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# The cohomology algebra of certain free loop spaces

by

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**Abstract.** Let $X$ be a simply connected space and $LX$ the space of free loops on $X$. We determine the mod $p$ cohomology algebra of $LX$ when the mod $p$ cohomology of $X$ is generated by one element or is an exterior algebra on two generators. We also provide lower bounds on the dimensions of the Hodge decomposition factors of the rational cohomology of $LX$ when the rational cohomology of $X$ is a graded complete intersection algebra. The key to both of these results is the identification of an important subalgebra of the Hochschild homology of a graded complete intersection algebra over a field.

**0. Introduction.** Let $p$ be a prime number or zero, $X$ a simply connected space and $LX$ the space of free loops on $X$. In this paper, $\mathbb{Z}/p$ means the rational number field $\mathbb{Q}$ if $p = 0$. In order to calculate the mod $p$ cohomology $H^*(LX;\mathbb{Z}/p)$ from $H^*(X;\mathbb{Z}/p)$, one may use the Eilenberg–Moore spectral sequence ([5], [13], [15]) for the fiber square $\mathcal{F}(X)$:

$$\begin{array}{ccc} LX & \to & X \\ \downarrow & & \downarrow \Delta \\ X & \underset{\Delta}{\to} & X \times X \end{array}$$

where $\Delta$ is the diagonal map. In the procedure, the Hochschild homology $HH_*(H^*(X;\mathbb{Z}/p),0)$ of the commutative differential graded algebra (DGA) $(H^*(X;\mathbb{Z}/p),0)$ with the trivial differential appears. In fact, the $E_2$-term of the spectral sequence for $\mathcal{F}(X)$ is isomorphic to

$$\mathrm{Tor}_{H^*(X;\mathbb{Z}/p)\otimes H^*(X;\mathbb{Z}/p)}(H^*(X;\mathbb{Z}/p), H^*(X;\mathbb{Z}/p)),$$

that is, $HH_*(H^*(X;\mathbb{Z}/p),0)$ with an appropriate bigrading as a bigraded algebra. If $H^*(X;\mathbb{Z}/p)$ is a graded complete intersection algebra (GCI-algebra), there is a DGA ([15], [11]) whose cohomology is isomorphic to the $E_2$-term.

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If $p = 0$, we can use the Sullivan model of $LX$ to calculate its rational cohomology ([19]). From the argument in the proof of [3, Theorem B], we see that the homology of the model is isomorphic as an algebra to the Hochschild homology $HH_*(\Omega^*(X), \partial)$ of the Sullivan–de Rham complex $(\Omega^*(X), \partial)$. Therefore, if the space $X$ is formal [9], then $HH_*(H^*(X;\mathbb{Q}), 0)$ is isomorphic to $H^*(LX;\mathbb{Q})$ as an algebra since $(H^*(X;\mathbb{Q}), 0)$ is weakly equivalent to $(\Omega^*(X), \partial)$. This means that the above Eilenberg–Moore spectral sequence collapses at the $E_2$-term and that the extension problem is solved. It is well known that $X$ is formal if $H^*(X;\mathbb{Q})$ is a GCI-algebra ([9]).

In this paper we study the algebra structure of the Hochschild homology of a GCI-algebra and then apply our knowledge to calculating the cohomology algebra of certain free loop spaces. More precisely, our paper is organized as follows.

Let $\mathbb{k}_p$ be a field of characteristic $p$. In §1, we identify a subalgebra of the Hochschild homology $HH_*(\Xi, 0)$ of a simply connected GCI-algebra $\Xi$ over $\mathbb{k}_p$. The explicit form of the subalgebra appears in Proposition 1.1. The proposition also asserts that the subalgebra is isomorphic to $HH_*(\Xi, 0)$ if $\Xi$ is a tensor product of truncated polynomial algebras.

Let $X$ be a simply connected space. In §2, first we determine the explicit algebra structure of $H^*(LX;\mathbb{Z}/p)$ whose mod $p$ cohomology is generated by a single element, under some hypotheses on the prime $p$ and the degree of the single generator in $H^*(X;\mathbb{Z}/p)$. The main tool for calculating $H^*(LX;\mathbb{Z}/p)$ is the Eilenberg–Moore spectral sequence for the fiber square $\mathcal{F}(X)$. By degree arguments we can conclude that the spectral sequence collapses at the $E_2$-term. Moreover, arguments based on total degrees and filtration degrees of elements in the $E_0$-term enable us to solve all extension problems.

Second, we will consider the algebra structure of $H^*(LX;\mathbb{Z}/p)$ when $H^*(X;\mathbb{Z}/p)$ is an exterior algebra generated by two elements. In particular, we treat the case in which the collapsing at the $E_2$-term of the Eilenberg–Moore spectral sequence is guaranteed by the $p$-formality of $X$ ([2], [6]) or by [17, Theorem]. We then solve the extension problems by application of the Steenrod operations on the Eilenberg–Moore spectral sequence. After each calculation of $H^*(LX;\mathbb{Z}/p)$ for some class of spaces $X$, we point out the extension problems that cannot be solved with our tools.

Burghelea, Fiedorowicz and Gajda [3] clarified the connection between the minimal model of $LX$ ([19]) and the Hodge decomposition of $H^*(LX;\mathbb{Q})$. The connection implies that the dimension of each Hodge decomposition factor can be determined from the algebra structure of $H^*(X;\mathbb{Q})$. In §3 we provide lower bounds on the dimensions of the Hodge decomposition factors of $H^*(LX;\mathbb{Q})$ when $H^*(X;\mathbb{Q})$ is a GCI-algebra.

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## 1. A useful subalgebra of Hochschild homology

NOTATIONS. Before we state the main results in this section, we prepare some notations which are used in this paper.

Let $\mathbb{k}_p$ be a field of characteristic $p$ and $(C, d)$ a differential graded commutative algebra (DGA) over $\mathbb{k}_p$ endowed with a differential $d$ of degree $+1$. Then we denote the Hochschild homology of $(C, d)$ ([7], [3], [4]) by $HH_*(C, d)$.

Let $\Gamma[\omega_1, \ldots, \omega_m]$ be the divided power algebra over $\mathbb{k}_p$. Note that, as a vector space, $\Gamma[\omega]$ is generated by elements $\gamma_i(\omega)$ $(i > 0)$ and a unit $\gamma_0(\omega) = 1$, and the multiplication is defined by $\gamma_k(\omega_i)\gamma_l(\omega_i) = \binom{k+l}{k}\gamma_{k+l}(\omega_i)$. Furthermore, $\Gamma^+[\omega_1, \ldots, \omega_s]$ denotes the subalgebra of $\Gamma[\omega_1, \ldots, \omega_s]$ generated by the monomials $\{\gamma_{k_1}(\omega_1) \ldots \gamma_{k_s}(\omega_s) : k_1 > 0, \ldots, k_s > 0\}$ (cf. [1]). When $p = 0$, we regard the algebra $\Gamma[\omega_1, \ldots, \omega_m]$ and an element $\gamma_k(\omega_i)$ in $\Gamma[\omega_1, \ldots, \omega_m]$ as the polynomial algebra $\mathbb{k}_0[\omega_1, \ldots, \omega_m]$ and $\omega_i^k$, respectively.

For any algebra $B$, let $A$, $I$ and $S$ be a subalgebra, an ideal and a subset of $B$, respectively. Then $A/I$ denotes the quotient algebra of $A$ by the ideal $A \cap I$ and $(S)_A$ denotes the sub-$A$-module of $B$ generated by $S$ when we regard $B$ as an $A$-module. If $A = B$, then $(S)_A$ is the ideal of $A$ generated by $S$. For an algebra $A$ and elements $a_1, \ldots, a_s$ of $A$, we denote by $\mathrm{Ann}_A(a_1, \ldots, a_s)$ the ideal of $A$ generated by the elements $\{a : a \cdot a_i = 0 \text{ for } 1 \le i \le s\}$.

A *graded complete intersection algebra* (GCI-algebra) is a commutative graded algebra $\Xi = \Lambda(y_1, \ldots, y_l) \otimes \mathbb{k}_p[x_1, \ldots, x_n]/(\varrho_1, \ldots, \varrho_m)$ where $\varrho_1, \ldots, \varrho_m$ is a regular sequence (or $m = 0$) and where $\deg y_j$ is odd and $\deg x_i$ is even if $p \neq 2$. We say that $\Xi$ is *simply connected* if $\Xi^1 = 0$.

