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Choice principles in Węglorz' models

by

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Abstract. Węglorz' models are models for set theory without the axiom of choice. Each one is determined by an atomic Boolean algebra. Here the algebraic properties of the Boolean algebra are compared to the set theoretic properties of the model.

1. Introduction. We investigate a class of permutation models $\mathcal{W}_{\mathbb{B}}^{\text{fin}}$ (where $\mathbb{B}$ is an atomic Boolean algebra) due to B. Węglorz [28]. The aim of [28] has been the proof that in the absence of the axiom of choice (AC) the atomicity of the powerset algebras is the only restriction on the structure of these Boolean algebras.

Subsequently M. Boffa [2] has applied $\mathcal{W}_{\mathbb{B}}^{\text{fin}}$ to constructions of models of second order Peano arithmetic and of fragments of Quine's NF. Boffa's main results depend on the additional requirement that the atomic Boolean algebra $\mathbb{B} \subseteq \mathcal{P}(A)$ is *structured*: Each infinite $b \in \mathbb{B}$ can be split into two infinite elements of $\mathbb{B}$. An equivalent condition is: $\mathbb{B}/\mathcal{I}_{\text{finite}}$ has no atoms; $\mathcal{I}_{\text{finite}}$ is the ideal which is generated by the atoms of $\mathbb{B}$. The purpose of the present paper is a characterization of this property in terms of the internal structure of $\mathcal{W}_{\mathbb{B}}^{\text{fin}}$ and related models. (See, for example, Theorems 3 and 5 below.)

1.1. *The model*. In the following definition of a slight generalization of the model $\mathcal{W}_{\mathbb{B}}^{\text{fin}}$ we shall use the notation of [16] and [7]. First one defines, within the class V of the pure sets of ZFC, a model $V(X)$ of $ZFA + AC$ whose set of atoms (objects without elements) is a copy of $X \in V$. Each permutation $\pi \in \mathcal{S}(X)$ (the symmetric group) extends to an $\in$-automorphism of $V(X)$. A Fraenkel–Mostowski model $\mathcal{M} \subseteq V(X)$ is *generated by the topological group* $G < \mathcal{S}(X)$ if for every m in $V(X)$, $m \in \mathcal{M}$ if and only if for every $x \in \text{TrCl}(\{m\})$, the stabilizer of x, $\text{stab}(x) = \{\pi \in G : \pi(x) = x\}$, is open in G. ($\text{TrCl}(x)$ is the transitive closure of x.)

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We may represent $\mathbb{B}$ as a subalgebra of $\mathcal{P}(A)$, where A is the set of the atoms. [Then $\mathcal{I}_{\text{finite}} = [A]^{<\omega}$ is the set of all finite subsets of A.] Moreover, in the following construction we shall identify A with the atoms (in the sense of set theory) of the model. Since a model satisfies AC if the set of the atoms is finite, we shall only consider infinite Boolean algebras $\mathbb{B}$.

We use the following terminology: $\mathrm{Aut}(\mathbb{B})$ is the group of all automorphisms of $\mathbb{B}$; it is a subgroup of the symmetric group $\mathcal{S}(A)$. $\mathcal{S}_{\text{finite}}(A)$ is the group of all finite permutations of A (i.e., each permutation only moves a finite number of elements); it is a subgroup of $\mathrm{Aut}(\mathbb{B})$. We shall define a group topology on a group Γ of automorphisms by means of a filterbase $\mathcal{F}$ which generates a normal filter of groups (i.e. a neighbourhoodbase of the identity of a T_2 group topology; cf. [20]).

Definition 1. $\mathbb{B}$ is an atomic Boolean algebra whose set of atoms is A and Γ is a group of permutations on A such that $\Gamma < \mathrm{Aut}(\mathbb{B})$. Then the model $\mathcal{W}_{\mathbb{B}}^{\Gamma} \subseteq V(A)$ is generated by the group Γ whose group topology is induced by the filterbase $\mathcal{F} = \{\mathrm{stab}(b) : b \in \mathbb{B}\}$.

If $x \in \mathcal{W}_{\mathbb{B}}^{\Gamma}$, then for some finite $\mathbb{D} \subseteq \mathbb{B}$ we have $\mathrm{stab}(x) \supseteq \mathrm{fix}(\mathbb{D}) = \bigcap\{\mathrm{stab}(d) : d \in \mathbb{D}\}$. We may assume that $\mathbb{D}$ is a subalgebra of $\mathbb{B}$ (in particular, $A \in \mathbb{D}$). Then the atoms of $\mathbb{D}$ form a finite ordered partition $\Pi = \langle P_1, \ldots, P_m\rangle$ of A into elements of $\mathbb{B}$ and

$$\mathrm{fix}(\mathbb{D}) = \{\pi \in \Gamma : (\forall 1 \le i \le m)(\forall p \in P_i)(\pi(p) \in P_i)\} = \mathrm{stab}(\Pi).$$

We will refer to such a partition as a *support* of x. By refining Π we may assume that each P_i is either infinite or a singleton.

1.2. *Examples.* In the trivial case $\Gamma = 1$ the model $\mathcal{W}_{\mathbb{B}}^{\Gamma} = V(\mathbb{B})$ satisfies AC. Węglorz' original model is $\mathcal{W}_{\mathbb{B}}^{\text{fin}} = \mathcal{W}_{\mathbb{B}}^{\Gamma}$ with $\Gamma = \mathcal{S}_{\text{finite}}(A)$. Its variant $\mathcal{W}_{\mathbb{B}}^{\text{Aut}}$ is defined in terms of the larger group $\Gamma = \mathrm{Aut}(\mathbb{B})$. It has been investigated in physics (cf. [8]). Węglorz' construction will provide the means for the unified treatment of independence proofs about atomic Boolean algebras. In Table 1 we survey sample properties of the algebra $\mathcal{P} = \mathcal{P}(A)$ in typical models $\mathcal{W}_{\mathbb{B}}^{\Gamma}$. (The notation will be explained later in the paper. The numbering of the models is provisional only. + means that an assertion is true in the model, − that it is false.)

$\mathcal{N}_1 = V(\omega) = \mathcal{W}_{\mathbb{B}}^{\Gamma}$ with $\mathbb{B} = \mathcal{P}(\omega)$ and $\Gamma = 1$ represents the ZFC theory of $\mathcal{P}(\omega)$.

$\mathcal{N}_2 = \mathcal{W}_{\mathbb{B}}^{\Gamma}$ is the ordered Mostowski model, where $\mathbb{B} \subseteq \mathcal{P}(\mathbb{Q})$ is the algebra which is generated by the open intervals and the finite sets (this is not the interval algebra) and $\Gamma = \mathrm{Aut}(\mathbb{Q}, <)$ consists of the increasing bijections. This model is atypical, insofar as it does not satisfy $\mathcal{S}_{\text{finite}}(\mathbb{Q}) < \Gamma$.

Table 1

Statements	Models					
	$\mathcal{N}_1$	$\mathcal{N}_2$	$\mathcal{N}_3$	$\mathcal{N}_4$	$\mathcal{N}_5$	$\mathcal{N}_6$
$\mathcal{P}$ is structured	+	+	−	+	+	+
$\lvert\mathcal{P}\rvert \geq \aleph_1$	+	−	−	−	−	−
$\mathcal{P} \oplus \mathcal{P}$ is σ-complete	−	+	+	+	+	+
$\mathcal{P}$ is (completely, 2)-distributive	+	+	−	−	−	−
$\mathcal{P}$ is compact	+	+	+	+	+	+
$\exists$ non-principal prime ideal on $\mathcal{P}$	+	+	+	−	+	−
$\exists$ non-principal σ-measure on $\mathcal{P}$	−	+	+	+	+	−

$\mathcal{N}_3 = \mathcal{W}_{\mathbb{B}}^{\Gamma}$ is the basic Fraenkel model, where $\mathbb{B}$ is the algebra of the finite and cofinite subsets of ω and $\Gamma = \mathcal{S}_{\text{finite}}$. The algebra $\mathbb{B}$ is not structured.

$\mathcal{N}_4 = \mathcal{W}_{\mathbb{B}}^{\Gamma}$, where $\mathbb{B}$ is the countable structured algebra which is generated by the arithmetic sequences and the finite subsets of $\mathbb{Z}$ and Γ is generated by the translations and the finite permutations.

$\mathcal{N}_5 = \mathcal{W}_{\mathbb{B}}^{\Gamma}$, where $\mathbb{B} = \mathcal{P}(\omega)$ and $\Gamma = \mathcal{S}_{\text{finite}}(\omega)$.

$\mathcal{N}_6 = \mathcal{W}_{\mathbb{B}}^{\Gamma}$, where $\mathbb{B} = \mathcal{P}(\omega)$ and $\Gamma = \mathcal{S}(\omega)$.

Unless stated otherwise, we shall assume $\mathcal{S}_{\text{finite}}(A) < \Gamma$ in order to exclude the following complication: Although Węglorz' model appears to be a rather limited construction, in a certain sense it is very general.

Let us recall from [6] that finite support models are the most general models in the following sense: Each Boolean combination of Jech–Sochor-bounded statements which is true in some permutation model is true in some finite support model. Here $\mathcal{M} \subseteq V(X)$ is a *finite support model* if its generating group $G < \mathcal{S}(X)$ carries the topology of pointwise convergence with respect to the discrete topology on X; i.e. the topology is induced by the filterbase $\mathcal{F} = \{\text{fix}(e) : e \subseteq X \text{ finite}\}$. Such a model may be identified with the following $\mathcal{W}_{\mathbb{B}}^{\Gamma}$. In $V(X)$ we set $\mathbb{B} = \mathbb{B}_{\mathcal{M}} = \mathcal{P}^{\mathcal{M}}(A)$, the powerset (in the sense of the model) of the atoms, and $\Gamma = \Gamma_{\mathcal{M}} = G$. Then $\mathcal{M} = \mathcal{W}_{\mathbb{B}}^{\Gamma}$, since both models are constructed within $V(X)$ by means of the same group with the same group action and the same topologies. [If $e = \{e_1, \ldots, e_n\} \subseteq X$ is finite, then the open group $\text{fix}(e) < G$ is open in Γ, since $\text{fix}(e) = \text{fix}(\Pi)$ for the support $\Pi = \langle \{e_1\}, \ldots, \{e_n\}, X \setminus e\rangle$. Conversely, if $\Pi = \langle P_1, \ldots, P_m\rangle$ is a support and $P_i \in \mathbb{B}$, then by the definition of $\mathbb{B}$ there are finite sets $e_i \subseteq X$ such that $\text{stab}(P_i) \supseteq \text{fix}(e_i)$ and the open group $\text{fix}(\Pi) < \Gamma$ is open in G, since $\text{fix}(\Pi) \supseteq \bigcap_{1\leq i\leq m} \text{stab}(P_i) \supseteq \bigcap_{1\leq i\leq m} \text{fix}(e_i)$.]

Following a suggestion by the referee we shall compare the models $\mathcal{W}_{\mathbb{B}}^{\Gamma}$ which satisfy $\mathcal{S}_{\text{finite}}(A) < \Gamma$ with the following restricted class of finite support models.

The structure $\mathcal{X}$ has an $\aleph_0$-*categorical* theory $\text{Th}(\mathcal{X})$ if each infinite countable model of $\text{Th}(\mathcal{X})$ is isomorphic to $\mathcal{X}$. The standard examples are $\mathcal{X}_1 = \aleph_0$, a countable set without any structure, and the rationals $\mathcal{X}_2 = (\mathbb{Q}, <)$.

Definition 2. We assume that $\mathcal{X}$ is a first order structure on the countable set X in a countable language with an $\aleph_0$-categorical theory. The topology of the automorphism group $G = \mathrm{Aut}(\mathcal{X}) < \mathcal{S}(X)$ is generated by the groups $\mathrm{fix}(e)$ for $e \subseteq X$ finite. Then $\mathcal{M}(\mathcal{X}) \subseteq V(X)$ is the permutation model which is generated by G (and the action id).

