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Are initially ω_1-compact separable regular spaces compact?

by

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Abstract. We investigate the question of the title. While it is immediate that CH yields a positive answer we discover that the situation under the negation of CH holds some surprises.

1. Introduction. A topological space is said to be *initially* ω_1*-compact* if every open cover of cardinality at most ω_1 has a finite subcover. This is well-known to be equivalent (in the context of Hausdorff spaces) to the property that the space is countably compact and each subset A of cardinality ω_1 has a *complete accumulation point*, that is, a point whose every neighbourhood contains a set of points of A with cardinality $|A|$.

Since a regular separable space has a base for the topology of cardinality at most $\mathfrak{c}$, it follows immediately from CH that a separable regular space is compact if it is initially ω_1-compact. If the space is in addition first countable (or even has countable tightness) then this implication has been shown to hold in several models of $\neg$CH and only recently been shown to consistently fail. However, there is a very well known class of examples, discussed below, which in many models of $\neg$CH are themselves non-compact initially ω_1-compact separable regular spaces. Hence it seemed reasonable to suppose that the failure of CH would imply the existence of such a space. In fact, with some help from Shelah (in the form of Theorem 3.7), we show that if $\mathfrak{c}$ has cofinality greater than ω_1 then this is indeed the case. However, we also show that the question is not decided by the assertion that $\mathfrak{c}$ has cofinality ω_1. Another interesting result (Theorem 2.6) is that it is consistent, with any value of $\mathfrak{c}$, to have that all locally compact such spaces (separable regular and initially ω_1-compact) are indeed compact.

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For the remainder of the paper we let $(I)_{\mathcal{C}}$ denote the assertion "every separable initially ω_1-compact regular space in the class $\mathcal{C}$ is compact". If we omit the subscript $\mathcal{C}$ then we interpret $\mathcal{C}$ as the entire class of regular spaces. We will use LC to denote the class of locally compact spaces, SC to denote the class of sequentially compact spaces, and CT will be the class of spaces with countable tightness. A space is *sequentially compact* if every infinite set contains a converging sequence. A space is said to have *countable tightness* if the closure of every set is equal to the union of the closures of its countable subsets.

2. On $(I)_{\mathcal{C}}$ for special $\mathcal{C}$. As mentioned above the following result is immediate.

PROPOSITION 2.1 (CH). (I), *i.e. every initially ω_1-compact separable regular space is compact.*

Part of the interest in this topic stems from the recent investigations into spaces of countable tightness which we now summarize.

PROPOSITION 2.2. $(I)_{\mathrm{CT}}$ *follows from PFA* ([2]) *and also holds in any model obtained by adding Cohen reals to a model of CH* ([4]). *However*, $(I)_{\mathrm{CT}}$ *can fail* ([10]) *and there can even be a first countable example* ([8]).

A well-known class of spaces are the so-called Franklin–Rajagopalan spaces [11], see also [13, §7] or [14, 2.11].

These are constructed as follows.

EXAMPLE 1. Fix a maximal strictly increasing mod finite well-ordered chain of co-infinite subsets of ω, say $\{a_\alpha : \alpha < \kappa\}$. We assume that κ is a regular cardinal. Let B be the Boolean subalgebra of $\mathcal{P}(\omega)$ which is generated by this chain together with the finite subsets of ω and set K to be the Stone space of ultrafilters on B. The space K can be regarded as $\omega \cup \{x_\alpha : \alpha \le \kappa\}$ in which ω is dense, open and discrete, while $\{x_\alpha : \alpha \le \kappa\}$ is homeomorphic to the ordinal space $\kappa + 1$. By the maximality of the chain, it follows that every infinite subset of ω has a subsequence converging to some x_α with $\alpha < \kappa$. It follows that both K and $K \setminus \{x_\kappa\}$ are sequentially compact.

PROPOSITION 2.3. *If there is a maximal chain mod finite of co-infinite subsets of ω with cofinality greater than ω_1 then there is a non-compact initially ω_1-compact separable regular space which is also locally compact and sequentially compact*, *i.e.* $(I)_{\mathrm{LC}}$ *and* $(I)_{\mathrm{SC}}$ *both fail.*

Proof. If there is such a chain of length κ such that $\mathrm{cf}(\kappa) > \omega_1$ then let K be the space constructed in Example 1 and let x_κ denote the same point as above. Simply set X equal to the space K with the point x_κ removed.

