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Property C'', strong measure zero sets and subsets of the plane

by

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Abstract. Let X be a set of reals. We show that

- X has property C'' of Rothberger iff for all closed $F \subseteq \mathbb{R} \times \mathbb{R}$ with vertical sections F_x ($x \in X$) null, $\bigcup_{x\in X} F_x$ is null;
- X has strong measure zero iff for all closed $F \subseteq \mathbb{R} \times \mathbb{R}$ with all vertical sections F_x ($x \in \mathbb{R}$) null, $\bigcup_{x\in X} F_x$ is null.

1. Introduction. Let Y be a separable metric space. Galvin [G] defined the following game $G^*(Y)$. In the nth round, $n \in \omega$, White chooses an open cover $\{U^n_k : k \in \omega\}$ of Y, then Black responds with $a_n \in \omega$. Black wins if every $y \in Y$ is in some $U^n_{a_n}$. Let $G^{*\sigma}(Y)$ be Galvin's game with "some" changed to "infinitely many".

Recław [R] showed that White has a winning strategy in $G^*(Y)$ iff for some closed set $D \subseteq Y \times \omega^\omega$ with meager vertical sections D_y ($y \in Y$), $\bigcup_{y\in Y} D_y = \omega^\omega$. His result easily extends to $G^{*\sigma}(Y)$ and $\mathbf{F}_\sigma$ sets. In [P], I showed that the following are equivalent:

- White has no winning strategy in $G^*(Y)$;
- White has no winning strategy in $G^{*\sigma}(Y)$;
- Y has property C'' of Rothberger.

Thus, $Y \in C''$ iff $\bigcup_{y\in Y} D_y \neq \omega^\omega$ for all closed (equivalently, all $\mathbf{F}_\sigma$) $D \subseteq Y \times \omega^\omega$ with meager sections D_y, $y \in Y$. It is not hard to see that ω^ω can be replaced here by any Polish space without isolated points (Lemma 4.2). We show that if T carries a nonzero nonatomic σ-finite Borel measure, then $Y \in C''$ iff $\bigcup_{y\in Y} F_y$ is null for all closed $F \subseteq Y \times T$ with null F_y, $y \in Y$. We also give a similar characterization of property C (i.e., strong measure zero sets).

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Throughout the paper T is the Cantor set 2^ω (see Note (1) in Section 6 for how to pass to arbitrary T); $\mathcal{M}$ and $\mathcal{N}$ are the ideals of meager and null (i.e. outer measure zero) subsets of T; $\mathcal{E}$ is the ideal of subsets of T coverable by null $\mathbf{F}_\sigma$ sets. For $X \subseteq Y$ and $F \subseteq Y \times T$, let $F[X] = \bigcup_{x\in X} F_x$. Let $F \in \mathcal{M}^Y$ mean that $F \subseteq Y \times T$ and for all $y \in Y$, $F_y \in \mathcal{M}$. We similarly use $\mathcal{N}^Y$ and $\mathcal{E}^Y$.

Let $X \subseteq Y$, with Y a separable metric space. We say that X has property

- G^σ_Y if $F[X] \neq T$ for all $\mathbf{F}_\sigma$ $F \in \mathcal{M}^Y$;
- G_Y if $F[X] \neq T$ for all closed $F \in \mathcal{M}^Y$;
- E_Y if $F[X] \in \mathcal{N}$ for all $\mathbf{F}_\sigma$ (equivalently, closed) $F \in \mathcal{N}^Y$;
- C''_Y (resp. H_Y, M_Y) if, given open covers $\{U^n_k : k \in \omega\}$ of Y, $n \in \omega$, there are $a_n \in \omega$ such that each $x \in X$ is in some $U^n_{a_n}$ (resp. in some $\bigcup_{k<a_n} U^n_k$, in all but finitely many $\bigcup_{k<a_n} U^n_k$).
- C_Y if, given $\varepsilon_n > 0$, $n \in \omega$, there are balls B_n of radius $< \varepsilon_n$ with $X \subseteq \bigcup_n B_n$.

Instead of $Y \in C''_Y$ we usually write $Y \in C''$. Similarly, $Y \in G$ means $Y \in G_Y$, etc.

We refer to [M] and [FM] for more information about C, C'', M and H. Here just note that for $X \subseteq Y$, $X \in C$ iff $X \in C_Y$, and that "some" in the definition of C''_Y and M_Y can be changed to "infinitely many" (split ω into infinitely many infinite sets).

We prove:

1.1. Theorem. (1) *$C'' = G^\sigma = G = E$.*

(2) *$C''_Y = G^\sigma_Y = G_Y = E_Y$ for finite-dimensional $Y \in M$.*

(3) *$C_Y = G^\sigma_Y = G_Y = E_Y$ for finite-dimensional $Y \in H$.*

Part (3) follows from part (2) since $H \subseteq M$ and $Y \in H \Rightarrow C''_Y = C_Y$, the last being an easy generalization of a result of Fremlin and Miller [FM, Thm. 8] that $H \cap C = H \cap C''$. For σ-compact Y, $C_Y = G^\sigma_Y = G_Y$ is an unpublished result of Galvin (see [AR]).

I do not know whether finite-dimensionality is essential. Dropping it I have to replace the last "=" in (2) and (3) by "$\subseteq$".

Question. Suppose $X \subseteq [0,1]^\omega$ and for all closed $F \subseteq [0,1]^\omega \times [0,1]$ with all vertical sections null, $F[X]$ is null. Does X have strong measure zero?

Theorem 1.1 is a special case of the following theorem, which specifies the role of M.

1.2. Theorem. (1) *$G^\sigma_Y \subseteq G_Y \subseteq C''_Y \subseteq E_Y$.*

(2) *$X \in C''_Z$ & $Z \in M_Y \Rightarrow X \in G^\sigma_Y$.*

(3) *$(X \in E_Z,\ Z$ finite-dimensional & $Z \in M_Y) \Rightarrow X \in G^\sigma_Y$.*

Consider two more properties. Let $X \subseteq Y$, with Y a separable metric space. We say that X has property

- G^+_Y if for all closed $F \in \mathcal{M}^Y$, $F[X] \in \mathcal{M}$;
- E^+_Y if for all closed $F \in \mathcal{N}^Y$, $F[X] \in \mathcal{E}$.

We prove:

1.3. Theorem. $H \cap C'' = G^+ = E^+$.

It is known ([GMS], [P1]) that

$$X \in C_T \Leftrightarrow D + X \neq T \text{ for all } D \in \mathcal{M} \Leftrightarrow D + X \in \mathcal{N} \text{ for all closed } D \in \mathcal{N}.$$

From this and Theorem 1.3, if $Y \subseteq T$ has property H then

$$Y \in C \Leftrightarrow D + Y \in \mathcal{M} \text{ for all } D \in \mathcal{M} \Leftrightarrow D + Y \in \mathcal{E} \text{ for all } D \in \mathcal{E}.$$

(Use $H \cap C = H \cap C''$ and $D + Y = F[Y]$ for $F = \bigcup_{x \in T}\{x\} \times (D + x)$.)

Theorem 1.3 is a special case of the following theorem, which specifies the role of H.

1.4. Theorem. (1) $G^+_Y \subseteq H_Y$.
(2) $X \in H_Z$ & $Z \in G_Y \Rightarrow X \in G^+_Y$.
(3) $E^+_Y \subseteq H_Y$ *for finite-dimensional* Y.
(4) $X \in H_Z$ & $Z \in E_Y \Rightarrow X \in E^+_Y$.

We turn to a Borel version of our theorems. Define $\widetilde{C}''_Y$ and $\widetilde{H}_Y$ like C''_Y and H_Y but replacing open sets by Borel ones. Clearly, for $X \subseteq Y$, we have $X \in \widetilde{C}''_Y$ iff $X \in \widetilde{C}''$, and $X \in \widetilde{H}_Y$ iff $X \in \widetilde{H}$. Also, it is not hard to see that $X \in \widetilde{C}''$ iff all Borel images of X into T have property C''; likewise for $\widetilde{H}$.