We let $\mathcal{A}$ be the structure which $\mathcal{X}$ induces on the set of atoms. By the definition of the model, $\mathcal{A} \in \mathcal{M}(\mathcal{X})$. Therefore each subset of A which in $\mathcal{A}$ is definable from finitely many parameters in A is an element of the model. For $\aleph_0$-categorical structures there are no other subsets of A in the sense of $\mathcal{M}$; cf. [7], proof of Theorem 16. We therefore may identify $\mathcal{M}(\mathcal{X})$ with the following model $\mathcal{W}_{\mathbb{B}}^{\Gamma}$: $\mathbb{B} = \mathcal{P}^{\mathcal{M}}(X)$ is the algebra of the sets which are definable from finitely many parameters in X, and $\Gamma = \mathrm{Aut}(\mathcal{X})$. If $\mathcal{X} = \mathcal{X}_1$, this is the basic Fraenkel model, and if $\mathcal{X} = \mathcal{X}_2$, then $\mathcal{W}_{\mathbb{B}}^{\Gamma}$ is the ordered Mostowski model; cf. [20] and [16]. The latter model does not satisfy $\mathcal{S}_{\mathrm{finite}}(A) < \Gamma$.

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2. Peculiarities of $\mathbb{B}$. In this section we investigate $\mathbb{B}$ from the point of view of the universe $\mathcal{W}_{\mathbb{B}}^{\Gamma}$ and derive arithmetical properties of this algebra which contradict the axiom of choice. If $\mathbb{B}$ is the algebra of the finite and cofinite subsets of ω and $\mathcal{S}_{\mathrm{finite}} < \Gamma$, then $\mathcal{W}_{\mathbb{B}}^{\Gamma}$ coincides with the basic Fraenkel model (cf. [7], [16]). For this particular model the algebraic structure of $\mathbb{B}$ is analyzed by Hodges [14], ring 3. To be accurate, the model of [14] is the ZF model which results from an application of the Jech–Sochor transfer theorem to the model of Definition 3 below (cf. [8]).

Definition 3. $\mathbb{B}$ is a Boolean algebra and $\Gamma < \mathrm{Aut}(\mathbb{B})$ is a subgroup of $\mathcal{S}(\mathbb{B})$. The model $\mathcal{M}_{\mathbb{B}}^{\Gamma} \subseteq V(\mathbb{B})$ is generated by the group Γ whose topology is induced by the filterbase $\mathcal{F} = \{\mathrm{stab}(b) : b \in \mathbb{B}\}$.

The models $\mathcal{M}_{\mathbb{B}}^{\Gamma}$ and $\mathcal{W}_{\mathbb{B}}^{\Gamma}$ differ by the sets of their atoms. From the point of view of the real world, the atoms of $\mathcal{M}_{\mathbb{B}}^{\Gamma}$ are a copy of the algebra $\mathbb{B}$, while the atoms of $\mathcal{W}_{\mathbb{B}}^{\Gamma}$ are a copy of the atoms of $\mathbb{B}$. This entails the following technical difference: The model $\mathcal{M}_{\mathbb{B}}^{\Gamma}$ is defined in terms of "finite supports".

If $\Gamma = \mathrm{Aut}(\mathbb{B})$, then $\mathcal{M}_{\mathbb{B}}^{\Gamma} = \mathcal{M}_{\mathbb{B}}^{\mathrm{Aut}}$. In [14], ring 2, there is an application of $\mathcal{M}_{\mathbb{B}}^{\mathrm{Aut}}$ to a non-atomic algebra. If $\mathbb{B}$ is atomic with the set A of atoms and $\Gamma = \mathcal{S}_{\mathrm{finite}}(A)$, then we set $\mathcal{M}_{\mathbb{B}}^{\Gamma} = \mathcal{M}_{\mathbb{B}}^{\mathrm{fin}}$. As follows from [7], Subsection

3.1, if $\mathbb{B}$ is atomic, then the model $\mathcal{M}_{\mathbb{B}}^{\Gamma}$ is isomorphic to $V(\mathbb{B})$, when this structure is constructed within $\mathcal{W}_{\mathbb{B}}^{\Gamma}$. Therefore the results of this paper, with the exception of Lemma 1, are immediately extended from $\mathcal{W}_{\mathbb{B}}^{\Gamma}$ to the corresponding $\mathcal{M}_{\mathbb{B}}^{\Gamma}$.

Węglorz' result is a special case of the following lemma. [The proof of [28] uses only permutations with finite supports and generalizes immediately.] Lemma 1 appears to be the characteristic property of Węglorz' construction. If $\mathcal{M}$ is a finite support model, $\mathbb{B} = \mathbb{B}_{\mathcal{M}}$ and $\Gamma = \Gamma_{\mathcal{M}}$, then $\mathcal{W}_{\mathbb{B}}^{\Gamma}$ satisfies the conclusion by its definition, but not necessarily the premise.

LEMMA 1 ([28]). *If* $\mathcal{S}_{\text{finite}}(A) < \Gamma$, *then* $\mathcal{W}_{\mathbb{B}}^{\Gamma} \models \mathcal{P}(A) = \mathbb{B}$.

As an immediate application let us consider the following assertion.

(1) If A is infinite, then $\mathcal{P}(A)$ is structured.

The failure of (1) is related to a "defect" in the definition of finiteness (cf. [16], p. 52). For apply $\mathcal{W}_{\mathbb{B}}^{\Gamma}$ to an algebra $\mathbb{B}$ which is not structured, e.g. the algebra of the finite and cofinite subsets of ω, and consider an atom $B/\mathcal{I}_{\text{finite}}$ of $\mathbb{B}/\mathcal{I}_{\text{finite}}$. Then by Lemma 1 in $\mathcal{W}_{\mathbb{B}}^{\Gamma}$ it is an atom of $\mathcal{P}(A)/\mathcal{I}_{\text{finite}}$. Therefore in $\mathcal{W}_{\mathbb{B}}^{\Gamma}$ the infinite set $B \subseteq A$ is *amorphous*. [Each infinite subset of B is cofinite in B.]

It follows from Lemma 1 that in the sense of the models each atomic $\mathbb{B}$ becomes a *complete* algebra [existence of least upper and greatest lower bounds]. Therefore $\mathbb{B}$ satisfies the *countable separation property*. [Countable sets $X, Y \subseteq \mathbb{B}$ such that $x \le y$ whenever $x \in X$ and $y \in Y$ may be separated by some $b \in \mathbb{B}$; i.e. $x \le b \le y$ for $x \in X$, $y \in Y$.] In ZFC the following assertion (2) is true ([18], p. 79). In $\mathcal{W}_{\mathbb{B}}^{\text{fin}}$ it is false in view of the following Lemma 2, when applied to a counterexample $\mathbb{C}$ of the separation property. (In view of [18], p. 177, the countable atomless algebra $\mathbb{C}$ is such a counterexample; then we may let $\mathbb{B} \subseteq \mathcal{P}(\mathbb{Z})$ be the atomic algebra which is generated by the arithmetic sequences and the finite sets.)

(2) The countable separation property is preserved by quotients.

LEMMA 2. *For each Boolean algebra* $\mathbb{C} \in V$ *there is an atomic Boolean algebra* $\mathbb{B} \in V$ *such that in* $\mathcal{W}_{\mathbb{B}}^{\text{fin}}$ *the algebra* $\mathcal{P}(A)/\mathcal{I}_{\text{finite}}$ *is isomorphic to* $\mathbb{C}$.

Proof. In V we may apply AC to represent $\mathbb{C}$ as a quotient $\mathbb{B}/\mathcal{I}_{\text{finite}}$ for some atomic algebra $\mathbb{B}$ ([18], p. 85). In $\mathcal{W}_{\mathbb{B}}^{\text{fin}}$ the set $\mathcal{P}(A)/\mathcal{I}_{\text{finite}}$ is wellorderable, since no equivalence class can be moved by finite permutations $\pi \in \mathcal{S}_{\text{finite}}(A)$. Therefore this algebra is isomorphic to $\mathbb{C}$ in the sense of the model. ∎

For $\mathcal{W}_{\mathbb{B}}^{\Gamma}$ this isomorphism may exist in V only. The quotient algebra is therefore elementarily equivalent to $\mathbb{C}$.

A set is *Dedekind-finite* if it does not contain a countably infinite subset. The existence of infinite Dedekind-finite sets contradicts the countable axiom of choice. Dedekind-finite Boolean algebras are *σ-complete* [completeness with respect to countable sets] and they violate the following *ZFC* result (cf. [18], p. 40 and Koppelberg's theorem on p. 177).

(3) If $\mathbb{B}$ is a σ-complete Booolean algebra, then its cardinality is at least $\aleph_1$.

LEMMA 3 ([8]). *If* $\mathcal{S}_{\text{finite}}(A) < \Gamma$, *then* $\mathcal{W}_{\mathbb{B}}^{\Gamma} \models \mathbb{B}$ *is Dedekind-finite.*

Lemma 3 also applies to the finite support models of Definition 2. [The proof of Plotkin's Theorem 16 in [7] is modified as follows: On X each definable quasi-wellordering relation $\prec$ defines a finite partition of X into sets of equivalent elements, whence in $\mathcal{M}(\mathcal{X}) = \mathcal{W}_{\mathbb{B}}^{\Gamma}$ the powerset $\mathbb{B}_{\mathcal{M}} = \mathcal{P}(A)$ is Dedekind-finite.]

We conclude that $\mathcal{I}_{\text{finite}}$ is a *σ-ideal* [the span $\vee X$ of countable sets $X \subseteq \mathcal{I}_{\text{finite}}$ exists and $\vee X \in \mathcal{I}_{\text{finite}}$] in $\mathbb{B}$. Lemmas 2 (applied to $\mathbb{C}$ which is not σ-complete) and 3 imply that there are models $\mathcal{W}_{\mathbb{B}}^{\text{fin}}$ where the following *ZFC* assertion (the easy part of a theorem due to von Neumann, Loomis and Sikorski) is false ([25], p. 74). By contrast, no *AC* is needed in the proof that a retract of a complete Boolean algebra is complete ([18], p. 71).

(4) The quotient of a σ-complete Boolean algebra modulo a σ-ideal is σ-complete.

In *ZFC* the free product of Boolean algebras is defined as the algebra of the clopen subsets of the topological product of their Stone spaces. For families of set algebras $\mathbb{C}_i \subseteq \mathcal{P}(X_i)$, where $i \in I$, we may apply a remark of [18], p. 159, to avoid Stone's representation theorem. We define the *direct sum* $\bigoplus_{i\in I} \mathbb{C}_i$ as the subalgebra of $\mathcal{P}(\prod_{i\in I} X_i)$ which is generated by the $\mathbb{C}_i$-rectangles. [Rectangles are products $\prod_{i\in I} Y_i$ of $Y_i \in \mathbb{C}_i$, where with finitely many exceptions $Y_i = X_i$.] Then in *ZFC* the direct sum coincides with the free product. Lemma 3 together with Theorem 1 (about Δ_4) of [26] imply that in $\mathcal{W}_{\mathbb{B}}^{\Gamma}$ the algebra $\mathbb{B} \oplus \mathbb{B} \subseteq \mathcal{P}(A \times A)$ is Dedekind-finite. Therefore the following theorem depends on *AC* (cf. [18], p. 172).

(5) If $\mathbb{B}$ is an infinite Boolean algebra, then $\mathbb{B} \oplus \mathbb{B}$ is not σ-complete.

By the proof of (5) in [18], p. 163, if $\mathbb{B}$ is an infinite atomic Boolean algebra, then *AC* is not needed in order to verify that $\mathbb{B} \oplus \mathbb{B}$ is not complete. [The span $\vee\{a \times a : a$ an atom of $\mathbb{B}\}$ does not exist.] A modification of $\mathcal{W}_{\mathbb{B}}^{\text{fin}}$ shows, however, that the following assertion depends on *AC*.

(6) If $\mathbb{B}_1$ and $\mathbb{B}_2$ are infinite atomic Boolean algebras, then $\mathbb{B}_1 \oplus \mathbb{B}_2$ is not complete.

DEFINITION 4. If $\mathbb{B}_1 \subseteq \mathcal{P}(A^1)$ and $\mathbb{B}_2 \subseteq \mathcal{P}(A^2)$ are atomic Boolean algebras with disjoint sets of atoms $\{\{a\} : a \in A^i\}$, then $\mathcal{W}_{\mathbb{B}_1,\mathbb{B}_2} \subseteq V(A^1 \cup A^2)$ is generated by the group $\mathcal{S}_{\text{finite}}(A^1 \cup A^2)$ whose group topology is induced by the filterbase $\mathcal{F} = \{\text{stab}(b) : b \in \mathbb{B}_1 \cup \mathbb{B}_2\}$.