To see that X is initially ω_1-compact we just note that no subset of X of cardinality at most ω_1 converges, in K, to x_κ. ∎

Clearly then $\mathfrak{t} > \omega_1$, hence $\mathrm{MA}(\omega_1)$, implies the failure of $(I)_{\mathrm{SC}}$ and $(I)_{\mathrm{LC}}$. It is well known that if Cohen reals are added to a model of CH then there are no such well-ordered mod finite chains. In addition, $(I)_{\mathrm{SC}}$ holds in such models because of the following result which is taken from Theorem 6 of [5].

PROPOSITION 2.4. *If Cohen reals are added to a model of CH then every ω_1-compact, sequentially compact, separable regular space is compact. That is, $(I)_{\mathrm{SC}}$ holds in all such models.*

It is considerably easier to get $(I)_{\mathrm{LC}}$ to fail. First some simple notation.

Analogous to converging ω-sequences, we say that, for an infinite cardinal κ, a sequence $\langle x_\alpha : \alpha < \kappa\rangle$ converges to a point x if for each neighbourhood U of x, there is a $\beta < \kappa$ such that $\{x_\alpha : \beta < \alpha < \kappa\} \subset U$. We say that a point x is a *κ-limit* if there is some κ sequence of distinct points which converges to x.

If p is any point of $\beta\omega \setminus \omega$ then the space $\beta\omega \setminus \{p\}$ is an example of a locally compact initially ω_1-compact space which is not compact if and only if p is not an ω_1-limit in $\beta\omega$. More generally, if K is a separable compact space in which there is a non-isolated point, say x, which is not an ω-limit nor an ω_1-limit in K then $X = K \setminus \{x\}$ is also such an example. However, we have the following two results.

THEOREM 2.5. *If uncountably many Cohen reals are added* (*to any model of ZFC*) *then every point p of $\beta\omega \setminus \omega$ is an ω_1-limit in $\beta\omega$.*

THEOREM 2.6. *If any number of Cohen reals are added to a model of CH then $(I)_{\mathrm{LC}}$ holds. In fact, for every compact separable space K and point p of K, p is an ω_1-limit in K if and only if p does not have a countable local base.*

For a non-empty set S, we let $\mathrm{Fn}(S, 2)$ denote the usual Cohen poset consisting of finite partial functions from S into 2. For an $\mathrm{Fn}(S, 2)$-name, τ, of a subset of ω, let the *support* of τ be the smallest (countable) subset R of S such that for each $n \in \omega$ and $p \in \mathrm{Fn}(S, 2)$,

$$p \Vdash n \in \tau \text{ iff } p{\restriction}R \Vdash n \in \tau.$$

P r o o f o f T h e o r e m 2.5. We first use the standard factoring lemma of forcing to observe we may assume that we are forcing with the poset $\mathrm{Fn}(\omega_1, 2)$. Indeed, if I is any uncountable index set, then $\mathrm{Fn}(I, 2)$ is isomorphic to $\mathrm{Fn}(I \cup \omega_1, 2)$. Therefore we first assume that we are forcing with $\mathrm{Fn}(I \cup \omega_1, 2)$. Now if G is any generic filter for $\mathrm{Fn}(I \cup \omega_1, 2)$, we can pass to the model $V[G \cap \mathrm{Fn}(I \setminus \omega_1, 2)]$ and treat this as our ground model. Thus the final model is obtained by forcing over this model with $\mathrm{Fn}(\omega_1, 2)$.

In the extension, let p be an ultrafilter on ω and for each $\alpha < \omega_1$, let F_α denote the filter base consisting of all those members F of p for which there is an $\mathrm{Fn}(\omega_1, 2)$-name in the ground model with support contained in α (such that F is equal to the interpretation of the name by the generic filter). It is easily seen that we can, for each $\alpha < \omega_1$, choose an ultrafilter $q_\alpha \in \beta\omega \setminus \{p\}$ such that $F_\alpha \subset q_\alpha$. The sequence $\langle q_\alpha : \alpha < \omega_1 \rangle$ converges to p since for each $A \in p$, there is a γ (containing the support of a name for A) such that $A \in q_\alpha$ for all $\alpha \geq \gamma$. ■

P r o o f o f T h e o r e m 2.6. We first establish that it suffices to show the following lemma.