1.5. Theorem. *Let $X \subseteq Y$. Then*

$$X \in \widetilde{C}'' \Leftrightarrow \forall \text{ Borel } F \in \mathcal{M}^Y \; F[X] \neq T,$$
$$X \in \widetilde{H} \cap \widetilde{C}'' \Leftrightarrow \forall \text{ Borel } F \in \mathcal{M}^Y \; F[X] \in \mathcal{M}.$$

If Y is Polish, then also

$$X \in \widetilde{C}'' \Leftrightarrow \forall \text{ Borel } F \in \mathcal{E}^Y \; F[X] \in \mathcal{N},$$
$$X \in \widetilde{H} \cap \widetilde{C}'' \Leftrightarrow \forall \text{ Borel } F \in \mathcal{E}^Y \; F[X] \in \mathcal{E}.$$

We will see that the sets F above can be required to have closed vertical sections. I do not know, however, if the assumption that Y is Polish in the second part is essential.

Question. Let $Y \subseteq [0,1]$. Suppose $F \subseteq Y \times [0,1]$ is (relatively) Borel and has all sections F_y $(y \in Y)$ closed null. Can we cover F with a Borel subset of $[0,1] \times [0,1]$ whose vertical sections are all coverable by null $\mathbf{F}_\sigma$ sets?

It follows from Theorem 1.5 that

$$\forall \text{Borel } F \in \mathcal{M}^Y \ F[X] \in \mathcal{M}$$
$$\Leftrightarrow X \in \widetilde{H} \text{ and } \forall D \in \mathcal{M} \ \forall \text{Borel } f : X \to T \ D + f[X] \neq T.$$

Similarly, if Y is Polish, then

$$\forall \text{Borel } F \in \mathcal{E}^Y \ F[X] \in \mathcal{E}$$
$$\Leftrightarrow X \in \widetilde{H} \text{ and } \forall D \in \mathcal{E} \ \forall \text{Borel } f : X \to T \ D + f[X] \in \mathcal{N}.$$

(Use Fremlin–Miller's result to see that $H \cap C$ and $H \cap C''$ are equivalent for $f[X]$.)

The paper is organized as follows. After fixing some basic notation, we give a proof of $C'' = E$. Then we introduce a general framework which is suitable for open as well as for Borel properties.

2. Notation and folklore. Let K and L range over subsets of ω, and i, j, k, l, m, n over members of ω. Given sets B_k, $k \in K$, let

$$\bigvee_{k \in K} B_k = \{x : \exists^\infty k \in K \ x \in B_k\}, \qquad \bigwedge_{k \in K} B_k = \{x : \forall^\infty k \in K \ x \in B_k\},$$

where "$\exists^\infty$" stands for "there exist infinitely many" and "$\forall^\infty$" for "for all but finitely many". Let $1_K = K \times \{1\}$.

Given K, μ refers to the product measure in 2^K arising from assigning the weight $1/2$ to each point in $\{0,1\}$. Note that if $A \subseteq 2^K$ with K finite, then $\mu(A) = |A| \cdot 2^{-|K|}$.

Suppose $A \subseteq 2^K$, $L \subseteq K$, $\tau \in 2^L$. Let

$$A|L = \{\sigma|L : \sigma \in A\},$$
$$[A] = \bigcup_{\sigma \in A} [\sigma], \quad \text{where } [\sigma] = \{t \in 2^\omega : \sigma \subseteq t\},$$
$$A_\tau = \{\sigma \in 2^{K \setminus L} : \tau \cup \sigma \in A\}.$$

Let $[k,l) = \{i : k \le i < l\}$. For $a \in \omega^\omega$ let $a_n = a(n)$. Let $\omega^\omega\!\nearrow$ be the set of all increasing functions from ω to ω.

The following is folklore:

- $A \in \mathcal{M}$ iff for some $a \in \omega^\omega\!\nearrow$ and some $\sigma_n \in 2^{[a_n, a_{n+1})}$, $n \in \omega$,

  $$\bigvee_n [\sigma_n] \subseteq T \setminus A.$$

  For such a,

  $$\exists^\infty n \ [a_n, a_{n+1}) \cap K = \emptyset \Rightarrow A|(\omega \setminus K) \text{ is meager (in } 2^{\omega \setminus K}).$$

  Hence, given $b \in \omega^\omega$ there is a subsequence $\{a_{n_m}\}$ such that

  $$\exists^\infty m \ |a_{n_m} \cap K| \le b_n \Rightarrow A|(\omega \setminus K) \text{ is meager}.$$

- $A \in \mathcal{N}$ iff for all (equivalently, for some) $c \in \omega^\omega$ with $\sum_n 2^{-c_n} < \infty$ there exist $a \in \omega^\omega\!\nearrow$ and $B_n \subseteq 2^{a_n}$ of measure $\le 2^{-c_n}$, $n \in \omega$, such that
$$A \subseteq \bigvee_n [B_n].$$
For such a and c,
$$\sum_n 2^{|a_n \cap K| - c_n} < \infty \Rightarrow A|(\omega \setminus K) \text{ is null (in } 2^{\omega\setminus K}).$$

As a consequence of these facts we have:

2.1. Lemma. (1) *Given $b \in \omega^\omega$ and $A \in \mathcal{M}$, there is $a \in \omega^\omega\!\nearrow$ such that*
$$\exists^\infty n\ |a_n \cap K| \le b_n \Rightarrow A|(\omega \setminus K) \text{ is meager.}$$
(2) *Given $b \in \omega^\omega$ and $A \in \mathcal{N}$, there is $a \in \omega^\omega\!\nearrow$ such that*
$$\forall^\infty n\ |a_n \cap K| \le b_n \Rightarrow A|(\omega \setminus K) \text{ is null.}$$

The next lemma is extracted from [BS].

2.2. Lemma. *Let $A \in \mathcal{N}$. Suppose $a \in \omega^\omega\!\nearrow$ and $\prod_n \varepsilon_n > 0$, ε_n's positive. Then for each $\tau \in 2^{a_n}$ there is $\Sigma^\tau \subseteq 2^{[a_n, a_{n+1})}$ of measure $< \varepsilon_n$ such that for all closed $D \subseteq A$,*
$$D_\tau|[a_n, a_{n+1}) \subseteq \Sigma^\tau \quad \text{for some } n \text{ and } \tau \in D|a_n.$$

Proof. Fix a and ε_n's. Cover A with open $G \subseteq T$, $\mu(G) < \prod_n \varepsilon_n$. Define
$$\Sigma^\tau = \{\sigma \in 2^{[a_n, a_{n+1})} : \varepsilon_n \mu(G_{\tau^\frown\sigma}) > \mu(G_\tau)\}, \quad \tau \in 2^{a_n}.$$
Then $\mu(\Sigma^\tau) < \varepsilon_n$ (apply the Fubini theorem to $G_\tau \subseteq 2^{[a_n, a_{n+1})} \times 2^{[a_{n+1}, \omega)}$). Also, if $D \subseteq G$ is closed, then for some n and $\tau \in D|a_n$,
$$\tau^\frown\sigma \in D|a_{n+1} \Rightarrow \varepsilon_n \mu(G_{\tau^\frown\sigma}) > \mu(G_\tau) \quad \text{for all } \sigma.$$
Indeed, if not, define inductively $t \in T$ with
$$t|a_n \in D|a_n \quad \text{and} \quad \varepsilon_n \mu(G_{t|a_{n+1}}) \le \mu(G_{t|a_n}).$$
Then $t \in D$ and $\varepsilon_0 \varepsilon_1 \ldots \varepsilon_n \mu(G_{t|a_{n+1}}) \le \mu(G)$. Since $t \in D \subseteq G$ we have $\forall^\infty n\ [t|a_{n+1}] \subseteq G$, so $\forall^\infty n\ \mu(G_{t|a_{n+1}}) = 1$, and we get $\prod_n \varepsilon_n \le \mu(G)$. A contradiction. ∎