LEMMA 4. $\mathcal{W}_{\mathbb{B}_1,\mathbb{B}_2} \models \mathcal{P}(A^i) = \mathbb{B}_i$ *and* $\mathcal{W}_{\mathbb{B}_1,\mathbb{B}_2} \models \mathcal{P}(A^1 \times A^2) = \mathbb{B}_1 \oplus \mathbb{B}_2$.

Proof. The stabilizer $\text{stab}(A^i)$ is open, since $A^i = 1^{\mathbb{B}_i} \in \mathbb{B}_i \in \mathcal{W}_{\mathbb{B}_1,\mathbb{B}_2}$. Therefore in $\mathcal{W}_{\mathbb{B}_1,\mathbb{B}_2}$ the set A of the atoms is partitioned into $A = A^1 \cup A^2$. The proof of Lemma 1 (in [28]) shows that $\mathcal{W}_{\mathbb{B}_1,\mathbb{B}_2} \models \mathcal{P}(A^i) = \mathbb{B}_i$. The assertion about the free product is verified by a similar argument.

We consider an element $S \in \mathcal{P}(A^1 \times A^2) \cap \mathcal{W}_{\mathbb{B}_1,\mathbb{B}_2}$. Then for some finite set $\mathbb{D} \subseteq \mathbb{B}_1 \cup \mathbb{B}_2$ we have $\text{stab}(S) \supseteq \text{fix}(\mathbb{D})$. We may assume that $\mathbb{D} = \mathbb{D}_1 \cup \mathbb{D}_2$ is the union of finite subalgebras $\mathbb{D}_i$ of $\mathbb{B}_i$. Their atoms form finite partitions $\Pi^i = \langle P^i_1, \ldots, P^i_{m^i} \rangle$ of A^i. If we consider an element $r = (p^1, p^2)$ of the rectangle $R = P^1_{j^1} \times P^2_{j^2}$, then the orbit

$$\text{orb}_{\text{fix}(\mathbb{D})}(r) = \{\langle \pi^1(p^1), \pi^2(p^2) \rangle : \pi^i \in \mathcal{S}_{\text{finite}}(P^i_{j^i})\} = R.$$

Hence S is the union of the finitely many rectangles of the partition which intersect S. By its definition, this union is an element of $\mathbb{B}_1 \oplus \mathbb{B}_2$. ∎

We next investigate distributive laws and construct counterexamples to Tarski's reformulation (7) of AC (cf. [23], p. 18).

(7) $\mathcal{P}(A)$ is completely distributive.

LEMMA 5. *If* $\mathcal{S}_{\text{finite}}(A) < \Gamma$, *then* $\mathcal{W}^{\Gamma}_{\mathbb{B}} \models \mathbb{B}$ *is not* (*completely*, 2)-*distributive.*

Proof. We first observe that there is no choice function on $[A]^2$. Suppose that Π is a support of a choice function f and P is an infinite equivalence class of Π. Let us consider a two-element set $\{a, b\} \in [P]^2$ and the transposition $\pi = (a; b)$. Then $(\pi f)(\pi\{a, b\}) = \pi(f(\{a, b\})) \neq f(\{a, b\})$ by the definition of f as a choice function and π as a transposition. But $(\pi f)(\pi\{a, b\}) = f(\{a, b\})$, since $\pi \in \text{stab}(\Pi) \cap \text{stab}(\{a, b\})$, a contradiction.

That $\mathbb{B}$ is not (completely, 2)-distributive follows from a proof in [23], p. 18: Fix $a_0 \in A$ and consider the sets $S_F = \{\{a, a_0\}, \{b, a_0\}\} \in [\mathbb{B}]^2$, where $F = \{a, b\} \in \mathcal{F} = [A \setminus \{a_0\}]^2$. The distributive law is formulated in terms of the set C of functions $f : \mathcal{F} \to [A]^2 \subseteq \mathbb{B}$ such that $f(F) \in S_F$. It asserts that for all functions $F \mapsto S_F$ where $|S_F| \leq 2$,

$$\bigcap_{F \in \mathcal{F}} \bigcup_{x \in S_F} x = \bigcup_{f \in C} \bigcap_{F \in \mathcal{F}} f(F).$$

Since $\bigcap_{F \in \mathcal{F}} \bigcup S_F \neq \emptyset$, there exists some $f \in C$ which defines a choice function $\{g(\{a, b\})\} = f(\{a,\ b\}) \setminus \{a_0\}$ on $[A \setminus \{a_0\}]^2$. ∎

As follows from the above proof, if $\mathcal{S}_{\mathrm{finite}}(A) < \Gamma$, then in $\mathcal{W}_{\mathbb{B}}^{\Gamma}$ the *ordering principle* [each set is the domain of a linear ordering relation] is false. On the other hand, the ordered Mostowski model is known to satisfy this principle ([20]) and therefore the powerset algebras are (completely, 2)-distributive.

For $\mathcal{W}_{\mathbb{B}}^{\mathrm{fin}}$ the failure of the ordering theorem may be explained in terms of the following principle which in the presence of the axiom of foundation is equivalent to AC. Halpern's [13] proof of the next lemma in the special case of the basic Fraenkel model ($\mathbb{B}$ is the algebra of the finite and cofinite sets) extends immediately to the general situation.

LEMMA 6 ([13]). $\mathcal{W}_{\mathbb{B}}^{\mathrm{fin}} \models$ *Each linearly orderable set is wellorderable.*

In order to prove an analogy of Lemma 6 for a model $\mathcal{W}_{\mathbb{B}}^{\mathrm{Aut}}$ we need the following lemma about *reflections* [permutations ϱ on a set A for which $\varrho^2 = 1_A$, the identity]. Degen [9] has considered (i) as a weak axiom of choice.

LEMMA 7. *If ψ is a permutation of a non-empty set A then there are two reflections ϱ_1 and ϱ_2 satisfying*

(i) $\psi = \varrho_2 \circ \varrho_1$ *and*
(ii) *for all $B \subseteq A$, if $\psi(B) = B$ then $\varrho_1(B) = B$ and $\varrho_2(B) = B$.*

Proof of (i). Assume ψ is a permutation of A. Write $\psi = \prod_{i\in K} c_i$ as a product of disjoint cycles. Each c_i can be written as a product of reflections $c_i = \varrho_{2,i} \circ \varrho_{1,i}$ as follows.

If c_i has odd length, then

$$\begin{aligned} c_i &= (a_{-n}; a_{-(n-1)}; \ldots; a_{-1}; a_0; a_1; \ldots; a_n), \\ \varrho_{1,i} &= (a_{-1}; a_1)(a_{-2}; a_2) \ldots (a_{-n}; a_n), \\ \varrho_{2,i} &= (a_0; a_1)(a_{-1}; a_2) \ldots (a_{-(n-1)}; a_n). \end{aligned}$$

If c_i has even length, then

$$\begin{aligned} c_i &= (a_{-(n-1)}; a_{-(n-2)}; \ldots; a_{-1}; a_0; a_1; \ldots; a_n), \\ \varrho_{1,i} &= (a_{-1}; a_1)(a_{-2}; a_2) \ldots (a_{-(n-1)}; a_{n-1}), \\ \varrho_{2,i} &= (a_0; a_1)(a_{-1}; a_2) \ldots (a_{-(n-1)}; a_n). \end{aligned}$$

If c_i is infinite, then

$$\begin{aligned} c_i &= (\ldots; a_{-2}; a_{-1}; a_0; a_1; a_2; \ldots), \\ \varrho_{1,i} &= \prod_{n=1}^{\infty} (a_{-n}; a_n), \\ \varrho_{2,i} &= \prod_{n=0}^{\infty} (a_{-n}; a_{n+1}). \end{aligned}$$

Then, if $\varrho_1 = \prod_{i\in K} \varrho_{1,i}$ and $\varrho_2 = \prod_{i\in K} \varrho_{2,i}$, we have $\varrho_1^2 = 1_A$, $\varrho_2^2 = 1_A$ and $\psi = \varrho_2 \circ \varrho_1$.

Proof of (ii). We first note that if η is a permutation whose expression as a product of disjoint cycles is $\eta = \prod_{j\in J} \sigma_j$ and $B \subseteq \mathrm{dom}(\eta)$ then the following is true:

$$(*)\qquad \eta(B) = B \Leftrightarrow (\forall j \in J)(\{x : \sigma_j(x) \neq x\} \subseteq B \text{ or } \{x : \sigma_j(x) \neq x\} \cap B = \emptyset).$$

Now assume that $B \subseteq A$ and that $\psi(B) = B$. By $(*)$ for each $i \in K$, either $\{a \in A : c_i(a) \neq a\} \subseteq B$ or $\{a \in A : c_i(a) \neq a\} \cap B = \emptyset$. When ϱ_1 is written as a product of disjoint cycles each cycle is a transposition $(a; b)$ which is a cycle of $\varrho_{1,i}$ for some $i \in K$. Hence by the definition of $\varrho_{1,i}$, $\{a, b\} \subseteq \{a \in A : c_i(a) \neq a\}$. Therefore $\{a, b\} \subseteq B$ (if $\{a \in A : c_i(a) \neq a\} \subseteq B$) or $\{a, b\} \cap B = \emptyset$ (if $\{a \in A : c_i(a) \neq a\} \cap B = \emptyset$). It follows from $(*)$ that $\varrho_1(B) = B$. Similarly, $\varrho_2(B) = B$. ∎

THEOREM 1. *If* $\mathbb{B} = \mathcal{P}(\omega)$, *then*

$$\mathcal{W}_{\mathbb{B}}^{\mathrm{Aut}} \models \textit{Every linearly orderable set is well orderable.}$$

Proof. Let X be linearly orderable in $\mathcal{W}_{\mathbb{B}}^{\mathrm{Aut}}$ and assume that the ordered partition $\Pi = \langle P_1, \ldots, P_m\rangle$ is a support of a linear ordering $<$ on X. We shall show that if $\psi \in \mathrm{stab}(\Pi)$, then ψ fixes X pointwise and hence Π is a support of a well ordering of X.

Assume $\psi \in \mathrm{stab}(\Pi)$. By Lemma 7, $\psi = \varrho_2 \circ \varrho_1$ where ϱ_1 and ϱ_2 are reflections and (by part (ii)) ϱ_1 and ϱ_2 are in $\mathrm{stab}(\Pi)$. The permutations ϱ_1 and ϱ_2 are therefore order automorphisms of $(X, <)$. Since no order automorphism of a linearly ordered set can have a finite cycle of length greater than 1, we conclude that for all $x \in X$, $\varrho_1(x) = \varrho_2(x) = x$ and therefore also $\psi(x) = x$. ∎

The following topological version of (7) due to Strauss (cf. [17], p. 285) depends on AC, too.

(8) If $\mathbb{B}$ is a compact zero-dimensional Hausdorff topological Boolean algebra, then $\mathbb{B}$ is completely distributive.

In view of Lemma 1, $\mathbb{B} = \mathcal{P}(A)$ carries the product topology of 2^A. It is zero-dimensional (i.e. the clopen sets form a base) and T_2 [consider the rectangles which form a base for the open sets] and with this topology $\mathbb{B}$ is a *topological Boolean algebra* [i.e. complementation, union and intersection are continuous; this can be seen by expressing these properties in terms of the topological group $(\mathbb{Z}/2\mathbb{Z})^A$]. Compactness is non-trivial. As follows from the failure of the axiom of choice for families of two-element sets (Lemma 5), $\mathcal{P}(A \times A)$ is not compact (cf. [4], p. 123). By contrast, in the ordered Mostowski model which does not satisfy $\mathcal{S}_{\text{finite}}(A) < \Gamma$ this space is compact, too.