The proof of the following proposition is based upon the projective resolution of $\Xi$ as a $(\Xi \otimes \Xi)$-module, constructed in [15] (see also [11]).

PROPOSITION 1.1. (i) *Suppose*

$$\Xi = \Lambda(y_1, \ldots, y_l) \otimes \mathbb{k}_p[x_1, \ldots, x_n]/(\varrho_1, \ldots, \varrho_m)$$

*is a simply connected GCI-algebra, where $\varrho_i$ is decomposable for any $i$. Then there exists a monomorphism of algebras*

$$\begin{aligned} \psi : B = {} & \Lambda(y_1, \ldots, y_l) \otimes \Gamma[\overline{y}_1, \ldots, \overline{y}_l] \\ & \otimes \Big\{ \Big( A \oplus \sum_{s=1}^{m} \sum_{i_1 < \ldots < i_s} \mathrm{Ann}_A(d(\omega_{i_1}), \ldots, d(\omega_{i_s})) \\ & \qquad \otimes \Gamma^+[\omega_{i_1}, \ldots, \omega_{i_s}] \Big) \Big/ (d\Gamma[\omega_1, \ldots, \omega_m])_A \Big\} \\ & \hookrightarrow \mathrm{Tor}_{\Xi \otimes \Xi}(\Xi, \Xi) = HH_*(\Xi, 0), \end{aligned}$$

*where* $A = \mathbb{k}_p[x_1, \ldots, x_n]/(\varrho_1, \ldots, \varrho_m) \otimes \Lambda(\overline{x}_1, \ldots, \overline{x}_n)$, $d(\omega_j) = \sum_{i=1}^{n} \frac{\partial \varrho_j}{\partial x_i} \overline{x}_i$, $\deg \overline{y}_i = \deg y_i - 1$, $\deg \overline{x}_i = \deg x_i - 1$ *and* $\deg \omega_j = \deg \varrho_j - 2$.

(ii) *In the case*

$$\Xi = \Lambda(y_1, \ldots, y_l) \otimes \mathbb{k}_p[z_1, \ldots, z_m] \otimes \mathbb{k}_p[x_1, \ldots, x_n]/(x_1^{s_1+1}, \ldots, x_n^{s_n+1}),$$

*there exists an isomorphism of algebras*

$$\begin{aligned}\psi : B = \Lambda(y_1, \ldots, y_l) &\otimes \Gamma[\overline{y}_1, \ldots, \overline{y}_l] \otimes \mathbb{k}_p[z_1, \ldots, z_m] \otimes \Lambda(\overline{z}_1, \ldots, \overline{z}_m) \\ &\otimes \bigotimes_{i=1}^{n} \{A_i/((s_i+1)x_i^{s_i}\overline{x}_i) \oplus ((\varepsilon_i, x_i, \overline{x}_i)_{A_i}/((s_i+1)x_i^{s_i}\overline{x}_i)_{A_i}) \otimes \Gamma^+[\omega_i]\} \\ &\cong \mathrm{Tor}_{\Xi\otimes\Xi}(\Xi, \Xi) = HH_*(\Xi, 0),\end{aligned}$$

*where* $A_i = \mathbb{k}_p[x_i]/(x_i^{s_i+1}) \otimes \Lambda(\overline{x}_i)$, *and the element* $\varepsilon_i$ *is the unit* $1$ *in* $A_i$ *if* $s_i + 1 = 0$ *in* $\mathbb{k}_p$; *otherwise*, *it is zero.*

Proof. (i) The Koszul–Tate complex associated to the GCI-algebra $\Xi$ is

$$\mathcal{K} = (\Lambda(y_1, \ldots, y_l) \otimes \Gamma[\overline{y}_1, \ldots, \overline{y}_l] \otimes A \otimes \Gamma[\omega_1, \ldots, \omega_m], d),$$

where $d(y_i) = d(\overline{y}_i) = d(A) = 0$ and $d(\omega_j) = \sum_{i=1}^n \frac{\partial \varrho_j}{\partial x_i}\overline{x}_i$. Since this DGA $\mathcal{K}$ is regarded as the complex where $H^*(\mathcal{K}) \cong HH_*(\Xi, 0)$ (cf. [15], [11]), it follows that there exists a natural inclusion $\psi$. Let $C$ be

$$A \oplus \sum_{s=1}^{m} \sum_{i_1<\ldots<i_s} \mathrm{Ann}_A(d(\omega_{i_1}), \ldots, d(\omega_{i_s})) \otimes \Gamma^+[\omega_{i_1}, \ldots, \omega_{i_s}].$$

Then we note that the ideal $(d\Gamma[\omega_1, \ldots, \omega_m])_{A\otimes\Gamma[\omega_1,\ldots,\omega_m]} \cap C$ of $C$ is equal to $(d\Gamma[\omega_1, \ldots, \omega_m])_A \cap C$ in $A \otimes \Gamma[\omega_1, \ldots, \omega_m]$.

(ii) Let $\Xi$ be a truncated algebra $\mathbb{k}_p[x_i]/(x_i^{s_i+1})$. By direct calculation, we see that $HH_*(\Xi, 0) \cong A_i/((s_i+1)x_i^{s_i}\overline{x}_i) \oplus (\varepsilon_i, x_i, \overline{x}_i)_{A_i}/((s_i+1)x_i^{s_i}\overline{x}_i)_{A_i} \otimes \Gamma^+[\omega_i]$ as an algebra. From the Künneth theorem for the Hochschild homology, we have the isomorphism $\psi$. ■

The following example shows that the monomorphism $\psi$ in Proposition 1.1(i) is not an isomorphism in general. Consider the algebra $A = K[x, y]/(x^4 + y^2, y^4) \otimes \Lambda(\overline{x}, \overline{y})$ over a field $K$ of characteristic zero, where $\deg x = 2$, $\deg y = 4$, $\deg \overline{x} = 1$ and $\deg \overline{y} = 3$. Let $(C, d)$ be a differential graded algebra $(A \otimes K[\omega_1, \omega_2], d)$ endowed with a differential $d$ of degree $+1$, satisfying

$$d(\omega_1) = \left(\frac{\partial}{\partial x}\overline{x} + \frac{\partial}{\partial y}\overline{y}\right)(x^4 + y^2) = 4x^3\overline{x} + 2y\overline{y} \quad \text{and}$$

$$d(\omega_2) = \left(\frac{\partial}{\partial x}\overline{x} + \frac{\partial}{\partial y}\overline{y}\right)y^4 = 4y^3\overline{y},$$

where $\deg \omega_1 = 6$ and $\deg \omega_2 = 14$. The element

$$\alpha = 2y^2\overline{x}\omega_1 - \overline{x}\omega_2$$

is a cycle element with degree 15 in $C$. In fact,

$$d(\alpha) = -2y^2\overline{x}(4x^3\overline{x} + 2y\overline{y}) + \overline{x}(4y^3\overline{y}) = -4y^3\overline{xy} + 4y^3\overline{xy} = 0.$$

If there exists an element $\beta$ such that $d(\beta) = \alpha$, then $\beta$ must have the elements $\omega_2^k$ $(k \geq 2)$ or $\omega_2^k\omega_1^s$ $(s \geq 1,\ k \geq 1)$ as terms since $\alpha$ has the non-zero term $\overline{x}\omega_2$. Though the degree of $\beta$ is 14, $\deg \omega_2^k = 14k > 14$ and $\deg \omega_2^k\omega_1^s = 14k + 6s > 14$. Therefore $\alpha$ represents a non-zero element of $H^*(C, d)$. Let

$$\begin{aligned}\Gamma = A &\oplus \mathrm{Ann}_A(d\omega_1) \otimes K^+[\omega_1]\\ &\oplus \mathrm{Ann}_A(d\omega_2) \otimes K^+[\omega_2] \oplus \mathrm{Ann}_A(d\omega_1, d\omega_2) \otimes K^+[\omega_1, \omega_2].\end{aligned}$$

If the monomorphism $\psi$ is an isomorphism, then there exists an element $\gamma \in \Gamma$ which maps $\alpha + d(\beta)$ by the lifting map of $\psi : \Gamma \to H^*(C, d)$ for some element $\beta \in C$. Since the degree of $\gamma$ is 15, $\gamma$ can be written as $b_0 + b_1\omega_1 + b_2\omega_1^2 + b_3\omega_2$, where $b_0 \in A$ and $b_1, b_2 \in \mathrm{Ann}_A(d(\omega_1))$ and $b_3 \in \mathrm{Ann}_A(d(\omega_2))$. Then $\psi^{-1}d(\beta) = \gamma - \psi^{-1}(\alpha) = b_0 + (b_1 - 2y^2\overline{x})\omega_1 + b_2\omega_1^2 + (b_3 + \overline{x})\omega_2$. Applying the above argument about degrees again, we have $b_3 = -\overline{x}$. On the other hand, $\overline{x} \notin \mathrm{Ann}_A(d(\omega_2)) = \mathrm{Ann}_A(4y^3\overline{y})$, which is a contradiction.