LEMMA 2.7. *V be a model of CH and G be an $\mathrm{Fn}(\kappa, 2)$-generic filter. If, in $V[G]$, $\mathcal{F}$ is a filter on ω which is not countably generated then there are filters $\mathcal{F}_\alpha$ $(\alpha < \omega_1)$ such that every member of $\mathcal{F}$ is a member of all but countably many of the $\mathcal{F}_\alpha$ and there is some $a \in \mathcal{F}_\alpha$ such that $\omega - a \in \mathcal{F}$.*

Indeed, suppose that X is a compact space and that ω is a dense subset of X. Also suppose that x in X does not have a countable local base. Set $\mathcal{F}$ to be the filter of all subsets F of ω such that x is not in the closure of $\omega \setminus F$, i.e. $\mathcal{F}$ is the trace of the neighbourhood filter of x on ω. It should be clear that $\mathcal{F}$ is not countably generated. Let $\{\mathcal{F}_\alpha : \alpha \in \omega_1\}$ be as above. For each α, choose a point x_α in X which is in the intersection $\bigcap\{\mathrm{cl}_X(a) : a \in \mathcal{F}_\alpha\}$. We now verify that the sequence $\langle x_\alpha : \alpha \in \omega_1 \rangle$ converges to x. We prove this by showing that $x_\alpha \in U$ for all but countably many α (it should be clear that $x_\alpha \neq x$ for each α). Of course if $W \subset U$ is a closed neighbourhood of x, then $a = W \cap \omega$ is a member of $\mathcal{F}$. In addition, for each α such that $a \in \mathcal{F}_\alpha$, we have $x_\alpha \in \mathrm{cl}_X(a) \subset W$, hence $x_\alpha \in U$. ■

P r o o f o f L e m m a 2.7. Let $\mathcal{F}$ be an $\mathrm{Fn}(\kappa, 2)$-name of a filter on ω and, with no loss of generality, assume that 1 forces that $\mathcal{F}$ has no base of cardinality less than or equal to ω_1. Recursively choose ω_2 names for members of $\mathcal{F}$ as follows. For each α, let 1 force that $\dot{F}_\alpha \in \mathcal{F}$ and $\dot{F}_\alpha$ does not contain any member of $\mathcal{F}$ which has an $\mathrm{Fn}(I_\alpha, 2)$-name, where I_α is a set of size at most $\aleph_1$ which contains the support of each $\dot{F}_\beta$ for $\beta < \alpha$.

Since $V \models \mathrm{CH}$, we can assume that the above sequence has been thinned down so that the supports of the $\dot{F}_\alpha$ form a Δ-system with root R and that all the $\dot{F}_\alpha$ are identical with respect to R. Let J_α denote the countable set which is equal to the support of $\dot{F}_\alpha$ with R removed. Hence the sets of the family $\{J_\alpha : \alpha < \omega_1\}$ are pairwise disjoint. Note that our assumption on the enumeration ensures that $\dot{F}_\alpha$ does not contain any member of $\mathcal{F}$ whose support is contained in R. By passing to the extension obtained by forcing with $\mathrm{Fn}(R,2)$ we can assume that the support of $\dot{F}_\alpha$ is equal to J_α (hence the supports are pairwise disjoint).

We are ready to define $\mathcal{F}_\xi$ for $\xi < \omega_1$.

$\mathcal{F}_\xi$ is generated by those $\dot{a}$ such that either $\dot{a} = \omega - \dot{F}_\xi$ or some condition forces that $\dot{a} \in \mathcal{F}$ and the support of $\dot{a}$ is disjoint from J_ξ.

