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# $\mathsf{G}_\delta$-sets in topological spaces and games

by

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**Abstract.** Players ONE and TWO play the following game: In the $n$th inning ONE chooses a set $O_n$ from a prescribed family $\mathcal{F}$ of subsets of a space $X$; TWO responds by choosing an open subset $T_n$ of $X$. The players must obey the rule that $O_n \subseteq O_{n+1} \subseteq T_{n+1} \subseteq T_n$ for each $n$. TWO wins if the intersection of TWO's sets is equal to the union of ONE's sets. If ONE has no winning strategy, then each element of $\mathcal{F}$ is a $\mathsf{G}_\delta$-set. To what extent is the converse true? We show that:

(A) For $\mathcal{F}$ the collection of countable subsets of $X$:

1. There are subsets of the real line for which neither player has a winning strategy in this game.
2. The statement "If $X$ is a set of real numbers, then ONE does not have a winning strategy if, and only if, every countable subset of $X$ is a $\mathsf{G}_\delta$-set" is independent of the axioms of classical mathematics.
3. There are spaces whose countable subsets are $\mathsf{G}_\delta$-sets, and yet ONE has a winning strategy in this game.
4. For a hereditarily Lindelöf space $X$, TWO has a winning strategy if, and only if, $X$ is countable.

(B) For $\mathcal{F}$ the collection of $\mathsf{F}_\sigma$-subsets of a subset $X$ of the real line the determinacy of this game is independent of ZFC.

**1. Definitions and conventions.** A subset of a topological space is a $\mathsf{G}_\delta$-*set* if it is the intersection of countably many open sets; it is an $\mathsf{F}_\sigma$-*set* if it is the complement of a $\mathsf{G}_\delta$-set. Let $\mathcal{F}$ be a family of subsets of a topological

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space such that $A \cup B$ is in $\mathcal{F}$ whenever $A$ and $B$ are, and all one-element subsets are elements of $\mathcal{F}$. Some classes of topological spaces are defined by specifying such a family $\mathcal{F}$ and then requiring that each element of $\mathcal{F}$ is a $\mathsf{G}_\delta$-set. For example:

1. According to K. Kuratowski, a topological space is a *rarified space* (or $\lambda$-*space*) if all its countable subsets are $\mathsf{G}_\delta$-sets [9].
2. According to W. Sierpiński and E. Szpilrajn (Marczewski), a space is a $\sigma$-*space* if every $\mathsf{F}_\sigma$-set is a $\mathsf{G}_\delta$-set.
3. According to F. Rothberger, a space is a $Q$-*space* if every subset is a $\mathsf{G}_\delta$-set [15].
4. A space is *perfect* if every closed subset is a $\mathsf{G}_\delta$-set [18], p. 162. According to E. Čech, a normal space which is perfect is called *perfectly normal* [4].

Since countable operations are involved in defining these concepts, they are susceptible to game-theoretic analysis. We introduce such an analysis by using the game defined in the abstract. For a family $\mathcal{F}$ this game is denoted by $\mathsf{G}(\mathcal{F})$.

A space $X$ has *property* $\mathsf{C}''$ if for every sequence $(\mathcal{U}_n : n \in \omega)$ of open covers of $X$ there is an open cover $(U_n : n \in \omega)$ such that $U_n \in \mathcal{U}_n$ for each $n$. This property was introduced by Rothberger in [14].

The symbol ${}^\omega\omega$ denotes the set of functions from $\omega$ to $\omega$. Define the binary relation $\prec$ on ${}^\omega\omega$ by $f \prec g$ if for all but finitely many $n$, $f(n) < g(n)$. Then $\prec$ is a pre-ordering on ${}^\omega\omega$. When $f \prec g$, we say that "$g$ eventually dominates $f$". A subset $\mathcal{S}$ of ${}^\omega\omega$ is *unbounded* if there is no $g$ such that for each $f \in \mathcal{S}$ we have $f \prec g$. The least cardinality of an unbounded subset of ${}^\omega\omega$ is denoted by $\mathfrak{b}$. A subset $\mathcal{S}$ is *dominating* if for each $g$ there is an $f \in \mathcal{S}$ such that $g \prec f$. The least cardinality of a dominating subset of ${}^\omega\omega$ is denoted by $\mathfrak{d}$. It is well known that $\aleph_1 \leq \mathfrak{b} \leq \mathfrak{d}$ and that there is always a subset of ${}^\omega\omega$ which is of cardinality $\mathfrak{b}$ and well-ordered by $\prec$. When $\mathfrak{b} = \mathfrak{d}$ one has a chain of length $\mathfrak{d}$ which is well-ordered by the eventual domination order and is cofinal in ${}^\omega\omega$. According to Hausdorff such a chain is said to be a $\mathfrak{d}$-*scale*.

Consider $\omega$ as a discrete topological space. Then ${}^\omega\omega$, endowed with the Tychonoff product topology, is homeomorphic to the space of irrational numbers. For a finite sequence $\sigma$ of finite ordinals, $[\sigma]$ denotes the subset $\{f \in {}^\omega\omega : \sigma \subset f\}$ of ${}^\omega\omega$. Subsets of this form constitute a basis for the Tychonoff product topology of ${}^\omega\omega$. We shall also borrow the following notation from Descriptive Set Theory: The collection of closed subsets of a space is denoted by $\Pi^0_1$, while the collection of $\mathsf{F}_\sigma$-subsets is denoted by $\Sigma^0_2$.

Next we recall some notions from [2]: the space $X$ is an $A_1$-*space* if for every Borel function $\Psi : X \to {}^\omega\omega$, $\Psi[X]$ has property $\mathsf{C}''$; it is an $A_2$-*space* if for every Borel function $\Psi$ from $X$ to ${}^\omega\omega$, $\Psi[X]$ is a bounded subset of ${}^\omega\omega$; it is an $A_3$-*space* if for any Borel function $\Psi$ from $X$ to ${}^\omega\omega$, $\Psi[X]$ is not a dominating family.

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## 2. Strategies for player One

LEMMA 1. *If* ONE *does not have a winning strategy in the game* $\mathsf{G}(\mathcal{F})$, *then every element of* $\mathcal{F}$ *is a* $\mathsf{G}_\delta$*-set.*

Proof. Let $A$ be an element of $\mathcal{F}$ and consider the strategy for ONE which calls on ONE to choose $A$ each inning. Since ONE has no winning strategy, this is not a winning strategy. Look at a play which defeats it; the sequence of open sets chosen by TWO during such a play witnesses that $A$ is a $\mathsf{G}_\delta$-set. ■

THEOREM 2. *If* $(X,\tau)$ *is an* $A_3$*-space*, *then the following are equivalent*:

1. *Each element of* $\mathcal{F}$ *is a* $\mathsf{G}_\delta$*-set.*
2. ONE *does not have a winning strategy in the game* $\mathsf{G}(\mathcal{F})$.