THEOREM 2. *If* $\mathcal{S}_{\text{finite}}(A) < \Gamma$ *and* $\mathbf{X} \in \mathcal{W}_{\mathbb{B}}^{\Gamma}$ *is a compact Hausdorff space which is wellorderable as a set*, *then* $\mathcal{W}_{\mathbb{B}}^{\Gamma} \models \mathbf{X}^A$ *is compact.*

Proof. As in [4] it suffices to consider only spaces of the form $\mathbf{X} = 2^{\alpha}$, where α is an ordinal number. [Each wellorderable space is a continuous image of a closed subspace of such an $\mathbf{X}$.] Given an open covering $\mathcal{O}$ of $\mathbf{X}^A$ we shall construct a finite refinement. Let $\Pi = \langle P_1, \ldots, P_m\rangle$ be a support of $\mathcal{O}$. If we are given finite sets $\Lambda \subseteq \alpha$ and $E \subseteq A$ such that $E \cap P_i \neq \emptyset$ for each equivalence class P_i of Π, then we set, for each $\phi \in 2^{\Lambda \times \{P_1, \ldots, P_m\}}$,

$$U(\Lambda, E, \phi) = \{f \in \mathbf{X}^A = 2^{\alpha \times A} : (\forall \lambda \in \Lambda)(\forall x \in E \cap P_i)(f(\lambda, x) = \phi(\lambda, P_i))\}.$$

Step 1: There exist integers $1 \leq k_i \leq |P_i|$ and a finite set $\Lambda \subseteq \alpha$ such that for all $\phi \in 2^{\Lambda \times \{P_1, \ldots, P_m\}}$ and each finite $E \subseteq A$ the following implication is true:

$$(|E \cap P_1| = k_1 \wedge \ldots \wedge |E \cap P_m| = k_m) \Rightarrow (\exists O \in \mathcal{O})(U(\Lambda, E, \phi) \subseteq O).$$

For the proof we define an open covering of $\mathbf{Y} = 2^{\alpha \times \{P_1, \ldots, P_m\}}$. No choice principle is needed in the proof that this space is compact; cf. [19]. Given $O \in \mathcal{O}$ we set

$$R(\Lambda, E, O) = \{\phi \in 2^{\alpha \times \{P_1, \ldots, P_m\}} : U(\Lambda, E, \phi/(\Lambda \times \{P_1, \ldots, P_m\})) \subseteq O\}.$$

$R(\Lambda, E, O)$ is open: for if $\phi \in R(\Lambda, E, O)$ and $\psi/(\Lambda \times \{P_1, \ldots, P_m\}) = \phi/(\Lambda \times \{P_1, \ldots, P_m\})$, then $\psi \in R(\Lambda, E, O)$.

The sets $R(\Lambda, E, O)$ cover $\mathbf{Y}$, where $E \subseteq A$ and $\Lambda \subseteq \alpha$ are finite, E meets each equivalence class of Π and $O \in \mathcal{O}$. For given $\phi \in Y$ we define the following $f \in \mathbf{X}^A$: If $x \in P_i$, then we set $f(\lambda, x) = \phi(\lambda, P_i)$. Since $\mathcal{O}$ covers $\mathbf{X}^A$, we have $f \in O$ for some $O \in \mathcal{O}$. As O is open, there are finite sets $\Lambda \subseteq \alpha$ and $E \subseteq A$ such that $g \in O$ if $g/(\Lambda \times E) = f/(\Lambda \times E)$. We may enlarge E so as to ensure $E \cap P_i \neq \emptyset$ for each equivalence class P_i of Π. Then $U(\Lambda, E, \phi) \subseteq O$ and $\phi \in R(\Lambda, E, O)$.

Since $\mathbf{Y}$ is compact, it is covered by finitely many $R(\Lambda_j, E_j, O_j)$. We set $\Lambda = \bigcup_j \Lambda_i$ and $k_i = |P_i \cap (\bigcup_j E_j)|$ and verify the claim of Step 1. Suppose that $\phi \in 2^{\Lambda \times \{P_1, \ldots, P_m\}}$ and let ψ be an arbitrary continuation of ϕ to $\mathbf{Y} = 2^{\alpha \times \{P_1, \ldots, P_m\}}$. If $E \subseteq A$ satisfies $|E \cap P_i| = k_i$, then we let $\pi_i \in \mathcal{S}_{\text{finite}}(P_i)$ be permutations such that $\pi_i''((\bigcup_j E_j) \cap P_i) = E \cap P_i$. So $\pi = \prod_i \pi_i \in \operatorname{stab}(\Pi)$ satisfies $\pi'' \bigcup_j E_j = E$. There is some j such that $\psi \in R(\Lambda_j, E_j, O_j)$. Hence $U(\Lambda, \bigcup_j E_j, \phi) \subseteq U(\Lambda_j, E_j, \phi) \subseteq O_j$ and $U(\Lambda, E, \phi) = \pi U(\Lambda, \bigcup_j E_j, \phi) \subseteq \pi O_j \in \mathcal{O}$.

This concludes Step 1 and shows that

$$\mathcal{V} = \{U(\Lambda, E, \phi) : \phi \in 2^{\Lambda \times \{P_1,\ldots,P_m\}} \text{ and } (\forall i)(|E \cap P_i| = k_i)\}$$

is a refinement of $\mathcal{O}$.

Step 2: We construct a finite subcover of $\mathcal{V}$.

Assume that P_i is infinite for $1 \le i \le n$ and $|P_i| = k_i = 1$ for $n < i \le m$. For $1 \le i \le n$ we choose finite subsets $Q_i \subseteq P_i$ such that $|Q_i| > k_i \cdot 2^{|\Lambda|}$ and set $Q = \bigcup_{1\le i\le n} Q_i \cup \bigcup_{i>n} P_i$. Then the following finite subfamily $\mathcal{V}_{\text{fin}}$ of $\mathcal{V}$ covers the space $\mathbf{X}^A$:

$$\mathcal{V}_{\text{fin}} = \{U(\Lambda, E, \phi) : \phi \in 2^{\Lambda \times \{P_1,\ldots,P_m\}},\ E \subseteq Q \text{ and } (\forall i)(|E \cap P_i| = k_i)\}.$$

For consider $f \in \mathbf{X}^A$ and define a sequence $E_i \subseteq P_i \cap Q$ as follows.

(i) If $i > n$, then $P_i = \{x\}$ and for $\lambda \in \Lambda$ we have $f(\lambda, x) = \phi(\lambda, P_i)$; we set $E_i = P_i$.

(ii) If $i \le n$, then $Q \cap P_i = Q_i \subseteq \bigcup_{\chi\in 2^\Lambda} \{x : (\forall \lambda \in \Lambda)(f(\lambda, x) = \chi(\lambda))\}$. The pigeon-hole principle implies that for some χ and different $x_1, \ldots, x_{k_i}$ in Q_i we have $(\forall \lambda \in \Lambda)(f(\lambda, x_1) = \ldots = f(\lambda, x_{k_i})\ (= \chi(\lambda)))$. We set $E_i = \{x_1, \ldots, x_{k_i}\}$.

We conclude that $f \in U(\Lambda, \bigcup_i E_i, \phi) \in \mathcal{V}_{\text{fin}}$. ■

As in [4] this theorem may be improved by means of Ramsey theory: $\mathbf{X}^{[A]^n}$ is compact. The premise $\mathcal{S}_{\text{finite}}(A) < \Gamma$ is not optimal. For if $\mathbb{B} = \mathcal{P}(\omega)$, then the compactness of $\mathbb{B}$ in $\mathcal{W}_{\mathbb{B}}^{\Gamma}$ may be verified directly and without restrictions on $\Gamma < \operatorname{Aut}(\mathbb{B})$: The topological spaces $\mathbb{B}$ in V and $\mathbb{B}$ in $\mathcal{W}_{\mathbb{B}}^{\Gamma}$ consist of the same points, but in $\mathcal{W}_{\mathbb{B}}^{\Gamma}$ there are fewer open coverings. A similar argument proves that $\mathbb{B}$ is a Baire space in this model. On the other hand, if $\mathbb{B}$ is the algebra of the finite and cofinite subsets of ω and $\mathcal{S}_{\text{finite}} < \Gamma$, then in the basic Fraenkel model $\mathcal{W}_{\mathbb{B}}^{\Gamma}$ the space $\mathbb{B}$ is of the first category. [The families of the sets of cardinality n and of the sets whose complements are of cardinality n are closed and nowhere dense.]

The following assertion (9) concerns a "defect" in a topological definition of finiteness due to J. Aczél [1]; cf. (13) below.

(9) Each infinite set A carries a Hausdorff topology $\mathbf{A}$ with infinitely many non-isolated points.

If $\mathcal{S}_{\text{finite}}(A) < \Gamma$, then in $\mathcal{W}_{\mathbb{B}}^{\Gamma}$ for each T_2 topology $\mathbf{A}$ on A the set of the isolated points is cofinite. For if Π is a support of $\mathbf{A}$, then each of its infinite equivalence classes P_i consists of isolated points. [If the points $p_1 \ne p_2$ of P_i are contained in the disjoint open neighbourhoods $p_j \ni U_j \in \mathbf{A}$, then $\pi = (a_1; a_2) \in \operatorname{stab}(\Pi)$ satisfies $\pi(U_2) = (U_2 \cup \{a_1\}) \setminus \{a_2\} \in \mathbf{A}$ and therefore

$\{a_1\} = U_1 \cap \pi(U_2) \in \mathbf{A}$.] Hence each non-isolated point is contained in one of the finitely many one-element classes of Π. By contrast, in the ordered Mostowski model the atoms carry a dense-in-itself Hausdorff topology.

3. Maximal ideals. In this section we imitate a classical permutation argument due to Feferman (cf. [11], p. 343, and [27]) which will prove the independence of weakenings of the *Boolean prime ideal theorem* (10) by means of the counterexample $\mathbb{B} \in \mathcal{W}_{\mathbb{B}}^{\Gamma}$.

(10) Each non-principal ideal of $\mathbb{B}$ is contained in a maximal non-principal ideal.

The ordered Mostowski model is known to satisfy (10); cf. [15]. In $\mathcal{W}_{\mathbb{B}}^{\mathrm{fin}}$, (10) fails in view of the remark subsequent to Lemma 5 [the Boolean prime ideal theorem implies the ordering theorem]. However, $\mathbb{B} \in \mathcal{W}_{\mathbb{B}}^{\mathrm{fin}}$ satisfies (10). [For consider a non-principal ideal $\mathcal{I} \in \mathcal{W}_{\mathbb{B}}^{\mathrm{fin}}$ of $\mathbb{B} = \mathcal{P}(A)$. Its image $\mathcal{I}/\mathcal{I}_{\mathrm{finite}}$ under the quotient map is an ideal in the quotient algebra. The latter is wellorderable as a set by Lemma 2, whence the quotient ideal is contained in a maximal ideal whose preimage is a maximal ideal which extends $\mathcal{I}$.]

THEOREM 3. *If* $\mathbb{B}$ *is a countable atomic algebra*, *then the following assertions are equivalent.*

(i) $\mathbb{B}$ *is structured*;

(ii) $\mathcal{W}_{\mathbb{B}}^{\mathrm{Aut}} \models$ *Each prime ideal of* $\mathbb{B}$ *is principal.*

Proof. For "(i)→(ii)" we first observe that, if $\mathbb{C}$ is a countable structured algebra and $\mathbb{B} \subseteq \mathcal{P}(\mathbb{Z})$ is the algebra which is generated by the arithmetic sequences and the finite sets, then both the quotient algebras $\mathbb{B}/\mathcal{I}_{\mathrm{finite}}$ and $\mathbb{C}/\mathcal{I}_{\mathrm{finite}}$ are countable and atomless and therefore isomorphic (cf. [14], p. 224). Hence by a theorem of Vaught, the countable atomic algebras $\mathbb{B}$ and $\mathbb{C}$ are also isomorphic (cf. [22], p. 1105). Therefore it suffices to consider the algebra $\mathbb{B}$. We shall use the following notation: $S(a, b) = \{az + b : z \in \mathbb{Z}\}$, where $a > 0$ and b are integers, is an infinite arithmetic sequence. The group $\Gamma < \mathrm{Aut}(\mathbb{B})$ is generated by the finite permutations and the translations $T_c(x) = x + c$, where $c \in \mathbb{Z}$.