R e m a r k 1.2. Let $(\Omega^*(X), \partial)$ be the Sullivan–de Rham complex over a field $\mathbb{k}_0$ where $H_*(\Omega^*(X), \partial) = H^*(X; \mathbb{k}_0)$ is isomorphic to a GCI-algebra $\Lambda = \Lambda(y_1, \ldots, y_l) \otimes \mathbb{k}_0[x_1, \ldots, x_n]/(\varrho_1, \ldots, \varrho_m)$, where $\varrho_i$ is decomposable. Then the DGA has minimal model $\mathcal{M} = (\wedge V, \widetilde{\partial})$ defined by $\wedge V = \Lambda(y_1, \ldots$ $\ldots, y_l) \otimes \mathbb{k}_0[x_1, \ldots, x_n] \otimes \Lambda(\tau_1, \ldots, \tau_m)$, $\widetilde{\partial}(y_i) = \widetilde{\partial}(x_i) = 0$ and $\widetilde{\partial}(\tau_j) = \varrho_j$. Here $\wedge V$ denotes the free commutative graded algebra over a graded vector space $V = \bigoplus_{i>1} V^i$. To calculate the cohomology of $LX$ over $\mathbb{k}_0$, one can use the complex $\varepsilon(\mathcal{M}) = (\wedge V \otimes \wedge \overline{V}, \delta)$ defined in [4], [19]. This DGA has the following properties:

(i) $\overline{V}^i = V^{i+1}$, that is, we have $\wedge \overline{V} = \mathbb{k}_0[\overline{y}_1, \ldots, \overline{y}_l] \otimes \Lambda(\overline{x}_1, \ldots, \overline{x}_n) \otimes \mathbb{k}_0[\overline{\tau}_1, \ldots, \overline{\tau}_m]$.

(ii) When $\beta$ is the unique derivation of degree $-1$ extending the maps $\beta(x_i) = \overline{x}_i$, $\beta(y_j) = \overline{y}_j$, $\beta(\tau_k) = \overline{\tau}_k$ and $\beta(\overline{x}_i) = \beta(\overline{y}_j) = \beta(\overline{\tau}_k) = 0$, then the differential $\delta$ is the unique derivation of degree $+1$ with $\delta|_{\wedge V} = \widetilde{\partial}$ and $\delta\beta + \beta\delta = 0$, that is, $\delta(y_i) = \delta(x_i) = \delta(\overline{y}_i) = \delta(\overline{x}_i) = 0$, $\delta(\tau_j) = \varrho_j$ and $\delta(\overline{\tau}_j) = -\sum_{i=1}^n \frac{\partial \varrho_j}{\partial x_i}\overline{x}_i$.

(iii) $H_*(\varepsilon(\mathcal{M})) \cong H^*(LX; \mathbb{k}_0)$.

Let $\mathcal{K}$ be the (not free) DGA over $\mathbb{k}_0$ defined in the proof of Proposition 1.1. In this case, we see that there is a natural map from $\varepsilon(\mathcal{M})$ to $\mathcal{K}$ which induces $H_*(\varepsilon(\mathcal{M})) \cong H_*(\mathcal{K})$. This is given by the correspondences: $x_i \mapsto x_i$, $y_i \mapsto y_i$, $\tau_j \mapsto 0$, $\overline{x}_i \mapsto \overline{x}_i$, $\overline{y}_i \mapsto \overline{y}_i$, $\overline{\tau}_j \mapsto \omega_j$ and $\delta \mapsto -d$.

**2. The mod $p$ cohomology of $LX$.** In this section, we denote the Eilenberg–Moore spectral sequence for the fiber square $\mathcal{F}(X)$ by $\{E_r^{*,*}, d_r\}$ with $d_r$ of bidegree $(r, 1 - r)$. The spectral sequence is lying in the second quadrant, that is, $E_r^{p,q}$ is bigraded with $p \le 0$ and $q \ge 0$. We may call the indices $p$ and $p+q$ the *filtration degree* and the *total degree*, respectively. The $E_2$-term is isomorphic to $\mathrm{Tor}_{H^*(X;\mathbb{Z}/p)\otimes H^*(X;\mathbb{Z}/p)}(H^*(X;\mathbb{Z}/p), H^*(X;\mathbb{Z}/p))$, that is, to the Hochschild homology $HH_*(H^*(X;\mathbb{Z}/p),0)$. Notice that the target of the spectral sequence is $H^*(LX;\mathbb{Z}/p)$. To be exact, there exists a decreasing filtration $\{F^iH^*(LX;\mathbb{Z}/p)\}_{i\le 0}$ on $H^*(LX;\mathbb{Z}/p)$ and the limit term $E_\infty^{*,*}$ is isomorphic to $E_0^{*,*}$ as bigraded algebras. Here $E_0^{i,j}$ is defined by $F^iH^{i+j}(LX;\mathbb{Z}/p)/F^{i+1}H^{i+j}(LX;\mathbb{Z}/p)$.

By virtue of Proposition 1.1(ii), we can determine the mod $p$ cohomology of a space $LX$ of free loops on a space $X$ whose mod $p$ cohomology ring is generated by one element.

Theorem 2.1. *Let $X$ be a simply connected space whose mod $p$ cohomology is isomorphic to $\Lambda(y)$, where $\deg y$ is odd. Then*

$$H^*(LX;\mathbb{Z}/p) \cong \Lambda(y)\otimes \Gamma[\overline{y}]$$

*as an algebra, where $\deg\overline{y} = \deg y - 1$.*

Theorem 2.2. *Let $X$ be a simply connected space whose mod $p$ cohomology is isomorphic to $\mathbb{Z}/p[x]/(x^{s+1})$.*

(i) *When $s+1\equiv 0 \bmod p$ and when $p\ne 2$ or $\deg x\ne 2$,*

$$H^*(LX;\mathbb{Z}/p)\cong \mathbb{Z}/p[x]/(x^{s+1})\otimes\Lambda(\overline{x})\otimes\Gamma[\omega]$$

*as an algebra, where $\deg\overline{x}=\deg x-1$ and $\deg\omega=(s+1)\deg x-2$.*

(ii) *When $s+1\not\equiv 0 \bmod p$ and when $s>1$ or $\deg x\ne 2$,*

$$\begin{aligned}H^*(LX;\mathbb{Z}/p)\cong \{(&\mathbb{Z}/p[x]/(x^{s+1})\otimes\Lambda(\overline{x}))/(x^s\overline{x})_A\}\\ &\oplus\{(x,\overline{x})_A/(x^s\overline{x})_A)\}\otimes\Gamma^+[\omega]\end{aligned}$$

*as an algebra, where $A=\mathbb{Z}/p[x]/(x^{s+1})\otimes\Lambda(\overline{x})$, $\deg\overline{x}=\deg x-1$ and $\deg\omega=(s+1)\deg x-2$.*

Next we consider the algebra structure of $H^*(LX;\mathbb{Z}/p)$ in the case when mod $p$ cohomology of a simply connected space $X$ is an exterior algebra generated by two elements $x_t$ and $x_u$ with $t\le u$. If $\widetilde{H}^i(X;\mathbb{Z}/p)$ is zero whenever $i$ is outside an interval of the form $[k+1,3k+1]$, that is, $t\le u\le 2t-2$, then $X$ is $p$-formal [2, Lemma 9]. Therefore, consideration of the proof of [10, Proposition 3.1] enables us to conclude that the singular cochains $(C^*(X;\mathbb{Z}/p),d)$ and the DGA $(H^*(X;\mathbb{Z}/p),0)$ are connected by a chain of DGA-quasi-isomorphisms. Moreover, since the Eilenberg–Moore map induces an isomorphism of algebras from $\mathrm{Tor}_{C^*(X)\otimes C^*(X)}(C^*(X),C^*(X))$ to $H^*(LX;\mathbb{Z}/p)$, it follows that $H^*(LX;\mathbb{Z}/p)$ is isomorphic to

$$\mathrm{Tor}_{H^*(X;\mathbb{Z}/p)\otimes H^*(X;\mathbb{Z}/p)}(H^*(X;\mathbb{Z}/p),H^*(X;\mathbb{Z}/p))$$

as a vector space. Consequently, the Eilenberg–Moore spectral sequence

$\{E_r, d_r\}$ collapses at the $E_2$-term if $t \le u \le 2t-2$. By solving the extension problem of the Eilenberg–Moore spectral sequence, we have