Proof. We must prove 1⇒2: Let $\sigma$ be a strategy for ONE. For each $F\in\mathcal{F}$ fix a descending sequence of open sets $(V_n(F): n\in\omega)$ such that $\bigcap_{n\in\omega} V_n(F) = F$. Let $F_\emptyset = \sigma(\emptyset)$ be ONE's opening move in the game. For each $n$, put $F_{(n)} = \sigma(V_{(n)}(F_\emptyset))$. Let $0<k<\omega$ be given, and assume that for each $(n_1,\dots,n_k)$ in ${}^k\omega$ we have defined $V_{(n_1,\dots,n_k)}$ and $F_{(n_1,\dots,n_k)}$ such that

1. $V_{(n_1,\dots,n_{k-1},m)} = V_m(F_{(n_1,\dots,n_{k-1})})$ for each $m<\omega$,
2. $F_{(n_1)} \subseteq F_{(n_1,n_2)} \subseteq \dots \subseteq F_{(n_1,\dots,n_k)}$, and
3. $F_{(n_1,\dots,n_k)} = \sigma(V_\emptyset, V_{(n_1)},\dots,V_{(n_1,\dots,n_k)})$.

Let $F_{(n_1,\dots,n_k,m)}$ be the set $\sigma(V_\emptyset, V_{(n_1)},\dots,V_{(n_1,\dots,n_k)}, V_{(n_1,\dots,n_k,m)})$.

This defines $F_\tau$ and $V_\tau$ for each $\tau$ in ${}^{<\omega}\omega$ such that:

- $F_\tau$ is an element of $\mathcal{F}$ and $V_\tau$ is an open subset of $X$,
- if $\tau$ is extended by $\nu$, then $F_\tau \subseteq F_\nu \subseteq V_\nu \subseteq V_\tau$,
- $F_\tau = \bigcap_{n<\omega} V_{\tau\frown(n)}$, and
- if $m<n$ then $V_{\tau\frown(n)} \subseteq V_{\tau\frown(m)}$.

Defeating the given strategy for ONE amounts to finding a $g\in{}^\omega\omega$ such that

$$\bigcup_{n\in\omega} F_{g|n} = \bigcap_{n\in\omega} V_{g|n}.$$

For this we use the hypothesis that $X$ is an $A_3$-space: For each $x\notin F_\emptyset$, define $f_x$ in ${}^\omega\omega$ as follows:

1. $f_x(0) = \min\{n > 0 : x \notin V_{(n)}\}$.
2. $f_x(n+1)$ is the least $m$ larger than $f_x(n)$ such that

$(\forall i \le f_x(n))(\forall n_1, \ldots, n_i \le f_x(n))(x \notin F_{(n_1,\ldots,n_i)} \Rightarrow x \notin V_{(n_1,\ldots,n_i,m)})$.

The mapping which assigns $f_x$ to $x$ is a Borel mapping from $X$ to ${}^\omega\omega$. Since $X$ is an $A_3$-space, we find an $f$ in ${}^\omega\omega$ such that:

1. $f$ is strictly increasing,
2. $1 < f(0)$, and
3. for each $x \in X$ there are infinitely many $n$ such that $f_x(n) < f(n)$.

For each $n < \omega$ put $g(n) = f^{n+1}(0)$. Then the play

$$(F_\emptyset, V_{g|1}, F_{g|1}, V_{g|2}, F_{g|2}, \ldots)$$

is lost by ONE: For consider a point $x \notin \bigcup_{n<\omega} F_{g|n}$. Pick the smallest positive $n$ such that $f_x(n) < f^n(0)$. Then we have $f^1(0) < \ldots < f^{n-1}(0) \le f_x(n-1)$. We see that $x \notin V_{(f^1(0),\ldots,f^{n-1}(0),f_x(n))}$. But then $x \notin V_{(f^1(0),\ldots,f^n(0))}$, and so we have $x \notin \bigcap_{n<\omega} V_{g|n}$. It follows that $\bigcup_{n<\omega} F_{g|n} = \bigcap_{n<\omega} V_{g|n}$. ∎

We shall later give an example which shows that the hypothesis that the space is an $A_3$-space, though sufficient, is not necessary.

*Lusin sets and Sierpiński sets.* A set of real numbers is said to be a *Lusin set* if it is uncountable but its intersection with every first category set of real numbers is countable. It is well known that Lusin sets have Rothberger's property. It is also well known that if $X$ is a subset of ${}^\omega\omega$ and has Rothberger's property, then there is a $g$ in ${}^\omega\omega$ such that for each $x \in X$ the set $\{n : x(n) = g(n)\}$ is infinite. The following theorem is well known; its proof is included for completeness.

THEOREM 3. *Every Lusin set is an $A_3$-space.*

Proof. Let $X \subset \mathbb{R}$ be a Lusin set, and let $f : X \to {}^\omega\omega$ be a Borel function. Then there is a first category set $K \subset L$ such that $f$ restricted to $L\setminus K$ is continuous ([8], Chapter II, §32.II). But $K$ is a countable subset of $L$ since $L$ is a Lusin set. Then $L\setminus K$ is still a Lusin set and thus has Rothberger's property. Rothberger's property is preserved by continuous images. Thus, $f[L \setminus K]$ is a subset of ${}^\omega\omega$ which has Rothberger's property. Since $f[K]$ is countable, it also has Rothberger's property, and so $f[X]$ has Rothberger's property. Then there is a $g$ in ${}^\omega\omega$ such that for each $x \in X$ there are infinitely many $n$ such that $f(x)(n) = g(n)$. This $g$ is not eventually dominated by any $f(x)$. ∎

COROLLARY 4. *If $X$ is a Lusin set and $\mathcal{F}$ is a collection of subsets of $X$, then the following are equivalent*:

1. *Each element of $\mathcal{F}$ is a $\mathsf{G}_\delta$-set.*
2. ONE *does not have a winning strategy in $\mathsf{G}(\mathcal{F})$.*

A set of real numbers is said to be a *Sierpiński set* if it is uncountable but its intersection with every set of Lebesgue measure zero is countable. Sierpiński showed that the Continuum Hypothesis implies that Sierpiński sets exist [16]. The following theorem is well known (see [6]).

THEOREM 5. *Every Sierpiński set is an* $A_2$*-set.*

COROLLARY 6. *For a collection* $\mathcal{F}$ *of subsets of a Sierpiński set* $X$, *the following are equivalent*:

1. *Each element of* $\mathcal{F}$ *is a* $\mathsf{G}_\delta$*-set.*
2. ONE *does not have a winning strategy in* $\mathsf{G}(\mathcal{F})$.