Since the generators of the $\mathcal{F}_\xi$ of the latter kind are members of $\mathcal{F}$, they are closed under finite intersections (the support of an intersection is contained in the union of the supports). Therefore to show that $\mathcal{F}_\xi$ has the finite intersection property it suffices to show that $\dot{a} - \dot{F}_\xi$ is not empty for each $\dot{a} \in \dot{\mathcal{F}}$ such that the support of $\dot{a}$ is disjoint from J_ξ. Otherwise there is a condition p which forces that $\dot{a}$ is contained in F_ξ. Since these sets have disjoint supports, it follows that p forces that F_ξ contains the set $\{n : (\exists q < p) \ \ q \Vdash n \in \dot{a}\}$. However, this latter set is in the ground model $V[G \cap \mathrm{Fn}(R,2)]$ and is a member of $\mathcal{F}$. By our assumption, $1 \Vdash F_\xi$ does not contain any member of $\mathcal{F}$ which is a member of $V[G \cap \mathrm{Fn}(R,2)]$. ∎

3. More on the failure of (I)**.** This section culminates with the result that (I) fails whenever the cofinality of $\mathfrak{c}$ is greater than ω_1. However, we first study some other conditions which imply that (I) fails and which seem to be of independent interest. When we use the symbol 2^κ we will mean the usual product topological space. In this context we will use $(2^\kappa)_0$ to denote all those members of the product which are eventually equal to 0 and the obvious analogue, $(2^\kappa)_f$, for arbitrary $f \in 2^\kappa$.

The following lemma is our main tool for constructing examples to witness the failure of (I).

LEMMA 3.1. *Suppose that* $\mathrm{cf}(\kappa) > \omega_1$ *and that* $S \in [2^\kappa]^\omega$ *is such that*

1. *for all infinite* $T \subset S$, $\mathrm{cl}(T) \cap (2^\kappa)_0$ *is not empty*, *and*
2. S *has a limit point not in* $(2^\kappa)_0$.

Then, *for some* $g \in 2^\kappa$, *the subspace*

$$X = (S \setminus \{g\}) \cup (\mathrm{cl}(S) \cap (2^\kappa)_0)$$

is a witness to the failure of (I).

Proof. We choose $g \in 2^\kappa \setminus (2^\kappa)_0$ to be any limit of S (which exists by 2). It follows that X is not compact since it is not closed in 2^κ. Obviously X is separable since $S \setminus \{g\}$ is dense in X. By hypothesis 1, every infinite subset of S has a limit point in $X \cap (2^\kappa)_0$. Therefore, to show that X is initially ω_1 compact, it suffices to show that $(2^\kappa)_0$ is initially ω_1 compact. This follows immediately from the fact that if $A \subset (2^\kappa)_0$ has cardinality at most ω_1 then $\overline{A} \subset (2^\kappa)_0$. Indeed, since $\mathrm{cf}(\kappa) > \omega_1$, there is a $\lambda < \kappa$ such that every member of A is constantly 0 on $[\lambda, \kappa)$. Therefore A is contained in the closed set $2^{\lambda} \times \langle 0, 0, 0, \ldots \rangle$. ∎

It is a natural strengthening of the hypotheses of the previous lemma to ask that $\mathrm{cl}(T) \cap (2^{\kappa})_f \neq \emptyset$ for all infinite $T \subset S$, for all $f \in 2^{\kappa}$. This is very closely related to the notion of an *independent splitting family* on ω (see [12]). As we see by the following definition we also have a very close parallel to the notion of a κ-HFD (see [7, 6.1]).

Definition 3.2. An infinite subset S of 2^{κ} is a (κ, λ)-HFD if for every infinite $T \subset S$, there is set $J \subset \kappa$ such that $|\kappa \setminus J| < \lambda$ and $\{t{\restriction}J : t \in T\}$ is dense in 2^J.

We will say that S is a $(\kappa, \rightarrow)$-HFD if for every infinite $T \subset S$, there is an $\alpha < \kappa$ such that $\{t{\restriction}(\kappa \setminus \alpha) : t \in T\}$ is dense in $2^{(\kappa \setminus \alpha)}$. Of course the notion of a (κ, κ)-HFD coincides with that of a $(\kappa, \rightarrow)$-HFD in the case where κ is regular.

The hypotheses of Lemma 3.1 are easily seen to be fulfilled if S is a $(\kappa, \rightarrow)$-HFD, hence we have the following result.