THEOREM 2.3. *Suppose that the mod $p$ cohomology of a simply connected space $X$ is isomorphic to the exterior algebra $\Lambda(x_t, x_u)$, where $t \le u \le 2t-2$. If $p > 3$ or $u \ne 3$, $u \ne 2t-3$ and $p = 3$, then, as an algebra,*

$$H^*(LX; \mathbb{Z}/p) \cong \Lambda(x_t, x_u) \otimes \Gamma[\overline{x}_t, \overline{x}_u].$$

Remark 2.4. Under the condition that $t \le u \le 2t-2$, the $p$-formality of $X$ enables us to conclude that the spectral sequence $\{E_r, d_r\}$ collapses at the $E_2$-term. Note that it is not easy to deduce the above fact under the conditions $t \le u \le 2t-2$ from degree considerations as the proof of Theorem 2.1 or 2.2. In fact, in the case $p = 3$, $t = 5$ and $u = 7$, simple degree considerations do not suffice to eliminate the possibility that $d_2(\gamma_3(\overline{x}_5)) = x_7\overline{x}_7 + \ldots$ in the $E_2$-term.

Suppose that $H^*(X; \mathbb{Z}/2)$ is isomorphic to the truncated polynomial algebra $\mathbb{Z}/2[x_1, \ldots, x_n]/(x_1^{2^{u_1}}, \ldots, x_n^{2^{u_n}})$. Then [17, Theorem] asserts that the Eilenberg–Moore spectral sequence collapses at the $E_2$-term if $\mathrm{Sq}^1 \equiv 0$ on $H^*(X; \mathbb{Z}/2)$. Moreover, from the argument of the proof, we see that the same conclusion holds if the vector space $\mathrm{Im}(\mathrm{Sq}^1)^{2^{k+1}m_i+2}$ is zero for any $k \ge 0$ and $1 \le i \le n$, where $m_i = 2^{u_i-1}i - 1$. In consequence, we have

THEOREM 2.5. *Suppose that the mod 2 cohomology of a simply connected space $X$ is isomorphic to the exterior algebra $\Lambda(x_t, x_{2t-1})$.*

(i) *If $\mathrm{Sq}^{t-1} x_t = 0$ and $t > 3$, then, as an algebra,*

$$H^*(LX; \mathbb{Z}/2) \cong \Lambda(x_t, x_{2t-1}) \otimes \Gamma[\overline{x}_t, \overline{x}_{2t-1}].$$

(ii) *If $\mathrm{Sq}^{t-1} x_t = x_{2t-1}$ and $t > 3$ or $\mathrm{Sq}^1 x_2 = x_3$, $\mathrm{Sq}^2 x_3 = 0$ and $t = 2$, then, as an algebra,*

$$H^*(LX; \mathbb{Z}/2) \cong \Lambda(x_t, x_{2t-1}) \otimes \bigotimes_{i \ge 0} \mathbb{Z}/2[\gamma_{2^i}(\overline{x}_t)]/(\gamma_{2^i}(\overline{x}_t)^4).$$

Let $V$ be a vector space and $x$, $y$ elements of $V$. In the proofs of Theorems 2.1, 2.2, 2.3 and 2.5, we will say that $x$ *contains* $y$ if the element $x$ can be represented by a linear combination in which the element $y$ has a non-zero coefficient.

Proof of Theorem 2.2. (i) By Proposition 1.1(ii), we have

$$E_2^{*,*} \cong \mathbb{Z}/p[x]/(x^{s+1}) \otimes \Lambda(\overline{x}) \otimes \Gamma[\omega],$$

where $\mathrm{bideg}\, x = (0, \deg x)$, $\mathrm{bideg}\, \overline{x} = (-1, \deg x)$ and $\mathrm{bideg}\, \gamma_i(\omega) = (-2i, i(s+1)\deg x)$.

First we prove the theorem under the assumption that $\deg x$ is even. Suppose that $d_r(\gamma_i(\omega))$ contains the element $x^l\overline{x}\gamma_j(\omega)$. Then we have

- $i((s+1)\deg x-2)+1=(l+1)\deg x-1+j((s+1)\deg x-2)$ and
- $-2i+r=-2j-1$

by an argument on total degrees and filtration degrees respectively. We have $i=j+(r+1)/2>j+1$ from the latter. This contradicts the former since $s>l$. Thus we can conclude that $E_2^{*,*}\cong E_\infty^{*,*}\cong E_0^{*,*}$ as bigraded algebras.

Let us solve extension problems. In this case, it suffices to prove that $\overline{x}\cdot\overline{x}$ does not contain $x$ and $\gamma_{p^f}(\omega)^p$ does not contain $x^l\gamma_k(\omega)$ since $E_0$ contains $\Gamma[\omega]$ as a subalgebra and the relations of $\Gamma[\omega]$ as an algebra are $\{\gamma_{p^f}(\omega)^p=0 : f\geq 0\}$. If $\overline{x}\cdot\overline{x}$ contains $x$, we have $\deg x=2$. Then there is a contradiction since $p\neq 2$ from the assumption. Next suppose that $\gamma_{p^f}(\omega)^p$ contains $x^l\gamma_k(\omega)$. Then we have an equality of the total degrees:

$$\text{(T)}\qquad p^{f+1}((s+1)\deg x-2)=l\deg x+k((s+1)\deg x-2).$$

Since the filtration induced on $H^*(LX;\mathbb{Z}/p)$ as the limit term of the Eilenberg–Moore spectral sequence is invariant under the action of the Steenrod operations, it follows that $\gamma_{p^f}(\omega)^p$ is in the filtration $F^{-2p^f}H^*(LX;\mathbb{Z}/p)$. Thus we have an inequality of the filtration degrees:

$$\text{(F)}\qquad p^f\geq k.$$

From (T) and (F), we have $p=2$ and $\deg x=2$. For, we have

$$\begin{aligned} p^{f+1}((s+1)\deg x-2) &\geq_{\text{(a)}} (k+1)((s+1)\deg x-2)\\ &= (s+1)\deg x-2+k((s+1)\deg x-2)\\ &\geq_{\text{(b)}} l\deg x+k((s+1)\deg x-2)\end{aligned}$$

in general. Here (a) follows from (F) and (b) follows from $s+1>l$. Then (a) and (b) are equalities by (T). The inequality of (F) and the equality of (a) imply that $p=2$, $f=0$ and $k=1$. The equality of (b) shows that $s=l$ and $\deg x=2$. Thus the assumption of Theorem 2.2(i) implies that $\gamma_{p^f}(\omega)^p$ does not contain $x^l\gamma_k(\omega)$, that is, $\gamma_{p^f}(\omega)^p=0$ in $H^*(LX;\mathbb{Z}/p)$.

Second, we prove the theorem under the assumption that $\deg x$ is odd and $p=2$. The fact that $d_r(\gamma_i(\omega))$ does not contain $x^l\overline{x}\gamma_j(\omega)$ follows by the same argument as above. Suppose that $d_r(\gamma_i(\omega))$ contains $x^l\gamma_j(\omega)$. Then we have

- $i((s+1)\deg x-2)+1=l\deg x+j((s+1)\deg x-2)$ and
- $-2i+r=-2j$

by an argument on total degrees and column degrees respectively. We have $i=j+r/2\geq j+1$ from the latter. This contradicts the former as $s+1>l$. Thus we can conclude that $E_2^{*,*}\cong E_\infty^{*,*}\cong E_0^{*,*}$ as bigraded algebras.