We suppose dually that in $\mathcal{W}_{\mathbb{B}}^{\mathrm{Aut}}$ there is a non-principal prime filter $\mathcal{U}$ of $\mathcal{P}(A) = \mathbb{B}$; then $A \in \mathcal{U}$, and $\{a\} \notin \mathcal{U}$ for all $a \in A$. We let Π be a support of $\mathcal{U}$ and $\mathbb{C}$ be the atomic subalgebra of $\mathbb{B}$ which is generated by Π. By refining Π, we may assume the existence of a finite set $E = \{e_1, \ldots, e_m\} \subseteq \mathbb{Z}$ and of an even $a > 0$ in $\mathbb{Z}$, where $|e_i| < a/2$, such that Π is of the form

$$\Pi = \langle \{e_1\}, \ldots, \{e_m\}, S(a, 0) \setminus E, \ldots, S(a, a-1) \setminus E \rangle.$$

CLAIM. *For each automorphism ϕ of $\mathbb{C}/\mathcal{I}_{\text{finite}}$ there exist a finite subalgebra $\mathbb{D} \subseteq \mathbb{B}$, an isomorphism $\Phi : \mathbb{D}/\mathcal{I}_{\text{finite}} \to \mathbb{C}/\mathcal{I}_{\text{finite}}$, and a permutation $\pi \in \Gamma \cap \operatorname{fix}(\mathbb{C})$ such that*

$$(L) \qquad \phi \circ \Phi(d/\mathcal{I}_{\text{finite}}) = \Phi(\pi(d)/\mathcal{I}_{\text{finite}}) \quad \textit{whenever } d \in \mathbb{D}.$$

By this weak "lifting" condition we mean that the automorphism ϕ of $\mathbb{C}/\mathcal{I}_{\text{finite}}$ may be imitated by some $\pi \in \operatorname{fix}(\mathbb{C})$. We note that the restriction of ϕ to the atoms of $\mathbb{C}/\mathcal{I}_{\text{finite}}$ is a permutation of some finite order n which we may represent by $f \in \mathcal{S}(\{0, 1, \ldots, a-1\})$ as follows:

$$\phi(S(a,k)/\mathcal{I}_{\text{finite}}) = S(a, f(k))/\mathcal{I}_{\text{finite}}.$$

We first define a copy $\mathbb{D} \subseteq \mathbb{B}$ of the algebra $\mathbb{C}/\mathcal{I}_{\text{finite}}$ and the isomorphism Φ. On the atoms we set

$$\Phi(D_k/\mathcal{I}_{\text{finite}}) = S(a,k)/\mathcal{I}_{\text{finite}},$$

where D_k is an atom of $\mathbb{D}$. For $0 \le k < a$ it is defined by means of the following partition of A:

$$D_k = \bigcup_{i=0}^{n-1} S(na, ia + f^{[-i]}(k)),$$

where for $z \in \mathbb{Z}$ the mapping $f^{[z]}$ is the zth iterate of f [$f^{[0]}(x) = x$, $f^{[1]} = f$ and $f^{[-1]}$ is the inverse mapping]. $\mathbb{D}$ is the subalgebra of $\mathbb{B}$ which is generated by these atoms.

Now we define π. The translation T_a leaves $\mathbb{C}$ invariant and it satisfies (L):

$$\begin{aligned} T_a``(D_k) &= \bigcup_{i=0}^{n-1} S(na, (i+1)a + f^{[-i]}(k)) \\ &= \bigcup_{i=0}^{n-1} S(na, ia + f^{[1-i]}(k) = D_{f(k)}. \end{aligned}$$

In order to construct a $\pi \in \operatorname{stab}(\Psi)$ it suffices to remove the finitely many deviations of T_a from $\operatorname{stab}(\Psi)$ by means of a finite permutation $\eta \in \mathcal{S}_{\text{finite}}(\mathbb{Z})$ (this does not influence (L)); i.e. $\pi = \eta \circ T_a$, where η is a product of transpositions

$$\eta = (e_1; e_1 + a) \circ \ldots \circ (e_m; e_m + a).$$

It is easy to verify that $\pi(e_i) = e_i$, since always $e_i + a \neq e_j$. [For otherwise $0 < a = |e_i - e_j| \le |e_i| + |e_j| < 2(a/2)$.] Therefore also $\pi(S(a,i) \setminus E) = S(a,i) \setminus E$ and $\pi \in \operatorname{stab}(\Pi) = \operatorname{fix}(\mathbb{C})$.

This completes the proof of the claim.

We apply the claim and consider the automorphism ϕ of $\mathbb{C}/\mathcal{I}_{\text{finite}}$ which is induced by the following product of disjoint transpositions of atoms of $\mathbb{C}/\mathcal{I}_{\text{finite}}$:

$$\phi = \prod_{k=0}^{a/2-1} \tau_k \quad \text{where} \quad \tau_k = (S(a,k)/\mathcal{I}_{\text{finite}}; S(a,k+a/2)/\mathcal{I}_{\text{finite}}).$$

As in the proof of the claim we set $\Phi(D_b/\mathcal{I}_{\text{finite}}) = S(a,b)/\mathcal{I}_{\text{finite}}$. We now consider the following sets $S_i \in \mathbb{D} \subseteq \mathbb{B}$:

$$S_1 = \bigcup_{k=0}^{a/2-1} D_k \quad \text{and} \quad S_2 = \bigcup_{k=a/2}^{a} D_k.$$

Then in $\mathbb{D}/\mathcal{I}_{\text{finite}} \subseteq \mathbb{B}/\mathcal{I}_{\text{finite}}$ the equivalence classes $S_i/\mathcal{I}_{\text{finite}}$ form a partition of unity such that for the permutation π (whose existence is assured by the claim) we have

$$\pi(S_1)/\mathcal{I}_{\text{finite}} = \Phi^{-1}\Big(\Big(\bigcup_{k=0}^{a/2-1} S(A,k+a/2)\Big)/\mathcal{I}_{\text{finite}}\Big) = S_2/\mathcal{I}_{\text{finite}} \quad \text{and}$$

$$\pi(S_2)/\mathcal{I}_{\text{finite}} = \ldots = S_1/\mathcal{I}_{\text{finite}}.$$

In terms of the algebra $\mathcal{P}(A) = \mathbb{B}$ this means that the symmetric differences $S_2 \triangle \pi(S_1)$ and $S_1 \triangle \pi(S_2)$ are finite. As $A \setminus (S_1 \cup S_2)$ is finite, some S_i is in the prime filter $\mathcal{U}$, say $S_1 \in \mathcal{U}$. Then also $S_2 \in \mathcal{U}$, since $\pi \in \operatorname{stab}(\Pi) \subseteq \operatorname{stab}(\mathcal{U})$ and $\pi(S_1) \in \mathcal{U}$ differs from S_2 by just a finite set not in $\mathcal{U}$ [which is non-principal]. We thereby derive the contradiction that $\mathcal{U}$ contains the finite set $S_1 \cap S_2$.

For the proof of "(ii)→(i)" we consider an atomic algebra $\mathbb{B} \subseteq \mathcal{P}(A)$ such that $B/\mathcal{I}_{\text{finite}}$ is an atom in $\mathbb{B}/\mathcal{I}_{\text{finite}}$. By the discussion following the assertion (1), independently of Γ in $\mathcal{W}_{\mathbb{B}}^{\Gamma}$ the set $B \subseteq A$ of atoms is amorphous and we may define a non-principal maximal ideal $\mathcal{I}$ on A: For $S \subseteq A$ we set $S \in \mathcal{I}$ if $S \cap B$ is finite. ∎

We conclude that, if $\mathbb{B}$ is the countable structured atomic algebra, then in $\mathcal{W}_{\mathbb{B}}^{\mathrm{Aut}}$ the following choice principle (11) fails. On the other hand, if $\mathbb{B}$ is the algebra of the finite and cofinite subsets of ω, then the basic Fraenkel model $\mathcal{W}_{\mathbb{B}}^{\mathrm{Aut}}$ satisfies (11) by [13] who calls it *SPI*.

(11) Each infinite set A carries a non-principal ultrafilter.

In the above model $\mathcal{W}_{\mathbb{B}}^{\mathrm{Aut}}$ the failure of (10) is combined with a strong version of the premise of the usual Zorn lemma proof of (10): each family of ideals of $\mathbb{B}$ which is wellordered by $\subseteq$ has a maximal element.

Lemma 8. *If $\mathbb{B}$ is the countable structured atomic Boolean algebra, then in $\mathcal{W}_{\mathbb{B}}^{\mathrm{Aut}}$ the set of the ideals of $\mathbb{B}$ is Dedekind-finite.*

Proof. As was observed in the proof of Theorem 3, $\mathbb{B}$ is isomorphic to the algebra of $\mathbb{B} \subseteq \mathcal{P}(\mathbb{Z})$ which is generated by the arithmetic sequences and the finite sets. Let us consider a support Π and an ideal $\mathcal{I}$ which is supported by Π. As in the proof of Theorem 3 we may assume Π consists of finitely many one-element sets $\{a\}$, where $a \in E$, and arithmetic sequences $S(a,b) \setminus E$, where $0 \le b < a$. If $P_i = S(a,b) \setminus E$ is an infinite equivalence class of Π, we investigate the ideal in the algebra $\mathbb{B}|P_i$ of [25], p. 30. In view of Lemma 1 we have $\mathbb{B}|P_i = \mathcal{P}(P_i)$. Then $\mathcal{I} \cap (\mathbb{B}|P_i)$ is one of the following ideals:

- zero ideal $\mathcal{I} \cap (\mathbb{B}|P_i) = \{\emptyset\}$,
- finite-subsets ideal $\mathcal{I} \cap (\mathbb{B}|P_i) = [P_i]^{<\omega}$ or
- degenerate case $\mathcal{I} \cap (\mathbb{B}|P_i) = \mathbb{B}|P_i$.

For if $s \in S \in \mathcal{I} \cap (\mathbb{B}|P_i)$, then by the argument in [14], pp. 225–226, the ideal $\mathcal{I} \cap (\mathbb{B}|P_i)$ contains the finite-subsets ideal. If some $S \in \mathcal{I} \cap (\mathbb{B}|P_i)$ is infinite, then $S \supseteq S(ka, b+ha) \setminus F$ for some $h \in \mathbb{Z}$, $k \in \mathbb{Z} \setminus \{0\}$ and a finite F. We combine the k translations $T_{ja} \in \mathrm{Aut}(\mathbb{B})$, where $0 \le j < k$, with finite permutations to define $\pi_j \in \mathrm{stab}(\Pi)$. Then there are finitely many permutations $\pi_{k+j} \in \mathcal{S}_{\mathrm{finite}}(P_i) \subseteq \mathrm{stab}(\Pi)$, where $j \ge 0$, such that $\bigcup_{l \ge 0} \pi_l(S) = P_i$, and we conclude that $P_i \in \mathcal{I} \cap (\mathbb{B}|P_i)$.

If P_i is a singleton set, then only the first and the third cases are possible. As in [14] it follows that there are only finitely many [namely $2^m - 1$] ideals which are supported by Π. Hence the set of all ideals is Dedekind-finite. ∎

The above argument is valid for $\mathcal{W}_{\mathbb{B}}^{\mathrm{Aut}}$ with $\mathbb{B} = \mathcal{P}(\omega)$, too. [14] has shown that, if $\mathbb{B}$ is the algebra of the finite and cofinite subsets of ω, then the basic Fraenkel model $\mathcal{W}_{\mathbb{B}}^{\mathrm{fin}}$ satisfies the conclusion of Lemma 8. By contrast, if $\mathbb{B} = \mathcal{P}(\omega)$ and $\langle S_i : i \in \omega \rangle$ is an infinite partition of ω into infinite pieces, then in $\mathcal{W}_{\mathbb{B}}^{\mathrm{fin}}$ there is a strictly increasing infinite sequence of ideals, namely $n \mapsto \mathcal{I}_n = \{y \subseteq A : y \setminus \bigcup_{i \le n} S_n \text{ is finite}\}$, which is supported by $\Pi = \langle A \rangle$.

We conclude from Theorem 3 that the following assertion (about injective cardinalities) depends on AC (cf. [12], p. 104).

(12) If $\mathbb{B}$ is an infinite Boolean algebra and $\beta(\mathbb{B})$ is the set of its maximal ideals, then $|\mathbb{B}| \le |\beta(\mathbb{B})|$.

For if $\mathbb{B} \subseteq \mathcal{P}(\mathbb{Z})$ is generated by the arithmetic sequences and finite sets of integers, then in $\mathcal{W}_{\mathbb{B}}^{\mathrm{Aut}}$ each maximal ideal $\mathcal{I}$ is generated by an atom $\{a\} \in \mathbb{B}$, i.e. $\mathcal{I} = \{b \subseteq A : a \notin b\}$, whence by Cantor's theorem $|\beta(\mathbb{B})| = |A| < |\mathcal{P}(A)| = |\mathbb{B}|$.