3. A proof of $E = C''$. Before presenting a general framework for our theorems we sketch a proof of $E = C''$.

$C'' \subseteq E$ is easy. Indeed, suppose $Y \in C''$. Let $F \in \mathcal{N}^Y$ be closed with the complement $\bigcup_i U_i \times O_i$, U_i's open in Y and O_i's open in T. For finite K with $\mu(\bigcup_{i\in K} O_i) > 1 - 2^{-n}$ let $U_K^n = \bigcap_{i \in K} U_i$. Then $\forall n\ Y = \bigcup_K U_K^n$. Since $Y \in C''$, $Y = \bigvee_n U_{K_n}^n$ for some K_n's. Then
$$F[Y] \subseteq \bigvee_n \Big(U \setminus \bigcup_{i \in K_n} O_i\Big) \in \mathcal{N}.$$

The reverse inclusion is longer. Let $Y \in E$.

First note that Y is zero-dimensional. Indeed, let δ be the metric. For $y_0 \in Y$, $\Delta = \{\langle y, \delta(y, y_0)\rangle : y \in Y\}$ is a closed subset of $Y \times [0, \infty)$ with null vertical sections. As $Y \in E$, $\Delta[Y]$ is null, so it avoids arbitrarily small $\varepsilon > 0$. (Note (1) in Section 6 explains why $[0, \infty)$ can replace 2^ω in property E.)

Next we show that $Y \in M$. For each n, let $\{U^n_k : k \in \omega\}$ be an open cover of Y. By zero-dimensionality we can, without loss of generality, restrict ourselves to covers whose members are pairwise disjoint. Let $\# : \omega \times \omega \to \omega$ be 1-1. For $y \in Y$ let $K_y = \{k^n_y : n \in \omega\}$, where $k^n_y = \#(n, k)$ for k such that $y \in U^n_k$. Note that each K_y is infinite. Define a closed $F \in \mathcal{N}^Y$ by

$$F_y = [1_{K_y}], \quad y \in Y.$$

As $Y \in E$, we have $F[Y] \in \mathcal{N}$. Use Lemma 2.1 to find $a \in \omega^\omega\nearrow$ such that $F[Y]|(\omega \subseteq K)$ is null for all K with $\forall n\ |K \cap a_n| \le n$. Then $Y \subseteq \bigcup_{k<a_n} U^n_k$. Otherwise, for some y, $\forall n\ k^n_y \ge a_n$. So

$$K_y \cap a_n \subseteq \{k^m_y : m < n\}$$

has size $\le n$ and thus $F_y|(\omega \setminus K_y)$ is null, which is absurd.

We can now start the main argument. Let, for each n, $\{U^n_k : k \in \omega\}$ be an open cover of Y that consists of pairwise disjoint sets. Let $c(n) = n + n2^n$ and let $\# : \bigcup_n \omega^{[n,c(n))} \to \omega$ be 1-1 such that $\#(\sigma) \ge n$ for $\sigma \in \omega^{[n,c(n))}$. Let

$$V^n_k = \bigcap_{n \le i < c(n)} U^i_{\sigma(i)} \quad \text{if } \sigma \in \omega^{[n,c(n))} \text{ and } \#(\sigma) = k,$$

and let $V^n_k = \emptyset$ otherwise. (So, $k < n \Rightarrow V^n_k = \emptyset$.)

Clearly, for each n, the V^n_k's cover Y. As $Y \in M$, find $a \in \omega^\omega\nearrow$ such that $a_{n+1} \ge c(a_n)$ and

$$Y = \bigvee_n \bigcup_{a_n \le k < a_{n+1}} V^{a_n}_k.$$

(This is possible: see Lemma 4.1.)

Let $A_n = [a_n, a_{n+1})$ and let

$$J_y = \bigcup_n \{k \in A_n : y \in V^{a_n}_k\}, \quad y \in Y.$$

Each J_y is infinite and has at most one point in each A_n.

Define a closed $F \in \mathcal{N}^Y$ by

$$F_y = [1_{J_y}], \quad y \in Y.$$

Let $A = F[Y]$. As $Y \in E$, we have $A \in \mathcal{N}$. Get Σ^τ's from Lemma 2.2 applied to A, a_n and $\varepsilon_n = 1 - 2^{-(n+1)}$.

For $\tau \in 2^{a_n}$ let

$$K^\tau = \{k \in A_n : [1_{\{k\}}]|A_n \subseteq \Sigma^\tau\} \quad \text{and} \quad K_n = \bigcup_{\tau \in 2^{a_n}} K^\tau.$$

By independence

$$2^{-|K^\tau|} \ge 1 - \mu(\Sigma^\tau) > 1 - \varepsilon_n = 2^{-(n+1)}.$$

So $|K^\tau| \le n$ and $|K_n| \le n2^{a_n}$.

CLAIM. $Y \subseteq \bigcup_{n,k\in K_n} V_k^{a_n}$.

Proof. Each $[1_{J_y}]$ is a closed subset of A. It follows from Lemma 2.2 that

$$[1_{J_y}]|A_n \subseteq \Sigma^\tau \quad \text{for some } n \text{ and } \tau \in 2^{a_n}.$$

As $\mu(\Sigma^\tau) < 1$, the left hand side cannot be 2^{A_n}, so it must be of the form $[1_{\{k\}}]|A_n$ for a unique $k \in J_y \cap A_n$. Then $k \in K^\tau \subseteq K_n$ and $y \in V_k^{a_n}$. ∎

Pick now $\tau_n \in \omega^{[a_n,c(a_n))}$ that meets each σ with $\#(\sigma)$ in K_n ($|K_n| \le n2^{a_n}$ and $|[a_n, c(a_n))| = a_n 2_n^a$). Then

$$\bigcup_{k\in K_n} V_k^{a_n} \subseteq \bigcup_{a_n \le i < c(a_n)} U^i_{\tau_n(i)},$$

which in view of the claim ends the proof.

4. Lemmas. Fix a set S and a family $\mathcal{S}$ of subsets of S which is closed under finite intersections and unions and contains $\emptyset$ and S. The intended interpretation is that S is a separable metric space and $\mathcal{S}$ is either $\mathbf{O}(S)$, the family of all open subsets of S, or $\mathbf{B}(S)$, the family of all Borel subsets of S.

Let (indices allowed)

- U and V range over $\mathcal{S}$;
- O range over open subsets of T;
- X, Y and Z range over subsets of S.

We say that Y is *$\le n$-dimensional* if for any U_k, $k \in K$, there are $V_k \subseteq U_k$ such that $Y \cap \bigcup_k V_k = Y \cap \bigcup_k U_k$ and each $y \in Y$ is in at most $n+1$ V_k's. Clearly, if $\mathcal{S} = \mathbf{B}(S)$ then any Y is zero-dimensional, and if $\mathcal{S} = \mathbf{O}(S)$ then Y is $\le n$-dimensional iff it is $\le n$-dimensional in the usual sense of dimension theory.

If $F \subseteq S \times T$ then we say that

- $F \in \mathcal{M}^Y$ iff $\forall y \in Y\ F_y \in \mathcal{M}$; similarly for $\mathcal{N}$ and $\mathcal{E}$;
- F is an *$\mathcal{F}$ set* defined by $\{U_i \times O_i : i \in \omega\}$ iff $F = S \times T \setminus \bigcup_i U_i \times O_i$;
- F is an *$\mathcal{F}_\sigma$ set* if it is a countable union of $\mathcal{F}$ sets.

