau}$ is $\pi_{\sigma\gamma}$ where $\overline{\sigma} <_1^* \sigma$, $\overline{\tau} <_1^* \tau$ and $\gamma = \pi_{\overline{\tau}\tau}(\overline{\sigma})$).

(c) Of course one could formulate "Gap 2 Morass with Square" but as we know of no applications of this principle, we have elected not to do so here for the sake of simplicity.

Proof. (a)–(e). This is just as in the gap 1 case, with one exception: we must show that if $\overline{\nu}$ is $<_0$-limit, $\overline{\nu} < \mu(\overline{\nu})$, $\overline{\nu} <_1 \nu$, $\pi_{\overline{\nu}\nu}\restriction\overline{\nu}$ not cofinal, $\lambda = \bigcup \mathrm{Range}(\pi_{\overline{\nu}\nu}\restriction\overline{\nu})$, $\mu_\lambda = \bigcup \mathrm{Range}(\pi_{\overline{\nu}\nu})$ and $\mu = \bigcup \mathrm{Range}(\pi_{\overline{\nu}\nu})$ then $\mu_\lambda <_1 \mu$ and $\pi_{\overline{\nu}\nu} = \pi_{\mu_\lambda\mu} \circ \pi_{\overline{\nu}\lambda}$. Consider $H_\lambda = H^{\beta(\nu),\sigma}_{n(\nu)}(\lambda \cup \{q(\nu)\})$ and $H = H^{\beta(\nu),\sigma}_{n(\nu)}(\nu \cup \{q(\nu)\})$ where $\sigma = \bigcup(\varrho^{\beta(\nu)}_{n(\nu)-1} \cap H^{\beta(\nu)}_{n(\nu)}(\alpha(\overline{\nu}) \cup \{q(\nu)\}))$. Then H_λ is $\Sigma^*_{n(\nu)}$-elementary in H and after transitive collapse yields $\pi_{\mu_\lambda\mu}$. And $\pi_{\overline{\nu}\nu} = \pi_{\mu_\lambda\mu} \circ \pi_{\overline{\nu}\lambda}$ follows from the fact that $\pi_{\overline{\nu}\nu}, \pi_{\overline{\nu}\lambda}$ are obtained respectively by collapsing the inclusion of $H^{\beta(\nu)}_{n(\nu)}(\alpha(\overline{\nu}) \cup \{q(\nu)\})$ in H, H_λ.

(f) The fact that $\overline{\mu} > \overline{\nu}$ iff $\mu > \nu$ is clear from $\Sigma^*_{n(\nu)}$-elementarity of $\widetilde{\pi}_{\overline{\nu}\nu}$. For the rest, first suppose that for some $\mu_0 <_1 \mu$, $\pi_{\mu_0\mu}$ is cofinal. If μ is a $<_1$-successor then we can take $\mu_0 <_1^* \mu$ and then we have $\nu(\mu_0) \in q(\nu)$ and therefore $\nu(\mu_0) = \pi(\overline{\nu}_0)$ for some $\overline{\nu}_0$, where $\pi = \widetilde{\pi}_{\overline{\nu}\nu}$. Then $H^{\beta(\overline{\nu})}_{n(\nu)}(\overline{\nu}_0 \cup \{q(\overline{\nu})\}) = H$ is cofinal in $\overline{\mu}$ and $H \cap \overline{\nu} = \overline{\nu}_0$. So we get $\overline{\mu}_0 <_1^* \overline{\mu}$ and $\nu(\overline{\mu}_0) = \overline{\nu}_0$. If μ is a $<_1$-limit then $n(\mu) < n(\nu)$ and π is therefore $\Sigma^*_{n(\mu)+1}$-elementary. Thus $\mathrm{Range}(\pi) \cap \{\nu(\mu_0) \mid \mu_0 <_1 \mu\}$ is unbounded in $\mathrm{Range}(\pi) \cap \nu$ so $\overline{\mu}$ is a $<_1$-limit as all maps $\pi_{\mu_0\mu}$, $\mu_0 <_1 \mu$ sufficiently large, are cofinal.