$\beta(\mathbb{B})$ carries the *Stone topology* which is generated by the subbase $\{r(b) : b \in \mathbb{B}\}$ of open sets $r(b) = \{\mathcal{I} \in \beta(\mathbb{B}) : b \notin \mathcal{I}\}$. As follows from [12], p. 101, AC is not needed in the proof of the following assertion: r is an isomorphism between $\mathbb{B}$ and the algebra of the compact open subsets of the *Stone space* $\beta(\mathbb{B})$. The compactness of $\beta(\mathbb{B})$ is known to be equivalent to (10), the Boolean prime ideal theorem; cf. [15]. As follows from Theorem 3, in $\mathcal{W}_{\mathbb{B}}^{\mathrm{Aut}}$ the Stone space of the arithmetic sequences algebra is A with the discrete topology. By Lemma 3 it is sequentially compact and countably compact. Hence even the following assertion depends on AC. (In Feferman's model [11] there are infinite discrete Stone spaces which, however, are not countably compact.)

(13) If $\beta(\mathbb{B})$ is countably compact and discrete, then $\mathbb{B}$ is finite.

The existence of non-principal two-valued measures on $\mathcal{P}(A) \in \mathcal{W}_{\mathbb{B}}^{\Gamma}$ is equivalent to the existence of maximal ideals of $\mathbb{B} \in \mathcal{W}_{\mathbb{B}}^{\Gamma}$. It does not follow from the existence of non-principal probability measures. For if $\mathbb{B} \subseteq \mathcal{P}(\mathbb{Z})$ is the algebra which is generated by the arithmetic sequences and the finite sets and Γ is generated by the translations and the finite permutations, then the permutations in Γ do not change the density $\delta(S) = \lim_{n\to\infty}(S \cap [-n,n])/(2n+1)$, whose restriction to $\mathbb{B}$ is therefore an element of $\mathcal{W}_{\mathbb{B}}^{\Gamma}$. Hence in $\mathcal{W}_{\mathbb{B}}^{\Gamma}$ it is a non-trivial non-principal measure on $\mathbb{B} = \mathcal{P}(A)$ and it is countably additive in view of Lemma 3. By the proof of Theorem 3 there are, however, no non-principal two-valued measures on $\mathcal{P}(A) \in \mathcal{W}_{\mathbb{B}}^{\Gamma}$. We next consider Pincus' strengthening of Feferman's argument (cf. [21]).

THEOREM 4. *If $\mathbb{B} = \mathcal{P}(\omega)$, then in $\mathcal{W}_{\mathbb{B}}^{\mathrm{Aut}}$ each bounded complex-valued finitely additive measure m on $\mathcal{P}(A)$ is concentrated on a finite set.*

Proof. We first consider the case where m is a real-valued non-principal measure on A which is supported by the partition $\Pi = \langle P_1,\ldots,P_n\rangle$ of A. Then m is finitely additive on disjoint sets, $m(A) \neq 0$, and $m(\{a\}) = 0$ for all $a \in A$. It follows that there must be an $i \leq n$ such that P_i is infinite and $m(P_i) \neq 0$. There exist pairwise disjoint infinite subsets, S_1, S_2, and S_3, of P_i such that $P_i = S_1 \cup S_2 \cup S_3$. Suppose $\sigma_1, \sigma_2 \in \mathrm{stab}(\Pi)$ are such that

(a1) $\sigma_1(S_1) = S_2 \cup S_3$ and $\sigma_1(S_2 \cup S_3) = S_1$ and
(b1) $\sigma_2(S_1) = S_2,\ \sigma_2(S_2) = S_3,$ and $\sigma_2(S_3) = S_1$.

Then we would have

(a2) $m(S_1) = m(S_2 \cup S_3) = m(S_2) + m(S_3)$, by (a1), and
(b2) $m(S_1) = m(S_2) = m(S_3)$, by (b1).

Consequently, $m(S_1) = 2m(S_1)$, which implies that $m(S_1) = 0$ [since $m(S_i) \neq \pm\infty$]. Thus, all $m(S_i) = 0$, and $m(P_i) = 0$, which is a contradiction.

In the general case we let v be the *total variation* of m [cf. [10], p. 97; v exists, since m is bounded]. It is a bounded, real-valued, positive finitely additive measure on $\mathcal{P}(A)$ such that $|m(S)| \leq v(S)$. In particular, v is monotonic. [If $S \subseteq T$, then $v(S) \leq v(S) + v(T \setminus S) = v(T)$.]

We now define a function $c : A \to \mathbb{R}$ as follows: $c(a) = v(\{a\}) = |m(\{a\})|$. It is in $\ell_1(A)$, for if $E \subseteq A$ is finite, then $\sum_{a\in E}|c(a)| = v(E) \leq v(A) < \infty$.

We set $E_0 = \{a \in A : c(a) \neq 0\}$; as $[A]^{<\omega}$ is Dedekind-finite (Lemma 3), E_0 is finite ([5], p. 3). Next we define a set function μ on $\mathcal{P}(A)$:

$$\mu(S) = v(S) - \sum_{a \in E_0 \cap S} c(a) = v(S) - v(E_0 \cap S).$$

It is finitely additive, positive (since v is monotonic) and non-principal; if $E \subseteq A$ is finite, then $\mu(E) = \sum_{a \in E \setminus E_0} c(a) = 0$. It follows that $\mu = 0$ and therefore $v(S) = v(S \cap E_0)$. Hence also $m(S \setminus E_0) = 0$ and $m(S) = \sum_{a \in S \cap E_0} m(\{a\})$. ∎

Theorem 4 implies that the Hahn–Banach theorem fails in $\mathcal{W}_{\mathbb{B}}^{\mathrm{Aut}}$; there is no non-zero continuous linear functional on the space $\ell_\infty(A)/c_0(A)$ (cf. the proof of Lemma 9; our notation in choiceless Banach-space theory follows [5]). While for $1 < p < \infty$ the spectral theory of linear operators on reflexive and *locally sequentially compact* [the closed unit ball is sequentially compact] spaces $\ell_p(D)$ is similar to the $\ell_p(\omega)$-theory (cf. [5]), not much is known for $\ell_\infty(A)$. Even the non-reflexivity varies with the model; i.e. the assertion (14) depends on *AC* (cf. [10], p. 339). Recall that a *Banach space* X [a sequentially complete normed vector space] is *reflexive* if the canonical isometry $\chi : X \to X^{\star\star}$ is onto the second dual; $\chi(x)(\phi) = \phi(x)$ for $x \in X$ and $\phi \in X^\star$. By Lemma 3 and Lemma 3.2 of [5] the spaces $\ell_\infty(A) \in \mathcal{W}_{\mathbb{B}}^{\varGamma}$ are locally sequentially compact.

(14) If $\ell_\infty(A)$ is reflexive and locally sequentially compact, then A is finite.

LEMMA 9. *If* $\mathbb{B} = \mathcal{P}(\omega)$ *then* $\mathcal{W}_{\mathbb{B}}^{\mathrm{Aut}} \models \ \ell_\infty(A)$ *is reflexive.*

Proof. If $a \in A$ and $f \in \ell_\infty(A)$ is a bounded complex-valued function on A, then $\mathrm{ev}_a(f) = f(a)$ is the evaluation functional.

We show first that each bounded linear functional $\phi : \ell_\infty(A) \to \mathbb{C}$ is a linear combination of evaluation functionals.

To this end we define a bounded additive set function μ on $\mathcal{P}(A)$, namely $\mu(S) = \phi(1_S)$, where 1_S is the characteristic function of $S \subseteq A$. By Theorem 4 this measure is concentrated on a finite set E_0. Next we compute $\phi(f)$ for $f \in \ell_\infty(A)$. By Lemma 3 the powerset of A is Dedekind-finite. Therefore f is finitely valued [$\mathbb{C}$ is wellorderable in all permutation models; cf. [3], p. 13], whence for some finite partition $\varPi$ of A and some $f_i \in \mathbb{C}$ we have $f = \sum_i f_i 1_{P_i}$. Then $\mathrm{ev}_a(f) = f_i$, whenever $a \in P_i$ and we may conclude that

$$\phi(f) = \sum_i f_i \mu(P_i) = \sum_i f_i \sum_{a \in P_i \cap E_0} \mu(\{a\}) = \sum_{a \in E_0} \mathrm{ev}_a(f)\mu(\{a\});$$

hence $\phi = \sum_{a \in E_0} d_a \,\mathrm{ev}_a$ for $d_a = \mu(\{a\}) \in \mathbb{C}$.

As an application of that observation let us consider any bounded linear functional $L : \ell_\infty(A)^\star \to \mathbb{C}$ in the second dual. We set $f(a) = L(\mathrm{ev}_a)$ and verify that $\chi(f) = L$.

Since $|f(a)| \le \|L\| \cdot \|\mathrm{ev}_a\|$ and $\|\mathrm{ev}_a\| \le 1$ we have $f \in \ell_\infty(A)$. Since $\chi(f)(\mathrm{ev}_a) = \mathrm{ev}_a(f) = f(a) = L(\mathrm{ev}_a)$ and the evaluation functionals span the dual $\ell_\infty(A)^\star$, for all $\phi \in \ell_\infty(A)^\star$ we have $\chi(f)(\phi) = L(\phi)$. ■

The same proof shows that in $\mathcal{W}_{\mathbb{B}}^{\mathrm{Aut}}$ the space $\ell_1(A)$ is reflexive. [If $L = \sum_i c_i \,\mathrm{ev}_{a_i} \in \ell_1(A)^{\star\star} = \ell_\infty(A)^\star$, then $L = \chi(\sum_i c_i e_{a_i})$, where $e_a \in \ell_1(A)$ is an element of the canonical unit vector base.] If $\mathbb{B}$ is the algebra of the finite and cofinite sets, then in the basic Fraenkel model $\mathcal{W}_{\mathbb{B}}^{\mathrm{Aut}}$ the space $\ell_\infty(A)$ is not reflexive [there is an additional functional $\mathrm{ev}_\infty(f) = c$ if $\{a \in A : f(a) = c\}$ is infinite, and $L \in \ell_\infty(A)^{\star\star}$ is in the range of χ iff for infinitely many $a \in A$ we have $L(\mathrm{ev}_a) = L(\mathrm{ev}_\infty)$]. Since in this model (by [3]) each infinite set contains a copy of ω or of $A \setminus F$ for some finite $F \subset A$, the model satisfies (14). However, the following topological variant of (14) is violated for $\ell_\infty(A)$ in all models $\mathcal{W}_{\mathbb{B}}^{\Gamma}$ which satisfy $\mathcal{S}_{\mathrm{finite}}(A) < \Gamma$ (cf. [10], p. 339).

(15) If the closed unit ball of $\ell_\infty(A)$ is weakly compact and sequentially compact, then A is finite.

The closed unit ball B of $\ell_\infty(A)$ in the weak topology is the product space $\mathbf{B} = \{z \in \mathbb{C} : |z| \le 1\}^A$. Since $\mathbb{C}$ is wellorderable in all permutation models, the product space is compact in view of Theorem 2.

In [8], Subsection 6.3, the model $\mathcal{M}_{\mathbb{B}}^{\mathrm{Aut}}$ of Definition 3 has been given the following empirical interpretation: $\mathbb{B}$ is the algebra of perceivable objects and the model consists of those empirical concepts which are not empirically meaningless for syntactical reasons. (The investigation of the countable structured algebra which is generated by the arithmetic sequences and the finite sets has been motivated by the algebra of the regular languages over a one-element alphabet which in turn corresponds to the algebra of events which are perceivable by nerve nets.) Independence proofs such as Theorem 4 therefore have an empirical interpretation. We consider the following special case of a theorem due to Schmeidler [24] about cooperative games as an example.

(16) Convex two-valued continuous games $v : \mathcal{P}(A) \to \mathbb{R}^+$ have a nonempty core.

The set function v is the worth function of a *convex game* if $v(\emptyset) = 0$ and for all coalitions S, T the following inequality is true: $v(S) + v(T) \le v(S \cup T) + v(S \cap T)$. *Continuity* asserts that for each increasing sequence $\langle S_i : i \in \omega\rangle$ of coalitions $v(\bigcup_i S_i) = \lim_i v(S_i)$. (This condition is not needed in [24].) The *core* of v consists of the finitely additive measures $m : \mathcal{P}(A) \to$

$\mathbb{R}$ (the payoffs of the game) such that $m(A) = v(A)$ and $m(S) \geq v(S)$ for all $S \in \mathcal{P}(A)$.