If $\mathcal{S} = \mathbf{O}(S)$, then $\mathcal{F}$ and $\mathcal{F}_\sigma$ subsets of $S \times T$ are just closed and $\mathbf{F}_\sigma$ subsets of $S\times T$. If S is a Polish space and $\mathcal{S} = \mathbf{B}(S)$ then $\mathcal{F}$ and $\mathcal{F}_\sigma$ sets are just Borel sets with closed and $\mathbf{F}_\sigma$ vertical sections (by theorems of Kunugui, Novikov and Saint-Raymond; see [Ke]). Also, every Borel set from $\mathcal{M}^Y$ can be covered by an $\mathcal{F}_\sigma$ set from $\mathcal{M}^Y$; similarly for $\mathcal{E}^Y$ if Y is itself Borel (I do not know whether this requirement is essential). (See e.g. [D], p. 290, Remarque (a). Remember that for Borel $F \subseteq S \times T$, $\{s \in S : F_s \in \mathcal{M}\}$ is Borel, but $\{s \in S : F_s \in \mathcal{E}\}$ may be true coanalytic.)

Definition. Let $X \subseteq Y$. We say that X has property

- C''_Y (resp. H_Y, M_Y) if, whenever $Y \subseteq \bigcap_n \bigcup_k U^n_k$, then $X \subseteq \bigcup_n U^n_{a_n}$ for some $a \in \omega^\omega$ (resp. $X \subseteq \bigwedge_n \bigcup_{k<a_n} U^n_k$, $X \subseteq \bigcup_n \bigcup_{k<a_n} U^n_k$);
- G_Y if $F[X] \neq T$ for all $F \in \mathcal{F} \cap \mathcal{M}^Y$;
- G^σ_Y if $F[X] \neq T$ for all $F \in \mathcal{F}_\sigma \cap \mathcal{M}^Y$;
- E_Y if $F[X] \in \mathcal{N}$ for all $F \in \mathcal{F} \cap \mathcal{N}^Y$;
- S_Y if, whenever $Y \subseteq \bigvee_k U_k$ and $a \in \omega^\omega\!\nearrow$, then $X \subseteq \bigvee_{k\in K} U_k$ for some K such that $\forall^\infty n\ |K \cap a_{n+1}| \le 2^{a_n}$.

Clearly, $C''_Y \cup H_Y \subseteq M_Y$. Also, in the definitions of M_Y and C''_Y we can replace "$\bigcup_n$" by "$\bigvee_n$" (split ω into infinitely many infinite sets).

As to S_Y note the following. For $a \in \omega^\omega\!\nearrow$ and $f \in (\omega \setminus 1)^\omega$ let

$$\Phi(a, f) = \{K : \forall^\infty n\ |K \cap a_{n+1}| \le f(n) \cdot 2^{a_n}\}.$$

Let $\Phi(a)$ be $\Phi(a, f)$ for $f \equiv 1$. Then $\forall f, a\ \Phi(a) \subseteq \Phi(a, f)$ and $\forall f, b\ \exists a\ \Phi(a, f) \subseteq \Phi(b)$. (For $b \in \omega^\omega\!\nearrow$, choose $c \in \omega^\omega\!\nearrow$ so that $a_n = b_{c_n} - f(n)$ (for $n \in \omega$) increase and $f(n+1)+f(n)\cdot 2^{a_n} \le 2^{b_{c_n}}$. If $\forall^\infty n\ |K\cap a_{n+1}| \le f(n)\cdot 2^{a_n}$, then $\forall^\infty n\ |K \cap b_{c_{n+1}})| \le f(n+1) + f(n) \cdot 2^{a_n} \le 2^{b_{c_n}}$.) So, $X \in S_Y$ iff for some f, whenever $Y \subseteq \bigvee_k U_k$ and $a \in \omega^\omega\!\nearrow$, then $\exists K \in \Phi(a, f)\ X \subseteq \bigvee_{k\in K} U_k$.

The following lemma is implicit in [FM].

4.1. Lemma. (1) *Suppose* $X \in M_Y$ *and* $Y \subseteq \bigcap_n \bigcup_k U^n_k$. *Then* $X \subseteq \bigvee_n \bigcup_{k<a_{n+1}} U^{a_n}_k$ *for some* $a \in \omega^\omega\!\nearrow$.

(2) *Suppose* $X \in C''_Y$ *and* $Y \subseteq \bigvee_n \bigcup_k U^n_k$. *Then* $X \subseteq \bigvee_n U^n_{a_n}$ *for some* $a \in \omega^\omega$.

Proof. (1) For increasing $\sigma \in \omega^{2n}$, $n > 0$, let

$$V_\sigma = \bigcap_{i<2n-1} \bigcap_{m\le\sigma(i)} \bigcup_{k<\sigma(i+1)} U^m_k.$$

Clearly, for each n, the V_σ's with $|\sigma| = 2n$ cover Y. As $X \in M_Y$, we can find finite $\Sigma_n \subseteq \omega^{2n}$ such that

$$X \subseteq \bigvee_n \bigcup_{\sigma\in\Sigma_n} V_\sigma.$$

We end the proof of (1) by taking $a \in \omega^{\omega}\nearrow$ such that $\Sigma_n \subseteq (a_n)^{2n}$. Indeed, given $x \in X$ and $N \in \omega$, choose $n > (a_N + n - N)/2$ such that $x \in V_\sigma$ for some $\sigma \in \Sigma_n$. Note that some interval $[a_m, a_{m+1})$ with $m \in [N, n)$ contains at least two $\sigma(i)$'s; otherwise $|\sigma| \le a_N + n - N < 2n$. Then $x \in \bigcup_{k<a_{m+1}} U_k^{a_m}$.

(2) For finite K and $\sigma \in \omega^K$, let $U_\sigma = \bigcap_{k\in K} U^n_{\sigma(k)}$. Then, for each n, the U_σ's with $|\sigma| = n$ cover Y. As $X \in C''_Y$ find σ_n of length n such that $X \subseteq \bigvee_n U_{\sigma_n}$. Inductively choose $m_n \in \mathrm{dom}(\sigma_n) \setminus \{m_0, \ldots, m_{n-1}\}$ and take $a \in \omega^\omega$ such that $a_{m_n} = \sigma_n(m_n)$. Then $X \subseteq \bigvee_m U^m_{a_m}$. ∎

Definition. Let $X \subseteq Y$. Using sets from $\mathcal{S}$ as Galvin used open sets define a game $G^*_Y(X)$ (resp. $G^{*\sigma}_Y(X)$) as follows. In the nth round White covers Y with $\{U^n_k : k \in \omega\}$, then Black picks $a_n \in \omega$. Black wins if $X \subseteq \bigcup_n U^n_{a_n}$ (resp. $X \subseteq \bigvee_n U^n_{a_n}$).

4.2. Lemma. *The following are equivalent*:

(a) *White has no winning strategy in* $G^{*\sigma}_Y(X)$.
(b) $X \in G^\sigma_Y$.

Similarly if σ is dropped.

Proof. (Cf. [R], Thm. 1.) (a)⇒(b). Suppose $F = \bigcup_i F_i$, $F_i \in \mathcal{F} \cap \mathcal{M}^Y$. We seek a point outside $F[X]$.