Second, suppose that there is no cofinal $\pi_{\mu_0\mu}$ with $\mu_0 <_1 \mu$. If $n(\mu) < n(\nu)$ then for $\mu_0 <_1 \mu$ we must have $\Sigma^*_{n(\mu)+1}$-elementarity for $\widetilde{\pi}_{\mu_0\mu}$ (see the proof of (g) from the gap 1 case). Thus if μ is a $<_1$-limit then $n(\nu) \geq n(\mu)+2$ and we see that $\overline{\mu}$ is a $<_1$-limit as $\{\nu(\mu_0) \mid \mu_0 <_1 \mu\}$ is $\Pi^*_{n(\mu)+1}$. If μ is not a $<_1$-limit then $\max\{\nu(\mu_0) \mid \mu_0 <_1 \mu\}$ belongs to $\mathrm{Range}(\pi)$ as it is either in $q(\nu)$ or is 0. Thus $\overline{\mu}$ is not a $<_1$-limit and if $\mu_0 <_1^* \mu$ then $\nu(\mu_0) = \nu(\overline{\mu}_0)$ where $\overline{\mu}_0 <_1^* \overline{\mu}$. If μ is $<_1$-minimal then so is $\overline{\mu}$. Finally, if $n(\mu) = n(\nu)$ then μ is not a $<_1$-limit; if $\nu_0 = \bigcup\{\nu' < \nu \mid \nu' = \nu \cap H^{\beta(\nu)}_{n(\nu)}(\nu' \cup \{q(\mu)\})\}$ then $\nu_0 \in \mathrm{Range}(\pi)$ but now since π is Q-elementary we get $H^{\beta(\nu)}_{n(\nu)}(\nu_0 \cup \{q(\mu)\}) \cap \mu$ bounded below $\mu^* = \bigcup(\mathrm{Range}(\pi) \cap \mu)$. It follows that $\mu_0 <_1 \mu$ iff $\mu_0 <_1 \mu^*$, so $\overline{\mu}$ is not a $<_1$-limit and if $\mu_0 <_1^* \mu$ then $\nu(\mu_0) = \pi(\nu(\overline{\mu}_0))$ where $\overline{\mu}_0 <_1^* \overline{\mu}$. If μ is $<_1$-minimal then so is $\overline{\mu}$.

Note that in the above argument we also verified the final statement of (f). The remaining claim in (f) is clear by Σ_1-elementarity.

(g) The argument in the proof of (f) showed that $\pi_{\bar{\nu}\nu}(\nu(\overline{\mu}_0)) = \nu(\mu_0)$ and $\pi_{\bar{\mu}_0\mu}$ cofinal iff $\pi_{\mu_0\mu}$ cofinal. Finally, if $\lambda = \bigcup \mathrm{Range}(\pi_{\mu_0\mu}) < \mu$ then we get $\lambda \in \mathrm{Range}(\pi_{\bar{\nu}\nu})$ by the argument in (f), and hence $\mu_0 \in \mathrm{Range}(\pi_{\bar{\nu}\nu})$. Then we must have $\pi^{-1}(\mu_0) = \overline{\mu}_0$.

(h) If μ is not a $<_1$-limit then either $n(\mu) = n(\nu)$ and the result follows easily or $H^{\beta(\nu)}_{n(\mu)}(\nu_0 \cup \{q(\mu)\}) \cap \mu$ is bounded in μ for each $\nu_0 < \nu$, which means that $X \cap \mu$ bounded in μ iff $X \cap \nu$ bounded in ν for any X which is $\Sigma^*_{n(\nu)}$-elementary in $\widetilde{J}_{\beta(\nu)}$.

(i) This is clear if $\overline{\mu}_1 < \overline{\mu}$. Otherwise it follows immediately when $\pi_{\bar{\nu}\nu}(\overline{\mu}_0) = \mu_0$ and otherwise by the fact that $\pi_{\bar{\mu}_0\mu}$ is given by $H^{\beta(\bar{\mu})}_{n(\bar{\mu})}(\nu(\overline{\mu}_0) \cup \{q(\overline{\mu})\})$, $\pi_{\mu_0\mu}$ is given by $H^{\beta(\mu)}_{n(\mu)}(\nu(\mu_0) \cup \{q(\mu)\})$ and $\pi_{\bar{\nu}\nu}(\nu(\overline{\mu}_0)) = \nu(\mu_0)$. ∎

References

A. Beller, R. B. Jensen and P. Welch [82], *Coding the Universe*, London Math. Soc. Lecture Note Ser. 47, Cambridge Univ. Press, 1982.

K. Devlin [84], *Constructibility*, Springer, 1984.

H.-D. Donder [85], *Another look at gap-1 morasses*, in: Recursion Theory, Proc. Sympos. Pure Math. 42, Amer. Math. Soc., 223–236.

H.-D. Donder, R. B. Jensen and L. J. Stanley [85], *Condensation-coherent global square systems*, ibid., 237–258.

S. D. Friedman [87], *Strong coding*, Ann. Pure Appl. Logic 35, 1–98.

S. D. Friedman [94], *A simpler proof of Jensen's coding theorem*, ibid. 70, 1–16.

R. B. Jensen [72], *The fine structure of the constructible hierarchy*, Ann. Math. Logic 4, 229–308.

R. B. Jensen [?], *Nonoverlapping extenders*, unpublished.

Department of Mathematics
Massachussets Institute of Technology
Cambridge, Massachussets 02139
U.S.A.
E-mail: sdf@math.mit.edu

Received 9 October 1996;
in revised form 7 February 1997