If $\mathbb{B} = \mathcal{P}(\omega)$, then in $\mathcal{M}_{\mathbb{B}}^{\mathrm{Aut}}$ the following set function is a convex game which is continuous by Lemma 3: $w(S) = 0$ if S is not cofinite, and $w(S) = 1$, otherwise. If m is in the core of w, then m is positive and non-principal. Hence by Theorem 4 the core is empty. In $V(\mathbb{B})$ the core is non-empty by AC. Therefore in view of the above interpretation of the model these solutions of the game are not empirically meaningful, even if all possible coalitions are perceivable.

4. Amorphous sets. The following result is a variant of Mostowski's [20] intersection lemma for the ordered Mostowski model.

LEMMA 10. *If* $\mathbb{B}_1$ *and* $\mathbb{B}_2$ *are finite subalgebras of* $\mathbb{B}$, *then the subgroup* $\mathrm{fix}(\mathbb{B}_1 \cap \mathbb{B}_2)$ *of* $\mathcal{S}_{\mathrm{finite}}(A)$ *is generated by* $\mathrm{fix}(\mathbb{B}_1) \cup \mathrm{fix}(\mathbb{B}_2)$.

Proof. Since it is obvious that the supposed generators are elements of the group, we just need to verify that each permutation $\pi \in \mathrm{fix}(\mathbb{B}_1 \cap \mathbb{B}_2) \subseteq \mathcal{S}_{\mathrm{finite}}(A)$ is composed of finitely many generators. We may, moreover, restrict our attention to transpositions $\pi = (a; b)$, where a and b are elements of an atom P of $\mathbb{B}_1 \cap \mathbb{B}_2$. We let C_j^i be the atoms of $\mathbb{B}_i$ which are subsets of P; then $P = \bigcup_j C_j^i$. The fact that the partition of A into the atoms of $\mathbb{B}_1 \cap \mathbb{B}_2$ is the join (in the partition lattice $\mathbb{P}\mathrm{art}(A)$; cf. [12], p. 192) of the partitions of A into the atoms of $\mathbb{B}_i$ has the following reformulation.

Any two elements $a \in P$ and $b \in P$ are connected by a finite chain $\langle C_m \rangle_{m=1}^n$ such that each C_m is either the atom $C_{j_m}^1$ of $\mathbb{B}_1$ or the atom $C_{j_m}^2$ of $\mathbb{B}_2$ for some j_m and such that $a \in C_0$, $b \in C_n$ and $C_m \cap C_{m+1} \neq \emptyset$ for $m < n$.

We let $n(a, b) \geq 1$ be the minimal length of such a chain and prove the lemma about $\{a, b\} \subseteq P$ by induction on $n(a, b)$; i.e. there exist permutations $\pi_i \in \mathrm{fix}(\mathbb{B}_1) \cup \mathrm{fix}(\mathbb{B}_2)$ such that $\pi = (a; b) = \pi_1 \circ \ldots \circ \pi_k$.

If $n(a, b) = 1$, then $\{a, b\} \subseteq C_0$ and $C_0 \in \mathbb{B}_i$, whence $\pi_1 = (a; b) \in \mathrm{fix}(\mathbb{B}_i)$. If the assertion is true for all $1 \leq n(x, y) < n(a, b)$, then we choose $c \in P$ such that $n(a, c) = n(a, b) - 1$ and $n(c, b) = 1$. Then $(a; c)$ and $(b; c)$ are composed of permutations in $\mathrm{fix}(\mathbb{B}_1) \cup \mathrm{fix}(\mathbb{B}_2)$, whence the same is true for $\pi = (a; b) = (b; c) \circ (a; c) \circ (b; c)$. ∎

It follows that for each $x \in \mathcal{W}_{\mathbb{B}}^{\mathrm{fin}}$ there exists a least subalgebra $\mathrm{supp}(x) = \mathbb{C}$ of $\mathbb{B}$ such that $\mathrm{stab}(x) \supseteq \mathrm{fix}(\mathbb{C})$, namely

$$\mathrm{supp}(x) = \bigcap \{\mathbb{D} \subseteq \mathbb{B} : \mathbb{D} \text{ is finite and } \mathrm{fix}(\mathbb{D}) \subseteq \mathrm{stab}(x)\}.$$

If $\pi \in \mathcal{S}_{\mathrm{finite}}(A)$, then $\mathrm{supp}(\pi x) = \pi\text{''}\,\mathrm{supp}(x)$. Therefore each $\pi \in \mathrm{stab}(x)$ is an automorphism of $\mathbb{C}$. If the infinite set $P \in \mathbb{C}$ is an atom of $\mathbb{C}$ (an "infinite atom"), then $\pi\text{''}P = P$. [$\pi\text{''}P$ is an atom of $\mathbb{C}$. Since P

is infinite and π moves only finitely many elements, $P \cap \pi"P \neq \emptyset$.] Hence, when considered as an automorphism of $\mathbb{C}$, the mapping $\pi \in \mathrm{stab}(x)$ does not move infinite atoms P of $\mathbb{C} = \mathrm{supp}(x)$.

The intersection lemma may be viewed as a weak choice principle. This is illustrated by the variant $\mathcal{M}_{\mathbb{B}}^{\mathrm{fin}}$ of $\mathcal{W}_{\mathbb{B}}^{\mathrm{fin}}$ where $\mathbb{B}$ is the set of the atoms (cf. Definition 3). As $\mathbb{B}$ is Dedekind-finite by Lemma 3, we may apply Theorem 4.3 of [3] and conclude that $\mathcal{M}_{\mathbb{B}}^{\mathrm{fin}}$ satisfies the axiom of choice for wellorderable families of non-empty wellorderable sets and the following partial choice principle: Each infinite family of wellorderable sets has an infinite subfamily which admits a choice function.

THEOREM 5. *An atomic algebra* $\mathbb{B}$ *is structured if and only if*

$$\mathcal{W}_{\mathbb{B}}^{\mathrm{fin}} \models \textit{There are no infinite amorphous sets.}$$

Proof. If $\mathbb{B}$ is not structured, then by the discussion following the assertion (1), there exists an amorphous subset of A. For the proof of the converse, we assume that X is infinite and $\mathbb{C} = \mathrm{supp}(X)$. We construct two infinite disjoint subsets X_i of X.

Let us consider two cases.

Case 1: There exists a $\mathbb{D} \supseteq \mathbb{C}$ such that for infinitely many x, say for $x \in Y \subseteq X$, the "$\mathbb{D}$-orbits" $\mathrm{orb}_{\mathrm{fix}(\mathbb{D})}(x)$ are finite. In this case Y is the union of a wellorderable family $\mathcal{F}$ of pairwise disjoint finite sets. If $\mathcal{F}_1$ and $\mathcal{F}_2$ partition this family into two infinite parts, then X is the disjoint union of the infinite sets $X_i = \bigcup \mathcal{F}_i$ and of $X \setminus Y$.

Case 2: For all finite algebras $\mathbb{D} \supseteq \mathbb{C}$ and all but finitely many $x \in X$ the $\mathbb{D}$-orbits of x are infinite. Suppose that the $\mathbb{C}$-orbit of $x \in X$ is infinite. We consider the subalgebra $\mathbb{D} = [\mathrm{supp}(x) \cup \mathbb{C}]$ of $\mathcal{P}(A) = \mathbb{B}$ which is generated by $\mathrm{supp}(x) \cup \mathbb{C}$ and verify the claim.

CLAIM. *There exists an infinite atom* Q *of* $\mathbb{C}$ *which properly contains an atom* P *of* $\mathbb{D}$.

We first observe that each $\pi \in \mathrm{fix}(\mathbb{C})$ is a composite $\pi = \pi_{C_1} \circ \ldots \circ \pi_{C_m}$ of mutually commuting permutations $\pi_{C_i} \in \mathcal{S}_{\mathrm{finite}}(C_i)$, where the C_i's are the atoms of $\mathbb{C}$. If the claim is false and each infinite atom of $\mathbb{C}$ is an atom of $\mathbb{D}$, then $\pi_{C_i}(x) = x$ for all infinite C_i and the $\mathbb{C}$-orbit of x reduces to the finite set $\{\phi(x) : \phi \in \mathcal{S}_{\mathrm{finite}}(\bigcup\{C_i : C_i \text{ finite}\})\}$, a contradiction which proves the claim.

As Q is partitioned into finitely many $\mathbb{D}$-atoms, we may assume that P is infinite. $\mathbb{B}$ is structured, whence P is the union of two disjoint infinite elements P_i of $\mathbb{B}$. We now choose $a \in Q \setminus P$ and $b_i \in P_i$ in order to define $\pi_i = (a; b_i) \in \mathrm{fix}(\mathbb{C})$ and $y_i = \pi_i(x) \in X$. Since the groups $G_i = \mathcal{S}_{\mathrm{finite}}(P_i)$ are sets in the model $\mathcal{W}_{\mathbb{B}}^{\mathrm{fin}}$, so are

$$X_i = \mathrm{orb}_{G_i}(y_i) \subseteq \mathrm{orb}_{\mathrm{fix}(\mathbb{C})}(y_i) \subseteq X.$$

X_1 and X_2 are disjoint. If $g_1(y_1) = g_2(y_2)$, where $g_i \in G_i$, then $\phi x = x$, where $\phi = \pi_2^{-1} \circ g_2^{-1} \circ g_1 \circ \pi_1 \in \mathrm{fix}(\mathbb{C})$. Therefore $\phi\text{"}\,\mathrm{supp}(x) = \mathrm{supp}(\phi x) = \mathrm{supp}(x)$; i.e. ϕ is an automorphism of $\mathbb{D}$ and $\mathbb{C}$ which does not move the infinite atoms P and Q. We now derive a contradiction: $a \in Q \setminus P$, but $\phi(a) = (g_2 \circ \pi_2)^{-1}(c_1) = c_1 \in P$, where $c_1 = g_1 \circ \pi_1(a) = g_1(b_1) \in P_1$.

X_i is infinite. If in the case $i = 1$, $g(y_1) = h(y_1)$ for some $g, h \in G_1$, then $\phi(x) = x$ for $\phi = \pi_2^{-1} \circ h^{-1} \circ g \circ \pi_1 \in \mathrm{fix}(\mathbb{C})$, whence as an automorphism of the Boolean algebras $\mathbb{D}$ and $\mathbb{C}$ the mapping ϕ does not move the atoms P and Q. This condition implies $\phi(a) = \pi_1^{-1}(c) \in Q \setminus P$, where $c = h^{-1} \circ g \circ \pi_1(a) = h^{-1} \circ g(b_1) \in P_1$. Therefore $c = b_1$ [since $\pi_1(c) \notin P_1$] and $g(b_1) = h(b_1)$. It follows that there is a surjective function of X_1 onto the infinite set $P_1 = \mathrm{orb}_{G_1}(b_1)$, namely the set

$$\Phi = \{\langle g(y_1), g(b_1)\rangle : g \in G_1\}.$$

$\Phi \in \mathcal{W}_{\mathbb{B}}^{\mathrm{fin}}$, because $G_1 \in \mathcal{W}_{\mathbb{B}}^{\mathrm{fin}}$. ■

THEOREM 6. *If $\mathbb{B} = \mathcal{P}(\omega)$, then in $\mathcal{W}_{\mathbb{B}}^{\mathrm{Aut}}$ there are no infinite amorphous sets.*

Proof. Assume that $X \in \mathcal{W}_{\mathbb{B}}^{\mathrm{Aut}}$ is infinite and that the ordered partition $\Pi = \langle P_1, \ldots, P_m\rangle$ of A is a support of X. We will show that there are infinite disjoint subsets $X_i \in \mathcal{W}_{\mathbb{B}}^{\mathrm{Aut}}$ of X.

If every element of X has support Π then X is well orderable in $\mathcal{W}_{\mathbb{B}}^{\mathrm{Aut}}$ and the proof is easy. We therefore assume that $y \in X$ and there is a $\phi \in \mathrm{stab}(\Pi)$ such that $\phi(y) \neq y$.

Step 1: We show that we may assume that y has a support

$$\Psi = \langle Q_1, Q_2, Q_3, \ldots, Q_{n_0}, Q_{n_0+1}, \ldots, Q_n\rangle$$

where

a. $n_0 \geq 3$ and Q_3 is infinite.
b. For $1 \leq j \leq n_0$, $Q_j \subseteq P_1$ and for $n_0 < j \leq n$, $Q_j \cap P_1 = \emptyset$.
c. $\phi(Q_1) = Q_2$, $\phi(Q_2) = Q_1$, and for every $a \in A \setminus (Q_1 \cup Q_2)$, $\phi(a) = a$.