Find nonempty rectangles $U_\sigma \times O_\sigma$, $\sigma \in \omega^{<\omega}$, so that

(1) $U_\sigma \times O_\sigma$ is disjoint from $\bigcup_{i<|\sigma|} F_i$ and $\mathrm{diam}(O_\sigma) < 2^{-|\sigma|}$;
(2) $Y \subseteq \bigcup_n U_{\sigma^\frown n}$;
(3) $\overline{O}_{\sigma^\frown n} \subseteq O_\sigma$;

Then some $\bigcap_n O_{s|n}$, $s \in \omega^\omega$, is disjoint from $F[X]$. Indeed, let White play according to U_σ's. He begins with $\{U_{\langle n\rangle} : n \in \omega\}$, against Black's choice of n he plays $\{U_{\langle n,m\rangle} : m \in \omega\}$, etc. This is not a winning strategy, so $X \subseteq \bigvee_n U_{s|n}$ for some $s \in \omega^\omega$. Let $t \in \bigcap_n O_{s|n}$. Then for each n, $\bigcup_{m>n} U_{s|m} \times \{t\}$ is disjoint from F_n. Hence $\bigvee_n U_{s|n} \times \{t\}$ is disjoint from $\bigcup_n F_n$.

(b)⇒(a). Suppose White has a winning strategy. Thus there exist U_σ, $\sigma \in \omega^{<\omega}$, such that for all σ, $Y = \bigcup_n U_{\sigma^\frown n}$, and for no $s \in \omega^\omega$, $X \subseteq \bigvee_n U_{s|n}$. Choose nonempty O_σ, $\sigma \in \omega^{<\omega}$, so that $O_\emptyset = T$ and for all σ, the $O_{\sigma^\frown n}$'s are pairwise disjoint subsets of O_σ with diameters $< 2^{-|\sigma|}$ and union dense in O_σ.

Let F be the complement of $\bigcap_m G_m$, $G_m = \bigcup_{|\sigma|\ge m} U_\sigma \times O_\sigma$. Each G_m has dense vertical sections for $y \in Y$. (Fix y. Given O, find $O_\sigma \subseteq O$, $|\sigma| \ge m$, next find n with $y \in U_{\sigma^\frown n}$. Then $O_{\sigma^\frown n} \subseteq (G_m)_y \cap O$.) Thus $F \in \mathcal{F}_\sigma \cap \mathcal{M}^Y$. Also, $F[X] = T$. Indeed, if $t \in T \setminus F[X]$, then $x \in X$ yields

$$\forall m\ \exists \sigma\ |\sigma| \ge m\ \&\ x \in U_\sigma\ \&\ t \in O_\sigma,$$

so $X \subseteq \bigvee_n U_{s|n}$ for $s = \bigcup\{\sigma : t \in O_\sigma\}$. ∎

4.3. Proposition.

$$G_Y^\sigma \subseteq G_Y \subseteq C''_Y \subseteq E_Y \cap S_Y \cap M_Y.$$

If Y is finite-dimensional then also

$$E_Y \subseteq S_Y \cap M_Y.$$

Proof.

• $G_Y^\sigma \subseteq G_Y$. Clear.

• $G_Y \subseteq C''_Y$. Use Lemma 4.2. If $X \notin C''_Y$ then White wins by playing covers that witness this.

• $C''_Y \subseteq E_Y$. (Cf. [M1], Thm. 2.1.) Let $X \in C''_Y$ and let $F \in \mathcal{F} \cap \mathcal{N}^Y$ be defined by $\{U_i \times O_i : i \in \omega\}$. For finite K such that $\mu(\bigcup_{i\in K} O_i) > 1 - 2^{-n}$ let $U_K^n = \bigcap_{i\in K} U_i$. Then $\forall n\ Y \subseteq \bigcup_K U_K^n$. As $X \in C''_Y$, $X \subseteq \bigvee_n U_{K_n}^n$ for some K_n's. Then

$$F[X] \subseteq \bigvee_n \Big(T \setminus \bigcup_{i\in K_n} O_i\Big) \in \mathcal{N}.$$

• $C''_Y \subseteq S_Y$. Let $Y \subseteq \bigvee_k U_k$ and $a \in \omega^\omega\nearrow$. Define U_k^n to be U_k if $k \in [a_n, a_{n+1})$ and $\emptyset$ otherwise. Use Lemma 4.1(2).

Now assume that Y is $<N$-dimensional.

• $E_Y \subseteq M_Y$. (Cf. [M1], Thms. 1.2 and 2.2.) Let $Y \subseteq \bigcap_n \bigcup_k U_k^n$. Let $\# : \omega\times\omega \to \omega$ be 1-1. For $s \in S$ let $K_s = \bigcup_n K_s^n$, where $K_s^n = \{\#(n,k) : s \in U_k^n\}$. Without loss of generality, $\forall n\ \forall y \in Y\ |K_y^n| \le N$. Define $F \in \mathcal{F} \cap \mathcal{N}^Y$ by

$$\forall s \in S\ F_s = [1_{K_s}].$$

Suppose $X \in E_Y$. Then $F[X] \in \mathcal{N}$. By Lemma 2.1, find $a \in \omega^\omega\nearrow$ such that for all K with $\forall n\ |K \cap a_n| \le nN$, $F[X]|(\omega \setminus K)$ is null. Fix $x \in X$. It suffices to prove

Claim. $\exists n\ K_x^n \cap a_n \ne \emptyset$.

Proof. Otherwise

$$\forall n\ K_x \cap a_n \subseteq \bigcup_{m<n} K_x^m$$

$(m \ge n\ \&\ K_x^m \cap a_m = \emptyset \Rightarrow K_x^m \cap a_n = \emptyset)$. Since $|K_x^m| \le N$, $|K_x \cap a_n| \le nN$. It follows that $F_x|(\omega \setminus K_x)$ is null, which is absurd. ∎

• $E_Y \subseteq S_Y$. (Cf. [BS], Thm. 2.1.) Suppose $X \in E_Y$, $Y \subseteq \bigvee_k U_k$ and $a \in \omega^\omega\nearrow$. For $s \in S$ let $J_s = \{k : s \in U_k\}$. Let $A_n = [a_n, a_{n+1})$. Without loss of generality, $\forall y \in Y\ |A_n \cap J_y| \le N$. Define $F \in \mathcal{F}_\sigma \cap \mathcal{N}^Y$ by

$$\forall s \in S\ F_s = \bigcup_n [1_{J_s \setminus a_n}].$$

As $X \in E_Y$, $F[X] \in \mathcal{N}$. Let Σ^τ's be obtained by Lemma 2.2 applied to $F[X]$, a and $\varepsilon_n = 1 - (1-2^{-N})^{(n+1)/N}$. Let $\{K_i^\tau : i < l\}$ be a maximal family of pairwise disjoint subsets of A_n of size $\le N$ such that $\forall i\ [1_{K_i^\tau}]|A_n \subseteq \Sigma^\tau$. Let $K^\tau = \bigcup_{i<l} K_i^\tau$. By independence

$$(1-2^{-N})^l \ge 1 - \mu(\Sigma^\tau) > 1 - \varepsilon_n = (1-2^{-N})^{(n+1)/N},$$

so $l \le n/N$ and $|K^\tau| \le lN \le n$.

Let

$$K_n = \bigcup_{\tau \in 2^{a_n}} K^\tau.$$

Then $|K_n| \le n2^{a_n}$.

Let $K = \bigcup_n K_n$. Clearly, $K_n = K \cap A_n$.

CLAIM. $\forall x \in X\ \exists^\infty n\ K_n \cap J_x \ne \emptyset$.

Proof. Fix $x \in X$, $m \in \omega$. As $D = [1_{J_x \setminus a_m}]$ is a closed subset of $F[X]$, by Lemma 2.2 find n and $\tau \in D|a_n$ such that $D_\tau|A_n \subseteq \Sigma^\tau$. Clearly, $n \ge m$ (otherwise $D_\tau|A_n = 2^{A_n}$, but $\mu(\Sigma^\tau) < 1$). It follows that

$$D_\tau|A_n = [1_{J_x}]|A_n, \quad \text{so} \quad [1_{J_x}]|A_n \subseteq \Sigma^\tau.$$

By the choice of K_i^τ's, some K_i^τ meets J_x, so K_n meets J_x. ∎

Proposition 4.3 is proved. ∎

4.4. PROPOSITION. $X \in S_Z\ \&\ Z \in M_Y \Rightarrow X \in G_Y^\sigma$.

Proof. (Cf. [P], Lemma 2.) Suppose $Y \subseteq \bigcap_{\sigma \in \omega^{<\omega}} \bigcup_i U_{\sigma^\frown i}$. We want $s \in \omega^\omega$ with $X \subseteq \bigvee_n U_{s|n}$.