**I.** $\mathcal{F}$ *is the collection of finite subsets of* $X$. If a space is a $\lambda$-space, then every countable subset (and in particular, every finite subset) is a $G_\delta$-set. This implies that $X$ is a $T_1$-space.

THEOREM 7. *If* $X$ *is a first countable* $\lambda$*-space*, *then* ONE *does not have a winning strategy in* $\mathsf{G}([X]^{<\aleph_0})$.

Proof. Let $\sigma$ be a strategy for ONE. Using the assumption that $X$ is a $\lambda$-space, for each countable subset $C$ of $X$ we choose a descending sequence of open sets, $(G_n(C) : n < \omega)$, with intersection equal to $C$. For finite $C$ we assume that $(G_n(C) : n < \omega)$ is a base at $C$ in $X$. Define sets $F_\tau$ and $G_\tau$ for $\tau \in {}^{<\omega}\omega$ by recursion over the length of $\tau$ as follows:

$$G_\emptyset = X, \quad F_\tau = \sigma(G_{\tau|0}, G_{\tau|1}, \ldots, G_\tau), \quad G_{\tau\frown m} = G_m(F_\tau).$$

Let $C = \bigcup_{\tau \in {}^{<\omega}\omega} F_\tau$, a countable subset of $X$. We may assume that $C$ is infinite (the case when $C$ is finite is even easier). Enumerate $C$ bijectively as $\{c_k : k < \omega\}$.

Here is how TWO defeats ONE's strategy $\sigma$: Let $m_1$ be the minimal $m$ such that $c_m \notin F_\emptyset$. Then choose $n_1$ so large that $c_{m_1} \notin G_{(n_1)}$, and $G_{(n_1)} \subset G_1(C)$ (we used $T_1$ here, and it is used similarly in the rest of the selection of the $n_i$'s). Then let $m_2$ be the least $m$ such that $c_m \in G_{(n_1)} \setminus F_{(n_1)}$, and choose $n_2$ so large that $c_{m_2} \notin G_{(n_1,n_2)}$, and $G_{(n_1,n_2)} \subset G_2(C)$. Continuing in this manner we find two infinite sequences $(m_1, m_2, \ldots)$ and $(n_1, n_2, \ldots)$ such that for each $k$,

1. $m_k$ is the least $m$ such that $c_m \in G_{(n_1,\ldots,n_{k-1})} \setminus F_{(n_1,\ldots,n_{k-1})}$, and
2. $n_k$ is so large that $c_{m_k} \notin G_{(n_1,\ldots,n_k)}$, and $G_{(n_1,\ldots,n_k)} \subset G_k(C)$.

Then $\bigcup_{0<k<\omega} F_{(n_1,\ldots,n_k)} = \bigcap_{0<k<\omega} G_{(n_1,\ldots,n_k)}$, and so ONE lost this play. ■

The assumption of first countability in the previous theorem is essential. In Section II.2 we will construct a $\lambda$-space on which ONE has a winning strategy in the game $\mathsf{G}([X]^{<\aleph_0})$.

Every first countable $T_1$-space which is also an $A_2$-space is a $\lambda$-space. Thus Theorem 7 includes all first countable $A_2$-spaces. By Theorem 2 we could have assumed that $X$ is an $A_3$-space instead of a $\lambda$-space. This is not subsumed by Theorem 7, because it is consistent that there are first countable $A_3$-spaces which are not $\lambda$-spaces: According to Besicovitch a set $X$ of real numbers is *concentrated* on a countable set $Y$ if for every open set $U$ which contains $Y$, $X \setminus U$ is countable [3]. In some sense, sets concentrated on a countable subset of itself are opposites of $\lambda$-sets. For example a Lusin set is an $A_3$-set which is concentrated on each of its countable dense subsets.

Some hypothesis besides first countability is needed in Theorem 7: According to Szpilrajn [21] a set $X$ of real numbers is said to have *property* $s_0$ if for every perfect set $P$ of real numbers there is a perfect set $Q$ such that $Q \subset P$ and $Q \cap X = \emptyset$.

THEOREM 8. *If $X$ is a set of real numbers for which* ONE *does not have a winning strategy in $G([X]^{<\aleph_0})$, then $X$ has property $s_0$.*

Proof. Let $P$ be a perfect subset of $\mathbb{R}$. If $X \cap P$ is not dense in $P$, then pick an open $U$ such that $U \cap P \neq \emptyset$ and $U \cap P \cap X = \emptyset$, and let $Q$ be a perfect subset of $U \cap P$. If $X \cap P$ is dense in $P$, then we need to find a perfect $Q \subset P$ such that $Q \cap X = \emptyset$. Consider the following strategy $\sigma$ for player ONE: In the first inning, $\sigma$ chooses $x_1 \in X \cap P$. In the $n$th inning suppose ONE has chosen $\{x_1, \ldots, x_{k-1}\}$. Let $U$ be any open set containing this set of points (typically, $U$ is TWO's response). Since $X \cap P$ is dense in $P$ and $P$ has no isolated points, we find for every $l < k$ two points $x_{k+2l}$ and $x_{k+2l+1}$ in $(x_l - 2^{-n}, x_l + 2^{-n}) \cap X \cap P \cap U$ such that the points $x_1, \ldots, x_{k-1}, x_k, \ldots, x_{3k-1}$ are all distinct. The strategy $\sigma$ chooses the set $\{x_1, \ldots, x_{3k-1}\}$ in the $n$th inning as the response to TWO's move $U$.

Since $\sigma$ is not a winning strategy for ONE, consider a $\sigma$-play

$$O_1, T_1, O_2, T_2, \ldots, O_n, T_n, \ldots$$

which is lost by ONE. Write $\bigcup_{n<\infty} O_n = \{x_1, x_2, \ldots\}$. Then $\bigcup_{n<\infty} O_n$ is a subset of $P$ and is dense in itself. Thus, $O = \mathrm{cl}(\bigcup_{n\in\omega} O_n)$ is perfect. Moreover, for each $n$, $T_n \cap O$ is a dense open subset of $O$; hence $\bigcap_{n<\infty} T_n \cap O$ is comeager in $O$ and so contains a perfect subset $Q$ that is disjoint from the countable set $\{x_0, x_1, \ldots\}$. Since $X \cap \bigcap_{n<\infty} T_n = \{x_1, x_2, \ldots\}$, we see that $Q \cap X = \emptyset$. Since $Q \subset O \subset P$, we are done. ∎

Sets of real numbers where ONE does not have a winning strategy in this game share some of the properties of $\lambda$-sets, but not all. For example: A set of real numbers is *perfectly meager* if its intersection with each perfect set is meager in the relative topology of that perfect set. Every $\lambda$-set is perfectly meager (cf. [11], pp. 118–119), and every perfectly meager set has property $s_0$.