Corollary 3.3. *The failure of* (I) *follows from the existence of a* $(\kappa, \rightarrow)$*-HFD for any cardinal* κ *with cofinality greater than* ω_1*.*

The existence of (κ, ω_1)-HFD's has been established in models obtained by Cohen forcing (see [6]).

Proposition 3.4. *Forcing with* $\mathrm{Fn}(\kappa, 2)$ *yields a model in which there is a* (κ, ω_1)*-HFD.*

By combining the previous two results we deduce that (I) fails in any model in which at least ω_2 Cohen reals have been added.

Corollary 3.5. *Forcing with* $\mathrm{Fn}(\omega_2, 2)$ *yields a model in which* (I) *fails. Therefore the failure of* (I) *is consistent with* $\mathfrak{c} > \omega_1$ *combined with any other cardinal arithmetic.*

It is also standard to show that Martin's Axiom for countable posets implies that there is a $(\mathfrak{c}, \rightarrow)$-HFD. This will follow from the next result. We would like to relate the existence of a $(\mathfrak{c}, \rightarrow)$-HFD to another less familiar cardinal function on ω. A π-*base* for an ultrafilter on ω is a family of infinite subsets of ω with the property that every member of the ultrafilter contains one (mod finite). The π-*character* of an ultrafilter is the least cardinal of a π-base. It was proven by Balcar and Simon [1] that the minimum possible π-character of a free ultrafilter on ω is equal to the *reaping number* $\mathfrak{r}$. A family $\mathcal{A} \subset [\omega]^{\omega}$ is *reaped* by $b \in [\omega]^{\omega}$ if $a \setminus b$ and $a \cap b$ are infinite for all $a \in \mathcal{A}$. Then $\mathfrak{r}$ is the least cardinal of a family that cannot be reaped. It is routine to check that Martin's Axiom for countable posets implies that $\mathfrak{r}$ is equal to $\mathfrak{c}$.

PROPOSITION 3.6. *The existence of a* $(\mathfrak{c}, \to)$*-HFD follows from* $\mathfrak{r} = \mathfrak{c}$.

Proof. Let $\{T_\alpha : \alpha \in \mathfrak{c}\}$ enumerate the infinite subsets of ω. We inductively choose a family $\{A_\alpha : \alpha \in \mathfrak{c}\}$ of subsets of ω. We will set $S = \{s_n : n \in \omega\} \subset 2^{\mathfrak{c}}$ where $s_n(\alpha) = 1$ if and only if $n \in A_\alpha$. In order to ensure that S is a $(\mathfrak{c}, \to)$-HFD we will ensure that, for each $\alpha < \mathfrak{c}$, the family $\{A_\beta \cap T_\alpha : \alpha < \beta < \mathfrak{c}\}$ is an independent family in $\mathcal{P}(T_\alpha)$. This is easily done. Let $\mathcal{B}_\alpha$ be the Boolean subalgebra of $\mathcal{P}(\omega)$ which is generated by $\{T_\gamma : \gamma < \alpha\}$ together with $\{A_\gamma : \gamma < \alpha\}$. Since $|\mathcal{B}_\alpha| < \mathfrak{r}$, there is a set A_α such that $b \cap A_\alpha$ and $b \setminus A_\alpha$ are infinite for each infinite $b \in \mathcal{B}_\alpha$. It follows easily by induction that S is a $(\mathfrak{c}, \to)$-HFD. ∎

QUESTION 1. Is there a $(\mathfrak{c}, \to)$-HFD? Is there a $(\kappa, \to)$-HFD for some κ?

Of course the existence of a $(\kappa, \to)$-HFD with $\mathrm{cf}(\kappa) > \omega_1$ is much stronger than the hypothesis of Lemma 3.1 and therefore of the failure of (I). We established that the failure of (I) followed from the existence of an ultrafilter with π-character equal to $\mathfrak{c}$ together with the assumption that $\mathrm{cf}(\mathfrak{c}) > \omega_1$ (recall that $\mathfrak{r} = \mathfrak{c}$, the hypothesis of Lemma 3.6, is the assertion that all free ultrafilters on ω have π-character equal to $\mathfrak{c}$). Let us also note that such an ultrafilter was shown to exist if $\mathfrak{c} > \omega_1$ is regular by Bell and Kunen [3]. (In fact, they showed that there is an ultrafilter of π-character $\geq \mathrm{cf}(\mathfrak{c})$.) Our result, however, follows from the following stronger result of S. Shelah. This result is included with Shelah's permission and was proven by him in the course of discussions with one of the authors about the results of this paper and will not be published elsewhere. The result can equally well be stated in terms of infinite subsets of ω or in the terms of Lemma 3.1.