Let $\Sigma = \bigcup_{n>0} n^n$. For $\sigma \in \Sigma$ and n let

$$U_\sigma^n = \bigcap_{\tau \in n^{\le n}} \bigcup_{0<i\le|\sigma|} U_{\tau^\frown \sigma|i}.$$

Clearly, for each n, $Y \subseteq \bigcup_\sigma U_\sigma^n$. Also, for $\sigma, \tau \in \Sigma$,

$$n \le m \Rightarrow U_\sigma^{m+|\tau|} \subseteq U_{\tau^\frown\sigma}^n.$$

Let $c(n) = 2^n$ and let $\# : \bigcup_n \Sigma^{c(n)} \to \omega$ be one-to-one and such that

$$\#(\sigma_1, \ldots, \sigma_{c(n)}) \ge n + |\sigma_1| + \ldots + |\sigma_{c(n)}|.$$

Let

$$V_k^n = U_{\sigma_1}^n \cap U_{\sigma_2}^{n+|\sigma_1|} \cap \ldots \cap U_{\sigma_{c(n)}}^{n+|\sigma_1|+\ldots+|\sigma_{c(n)-1}|}$$

if $k = \#(\sigma_1, \ldots, \sigma_{c(n)})$ and $V_k^n = \emptyset$ otherwise. (So $k < n \Rightarrow V_k^n = \emptyset$.) Clearly, for each n, $Y \subseteq \bigcup_k V_k^n$. Since $Z \in M_Y$, by Lemma 4.1, find $a \in \omega^\omega\nearrow$ such that

$$Z \subseteq \bigvee_n \bigcup_{a_n \le k < a_{n+1}} V_k^{a_n}.$$

As $X \in S_Z$, there exist $K_n \subseteq [a_n, a_{n+1})$ of size $\le c(a_n)$ such that

$$X \subseteq \bigvee_n \bigcup_{k \in K_n} V_k^n.$$

Now diagonalize. Pick σ_i^j so that

$$\#(\sigma_1^i, \ldots, \sigma_{c(a_n)}^i), \qquad i = 1, \ldots, c(a_n),$$

and enumerate (possibly with repetitions) K_n in such a way that $j \ge i \Rightarrow |\sigma_i^j| \ge |\sigma_i^i|$.

Let

$$\tau_n = \sigma_1^1 {}^\frown \sigma_2^2 {}^\frown \ldots {}^\frown \sigma_{c(a_n)}^{c(a_n)}.$$

Then

$$\bigcup_{k \in K_n} V_k^n \subseteq U_{\tau_n}^{a_n}.$$

Note that $|\sigma_1^1| + \ldots + |\sigma_{i-1}^{i-1}| \le |\sigma_1^i| + \ldots + |\sigma_{i-1}^i|$ yields

$$U_{\sigma_i^i}^{a_n + |\sigma_1^i| + \ldots + |\sigma_{i-1}^i|} \subseteq U_{\tau_n}^{a_n}.$$

As also $\forall m\ a_m + |\tau_m| \le a_{m+1}$, it follows that

$$\bigcup_{k \in K_n} V_k^n \subseteq U_{\tau_n}^{a_n} \subseteq U_{\tau_n}^{|\tau_0| + \ldots + |\tau_{n-1}|}.$$

Finally, since $\tau_0{}^\frown \ldots {}^\frown \tau_{n-1} \in \Sigma$,

$$U_{\tau_n}^{|\tau_0| + \ldots + |\tau_{n-1}|} \subseteq \bigcup_{0 < i \le |\tau_n|} U_{\tau_0{}^\frown \ldots {}^\frown \tau_{n-1}{}^\frown \tau_n | i}.$$

Thus

$$\bigcup_{k \in K_n} V_k^n \subseteq \bigcup_{0 < i \le |\tau_n|} U_{\tau_0{}^\frown \ldots {}^\frown \tau_{n-1}{}^\frown \tau_n | i},$$

hence $X \subseteq \bigvee_n U_{s|n}$ for $s = \tau_0{}^\frown \tau_1{}^\frown \ldots$ ∎

4.5. Corollary. *$G^\sigma = G = C'' = S \cap M \subseteq E$. For finite-dimensional sets, C'' and E are equivalent.*

Proof. By Proposition 4.4, $S \cap M \subseteq G^\sigma$. By Proposition 4.3, $G^\sigma \subseteq G \subseteq C'' \subseteq E \cap S \cap M$, and for finite-dimensional sets, $E \subseteq S \cap M$. ∎

Definition. Let $X \subseteq Y$. We say that X has property

- E_Y^+ if $F[X] \in \mathcal{E}$ for all $F \in \mathcal{F} \cap \mathcal{E}^Y$;
- G_Y^+ if $F[X] \in \mathcal{M}$ for all $F \in \mathcal{F} \cap \mathcal{M}^Y$.

4.6. Proposition. (1) *$G_Y^+ \subseteq H_Y$.*

(2) *$E_Y^+ \subseteq H_Y$ for finite-dimensional Y.*

(3) *$X \in H_Z$ & $Z \in G_Y \Rightarrow X \in G_Y^+$.*

(4) *$X \in H_Z$ & $Z \in E_Y \Rightarrow X \in E_Y^+$.*

Proof. For (1) and (2) let $Y \subseteq \bigcap_n \bigcup_k U^n_k$. Define K_s and K^n_s as in Proposition 4.3 in the proof of $E_Y \subseteq M_Y$. Without loss of generality, $\forall s \in S\ \forall n\ \min(K^n_s) \le \min(K^{n+1}_s)$. (Let the $(n+1)$th cover refine the nth cover, and define $\#$ so that $\forall k\ \exists l\ \#(n+1,k) \ge \#(n,l)$ and $U^{n+1}_k \subseteq U^n_l$.)

For (1) define $F \in \mathcal{F} \cap \mathcal{M}^Y$ by

$$\forall s \in S\ (t \notin F_s \Leftrightarrow \exists n\ \exists k\ s \in U^n_k\ \&\ |\{l < k : t(l) = 1\}| < n).$$

Then

$$\forall y \in Y\ (t \in F_y \Leftrightarrow \forall n\ |\{l < \min(K^n_y) : t(l) = 1\}| \ge n).$$

Let $X \in G^+_Y$, so that $F[X] \in \mathcal{M}$. Let $a \in \omega^\omega\nearrow$ be obtained from Lemma 2.1 applied to $F[X]$ and $b_n = n$. Fix $x \in X$. It is enough to prove

CLAIM. $\forall^\infty n\ K^n_x \cap a_n \ne \emptyset$.

Proof. Suppose otherwise, i.e., $\exists^\infty n\ a_n \le \min(K^n_x)$. Define $K = \{\min(K^n_x) : n \in \omega\}$. Then $\exists^\infty n\ |K \cap a_n| \le n$, so $F_x|(\omega \setminus K)$ is meager. Note, however, that $[1_K] \subseteq F_x$. A contradiction. ■

For (2) note first that, without loss of generality, $\forall n, y\ |K^n_y| \le N$, for some N. Define $F \in \mathcal{F} \cap \mathcal{E}^Y$ by

$$\forall s \in S\ F_s = [1_{K_s}].$$

Let $X \in E^+_Y$, so that $F[X] \in \mathcal{E} \subseteq \mathcal{M}$. Let $a \in \omega^\omega\nearrow$ be obtained from Lemma 2.1 applied to $F[X]$ and $b_n = nN$. Fix $x \in X$. It suffices to prove

CLAIM. $\forall^\infty n\ K^n_x \cap a_n \ne \emptyset$.