Let us solve extension problems. In this case, it suffices to prove that $\gamma_{p^f}(\omega)^p$ does not contain either $x^l\gamma_k(\omega)$ or $x^l\overline{x}\gamma_k(\omega)$. The fact that $\gamma_{p^f}(\omega)^p$ does not contain $x^l\gamma_k(\omega)$ follows from the same argument as above. Suppose that $\gamma_{p^f}(\omega)^p$ contains $x^l\overline{x}\gamma_k(\omega)$. Then we have

- $p^{f+1}((s+1)\deg x - 2) = (l+1)\deg x - 1 + k((s+1)\deg x - 2)$ and
- $2p^f > 2k+1$

by the argument as above on total degrees and filtration degrees respectively. Then these contradict each other since $s > l$. Thus we can conclude that $\gamma_{p^f}(\omega)^p$ does not contain $x^l\overline{x}\gamma_k(\omega)$, that is, $\gamma_{p^f}(\omega)^p = 0$ in $H^*(LX;\mathbb{Z}/p)$. Thus we have Theorem 2.2(i).

(ii) By Proposition 1.1(ii), we have

$$E_2^{*,*} \cong A/((s+1)x^s\overline{x})_A \oplus \{(x,\overline{x})_A/((s+1)x^s\overline{x})_A\} \otimes \Gamma^+[\omega],$$

where $A = \mathbb{Z}/p[x]/(x^{s+1}) \otimes \Lambda(\overline{x})$ as a bigraded algebra. Let

$$\begin{aligned} A_{l,i} &= \deg x^l\gamma_i(\omega) = l\deg x + i((s+1)\deg x - 2),\\ B_j &= \deg \overline{x}\gamma_j(\omega) = \deg x - 1 + j((s+1)\deg x - 2),\\ C_{m,k} &= \deg \overline{x}x^m\gamma_k(\omega) = (m+1)\deg x - 1 + k((s+1)\deg x - 2). \end{aligned}$$

Then we can conclude that the Eilenberg–Moore spectral sequence $\{E_r^{*,*}, d_r\}$ collapses at the $E_2^{*,*}$-term since the following inequalities hold:

$$\begin{aligned} A_{l,i}+1 &> B_j \quad (i > j+1), & B_j + 1 &> A_{l,i} \quad (j > i),\\ C_{m,k}+1 &> A_{l,i} \quad (k > i), & B_j + 1 &> C_{m,k} \quad (j > k),\\ A_{l,i}+1 &> A_{k,j} \quad (i > j), & B_j + 1 &> B_i \quad (j > i), \end{aligned}$$

$$\begin{aligned} A_{l,i}+1 &> C_{m,k} \quad (i > k+1),\\ C_{m,k}+1 &> B_j \quad (k > j),\\ C_{m,k}+1 &> C_{n,l} \quad (k > l). \end{aligned}$$

Here the inequalities in ( ) are induced by an argument on column degrees. Note that last five inequalities have meaning only in the case when $p = 2$ and $\deg x$ is odd. Thus we can conclude that $E_2^{*,*} \cong E_\infty^{*,*} \cong E_0^{*,*}$ as a bigraded algebra.

Let us consider extension problems. We must verify that the following equalities hold in $H^*(LX;\mathbb{Z}/p)$:

(1) $x^s \cdot x\gamma_i(\omega) = 0$,
(2) $x^s \cdot \overline{x}\gamma_i(\omega) = 0$,
(3) $\overline{x} \cdot \overline{x}\gamma_i(\omega) = 0$,
(4) $\overline{x}\gamma_j(\omega) \cdot \overline{x}\gamma_k(\omega) = 0$,
(5) $\overline{x}\gamma_j(\omega) \cdot x^l\gamma_k(\omega) = 0$ if $\binom{j+k}{j} \equiv \mathrm{mod} p$,

(6) $x^l\gamma_j(\omega)\cdot x^m\gamma_k(\omega)=0$ if $\binom{j+k}{j}\equiv 0 \bmod p$,

where $i,j,k,l,m>0$ for (1), (4), (5), (6) and $i\geq 0$ for (2), (3).

Let us first verify that (1) holds. It suffices to prove that $x^s\cdot x\gamma_i(\omega)$ does not contain either $x^m\gamma_j(\omega)$ or $x^m\overline{x}\gamma_j(\omega)$. Suppose that $x^s\cdot x\gamma_i(\omega)$ contains $x^m\gamma_j(\omega)$. Then we have

- $(s+1)\deg x+i((s+1)\deg x-2)=m\deg x+j((s+1)\deg x-2)$ and
- $i>j$

by an argument on total degrees and filtration degrees respectively. These contradict each other since $s+1>m$.

Suppose that $x^s\cdot x\gamma_i(\omega)$ contains $x^m\overline{x}\gamma_j(\omega)$, where $p=2$ and deg$x$ is odd. Then we have

- $(s+1)\deg x+i((s+1)\deg x-2)$
  $=(m+1)\deg x-1+j((s+1)\deg x-2)$ and
- $2i>2j+1$

by an argument on total degrees and filtration degrees respectively. These contradict each other since $s>m$. Thus the equality (1) holds. Applying the same argument as above, it follows that equalities (2), (5) and (6) hold.

Let us next verify that (3) holds. It suffices to prove that $\overline{x}\cdot\overline{x}\gamma_i(\omega)$ does not contain either $x^l\gamma_j(\omega)$ or $x^l\overline{x}\gamma_j(\omega)$.

Suppose that $\overline{x}\cdot\overline{x}\gamma_i(\omega)$ contains $x^l\gamma_j(\omega)$. Then we have

- $2(\deg x-1)+i((s+1)\deg x-2)=l\deg x+j((s+1)\deg x-2)$ and
- $i+1>j$

by an argument on total degrees and filtration degrees respectively. Then since $s+1>l$, we have $i=j$, $l=1$ and $\deg x=2$. In this case, it turns out that $\overline{x}\cdot\overline{x}\gamma_i(\omega)=\lambda x\gamma_i(\omega)$ for some constant $\lambda$. If $\overline{x}\cdot\overline{x}=0$ in $H^*(LX;\mathbb{Z}/p)$, then $\lambda\overline{x}\cdot x\gamma_i(\omega)=\overline{x}\cdot(\overline{x}\cdot\overline{x}\gamma_i(\omega))=(\overline{x}\cdot\overline{x})\cdot\overline{x}\gamma_i(\omega)=0$. Since $s>1$, it follows that $\overline{x}\cdot x\gamma_i(\omega)\neq 0$ in $E_0^{*,*}H^*(LX;\mathbb{Z}/p)$, and therefore in $H^*(LX;\mathbb{Z}/p)$ as well. Hence we have $\lambda=0$. Thus it suffices to show that $\overline{x}\cdot\overline{x}=0$ in $H^*(LX;\mathbb{Z}/p)$.

When $p\neq 2$, it is clear that $\overline{x}\cdot\overline{x}=0$. If $p=2$ and $\overline{x}\cdot\overline{x}\neq 0$, by the usual argument on total degrees and degrees of filtrations, we see that $\overline{x}\cdot\overline{x}=\mu x$ for some non-zero constant $\mu$. The indecomposable element $x$ in $H^*(LX;\mathbb{Z}/p)$ is the image of the indecomposable element $x$ in $H^*(X;\mathbb{Z}/p)$ by the map $\pi^*$ induced from the projection of the fibration $\pi: LX\to X$. Let $s^*$ be the homomorphism which is induced from the section $s: X\to LX$ defined by $s(a)(t)=a$ (for $a\in X$ and $t\in S^1$). Since $\mu x=\mu s^*\pi^*(x)=\mu s^*(x)=s^*(\overline{x})\cdot s^*(\overline{x})$ in $H^*(X;\mathbb{Z}/p)$, it follows that the element $x$ in $H^*(X;\mathbb{Z}/p)$ is decomposable, which is a contradiction.

Suppose that $\overline{x}\cdot\overline{x}\gamma_i(\omega)$ contains $x^l\overline{x}\gamma_j(\omega)$, where $p=2$ and $\deg x$ is odd. Then we have

- $2(\deg x-1)+i((s+1)\deg x-2)$
  $=(l+1)\deg x-1+j((s+1)\deg x-2)$ and
- $i+1>j$

by an argument on total degrees and filtration degrees respectively. If $i=j$, then from the equality of the total degrees we have $(l-1)\deg x=-1$, which is a contradiction. If $i>j$, then from $s>l$ we have

$$(i-j)((s+1)\deg x-2)\geq(s+1)\deg x-2>l\deg x>(l-1)\deg x+1,$$

which contradicts the equality of the total degrees.