THEOREM 3.7 (S. Shelah). *There is a set* $S \in [2^{\mathfrak{c}}]^\omega$ *such that*

1. *for all infinite* $T \subset S$, $\mathrm{cl}(T) \cap (2^{\mathfrak{c}})_0$ *is not empty*, *and*
2. *the constant function* 1 *is a limit point of* S.

Proof. As in Lemma 3.6, it is equivalent to find a special family of subsets of ω.

Indeed, we shall construct a centred family $\{A_\alpha : \alpha \in \mathfrak{c}\}$ such that for each infinite $T \subset \omega$, there is a $\gamma \in \mathfrak{c}$ such that $\{T \cap (\omega \setminus A_\alpha) : \alpha \in \mathfrak{c} \setminus \gamma\}$ is also centred.

Having done so, we set $S = \{s_n : n \in \omega\}$ with $s_n(\alpha) = 1$ if and only if $n \in A_\alpha$. We check that properties 1 and 2 of Lemma 3.1 hold. Since the family of A_α's is centred, it follows that the constant sequence $\vec{1} \in 2^{\mathfrak{c}}$ is a limit point of S. Similarly, for each infinite $\widetilde{T} \subset S$, set $T = \{n : s_n \in \widetilde{T}\}$. Let γ be such that $\{T \setminus A_\alpha : \alpha \in \mathfrak{c} \setminus \gamma\}$ is centred. It follows then that $\bigcap\{\mathrm{cl}(\{s_n : n \in T \setminus A_\alpha\}) : \gamma < \alpha < \mathfrak{c}\}$ is not empty, so fix any $g \in 2^{\mathfrak{c}}$ in this intersection. For each $\alpha \geq \gamma$, g is a limit of $\{s_n : n \notin A_\alpha\}$, hence $g(\alpha) = 0$. Therefore $g \in (2^{\mathfrak{c}})_0$.

We construct the family $\{A_\alpha : \alpha \in \mathfrak{c}\}$ by induction on α with the aid of Kunen's $\mathfrak{c}\times\mathfrak{c}$-independent matrix ([9]) (similar to, but easier than, Bell and Kunen's [3] construction of an ultrafilter of π-character $\geq \mathrm{cf}(\mathfrak{c})$). A family $\{a(\alpha,\gamma) : \alpha \in R, \gamma \in \mathfrak{c}\}$ is said to be an $R\times\mathfrak{c}$-*independent matrix* if for each $\alpha \in R$ and $\beta,\gamma \in \mathfrak{c}$, $a(\alpha,\gamma)\cap a(\alpha,\beta)$ is finite while for each finite $H \subset R$ and each function $\varphi \in \mathfrak{c}^H$, $\bigcap\{a(\alpha,\varphi(\alpha)) : \alpha \in H\}$ is infinite. Kunen proved in [9] that there is a $\mathfrak{c}\times\mathfrak{c}$-independent matrix; let $\{a(\alpha,\beta) : \alpha,\beta \in \mathfrak{c}\}$ be such a matrix.

Fix an enumeration, $\{T_\beta : \beta < \mathfrak{c}\}$, of $[\omega]^\omega$. Suppose that $\alpha < \mathfrak{c}$ and we have chosen $\varrho_\beta \in \mathfrak{c}$ for each $\beta < \alpha$ and set $A_\beta = a(\beta,\varrho_\beta)$ so that $\{T_\xi \cap (\omega\setminus A_\beta) : \beta \in \alpha\setminus\xi\}$ is centred for each $\xi < \alpha$. Note that $\{A_\beta : \beta < \alpha\}$ is centred because the matrix is independent.