COROLLARY 9. *If $X$ is a Lusin set, then*

(a) *$X$ is not meager (and hence not perfectly meager), and*
(b) ONE *has no winning strategy in* $\mathsf{G}([X]^{<\aleph_0})$.

Proof. Let $X$ be a Lusin set. Then $X$ is an $A_3$-space (Theorem 3), and as it is a metric space, each finite subset of it is a $\mathsf{G}_\delta$-set. Then Theorem 2 implies that ONE has no winning strategy in $\mathsf{G}([X]^{<\aleph_0})$. Being a Lusin set, $X$ is not meager and thus not perfectly meager. ∎

**II.** *$\mathcal{F}$ is the collection of countable subsets of $X$.* Our results so far show that an $A_3$-space $X$ is a $\lambda$-space if and only if ONE does not have a winning strategy in the game $\mathsf{G}([X]^{\le\aleph_0})$. We explore this a little further.

**II.1.** *The $A_3$-hypothesis is sufficient but not necessary*

THEOREM 10. *Let $\kappa$ be an infinite ordinal. For every subspace $X$ of ${}^\omega\omega$ which is of the form $\{f_\alpha : \alpha < \kappa\}$, where $f_\alpha \prec f_\beta$ when $\alpha < \beta$,* ONE *does not have a winning strategy in the game* $\mathsf{G}([X]^{\le\aleph_0})$.

Proof. ONE not having a winning strategy in $\mathsf{G}([X]^{\le\aleph_0})$ is hereditary so we may assume that $\mathrm{cof}(\kappa) > \omega$. Fix a strategy $\sigma$ for player ONE in the game. We define for each $F \in [X]^{\le\aleph_0}$, each open $U \supset F$, each $\Gamma \in [X]^{<\aleph_0}$, and each $n \in \omega$ an open set $U_{\Gamma,U,n}(F)$ containing $F$ and contained in $U$ as follows. For each $f \in F$ fix $n_f > n$ minimal so that

(a) For each $g \in \Gamma$, if $f \prec g$ then $f(l) < g(l)$ for each $l \ge n_f$.
(b) For each $g \in \Gamma$, if $g \prec f$ then $g(l) < f(l)$ for each $l \ge n_f$.
(c) $[f|n_f] \subset U$.

Now let

$$U_{\Gamma,U,n}(F) = \bigcup_{f\in F} [f|n_f + 1].$$

It is worth noting that $g(n_f) \ne f(n_f)$ whenever $f \in F$ and $g \in \Gamma \setminus F$. Therefore

(d) If $g \in \Gamma$ and $g \notin F$, then $g \notin U_{\Gamma,U,n}(F)$ for open $U$ and $n \in \omega$.

Next we choose a countable $G \subset X$ on which we may effectively restrict our play of the game. Fix a countable $G \subset X$ satisfying the following: Suppose $(F_0, U_0, \ldots, F_{m-1}, U_{m-1})$ is a play of the game such that

- $\sigma(F_0, U_0, \ldots, F_{i-1}, U_{i-1}) = F_i$ for each $i < m$, and
- $U_i = U_{\Gamma,U_{i-1},n}(F_{i-1})$ for each $i < m$ and for some finite $\Gamma \subset G$ and $n \in \omega$ (both depending on $i$).

Then

(e) $\sigma(F_0, U_0, \ldots, F_{n-1}, U_{n-1}) \subset G$, and
(f) if $\beta = \sup\{\alpha < \beta : f_\alpha \in F_i\}$ then $f_\beta \in G$.

In particular, $F_0 = \sigma(\emptyset) \subset G$. $G$ can be constructed by a simple closing off argument (or by letting $G = \mathcal{M} \cap X$ for $\mathcal{M}$ an appropriate elementary submodel).

Let $A = \{\alpha \in \kappa : f_\alpha \in G\}$ and let $B$ be the closure of $A$ in $\kappa$. Then $A$ and $B$ are countable subsets of $\kappa$. We now describe how Two should play to defeat One. Enumerate $A$ as $\{\alpha_i : i \in \omega\}$ and enumerate $B \setminus A$ as $\{\beta_i : i \in \omega\}$ (if $B \setminus A$ is finite the proof is the same).

Consider a partial play of the game $(F_0, U_0, \ldots, F_{m-1}, U_{m-1}, F_m)$. For each $i < m$ the set $\{\alpha < \beta_i : f_\alpha \in F_m\}$ is bounded below $\beta_i$ (since $\beta_i \notin A$). Therefore there is an $\alpha(\beta_i) \in A \cap \beta_i$ such that $\alpha < \alpha(\beta_i)$ whenever $\alpha < \beta_i$ and $f_\alpha \in F_m$. Let

$$\Gamma_m = \{f_{\alpha_i} : i \le m\} \cup \{f_{\alpha(\beta_i)} : i \le m\}.$$

Let $n(m) > m$ be large enough so that for any $f_\gamma, f_\delta \in \Gamma_m \cup \{f_{\beta_i} : i \le m\}$ if $\gamma < \delta$ then $f_\gamma(l) < f_\delta(l)$ for each $l > n(m)$. Two then plays the open set

$$U_m = U_{\Gamma_m, U_{m-1}, n(m)}(F_m).$$

Now suppose that $(F_0, U_0, \ldots, F_m, U_m, \ldots)$ is a play of the game where One uses the strategy $\sigma$ and Two responds as described above.

Claim 11. Two *wins the play* $(F_0, U_0, \ldots, F_m, U_m, \ldots)$.

Proof. Fix $f \in \bigcap_{m \in \omega} U_m$.

Case 1: $f \in G$. Then there exists an $i \in \omega$ such that $f = f_{\alpha_i}$. So $f \in \Gamma_{i+1}$. If $f \notin F_{i+1}$, we would deduce by (d) that $f \notin U_{i+1}$. Therefore $f \in \bigcup F_m$.

Case 2: $f \notin G$. Fix $\gamma$ such that $f = f_\gamma$. Let $\beta = \max B \cap \gamma + 1$ ($B$ is closed in $\kappa$) and let $\alpha = \min A \setminus \gamma$. We only consider the most difficult case where $\alpha$ exists and where $\beta \in B \setminus A$. We therefore have $\beta < \alpha$, and there are $i, j \in \omega$ such that $\beta = \beta_i$ and $\alpha = \alpha_j$. Fix $m \in \omega$ large enough so that $i, j < m$ and so that

(g) $f_\beta(l) \le f(l) < f_\alpha(l)$ for each $l > m$.

Claim 12. $f \notin U_m$.