First, we recall the assumption that each P_i is either infinite or a singleton.

Secondly, by writing $\phi = \phi_1 \circ \ldots \circ \phi_m$ where $\phi_i(a) = a$ for $a \in A \setminus P_i$ and replacing ϕ by one of the ϕ_i, we may assume that there is an $i \leq m$ such that $\phi(a) = a$ for $a \in A \setminus P_i$. [If $(\prod_{i=1}^m \phi_i)(y) \neq y$ then for some i, $\phi_i(y) \neq y$.] Further it is no loss of generality to assume that $i = 1$.

Let $\Psi = \langle Q_1, \ldots, Q_n\rangle$ be a support of y. By replacing each Q_j with the two sets $Q_j \cap P_1$ and $Q_j \setminus P_1$ (if necessary) and reordering the Q_j's we may assume that there is an $n_0 \leq n$ such that $Q_j \subseteq P_1$ for $j \leq n_0$ and $Q_j \cap P_1 = \emptyset$ for $j > n_0$.

As in the proof of Theorem 1 (using Lemma 7) we may assume that ϕ is a reflection. By replacing Ψ by a partition whose cells are

$$\{Q_i \cap \phi(Q_j) : 1 \leq i, j \leq n\} \setminus \{\emptyset\}$$

we may assume (since ϕ is a reflection) that

$$(\forall 1 \leq i \leq n)(\exists 1 \leq j \leq n)(\phi(Q_i) = Q_j \text{ and } \phi(Q_j) = Q_i).$$

(By our simplifying assumptions so far, $\phi(Q_j) = Q_j$ for $j > n_0$.)

We decompose ϕ into a product $\phi = \gamma_1 \circ \ldots \circ \gamma_m$, where for each $1 \leq r \leq m$ there are i and j such that

$$\gamma_r(a) = a \quad \text{for } a \in A \setminus (Q_i \cup Q_j), \qquad \gamma_r(Q_i) = Q_j \quad \text{and} \quad \gamma_r(Q_j) = Q_i.$$

Replacing ϕ by one of the γ_r, we may assume that

$$\phi(a) = a \quad \text{for } a \in A \setminus (Q_i \cup Q_j), \qquad \phi(Q_i) = Q_j \quad \text{and} \quad \phi(Q_j) = Q_i.$$

It is no loss of generality to assume that $i = 1$ and $j = 2$.

Finally, we may assume that for some $3 \leq i \leq n_0$ the set Q_i is infinite. Assuming that this is not the case and using the facts that $Q_1 \cup Q_2 \subseteq P_1$ and that P_1 is infinite we conclude that Q_1 and Q_2 are infinite. Let C' be any infinite subset of Q_1 whose complement in Q_1 is infinite. Note that Q_2 is the disjoint union $Q_2 = \phi(C') \cup \phi(Q_1 \setminus C')$. Therefore

$$\Psi' = \langle C', \phi(C'), Q_1 \setminus C', \phi(Q_1 \setminus C'), Q_3, \ldots, Q_n \rangle$$

is a refinement of Ψ which supports y. Define ϕ' and ϕ'' (both in $\operatorname{stab}(\Pi)$) as follows:

$$\phi'(a) = \phi(a) \quad \text{ if } a \in C' \cup \phi(C'), \quad \text{and } \ldots = a \text{ otherwise;}$$
$$\phi''(a) = \phi(a) \quad \text{ if } a \in (Q_1 \setminus C') \cup \phi(Q_1 \setminus C'), \quad \text{and } \ldots = a \text{ otherwise.}$$

Clearly $\phi = \phi' \circ \phi''$ and therefore one of ϕ' or ϕ'' moves y. Assume without loss of generality that $\phi'(y) \neq y$. We then replace Ψ with Ψ' and ϕ by ϕ'.

This completes Step 1.

Step 2: Use y to obtain infinite disjoint subsets X_1 and $X \setminus X_1$ of X. Our primary tool will be the following claim.

Claim. *Assume that* $\delta, \lambda \in \operatorname{stab}(\Pi)$, *that* $\lambda(a) = a$ *for all* $a \in A \setminus (\delta(Q_1) \cup \delta(Q_2))$, *that* $\lambda(\delta(Q_1)) = \delta(Q_2)$, *and that* $\lambda(\delta(Q_2)) = \delta(Q_1)$. *Then* $\lambda(\delta(y)) \neq \delta(y)$.

By the hypotheses, $\delta^{-1}\lambda\delta(Q_1) = Q_2$ and $\delta^{-1}\lambda\delta(Q_2) = Q_1$ and for all $a \in A \setminus (Q_1 \cup Q_2)$, $\delta^{-1}\lambda\delta(a) = a$. This means that $\delta^{-1}\lambda\delta(\Psi) = \phi(\Psi)$ and since Ψ is a support of y, $\delta^{-1}\lambda\delta(y) = \phi(y) \neq y$. It follows that $\lambda(\delta(y)) \neq \delta(y)$, proving the claim.

Now by using permutations in $\mathrm{stab}(\Pi)$ which move Q_1 and Q_2 we can obtain infinitely many "copies" of y, all of which are in X.

Partition Q_3 into sets $\{S_j, T_j, U_j, V_j : j \in \omega\}$ so that

$$|S_j| = |T_j| = |U_j| = |V_j| = |Q_1| \ (= |Q_2|).$$

Let Q_3' and Q_3'' denote the sets $\bigcup_{j\in\omega}(S_j \cup T_j)$ and $\bigcup_{j\in\omega}(U_j \cup V_j)$ respectively. Let $\tau \in \mathrm{stab}(\Pi)$ satisfy

$$\tau(a) = a \quad \text{for all } a \in A \setminus (Q_1 \cup Q_2 \cup S_0 \cup T_0),$$
$$\tau(Q_1) = S_0, \quad \tau(S_0) = Q_1, \quad \tau(Q_2) = T_0 \quad \text{and} \quad \tau(T_0) = Q_2.$$

Let $\nu \in \mathrm{stab}(\Pi)$ satisfy

$$\nu(a) = a \quad \text{for all } a \in A \setminus (Q_1 \cup Q_2 \cup U_0 \cup V_0),$$
$$\nu(Q_1) = U_0, \quad \nu(U_0) = Q_1, \quad \nu(Q_2) = V_0 \quad \text{and} \quad \nu(V_0) = Q_2.$$

For $j \geq 1$, let $\sigma_j \in \mathrm{stab}(\Pi)$ satisfy

$$\sigma_j(a) = a \quad \text{for } a \in A \setminus (S_0 \cup T_0 \cup S_j \cup T_j),$$
$$\sigma_j(S_0) = S_j, \quad \sigma_j(S_j) = S_0, \quad \sigma_j(T_0) = T_j \quad \text{and} \quad \sigma_j(T_j) = T_0.$$

Similarly, for $j \geq 1$, let $\gamma_j \in \mathrm{stab}(\Pi)$ satisfy

$$\gamma_j(a) = a \quad \text{for all } a \in A \setminus (U_0 \cup V_0 \cup U_j \cup V_j),$$
$$\gamma_j(U_0) = U_j, \quad \gamma_j(U_j) = U_0, \quad \gamma_j(V_0) = V_j, \quad \text{and} \quad \gamma_j(V_j) = V_0.$$

The partition

$$\Pi' = \langle Q_1, Q_2, Q_3', Q_3'', \ldots, Q_{n_0}, P_2, \ldots, P_m \rangle$$

is a refinement of Π and therefore is a support of X. Hence the two sets

$$X_1 = \{\psi(\tau(y)) : \psi \in \mathrm{stab}(\Pi')\} \quad \text{and} \quad X_2 = \{\psi(\nu(y)) : \psi \in \mathrm{stab}(\Pi')\}$$

are subsets of X. Further, since Π' is a support of X_1 and X_2, these sets are in $\mathcal{W}_{\mathbb{B}}^{\mathrm{Aut}}$. We shall also prove the following:

(i) $(\forall j \in \omega,\ j \geq 1)(\sigma_j(\tau(y)) \in X_1 \wedge \gamma_j(\nu(y)) \in X_2)$;

(ii) $(\forall j \in \omega,\ j \geq 1)(\sigma_j(\tau(y)) \notin X_2 \wedge \gamma_j(\nu(y)) \notin X_1)$;

(iii) $(\forall j,\ r \in \omega,\ j, r \geq 1)(j \neq r \rightarrow [\sigma_j(\tau(y)) \neq \sigma_r(\tau(y)) \wedge \gamma_j(\nu(y)) \neq \gamma_r(\nu(y))])$.

It will follow that X_1 and $X \setminus X_1$ are infinite disjoint subsets of X and the proof will be complete.

To prove (i) we observe that since σ_j and γ_j agree with the identity permutation outside of Q_3' and Q_3'' respectively for $j \geq 1$, they are both in $\mathrm{stab}(\Pi')$.

For (ii) we first note that since Ψ is a support of y, a support of $\nu(y)$ is given by

$$\begin{aligned}\psi(\nu(\Psi)) &= \langle \nu(Q_1), \nu(Q_2), \nu(Q_3), \ldots, \nu(Q_n)\rangle \\ &= \langle U_0, V_0, [Q_3 \setminus (U_0 \cup V_0)] \cup (Q_1 \cup Q_2), Q_4, \ldots, Q_n\rangle.\end{aligned}$$

Therefore if ψ fixes Π' then $\psi(\nu(y))$ has support

$$\begin{aligned}&\psi(\nu(\Psi))\psi(\nu(\Psi)) \\ &\quad = \langle \psi(U_0), \psi(V_0), [\psi(Q_3) \setminus \psi(U_0 \cup V_0)] \cup \psi(Q_1 \cup Q_2), \psi(Q_4), \ldots, \psi(Q_n)\rangle \\ &\quad = \langle \psi(U_0), \psi(V_0), [Q_3 \setminus \psi(U_0 \cup V_0)] \cup (Q_1 \cup Q_2), \psi(Q_4), \ldots, \psi(Q_n)\rangle.\end{aligned}$$

Since $Q_3' \subseteq Q_3 \setminus \psi(U_0 \cup V_0)$ we may conclude that for any $\lambda \in \mathrm{stab}(\Pi)$, if $\lambda(a) = a$ for all $a \in A \setminus Q_3'$ then $\lambda(\psi(\nu(\Psi))) = \psi(\nu(\Psi))$ and therefore $\lambda(\psi(\nu(y))) = \psi(\nu(y))$. Hence such a λ fixes X_2 pointwise. In particular, if we choose λ so that $\lambda(a) = a$ for all $a \in A \setminus (S_j \cup T_j)$, $\lambda(S_j) = T_j$ and $\lambda(T_j) = S_j$ then λ fixes X_2 pointwise and by the claim (with $\delta = \sigma_j \circ \tau$)

$$\lambda(\sigma_j(\tau(y))) \neq \sigma_j(\tau(y)).$$

It follows that $\sigma_j(\tau(y)) \notin X_2$. A similar argument shows that $\gamma_j(\nu(y)) \notin X_1$.

The argument for (iii) is similar. We shall show

$$(\forall j, r \geq 1)(j \neq r \rightarrow \sigma_j(\tau(y)) \neq \sigma_r(\tau(y)))$$

and leave the similar argument that $\gamma_j(\nu(y)) \neq \gamma_r(\nu(y))$ to the reader. First note that $\sigma_r(\tau(y))$ has support

$$\sigma_r(\tau(\Psi)) = \langle S_r, T_r, [Q_3 \setminus (S_r \cup T_r)] \cup (Q_1 \cup Q_2), Q_4, \ldots, Q_n\rangle.$$

Call this support Ψ'. Since $S_j \cup T_j \subseteq Q_3 \setminus (S_r \cup T_r)$ any λ satisfying $\lambda(a) = a$ for all $a \in A \setminus (S_j \cup T_j)$ fixes Ψ' and therefore fixes $\sigma_r(\tau(y))$. In particular, if we choose λ for which $\lambda(S_j) = T_j$ and $\lambda(T_j) = S_j$ then λ fixes $\sigma_r(\tau(y))$ and by the claim (with $\delta = \sigma_j \circ \tau$), $\lambda(\sigma_j(\tau(y))) \neq \sigma_j(\tau(y))$. We can therefore conclude that $\sigma_r(\tau(y)) \neq \sigma_j(\tau(y))$. ∎

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