Thus equality (3) holds. Applying the same argument as above, we see that (4) holds as well. Thus we have Theorem 2.2(ii). ∎

By a similar argument on total degrees and filtration degrees, we can prove Theorem 2.1. The details are left to the reader.

Remark 2.6. In the case where $s+1\equiv 0 \bmod p$, $p=2$ and $\deg x=2$ or $p$ is odd, $s=1$ and $\deg x=2$, we can see that the Eilenberg–Moore spectral sequence converging to $H^*(LX;\mathbb{Z}/p)$ collapses at the $E_2$-term. However, we cannot solve extension problems by using the usual argument on total degrees and column degrees of the associated bigraded algebra $E_0^{*,*}$. For example, there is no immediate contradiction to the existence of the relation $\omega^2=x^s\omega$ when $s+1\equiv 0 \bmod p$, $p=2$ and $\deg x=2$ or the relations $\overline{x}\cdot\overline{x}\gamma_i(\omega)=x\gamma_i(\omega)$ $(i>0)$ when $p$ is odd, $s=1$ and $\deg x=2$.

Proof of Theorem 2.3. It suffices to prove that the elements $\gamma_{p^f}(\overline{x}_t)^p$ and $\gamma_{p^f}(\overline{x}_u)^p$ do not contain the element $x_tx_u\gamma_i(\overline{x}_t)\gamma_j(\overline{x}_u)$, where $p^f>i+j\geq 0$ and $f\geq 0$, as in the proof of Theorem 2.2(i).

If $f=0$, then $i+j=0$. If $p>3$, we have

$$\deg x_tx_u<\deg\overline{x}_t^p\leq\deg\overline{x}_u^p.$$

Therefore we can conclude that $\overline{x}_u^p$ and $\overline{x}_t^p$ do not contain the element $x_tx_u$ if $p>3$. If $p=3$, we have $\deg\overline{x}_u^p=\deg x_tx_u$ if and only if $t=u=3$ and $\deg\overline{x}_t^p=\deg x_tx_u$ if and only if $u=2t-3$. So if $u\neq 3$ and $u\neq 2t-3$, we can conclude that $\overline{x}_u^p$ and $\overline{x}_t^p$ do not contain the element $x_tx_u$ since $t\leq u$.

If $f>0$, since $p\neq 2$ and $t\leq u\leq 2t-2$, we have

$$\deg x_tx_u\gamma_i(\overline{x}_t)\gamma_j(\overline{x}_u)<\deg\gamma_{p^f}(\overline{x}_t)^p\leq\deg\gamma_{p^f}(\overline{x}_u)^p.$$

Therefore $\gamma_{p^f}(\overline{x}_t)^p$ and $\gamma_{p^f}(\overline{x}_u)^p$ do not contain $x_tx_u\gamma_i(\overline{x}_t)\gamma_j(\overline{x}_u)$. It turns out that $\gamma_{p^f}(\overline{x}_t)^p=0=\gamma_{p^f}(\overline{x}_u)^p$ in $H^*(LX;\mathbb{Z}/p)$. Thus we have Theorem 2.3. ∎

Remark 2.7. In the case when $p = 2$, the Eilenberg–Moore spectral sequence $\{E_r^{*,*}, d_r\}$ collapses at the $E_2$-term because $X$ is $p$-formal. However, for instance, we cannot decide whether $\gamma_2(\overline{x}_4)^2$ is equal to $x_3x_4\overline{x}_3\overline{x}_4$ for $p = 2$, $t = u = 2$ by the usual consideration of degrees.

Let us compare two different resolutions of the GCI-algebra

$$\varXi = \bigotimes_k \varLambda(x_k),$$

an exterior algebra over $\mathbb{Z}/2$, before proving Theorem 2.5. Let $B^*(\varXi\otimes\varXi, \varXi)$ denote the bar resolution of $\varXi$, considered as a left $(\varXi \otimes \varXi)$-module. Let

$$\mathcal{F} = \Big(\varXi \otimes \varXi \otimes \bigotimes_k \varGamma[\overline{x}_k], d\Big)$$

where

$$d(\gamma_i(\overline{x}_k)) = (x_k \otimes 1 - 1 \otimes x_k)\gamma_{i-1}(\overline{x}_k).$$

Then $\mathcal{F} \xrightarrow{\mu} \varXi \to 0$ is a proper projective resolution of $\varXi$, considered as a left $(\varXi \otimes \varXi)$-module, where $\mu$ denotes the multiplication on $\varXi$.

Lemma 2.8 ([11, Lemma 1.5]). *There exists a morphism of resolutions from $B^*(\varXi \otimes \varXi, \varXi)$ to $\mathcal{F}$, inducing an automorphism $\phi$ of $\mathrm{Tor}_{\varXi\otimes\varXi}(\varXi, \varXi)$ such that*

$$\phi\Big(\overbrace{[x_k \otimes 1 - 1 \otimes x_k| \ldots |x_k \otimes 1 - 1 \otimes x_k]}^{i\ times}\Big) = \gamma_i(\overline{x}_k).$$

Proof. Since the elements $z = 1\otimes 1[x_k\otimes 1-1\otimes x_k| \ldots |x_k\otimes 1-1\otimes x_k]$ are part of a $(\varXi\otimes\varXi)$-basis of the bar resolution $B^*(\varXi\otimes\varXi, \varXi)$, we can define a morphism $\psi$ from $B^*(\varXi \otimes \varXi, \varXi)$ to $\mathcal{F}$ so that $\psi(z) = 1\otimes 1\otimes \gamma_i(\overline{x}_k)$. Then $\psi$ induces the required isomorphism $\phi$. ∎

Proof of Theorem 2.5. Using Proposition 1.1, we can determine the algebra structure of $E_2^{*,*}$ explicitly. If $t > 3$, then the spectral sequence $\{E_r, d_r\}$ collapses at the $E_2$-term by [17, Theorem] since a degree argument shows that $\mathrm{Sq}^1 = 0$.

To solve the extension problem, we use the Steenrod operations $\{\mathrm{Sq}^i_{\mathrm{EM}}\}_{i\ge 0}$ on the Eilenberg–Moore spectral sequence ([12], [14]), which are induced from operations on the bar construction. Notice that the operations $\mathrm{Sq}^i_{\mathrm{EM}}$ $(i \ge 0)$ on $E_\infty^{*,*}$ coincide with the operations on $E_0^{*,*}H^*(LX;\mathbb{Z}/2)$ induced from the ordinary Steenrod operations on $H^*(LX;\mathbb{Z}/2)$. Since

$$\begin{aligned}
&\mathrm{Sq}^{2^f(t-1)}_{\mathrm{EM}} \overbrace{[x_t \otimes 1 - 1 \otimes x_t| \ldots |x_t \otimes 1 - 1 \otimes x_t]}^{2^f \text{ times}} \\
&\quad = \overbrace{[\mathrm{Sq}^{t-1} x_t \otimes 1 - 1 \otimes \mathrm{Sq}^{t-1} x_t| \ldots | \mathrm{Sq}^{t-1} x_t \otimes 1 - 1 \otimes \mathrm{Sq}^{t-1} x_t]}^{2^f \text{ times}}
\end{aligned}$$

in $E_\infty^{*,*}$, it follows from Lemma 2.8 that $\gamma_{2^f}(\overline{x}_t)^2 = 0$ if $\mathrm{Sq}^{t-1} = 0$ and $\gamma_{2^f}(\overline{x}_t)^2 = \mathrm{Sq}^{2^f(t-1)}\gamma_{2^f}(\overline{x}_t) = \gamma_{2^f}(\overline{x}_{2t-1})$ if $\mathrm{Sq}^{t-1} \neq 0$ in $E_0^{*,*}$. Since $\mathrm{Sq}^{2t-2}x_{2t-1} = 0$ for $t > 3$, by the same argument as above, we see that $\gamma_{2^f}(\overline{x}_{2t-1})^2 = 0$ in $E_0^{*,*}$.