Proof. Otherwise there is a $\delta$ such that $f_\delta \in F_m$ and $f(i) = f_\delta(i)$ for each $i < n_{f_\delta} + 1$. (Recall that $U_m = \bigcup_{f \in F_m} [f | n_f + 1]$). Suppose that $\delta < \beta_i$; then there is an $\alpha(\beta_i) \in \Gamma_m$ such that $\delta < \alpha(\beta_i) < \beta_i$. By (a) and (g) and the fact that $m_{f_\delta} > m$, we have

$$f_\delta(l) < f_{\alpha(\beta_i)}(l) < f_{\beta_i}(l) \le f(l) \quad \text{for each } l > m_{f_\delta}.$$

In particular, $f_\delta(m_{f_\delta}) < f_{\beta_i}(m_{f_\delta}) \le f(m_{f_\delta})$; this shows that $f(i) = f_\delta(i)$ does not hold for each $i < m_{f_\delta} + 1$. In the case where $\delta > \alpha$ the proof is similar. ∎

Theorem 10 generalizes the well-known fact (already noted on p. 128 of [9]) that the $\prec$-well-ordered subsets of ${}^\omega\omega$ are $\lambda$-sets and thus perfectly meager sets ([10] and [11]).

Corollary 13. *If $\mathfrak{d} = \mathfrak{b}$, then there is a set $X$ of real numbers which is not an $A_3$-space, and yet* One *does not have a winning strategy in the game* $\mathsf{G}([X]^{\leq\aleph_0})$.

The Continuum Hypothesis or Martin's Axiom, each a statement which is consistent with classical mathematics, implies that $\mathfrak{b}$ is equal to $\mathfrak{d}$.

Lemma 1 gives for a subset $X$ of the real line

"One does not have a winning strategy in $\mathsf{G}([X]^{\leq\aleph_0}) \Rightarrow X$ is a $\lambda$-set".

While the converse of this implication is false for Tychonoff spaces, for sets of reals the converse is independent of ZFC. One direction is straightforward from the result of A. Miller that in the Cohen model every $\lambda$-set of real numbers is of size $< 2^{\aleph_0}$ (see Theorem 9.8 in [13]). Since every dominating family in this model is of size $2^{\aleph_0}$ it follows that every $\lambda$-set of real numbers is $A_3$ in the Cohen model. Therefore we have the following corollary to Theorem 2:

Corollary 14. *In the model obtained by adding $> 2^{\aleph_0}$ many Cohen reals to a model of set theory the following are equivalent for a set $X$ of reals*:

1. *$X$ is a $\lambda$-set.*
2. One *does not have a winning strategy in the game* $\mathsf{G}([X]^{\leq\aleph_0})$.

We shall now show:

(I) There exists a $\lambda$-space such that One has a winning strategy in $\mathsf{G}([X]^{\leq\aleph_0})$.

(II) The Continuum Hypothesis implies that there is a $\lambda$-set $X$ of real numbers such that One has a winning strategy in $\mathsf{G}([X]^{\leq\aleph_0})$.

**II.2.** *A $\lambda$-space for which* One *has a winning strategy in the game* $\mathsf{G}([X]^{<\aleph_0})$. The next example shows that the property of One not having a winning strategy in $G([X]^{\leq\aleph_0})$ is strictly stronger than being a $\lambda$-space. Also, this example shows that the assumption of first countability in Theorem 7 is essential.

Theorem 15. *Let $\kappa, \nu$ be uncountable cardinals such that $\kappa^{<\nu} = \kappa$. Then there exists a zero-dimensional $T_1$-space $(X, \tau)$ with $|X| = \kappa$ such that every subset of $X$ of cardinality $\nu$ is a $G_\delta$-set and* One *has a winning strategy in the game $G([X]^{<\aleph_0})$. In particular, there exists a zero-dimensional $T_1$ $\lambda$-space $(X, \tau)$ with $|X| = \mathfrak{c}$ such that* One *has a winning strategy in the game $G([X]^{<\aleph_0})$.*

P r o o f. Let $\kappa, \nu$ be as in the assumptions. The underlying set of the space will be the initial ordinal $\kappa$. In order to construct $\tau$, we will define for each $Y \in [\kappa]^{<\nu}$ a function $g_Y : \kappa \to \omega + 1$. For $Y \in [\kappa]^{<\nu}$ and $m \in \omega$ let

$$G_{Y,m} = g_Y^{-1}([m, \omega]) \quad \text{and} \quad H_{Y,m} = g_Y^{-1}\{m\}.$$

Let $\tau$ be the topology generated by the subbase

$$\{G_{Y,m} : Y \in [\kappa]^{<\nu},\ m \in \omega\} \cup \{H_{Y,m} : Y \in [\kappa]^{<\nu},\ m \in \omega\}.$$

We will require that for all $Y \in [\kappa]^{<\nu}$ and $x \in X$,

(1) $g_Y(x) = \omega$ if and only if $x \in Y$.

Moreover, for all countable subsets $\mathcal{Y}$ of $[\kappa]^{<\nu}$ and all functions $f : \mathcal{Y} \to \omega$ we will require that

(2) $|\bigcap_{Y \in \mathcal{Y}} H_{Y,f(Y)}| = \kappa$.

In order to construct the family $\{g_Y : Y \in [\kappa]^{<\nu}\}$, fix an enumeration $(f_\beta : \beta < \kappa)$ of all functions that map a countable subset of $[\kappa]^{<\nu}$ into $\omega$ such that each of these functions appears $\kappa$ times in the enumeration. Now let us define the $g_Y$'s. If $\beta \in Y$, then $g_Y(\beta) = \omega$. This will take care of (1). If $\beta \notin Y$ and $Y$ is in the domain of $f_\beta$, then let $g_Y(\beta) = f_\beta(Y)$. Since our assumptions imply that the union of each countable subset of $[\kappa]^{<\nu}$ has fewer than $\kappa$ elements, the latter clause ensures (2). Finally, if $\beta \notin Y$ and $Y \notin \mathrm{dom}(f_\beta)$, then let $g_Y(\beta) = 0$.

It is not hard to see that the space $(X, \tau)$ just defined is zero-dimensional and that every set $Y \in [\kappa]^{<\nu}$ is equal to $\bigcap_{m\in\omega} G_{Y,m}$, and is thus a $\mathsf{G}_\delta$-set. Moreover, considering $Y = \{x\}$, we see that the space is $T_1$.

It remains to define a winning strategy $\sigma$ for player ONE. For sets $A$ and $B$ let $\mathsf{Fn}(A, B)$ denote the set of all finite partial functions from $A$ to $B$. For $p \in \mathsf{Fn}([\kappa]^{<\nu}, \omega)$, let

$$W_p = \bigcap\{H_{Y,p(Y)} : Y \in \mathrm{dom}(p)\} \quad \text{and} \quad \mathcal{W} = \{W_p : p \in \mathsf{Fn}([\kappa]^{<\nu}, \omega)\}.$$

Each $W_p$ is a clopen subset of $X$, and (2) implies in particular that

(3) each element of $\mathcal{W}$ has size $\kappa$.