Proof. If $K^n_x \cap a_n = \emptyset$, then $K_x \cap a_n \subseteq \bigcup_{m<n} K^m_x$, so $|K_x \cap a_n| \le nN$. It follows that if $\exists^\infty n\ K^n_x \cap a_n = \emptyset$, then $\exists^\infty n\ |K_x \cap a_n| \le nN$. Thus $F_x|(\omega \setminus K_x)$ is meager, which is absurd. ■

(3) (Cf. [T], Thm. 5.3(iii).) Let $X \in H_Z$ and $Z \in G_Y$. Let $F \in \mathcal{F} \cap \mathcal{M}^Y$ be defined by $\{U_i \times O_i : i \in \omega\}$. Use Lemma 4.2 to find a dense set $\{r_n : n \in \omega\} \subseteq T \setminus F[Z]$. Let U^n_i be U_i if $r_n \in O_i$ and $\emptyset$ otherwise. Then for each n, the U^n_i's cover Z. Since $X \in H_Z$, find $a \in \omega^\omega$ with $X \subseteq \bigwedge_n \bigcup_{i<a_n} U^n_i$. Define $O^n = \bigcap\{O_i : r_n \in O_i,\ i < a_n\}$. Then $r_n \in O^n$ and $\forall x \in X\ \forall^\infty n\ O^n \cap F_x = \emptyset$. It follows that $\bigvee_n O^n$ is a dense $\mathbf{G}_\delta$ set disjoint from $F[X]$.

(4) Let $X \in H_Z$, $Z \in E_Y$. Let $F \in \mathcal{F} \cap \mathcal{N}^Y$ be defined by $\{U_i \times O_i : i \in \omega\}$. As $F[Z] \in \mathcal{N}$, find an increasing sequence of compact sets $C^n \subseteq T \setminus F[Z]$ such that $\mu(T \setminus \bigcup_n C^n) = 0$. For n and finite $K \subseteq \omega$ let U^n_K be $\bigcap_{i\in K} U_i$ if $C^n \subseteq \bigcup_{i \in K} O_i$ and $\emptyset$ otherwise. Then for each n, the U^n_K's cover Z (C^n are compact!). Since $X \in H_Z$, find finite $\mathcal{K}_n$ with $X \subseteq \bigwedge_n \bigcup_{K \in \mathcal{K}_n} U^n_K$. Let $O^n = \bigcap_{K \in \mathcal{K}_n} \bigcup_{i \in K} O_i$. Then $C^n \subseteq O^n$ and $\forall x \in X\ \forall^\infty n\ O^n \cap F_x = \emptyset$. It follows that $\bigwedge_n (T \setminus O^n)$ is a null $\mathbf{F}_\sigma$ cover of $F[X]$. ■

4.7. Corollary. *$G^+ = C'' \cap H$. Further, $G^+ \subseteq E^+$; for finite-dimensional sets, G^+ and E^+ are equivalent.*

Proof. Clearly, $G^+ \subseteq G$. By Proposition 4.6, $G^+ \subseteq H$ and $H \cap G \subseteq G^+$. Also, by Corollary 4.5, $G = C''$. Thus, $G^+ = H \cap C''$. Next, as $C'' \subseteq E$, we have $G^+ \subseteq H \cap E$. But $H \cap E \subseteq E^+$ by Proposition 4.6. So, $G^+ \subseteq E^+$. If $Y \in E^+$ is finite-dimensional, then $Y \in H \cap E$ by Proposition 4.6. As for finite-dimensional sets, E and C'' are equivalent, we get $Y \in H \cap C''$, hence $Y \in G^+$. ∎

The following lemma is straightforward.

4.8. Lemma. *If $\mathcal{S}$ is a σ-field then for any property P considered in this section, $X \in P_Y$ iff $X \subseteq Y$ and $X \in P$.*

5. Proofs of Theorems. Let Y be a separable metric space. Set $S = Y$ and $\mathcal{S} = \mathbf{O}(S)$. Theorem 1.2 follows from Propositions 4.3 and 4.4; and Theorem 1.4 from Proposition 4.6. For Theorems 1.1 and 1.3, use Corollaries 4.5 and 4.7. ($Y \in E$ implies that Y is zero-dimensional; see Section 3.)

For the Borel versions, let $Y \subseteq S$, S Polish, and $\mathcal{S} = \mathbf{B}(S)$. Then all subsets of Y are zero-dimensional. In view of Lemma 4.8, the results follow from Corollaries 4.5 and 4.7 and the fact that Borel sets from $\mathcal{M}^Y$ can be covered by $\mathcal{F}_\sigma$ sets from $\mathcal{M}^Y$, and if $Y = S$ then the same applies to $\mathcal{E}^Y$.

6. Notes. (1) T can be any Polish space with no isolated points and a nonzero and nonatomic (i.e., vanishing on points) σ-finite Borel measure μ. We get the same classes of sets. For G_Y and G_Y^σ this follows from the proof of Lemma 4.2.

For E_Y and E_Y^+, given such a space T, there exist a null $\mathbf{F}_\sigma$ set $F \subseteq T$, a countable $Q \subseteq [0,1]$, and a homeomorphism $f : T \setminus F \to [0,1] \setminus Q$ such that a subset of $T \setminus F$ is μ-null iff its image is λ-null, λ being the Lebesgue measure. (Change μ so that null sets are the same and $\mu(T) = 1$. Remove all open null sets. Next remove a countable dense subset C, and for each $c \in C$ and n remove a sphere which is null, has center c and radius $\leq 2^{-n}$. This can be done because the spheres with a fixed center are pairwise disjoint. We have removed a null $\mathbf{F}_\sigma$ set, and the remaining part T' can be identified with the irrationals of $[0,1]$. Define $f : [0,1] \to [0,1]$ by $f(x) = \mu([0,x) \cap T')$. Then f is a homeomorphism and for all $A \subseteq T'$, $\mu^*(A) = \lambda^*(f[A])$.)

For G_Y^+ just note that T contains a dense $\mathbf{G}_\delta$ copy of ω^ω.

(2) Let T be as above.

If $\mathcal{E} \subseteq \mathcal{M}$ (i.e., if open sets have positive measure), then in the definition of E_Y^+, "$F[X] \in \mathcal{E}$" can be replaced by "$F[X] \in \mathcal{M} \cap \mathcal{N}$".

In G_Y^σ (similarly in G_Y) instead of $F[X] \neq T$ we can require that for all nonmeager $B \subseteq T$ with the Baire property, $B \setminus F[X]$ contains a perfect set.

Indeed, let $X \in G^{\sigma}_{Y}$. Let $O \subseteq T$ be nonempty and let D_i, $i \in \omega$, be nowhere-dense subsets of T. Suppose $F = \bigcup_i F_i$, $F_i \in \mathcal{F} \cap \mathcal{M}^Y$. Choose nonempty rectangles $U_\sigma \times O_{\sigma,\tau}$, $\sigma \in \omega^{<\omega}$, $\tau \in 2^{<\omega}$, $|\sigma| = |\tau|$, so that

$$O_{\sigma,\tau} \subseteq O \setminus \bigcup_{i<|\sigma|} D_i \text{ and } \operatorname{diam}(O_{\sigma,\tau}) \le 2^{-|\sigma|};$$

$$U_\sigma \times O_{\sigma,\tau} \text{ is disjoint from each } F_i \text{ for } i < |\sigma|;$$

$$Y \subseteq \bigcup_n U_{\sigma^\frown n};$$

$$\overline{O}_{\sigma^\frown n,\tau^\frown 0} \cup \overline{O}_{\sigma^\frown n,\tau^\frown 1} \subseteq O_{\sigma,\tau};$$

$$O_{\sigma^\frown n,\tau^\frown 0} \cap O_{\sigma^\frown n,\tau^\frown 1} = \emptyset.$$

Let $O_\sigma = \bigcup_\tau O_{\sigma,\tau}$. Each $\bigcap_n O_{s|n}$, $s \in \omega^\omega$, is a perfect subset of $O \setminus \bigcup_i D_i$. To see that some $\bigcap_n O_{s|n}$ is disjoint from $F[X]$ suppose that White plays according to U_σ's. This is not a winning strategy, so $X \subseteq \bigvee_n U_{s|n}$ for some $s \in \omega^\omega$. Now $y \in \bigcap_n O_{s|n}$ yields $\forall x \in X\ \exists^\infty n\ \langle x, y\rangle \in U_{s|n} \times O_{s|n}$, hence $\forall x \in X\ \forall i\ \langle x, y\rangle \notin F_i$.