In order to complete the proof of Theorem 2.5(i), we must show that $\gamma_{2^f}(\overline{x}_t)^2 = 0$ and $\gamma_{2^f}(\overline{x}_{2t-1})^2 = 0$ in $H^*(LX;\mathbb{Z}/2)$ if $\mathrm{Sq}^{t-1} = 0$. To this end, we verify that $\gamma_{2^f}(\overline{x}_t)^2$ and $\gamma_{2^f}(\overline{x}_{2t-1})^2$ do not contain either $\gamma_i(\overline{x}_t)\gamma_j(\overline{x}_{2t-1})$, $x_t\gamma_i(\overline{x}_t)\gamma_j(\overline{x}_{2t-1})$, $x_{2t-1}\gamma_i(\overline{x}_t)\gamma_j(\overline{x}_{2t-1})$ or $x_tx_{2t-1}\gamma_i(\overline{x}_t)\gamma_j(\overline{x}_{2t-1})$, where $2^f > i + j$.

Suppose $\gamma_{2^f}(\overline{x}_t)^2$ contains $\gamma_i(\overline{x}_t)\gamma_j(\overline{x}_{2t-1})$. Then we have $2^{f+1}(t-1) = i(t-1)+j(2t-2)$ by an argument on total degrees. This contradicts $2^f > i+j$. Suppose $\gamma_{2^f}(\overline{x}_t)^2$ contains $x_tx_{2t-1}\gamma_i(\overline{x}_t)\gamma_j(\overline{x}_{2t-1})$. Then we have

$$2^{f+1}(t-1) = t + 2t - 1 + i(t-1) + j(2t-2)$$

by an argument on total degrees. Though $t-1$ divides the left-hand side of the equation, it does not divide the right-hand side, since $t > 3$. Thus we deduce that $\gamma_{2^f}(\overline{x}_t)^2$ does not contain $x_tx_{2t-1}\gamma_i(\overline{x}_t)\gamma_j(\overline{x}_{2t-1})$. By similar arguments, we can eliminate the other possibilities. Therefore we have $\gamma_{2^f}(\overline{x}_t)^2 = 0$ in $H^*(LX;\mathbb{Z}/2)$ if $\mathrm{Sq}^{t-1} = 0$. The usual argument on total degrees and filtration degrees allows us to conclude that $\gamma_{2^f}(\overline{x}_{2t-1})^2 = 0$ in $H^*(LX;\mathbb{Z}/2)$.

To prove Theorem 2.5(ii), we consider the case when $t = 2$. Though the action of $\mathrm{Sq}^1$ on $H^*(X;\mathbb{Z}/2)$ is not trivial, the vector space $\mathrm{Im}\,(\mathrm{Sq}^1)^{2^{k+1}m_i+2} = 0$ for any $k \geq 0$ and $1 \leq i \leq 2$ because $\mathrm{Im}\,(\mathrm{Sq}^1)^{\mathrm{even}} = 0$. Therefore the Eilenberg–Moore spectral sequence collapses at the $E_2$-term.

Furthermore, we can see $\gamma_{2^f}(\overline{x}_3) = \gamma_{2^f}(\overline{x}_2)^2 + P$ for any $f \geq 0$, where $P$ is a polynomial generated by the elements $x_2, x_3, \gamma_{2^{f-1}}(\overline{x}_2), \gamma_{2^{f-2}}(\overline{x}_2), \ldots$ $\ldots, \gamma_2(\overline{x}_2)$ and $\overline{x}_2$. From the usual argument on total degrees and filtration degrees it follows that $\gamma_{2^f}(\overline{x}_3)^2 = 0$ in $H^*(LX;\mathbb{Z}/2)$. Thus we can construct an isomorphism $\eta$ of algebras from

$$H^*(LX;\mathbb{Z}/2) \cong \Lambda(x_2, x_3) \otimes \bigotimes_{i\geq 0} \mathbb{Z}/2[\gamma_{2^i}(\overline{x}_2)]/(\gamma_{2^i}(\overline{x}_2)^4)$$

to $H^*(LX;\mathbb{Z}/2)$ with $\eta(\gamma_{2^f}(\overline{x}_2)) = \gamma_{2^f}(\overline{x}_2)$ and $\eta^{-1}(\gamma_{2^f}(\overline{x}_3)) = \gamma_{2^f}(\overline{x}_2)^2 + P$. The same argument works for $t > 3$. ∎

R e m a r k 2.9. In the case of $t = 2$ or $t = 3$, there are some extension problems which cannot be solved by a mere argument with the Steenrod operation on the Eilenberg–Moore spectral sequence and degree considerations as in the proof of Theorem 2.5. For example, there is the problem of whether $\gamma_2(\overline{x}_2)^2 = \overline{x}_2x_3$ in the case $t = 2$ or $\gamma_{2^2}(\overline{x}_3)^2 = x_3x_5\gamma_2(\overline{x}_5)$ in the case $t = 3$.

**3. The Hodge decomposition of the rational cohomology of $LX$.** Let $X$ be a simply connected space and $\varphi_n$ the power map $\varphi_n : LX \to LX$ defined by $\varphi_n(\gamma)(e^{i\theta}) = \gamma(e^{in\theta})$. Then we put $H^*(LX;\mathbb{Q}) = \bigoplus_{i\ge 0} HH_*^{(i)}$, where $HH_*^{(i)}$ is the eigenspace of the eigenvalue $n^i$ of the power operation $\varphi_n^*$ (see [3]). Here $HH_*^{(i)}$ is called the *i-factor* of the Hodge decomposition of the rational cohomology of $LX$. In general, for the minimal model $\mathcal{M} = (\wedge V, \widetilde{\partial})$ of $X$ there is a minimal model $\varepsilon(\mathcal{M}) = (\wedge V \otimes \wedge \overline{V}, \delta)$, where $H_*(\varepsilon(\mathcal{M})) \cong H^*(LX;\mathbb{Q})$ (see Remark 1.2). Here $\overline{V}^i = V^{i+1}$. Then we can decompose $\wedge V \otimes \wedge \overline{V}$ as $\bigoplus_i (\wedge V \otimes \wedge^i \overline{V})$. Since $\delta(\wedge V \otimes \wedge^i \overline{V}) \subset \wedge V \otimes \wedge^i \overline{V}$, we can put $H_*(\wedge V \otimes \wedge \overline{V}, \delta) = \bigoplus_i H_*(\wedge V \otimes \wedge^i \overline{V}, \delta)$ (cf. [4]). Then it is known that $HH_*^{(i)} \cong H_*(\wedge V \otimes \wedge^i \overline{V}, \delta)$ according to [3]. We will take advantage of this identification throughout the remainder of this section.

In this section, we consider only the case in which $H^*(X;\mathbb{Q})$ is a GCI-algebra, so that $\mathcal{M}$ is uniquely determined by $H^*(X;\mathbb{Q})$, since $H^*(X;\mathbb{Q})$ is then intrinsically formal ([9]). This $\mathcal{M}$ is isomorphic to the $\mathcal{M}$ of Remark 1.2 with $\Bbbk_0 = \mathbb{Q}$ and $l = 0$.

In the proofs of the following theorems, we use the notation of Proposition 1.1(i), in particular the correspondence of $\varepsilon(\mathcal{M})$ and $\mathcal{K}$, as made explicit in Remark 1.2.

THEOREM 3.1. *Let $H^*(X;\mathbb{Q})$ be a GCI-algebra*

$$\Xi = \mathbb{Q}[x_1, \dots, x_n]/(\varrho_1, \dots, \varrho_m)$$

*where $\varrho_i$ is decomposable and let $\overline{HH}_*^{(i)}$ denote the vector space*

$$HH_*^{(i)}/HH_*^{(0)} \cdot HH_*^{(i)}.$$

*Then*

(i) *For $m \le n$,*

$$\dim_{\mathbb{Q}} \overline{HH}_*^{(i)} \ge \begin{cases} \binom{m-n+i-1}{i-n} & \text{when } i > n, \\ \binom{n}{i} & \text{when } i \le n. \end{cases}$$

*In particular*, $\dim_{\mathbb{Q}} HH_*^{(i)} \ne 0$ *for any $i \ge 0$.*

(ii) *If $m = n$ then*

$$\dim_{\mathbb{Q}} HH_*^{(i)} \ge \begin{cases} \binom{i-1}{i-n} + \binom{n+i-1}{i} & \text{when } i > n, \\ \binom{n}{i} + \binom{n+i-1}{i} & \text{when } 1 \le i \le n. \end{cases}$$

THEOREM 3.2. *Suppose $m = n$. Let $[\Xi]$ be the fundamental class of the algebra $\Xi$* (see [16]). *If $\varrho_t$ is the element of the greatest degree in the regular sequence $\varrho_1, \dots, \varrho_n$, then for all $i$, $HH_j^{(i)} = 0$ whenever $j > \deg[\Xi] + i(\deg \varrho_t - 2)$. Moreover*, $\dim_{\mathbb{Q}} HH^{(i)}_{\deg[\Xi]+i(\deg \varrho_t - 2)} = 1$.