Moreover, it follows immediately from the definition of the topology that

(4) every nonempty open set in $X$ contains an element of $\mathcal{W}$ as a subset.

Fix $\alpha_0 \in X$ and let ONE's opening play of the game be $\sigma(\emptyset) = \{\alpha_0\}$. Suppose that

$$(\{\alpha_0\}, U_0, \{\alpha_0, \alpha_1\}, \ldots, \{\alpha_0, \ldots, \alpha_{n-1}\}, U_{n-1})$$

is a partial play of the game and that $p_0, \ldots, p_{n-1} \in \mathsf{Fn}([\kappa]^{<\nu}, \omega)\}$ are such that

(a) $W_{p_i} \subseteq U_i$ for all $i < n$,
(b) for each $i < j < n$, $p_j$ extends $p_i$.

Then fix $\alpha_n \in W_{p_{n-1}}$ so that $\alpha_n \notin \{\alpha_i : i < n\}$ and let

$$\sigma(\{\alpha_0\}, U_0, \{\alpha_0, \alpha_1\}, \ldots, \{\alpha_0, \ldots, \alpha_{n-1}\}, U_{n-1}) = \{\alpha_i : i \le n\}.$$

Suppose that Two responds with the open set $U_n$. We must still show how to define $p_n$ preserving the properties (a) and (b). Fix a basic open set $V$ containing $\alpha_n$ such that $V \subseteq U_n$. Then

$$V = \Big(\bigcap_{Y\in F_0} H_{Y,n_y}\Big) \cap \Big(\bigcap_{Y\in F_0} G_{Y,n_y}\Big)$$

for some finite disjoint sets $F_0, F_1 \subseteq [\kappa]^{<\nu}$ and sequence of integers $\{n_y : Y \in F_0 \cup F_1\}$. Notice that if $Y \in \operatorname{dom}(p_{n-1}) \cap F_0$ then $p_{n-1}(Y) = n_y$ and if $Y \in \operatorname{dom}(p_{n-1}) \cap F_1$ then $p_{n-1}(Y) \ge n_y$. Let $p_n = p_{n-1} \cup \{(Y, n_Y) : Y \in (F_0 \cup F_1) \setminus \operatorname{dom}(p_{n-1})\}$. Clearly, $p_n$ satisfies (a) and (b). Therefore if we let $f = \bigcup_{n\in\omega} p_n$ then

$$\bigcap_{Y\in\mathcal{Y}} H_{Y,f(Y)} = \bigcap_{n\in\omega} W_{p_n} \subseteq \bigcap U_n$$

and by construction $\bigcap_{Y\in\mathcal{Y}} H_{Y,f(Y)}$ is of size $\kappa$. Therefore $\sigma$ is a winning strategy for player One. ■

**II.3.** *A $\lambda$-set of real numbers where* One *has a winning strategy*. In this section we assume the Continuum Hypothesis. We partition $\omega$ into countably many pairwise disjoint infinite subsets $(a_n : n < \omega)$. For each $n$ we let $\mathbb{P}_n = \mathsf{Fn}(\omega, a_n)$. Partially order elements of $\mathbb{P}_n$ by the order $<_n$ so that if $p$ and $q$ are elements of $\mathbb{P}_n$, then $p <_n q$ if $p$ extends $q$.

Let $(\mathsf{M}_\eta : \eta < \omega_1)$ be a sequence of elementary submodels of $(\mathsf{H}_{\omega_2}, \in)$ such that:

1. for all $\eta < \omega_1$, $\mathsf{M}_\eta \in \mathsf{M}_{\eta+1}$,
2. $\{(a_n : n < \omega), \omega, {}^{<\omega}\omega\} \subseteq \mathsf{M}_0$, and
3. $\bigcup_{\eta\in\omega_1} \mathsf{M}_\eta \supseteq [\omega_1]^{\aleph_0}$.

For each subset $X$ of ${}^\omega\omega$ define what we call a *Skolem strategy* $\sigma_X$ for One in $\mathsf{G}([X]^{\le\aleph_0})$ as follows:

S.1. $O_1 = \sigma_X(\emptyset) = X \cap \mathsf{M}_0$;
S.$n$. Assume open subsets $T_1 \supseteq \ldots \supseteq T_n$ of ${}^\omega\omega$ are given such that

$$O_1 \subseteq \sigma_X(T_1) \subseteq \ldots \subseteq \sigma_X(T_1, \ldots, T_{n-1}) \subseteq X \cap T_n.$$

First choose the least $\varrho_{n+1} < \omega_1$ such that $\varrho_{n+1}$ is a limit ordinal and:

1. $T_n \in \mathsf{M}_{\varrho_{n+1}}$, and
2. $\sigma(T_1, \ldots, T_{n-1}) \subseteq \mathsf{M}_{\varrho_{n+1}}$.

Then define

$$\sigma_X(T_1, \ldots, T_n) = X \cap T_n \cap \mathsf{M}_{\varrho_{n+1}+\omega}.$$

The latter set will also be denoted by $X_n$. We shall construct an $X$ such that $\sigma_X$ is a winning strategy for ONE. $X$ will be of the form $\{f_\eta : \eta < \omega_1\}$, where the $f_\eta$'s will be selected recursively. Along with selecting the $f_\eta$'s we shall also select terms of an $\omega_1 \times \omega$ matrix $(H^n_\eta : n < \omega,\ \eta < \omega_1)$ such that each $H^n_\alpha$ is a function from $a_n$ to $\omega$. For further reference define for each $\eta < \omega_1$ and for each $n < \omega$ the set

$$F^n_\eta = \{f \in {}^\omega\omega : f \cap H^n_\eta = \emptyset\}.$$

In the course of this construction we need to consider sequences which are potential plays by player TWO and which are legitimate candidates for Skolem strategies. To this end fix an enumeration

$$(((W^n_\xi : n < \omega), (\varrho^\xi_{n+1} : n \in \omega)) : \xi \in \omega_1 \cap \mathbf{LIM})$$

of all pairs such that $(W^n_\xi : n \in \omega)$ is a nonincreasing sequence of open subsets of ${}^\omega\omega$ and $(\varrho^\xi_{n+1} : n \in \omega)$ is an increasing sequence of countable ordinals such that for each $n$, $W^n_\xi$ is in $\mathsf{M}_{\varrho^\xi_{n+1}}$ and $\varrho^\xi_{n+1} < \xi$.