For each $\xi \leq \alpha$ and for each $H \in [\alpha\setminus\xi]^{<\omega}$, choose $g(\xi,H) \in \mathfrak{c}$ so that, if possible,

$$T_\xi \cap \bigcap_{\zeta\in H}(\omega\setminus A_\zeta) \subset a(\alpha, g(\xi,H))\ .$$

Let ϱ_α be any member of $\mathfrak{c}$ which is not equal to $g(\xi,H)$ for any $\xi \leq \alpha$ and $H \in [\alpha\setminus\xi]^{<\omega}$. We finish by verifying that, for each $\xi \leq \alpha$, $\{T_\xi\}\cup\{\omega\setminus A_\beta : \xi \leq \beta \leq \alpha\}$ is centred mod finite. If it were not, then there would be some finite $H \subset (\alpha\setminus\xi)$ such that $T_\xi \cap \bigcap_{\zeta\in H}(\omega\setminus A_\zeta)$ is disjoint (mod finite) from $\omega\setminus A_\alpha$. Of course this means that $a(\alpha,\varrho_\alpha)$ contains mod finite this intersection. However, this contradicts the fact that $a(\alpha,g(\xi,H))$ is almost disjoint from $a(\alpha,\varrho_\alpha)$ since, by induction, $T_\xi \cap \bigcap_{\zeta\in H}(\omega\setminus A_\zeta)$ is infinite. ∎

As indicated, this section culminates with the following quite satisfactory result.

Corollary 3.8. *If $\mathfrak{c}$ has cofinality greater than ω_1, then (I) fails.*

4. (I) **does not imply CH.** In the previous section we established that (I) implies that $\mathfrak{c}$ has cofinality ω_1, and, by Corollary 3.5, (I) can fail even if $\mathrm{cf}(\mathfrak{c})$ is ω_1. We now establish that there is a model of the failure of CH in which (I) holds.

Theorem 4.1. *It is consistent with the negation of CH that every separable initially ω_1-compact regular space is compact.*

Proof. Consider the Bell–Kunen model constructed in [3]. In this model of the failure of CH, the following property holds (which implies that every ultrafilter on ω has π-character equal to ω_1):

For each filter F on ω, there is a family $\{a_\alpha : \alpha < \omega_1\}$ of infinite subsets of ω such that for each $A \in F$, there is a $\beta < \omega_1$ such that $a_\alpha \setminus A$ is finite for each $\alpha > \beta$.

The model is constructed by iteratively constructing models M_α ($\alpha < \omega_1$) of Martin's Axiom in which $M_\alpha \models \mathfrak{c} = \omega_{\alpha+1}$. The final model M is the union of these inner models. If F is a filter in the final model, then $F \cap M_\alpha$ will be "diagonalized" by some a_α in the model $M_{\alpha+1}$, i.e. $a_\alpha \setminus A$ is finite for each $A \in F \cap M_\alpha$. If A is any member of F, then A will appear in some model M_β and a_α will be almost contained in A for all $\alpha > \beta$.

Now suppose that X is an initially ω_1-compact regular space which has ω as a dense subset. If X is not compact, then let $\mathcal{W}$ be any open cover which has no finite subcover. Since X is regular we may assume that no finite subcollection of $\mathcal{W}$ has dense union. Let F be the following filter base on ω:

$$\Big\{\omega \setminus \bigcup \mathcal{W}' : \mathcal{W}' \in [\mathcal{W}]^{<\omega}\Big\}.$$

Let $\{a_\alpha : \alpha < \omega_1\}$ be the above mentioned converging π-base family for F. For each $\alpha < \omega_1$, there is an $x_\alpha \in X$ such that x_α is a limit point of a_α since X is countably compact. But now, we fix $x \in X$ and we show that x is not a complete accumulation point of $\{x_\alpha : \alpha < \omega_1\}$. Let $W \in \mathcal{W}$ be a neighbourhood of x and set $A = \omega \setminus W$. By the choice of the a_α's, there is a $\beta < \omega_1$ such that $x_\alpha \in \overline{A}$ for all $\alpha \geq \beta$. Therefore $W \cap \{x_\gamma : \gamma \in \omega_1\} \subset \{x_\gamma : \gamma < \beta\}$. ■

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