Proof of Theorem 3.1. (i) Since $\overline{x}_1 \ldots \overline{x}_n$ belongs to $\mathrm{Ann}_A(d(\omega_{i_1}), \ldots, d(\omega_{i_s}))$ for any $i_1, \ldots, i_s$, it follows that the elements $\overline{x}_1 \ldots \overline{x}_n \omega_1^{k_1} \ldots \omega_m^{k_m}$ $(k_1 \geq 0, \ldots, k_m \geq 0)$ represent elements of $\overline{HH}_*^{(i)}$, where $k_1 + \ldots + k_m + n = i$ from Proposition 1.1(i). Moreover, $(d\mathbb{Q}[\omega_1, \ldots, \omega_m])_A$ does not contain any linear combination of elements $\overline{x}_1 \ldots \overline{x}_n \omega_1^{k_1} \ldots \omega_m^{k_m}$. Therefore Proposition 1.1(i) also enables us to conclude that the elements $\overline{x}_1 \ldots \overline{x}_n \omega_1^{k_1} \ldots \omega_m^{k_m}$ $(k_1 \geq 0, \ldots, k_m \geq 0)$ are linearly independent in $\overline{HH}_*^{(i)}$. Thus $\dim_{\mathbb{Q}} \overline{HH}_*^{(i)} \geq \binom{m-1+i-n}{i-n}$ when $i > n$. Furthermore, since the elements $\overline{x}_{j_1} \cdot \overline{x}_{j_2} \ldots \overline{x}_{j_i}$ $(1 \leq j_1 < \ldots < j_i \leq n)$ are linearly independent in $\overline{HH}_*$, it follows that $\dim_{\mathbb{Q}} \overline{HH}_*^{(i)} \geq \binom{n}{i}$ when $i \leq n$.

(ii) Let $[\Xi]$ be the fundamental class of the GCI-algebra $\Xi$. Since $[\Xi]$ annihilates the augmentation ideal $\overline{\Xi}$, it follows from Proposition 1.1(i) that the elements $[\Xi]\omega_1^{k_1} \ldots \omega_n^{k_n}$ represent non-zero elements in $HH_*^{(k_1+\ldots+k_n)}$ from Proposition 1.1(i). Moreover, we see that the elements $\overline{x}_1 \ldots \overline{x}_n \omega_1^{k_1} \ldots \ldots \omega_n^{k_n}$ $(k_1 + \ldots + k_n + n = i)$ and $[\Xi]\omega_1^{l_1} \ldots \omega_n^{l_n}$ $(l_1 + \ldots + l_n = i)$ are linearly independent in $HH_*^{(i)}$ when $i > n$. In the case $i \leq n$, we can deduce that the elements $\overline{x}_{j_1} \ldots \overline{x}_{j_i}$ $(1 \leq j_1 < \ldots < j_i \leq n)$ and $[\Xi]\omega_1^{l_1} \ldots \omega_n^{l_n}$ $(l_1 + \ldots + l_n = i)$ are linearly independent in $HH_*^{(i)}$. ■

Proof of Theorem 3.2. By the same argument as in the proof of Theorem 3.1, we see that $[\Xi]\omega_t^i$ represents a non-zero element of $HH_*^{(i)}$. Any element $u$ of $\mathrm{Ann}_A(d(\omega_{i_1}), \ldots, d(\omega_{i_s})) \cdot \omega_1^{k_1} \ldots \omega_m^{k_m}$ can be written as $u = (\sum_l a_l b_l) \cdot \omega_1^{k_1} \ldots \omega_m^{k_m}$ with monomials $a_l \in \Xi$ and $b_l \in \Lambda(\overline{x}_1, \ldots, \overline{x}_n)$. Since the algebra $\Xi$ is a finite-dimensional vector space, it follows that $\deg x_i < \deg \varrho_t$ for any $i$. Therefore $\deg \overline{x}_i \leq \deg \varrho_t - 2 = \deg \omega_t$. So $\deg(b_l \omega_1^{k_1} \ldots \omega_n^{k_n}) \leq \deg \omega_t^i$ when $b_l = \overline{x}_{j_1} \ldots \overline{x}_{j_s}$ and $k_1 + \ldots + k_n + s = i$. The fact that $\Xi^k = 0$ for any $k > \deg[\Xi]$ enables us to conclude that $HH_j^{(i)} = 0$ whenever $j > \deg([\Xi]\omega_t^i)$. Moreover, since $\Xi^{\deg[\Xi]}$ is a 1-dimensional vector space generated by $[\Xi]$ (cf. [16]), it follows that $HH^{(i)}_{\deg[\Xi]+i(\deg \varrho_t - 2)}$ is generated by the element $[\Xi]\omega_t^i$. ■

Example 3.3. The minimal model $\mathcal{M}$ of the Sullivan–de Rham complex $(\Omega^*(X), \partial)$ for $X = U(2+2)/U(2) \times U(2)$ is

$$\mathcal{M} = (\mathbb{Q}[c_1, c_2] \otimes \Lambda(\tau_1, \tau_2), \widetilde{\partial}),$$

where $\deg c_i = 2i$, $\deg \tau_j = 2 \cdot 2 + 2j - 1$, $\widetilde{\partial}(c_i) = 0$, $\widetilde{\partial}(\tau_1) = \varrho_1 = 2c_1c_2 - c_1^3$ and $\widetilde{\partial}(\tau_2) = \varrho_2 = c_2^2 - 3c_1^2c_2 + c_1^4$ (see [11, Lemma 2.3]). Since

$$d(\overline{\tau}_1) = \frac{\partial \varrho_1}{\partial c_1}\overline{c}_1 + \frac{\partial \varrho_1}{\partial} c_2\overline{c}_2 = (2c_2 - 3c_1^2)\overline{c}_1 + 2c_1\overline{c}_2$$

in $\mathcal{K}$, it follows that the element $v = c_1^2\overline{c}_1 - c_1\overline{c}_2$ belongs to $\mathrm{Ann}(d\overline{\tau}_1)$. We can see that $v\overline{\tau}_1^{i-1} (i \geq 1)$ is a non-zero element of $HH_*^{(i)}$ for degree reasons. Indeed, suppose that $v\overline{\tau}_1^{i-1}$ is zero in $HH_*^{(i)}$. Then we can write $v\overline{\tau}_1^{i-1} = d(\sum_{j=0}^{i} a_j \overline{\tau}_1^{i-j} \overline{\tau}_2^{j})$ for some $a_j \in \mathbb{Q}[c_1, c_2]/(\varrho_1, \varrho_2) \otimes \wedge(\overline{c}_1, \overline{c}_2)$ in general. Since $\deg v = \deg d(\overline{\tau_1}) < \deg d(\overline{\tau}_2)$, it follows that $v\overline{\tau}_1^{i-1} = d(a_0 \overline{\tau}_1^{i})$ and $\deg(a_0) = 0$. Therefore we have $c_1^2 \overline{c}_1 - c_1 \overline{c}_2 = (2a_0 c_2 - 3a_0 c_1^2)\overline{c}_1 + 2a_0 c_1 \overline{c}_2$ in $\mathbb{Q}[c_1, c_2]/(\varrho_1, \varrho_2) \otimes \wedge(\overline{c}_1, \overline{c}_2)$, which is a contradiction. Thus we can conclude that $v\overline{\tau}_1^{i-1} \neq 0$ in $HH_*^{(i)}$.

In particular, $v\overline{\tau}_1^{i-1}$ $(i \geq 1)$ is different from a linear combination of the elements which we have chosen in the proof of Theorem 3.1(ii). Thus we have from Theorem 3.1(ii),

$$\dim_{\mathbb{Q}} HH_*^{(i)} > \begin{cases} 2i & \text{for } i > 2, \\ 4 & \text{for } i = 2, \\ 4 & \text{for } i = 1. \end{cases}$$

Since the degree of the fundamental class of $H^*(X; \mathbb{Q})$ is 8, from Theorem 3.2, we have $\dim_{\mathbb{Q}} HH_{8+6i}^{(i)} = 1$ and $HH_j^{(i)} = 0$ for $j > 8 + 6i$.

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