We require that for each $\eta$:

R.1. $(f_\eta, \{H^n_\eta\}_{n\in\omega}) \in \mathsf{M}_{\eta+1}$;

R.2. For each $n$ and $f \in {}^\omega\omega \cap \mathsf{M}_\eta$, the sets $\{m \in a_n : f(m) = H^n_\eta(m)\}$ and $\{m \in a_n : f(m) \neq H^n_\eta(m)\}$ are both infinite;

R.3. $(\forall \beta \geq \eta)(\exists n)(f_\beta \cap H^n_\eta = \emptyset)$;

R.4. $(\forall H \in [\eta]^{<\aleph_0})(\forall \pi : H \to \omega)(\{f_{\eta+k} : k \in \omega\} \cap \bigcap_{\alpha\in H} F^{\pi(\alpha)}_\alpha$ is dense in $\bigcap_{\alpha\in H} F^{\pi(\alpha)}_\alpha)$;

R.5. If $\eta$ is a limit and $(W^n_\eta)_{n\in\omega}$ is a possible sequence of TWO's moves in a game where ONE follows $\sigma_X$ and $(\varrho^\eta_{n+1})_{n\in\omega}$ is the corresponding sequence of $\varrho_{n+1}$'s, then $f_\eta \in \bigcap_{n\in\omega} W^n_\eta$.

PROPOSITION 16. *If* R.1 *through* R.5 *are satisfied*, *then* $X$ *is a* $\lambda$*-set.*

Proof. Observe that each $F^n_\eta$ is a closed set, whence its complement $G^n_\eta$ is open. By R.3, $X \cap (\bigcap_{n<\omega} G^n_\eta) \subseteq \{f_\delta : \delta < \eta\}$, and by R.2, $\bigcap_{n<\omega} G^n_\eta \cap X \supseteq \{f_\delta : \delta \leq \eta\}$. ■

PROPOSITION 17. *If there is a sequence* $(f_\eta, \{H^n_\eta\}_{n\in\omega})$ *so that* R.1–R.5 *are satisfied*, *then* $\sigma_X$ *is a winning strategy for* ONE *in* $\mathsf{G}([X]^{\leq\aleph_0})$.

Proof. Let $(X_n, W_n : n \in \omega)$ be a play of the game where ONE follows $\sigma_X$. Fix $\eta$ such that $W_n = W^n_\eta$ for all $n \in \omega$ and such that $f_\eta \notin \bigcup X_n$. Then R.5 implies that $\bigcap_{n\in\omega} W_n \neq \bigcup_{n\in\omega} X_n$, and hence ONE wins the game. ■

THEOREM 18. *There is a sequence* $(f_\eta, \{H_\eta^n\}_{n\in\omega})$ *so that* R.1–R.5 *are satisfied.*

Proof. Choose for each $\eta < \omega_1$ and $n \in \omega$ functions $H_\eta^n \in M_{\eta+1}$ that are $\mathbb{P}_n$-generic over $M_\eta$. This ensures that R.2 is satisfied.

For reasons that will become apparent later in the proof, we will also make sure that if $\beta_0 < \beta_1 < \ldots < \beta_k < \omega_1$ and $(n_0, \ldots, n_k) \in \omega^{k+1}$, then

$$(H_{\beta_0}^{n_0}, H_{\beta_1}^{n_1}, \ldots, H_{\beta_k}^{n_k}) \text{ is } \mathbb{P}_{n_0} \times \mathbb{P}_{n_1} \times \ldots \times \mathbb{P}_{n_k}\text{-generic over } M_{\beta_0}.$$

We will also require that for all $\beta \in \omega_1 \cap \mathbf{LIM}$ the set $\{f_{\beta+k} : k \in \omega\}$ is dense in ${}^\omega\omega$.

At limit stages $\eta$, consult the pair of sequences $(W_\eta^n : n < \omega)$ and $(\varrho_{n+1}^\eta : n < \omega)$. We may assume that $(W_\eta^n)_{n\in\omega}$ is a possible sequence of TWO's moves in a game where ONE follows $\sigma_X$ and $(\varrho_{n+1}^\eta)_{n\in\omega}$ is the corresponding sequence of $\varrho_{n+1}$'s (otherwise R.5 holds vacuously).

Let $\delta = \sup\{\varrho_k^\eta : k \in \omega\}$. We consider here only the case where $\delta < \eta$; the case $\delta = \eta$ is similar, and even easier. Let $\{\alpha_k : k \in \omega\}$ be a one-to-one enumeration of $\delta$ such that $\alpha_k < \varrho_{k+1}^\eta$ for every $k \in \omega$, and let $\{\beta_k : k \in \omega\}$ be a one-to-one enumeration of $\eta\backslash\delta$. We will construct recursively a sequence $(s_k)_{k\in\omega}$ of functions in ${}^{<\omega}\omega$ and a sequence $(n_k)_{k\in\omega}$ of natural numbers such that

(a) $s_k \subseteq s_{k+1}$ for all $k \in \omega$;
(b) $[s_k](= \{g \in {}^\omega\omega : s_k \subset g\}) \subset W_\eta^k$;
(c) $\min a_{n_k} > \max \operatorname{dom}(s_k)$ for all $k \in \omega$;
(d) $s_k \cap H_{\alpha_l}^{n_l} = \emptyset$ and $s_k \cap H_{\beta_l}^{n_l} = \emptyset$ for all $l \le k < \omega$.

To get the construction started, consider $W_\eta^0$. Since this set contains $X \cap M_0$, and since the extra requirement on the $f_{\beta+k}$'s mentioned at the beginning of this proof insures that $X$ is dense in ${}^\omega\omega$, $W_\eta^0$ is also a dense subset of ${}^\omega\omega$. Choose $s_0$ such that $[s_0] \subset W_\eta^0$. Then choose $n_0$ such that $\min a_{n_0} > \max \operatorname{dom}(s_0)$. Note that $F_{\alpha_0}^{n_0} \cap F_{\beta_0}^{n_0} \cap [s_0]$ is nonempty and perfect.

Having constructed $(s_i : i \le k)$ and $(n_i : i \le k)$ such that (a)–(d) hold, notice that the closed set $[s_k] \cap \bigcap_{i\le k} F_{\alpha_i}^{n_i}$ is nonempty, and hence perfect. By R.4, $\{f_{\varrho_{k+m}^\eta} : m \in \omega\}$ contains a dense subset $D$ of $[s_k] \cap \bigcap_{i\le k} F_{\alpha_i}^{n_i}$, and by (b), $D \subseteq W_\eta^k$. Since ONE plays $X \cap W_\eta^k \cap M_{\varrho_k^\eta+\omega}$ in inning number $k+1$, and since $W_\eta^{k+1}$ covers this set, $D$ is also a subset of $W_\eta^{k+1}$. Thus $W_\eta^{k+1} \cap [s_k]$ contains a dense subset of $[s_k] \cap \bigcap_{i\le k} F_{\alpha_i}^{n_i}$. Now we have to use the fact that $W_\eta^{k+1} \in M_\delta$ and $(H_{\beta_0}^{n_0}, H_{\beta_1}^{n_1}, \ldots, H_{\beta_k}^{n_k})$ is $\mathbb{P}_{n_0} \times \mathbb{P}_{n_1} \times \ldots \times \mathbb{P}_{n_k}$-generic over $M_\delta$.