(3) We call $A \subseteq \omega^\omega$ *diagonalized* (resp. *dominated*) if for some $x \in \omega^\omega$ for all $a \in A$, $\exists^\infty n\ a(n) = x(n)$ (resp. $\forall^\infty n\ a(n) < x(n)$). We call $f : Y \to \omega^\omega$ $\mathcal{S}$*-measurable* if for all n, m there is $V \in \mathcal{S}$ such that $y \in V \Leftrightarrow f(y)(n) = m$. Clearly, if Y is zero-dimensional, then X is in H_Y (resp. C''_Y) iff for all $\mathcal{S}$-measurable $f : Y \to \omega^\omega$, $f[X]$ is dominated (resp. diagonalized).

For $\mathcal{S} = \mathbf{O}(S)$ (resp. $\mathcal{S} = \mathbf{B}(S)$), $\mathcal{S}$-measurable means continuous (resp. Borel). From this, X is in $\widetilde{C}''$ (resp. $\widetilde{H}$) iff all Borel images of X into ω^ω are diagonalized (resp. dominated) iff all Borel images of X into a given Polish space T have property C'' (resp. H). Similarly for zero-dimensional separable metric spaces, properties C'' and H, and continuous images.

(4) Consistently, $S \not\subseteq M$ (even $S(\mathbf{B}(T)) \not\subseteq M(\mathbf{O}(T))$). Shelah ([Sh], Prop. 2.9) has an ω^ω bounding forcing which makes the ground model reals an $S(\mathbf{B}(T))$ set.

(5) The argument of Proposition 4.4 shows that if W is a model of ZFC, $x \in \omega^\omega$ is an unbounded real over W, and the union of closed null sets coded in $W[x]$ is null, then there is a Cohen real over W. Do we really need the intermediate unbounded real?

(6) Clearly, E_Y, E^+_Y and G^+_Y are σ-ideals. So are C''_Y, S_Y, H_Y and M_Y.

For C''_Y and M_Y just split ω into infinitely many infinite sets. (Let $\omega = \bigcup_i K_i$, with the K_i's infinite pairwise disjoint. Let $Y \subseteq \bigcap_n \bigcup_k U^n_k$. If $X_i \in C''_Y$, $i \in \omega$, there exists $a \in \omega^\omega$ such that $\forall i\ X_i \subseteq \bigcup_{n\in K_i} U^n_{a_n}$. Then $\bigcup_i X_i \subseteq \bigcup_n U^n_{a_n}$.)

For H_Y, suppose $X_i \in H_Y$, $i \in \omega$, and $Y \subseteq \bigcap_n \bigcup_k U_k^n$. Find $a^i \in \omega^\omega$ with $X_i \subseteq \bigwedge_n \bigcup_{k<a_n^i} U_k^n$. Let $a_n = \max_{i\le n} a_n^i$. Then $\bigcup_i X_i \subseteq \bigwedge_n \bigcup_{k<a_n} U_k^n$.

For S_Y, let $Y \subseteq \bigvee_k U_k$ and $a \in \omega^\omega\nearrow$. Suppose $X_i \in S_Y$, $i \in \omega$, and let $K_i \in \Phi(a)$ witness this. Set $K = \bigcup_n \bigcup_{i\le n} K_i \cap [a_n, a_{n+1})$. Then $X \subseteq \bigvee_{k\in K} U_k$ and $\forall n\ |K \cap [a_n, a_{n+1})| \le (n+1)2^{a_n}$.

I do not know whether G_Y and G_Y^σ are σ-ideals. (Yes, if $Y \in M_Y$, as then $G_Y = G_Y^\sigma = C''_Y$. Note that if $Y = \bigcup_n X_n$ and $\forall n,\ X_n \in G_Y$, then $\forall n\ X_n \in M_Y$, hence $Y \in M_Y$.)

(7) Assume that S is a separable metric space, sets from $\mathcal{S}$ have the Baire property, $X \subseteq Y$, and all sets meager in X have one of the properties C''_Y, E_Y, S_Y, M_Y. Then X has the respective property. (By σ-additivity, we can replace meager by nowhere-dense.)

We give a proof for S_Y. Let $Y \subseteq \bigvee_k U_k$ and let $a \in \omega^\omega\nearrow$. Consider

$$G = \bigvee_n \bigcup_{a_n\le<a_{n+1}} (U_k \cap X) \times \Big\{t \in \prod_n [a_n, a_{n+1}) : t_n = k\Big\}.$$

For each $t \in \prod_n [a_n, a_{n+1})$, the horizontal section G^t determined by t is covered by $\bigvee_n U_{t_n}$. All vertical sections of G are dense $\mathbf{G}_\delta$ sets. Also, G, as a subset of $X \times \prod_n [a_n, a_{n+1})$, has the Baire property. By the Kuratowski–Ulam theorem, find $t \in \prod_n [a_n, a_{n+1})$ such that $Z = X \setminus G^t$ is meager in X. Then $Z \in S_Y$, so $Z \subseteq \bigvee_{k\in K} U_k$ for some $K \in \Phi(a)$. Let $L = K \cup \operatorname{rng}(t)$. Then $X \subseteq \bigvee_{k\in L} U_k$ and $\forall^\infty n\ |L \cap a_{n+1}| \le n + 2^{a_n}$.

I do not know whether the above is true for G_Y or G_Y^σ. (It is if $X = Y$: if all nowhere-dense subsets of X have property G_X, then $X \in C''$.)

(8) If ν is a σ-finite measure on S, sets from $\mathcal{S}$ are measurable, $X \subseteq Y$, and all null subsets of X have property H_Y, then $X \in H_Y$.

Indeed, without loss of generality, $\nu^*(X) < \infty$ (ν is σ-finite and H_Y is σ-additive). Let $Y \subseteq \bigcap_n \bigcup_k U_k^n$. For each n pick k_n with

$$\nu^*\Big(X \setminus \bigcup_{k\le k_n} U_k^n\Big) < 2^{-n}.$$

Let $Z = X \setminus \bigwedge_n \bigcup_{k\le k_n} U_k^n$. Then Z is null, so $Z \in H_Y$, hence $Z \subseteq \bigwedge_n \bigcup_{k\le l_n} U_k^n$, for some l_n's. It follows that $X \subseteq \bigwedge_n \bigcup_{k\le \max(k_n,l_n)} U_k^n$.

(9) Sierpiński and Lusin sets destroy various extensions of the above. If Y is a Lusin set, then $Y \notin H_Y$, but meager subsets of Y are in $G_Y^+ \cap E_Y^+ \subseteq H_Y$. If Y is a Sierpiński set, then $G_Y^\sigma = G_Y = C''_Y = E_Y$ (Y is zero-dimensional, $Y \in H_Y$ and $H_Y \subseteq M_Y$). Also, $Y \notin E_Y$ (look at the identity function on Y), while null subsets of Y are in E_Y.

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