CLAIM. *There exists* $s_{k+1} \supset s_k$ *such that* $[s_{k+1}] \subset W_\eta^{k+1}$ *and* $[s_{k+1}] \cap \bigcap_{i\le k} F_{\alpha_i}^{n_i} \cap \bigcap_{i\le k} F_{\beta_i}^{n_i} \neq \emptyset$.

Proof. Let $p = (p_{n_0}, p_{n_1}, \ldots, p_{n_k}) \in \mathbb{P}_{n_0} \times \mathbb{P}_{n_1} \times \ldots \times \mathbb{P}_{n_k}$ be any condition such that $p \Vdash [s_k] \cap \bigcap_{i\le k} F^{n_i}_{\beta_i} \neq \emptyset$, i.e., such that $\mathrm{dom}(p_{n_i}) \subseteq \mathrm{dom}(s_k) \cap a_{n_i}$ and $p_{n_i} \cap s_k = \emptyset$ for all $i \le k$. We want to show that there are $q \le p$ and $s_{k+1} \supset s_k$ such that $[s_{k+1}] \subset W^{k+1}_\eta$ and $q \Vdash [s_{k+1}] \cap \bigcap_{i\le k} F^{n_i}_{\alpha_i} \cap \bigcap_{i\le k} F^{n_i}_{\beta_i} \neq \emptyset$. Let $m \in \omega$ be such that $\mathrm{dom}(p_{n_i}) \subseteq m$ for all $i \le k$, and let $t_k \in {}^m\omega$ be such that $s_k \subseteq t_k$ and $t_k(j) \neq p_{n_i}(j)$ and $t_k(j) \neq H^{n_i}_{\alpha_i}(j)$ for all $i \le k$ and all eligible $j$. Then $[t_k] \subseteq [s_k]$ and $[t_k] \cap \bigcap_{i\le k} F^{n_i}_{\alpha_i} \neq \emptyset$. Since $W^{k+1}_\eta \cap [s_k]$ is dense in $[s_k] \cap \bigcap_{i\le k} F^{n_i}_{\alpha_i}$, there exists $s_{k+1} \supseteq t_k$ such that $[s_{k+1}] \subset W^{k+1}_\eta$ and $[s_{k+1}] \cap \bigcap_{i\le k} F^{n_i}_{\alpha_i} \neq \emptyset$. Let $M = \mathrm{dom}(s_{k+1})$, and let $q_{n_i} = p_{n_i} \cup \{(j, s_{k+1}(j)+1) : j \in M \cap a_{n_i} \setminus \mathrm{dom}(p_{n_i})\}$. Define $q = (q_{n_0}, q_{n_1}, \ldots, q_{n_k})$. Then $q \le p$, and $q \Vdash [s_{k+1}] \cap \bigcap_{i\le k} F^{n_i}_{\alpha_i} \cap \bigcap_{i\le k} F^{n_i}_{\beta_i} \neq \emptyset$, as required.

Note that $s_{k+1} \cap H^{n_i}_{\alpha_i} = \emptyset = s_{k+1} \cap H^{n_i}_{\beta_i}$ for all $i \le k$. Choose $n_{k+1}$ such that $\min a_{n_{k+1}} > \max \mathrm{dom}(s_{k+1})$. Then $s_{k+1} \cap H^{n_{k+1}}_{\alpha_{k+1}} = \emptyset = s_{k+1} \cap H^{n_{k+1}}_{\beta_{k+1}}$.

Letting $f_\eta = \bigcup_{k\in\omega} s_k$, R.3 and R.5 are satisfied as required.

It only remains to construct $\{f_{\eta+k} : k > 0\}$ so that this set is dense in ${}^\omega\omega$ and R.4 holds. This is easily done by fixing an appropriate enumeration of pairs of finite subsets of $\eta$ and basic open sets in ${}^\omega\omega$ and defining $f_{\eta+k}$ recursively using the following lemma:

Lemma 19. *If $S$ is a finite subset of $\omega_1$ and $\pi : S \to \omega$ is a function, then $\bigcap_{\eta\in S} F^{\pi(\eta)}_\eta$ is a nonempty subspace of ${}^\omega\omega$ without isolated points.* ∎

**III.** *Other examples*. The special case of our game for which $\mathcal{F}$ is $\Pi^0_1$, the collection of closed subsets of $X$, is naturally associated with the notion of a perfect space. Since $\Pi^0_1$ is not closed under countable unions and since typically the set of points covered by One during a play is a countable union of closed sets, one would expect that our game is not an accurate instrument for detecting whether a space is perfect or not. We have only partial results in this direction. According to Theorem 2 we know the following:

Proposition 20. *If $X$ is an $A_3$-space, then the following are equivalent*:

1. *$(X,\tau)$ is a perfect space.*
2. *One does not have a winning strategy in the game $\mathsf{G}(\Pi^0_1)$.*

In particular, we see that if $X$ is a Lusin set of real numbers (and thus an $A_3$-set, by Theorem 3) then One does not have a winning strategy in the game $\mathsf{G}(\Pi^0_1)$. Lusin sets are not $\sigma$-sets.

For $T_1$-spaces, if One has a winning strategy in the game $\mathsf{G}([X]^{<\aleph_0})$, then One has a winning strategy in the game $\mathsf{G}(\Pi^0_1)$. We see from Theorem 8 that One has a winning strategy in the game $\mathsf{G}(\Pi^0_1)$ played on any set of real numbers which does not have property $s_0$. This shows that in Proposition 20 some hypothesis, perhaps weaker than being an $A_3$-space, is needed.

A perfect space which is also an $A_2$-space is a $\sigma$-space in which ONE does not have a winning strategy in the game $\mathsf{G}(\Pi^0_1)$. It is not clear if the hypothesis that $X$ is a $\sigma$-space plays as important a role in identifying the spaces for which ONE does not have a winning strategy in the game $\mathsf{G}(\Pi^0_1)$, as did the hypothesis that $X$ be a $\lambda$-space (in Theorem 7) in identifying spaces where ONE did not have a winning strategy in the game $\mathsf{G}([X]^{<\aleph_0})$.

PROBLEM 1. *Is there a $\sigma$-space for which* ONE *has a winning strategy in the game* $\mathsf{G}(\Pi^0_1)$?

The instance of our game when $\mathcal{F}$ is the set of $\mathsf{F}_\sigma$-sets is naturally associated with the notion of a $\sigma$-space, and is denoted by $\mathsf{G}(\Sigma^0_2)$. From our earlier results we deduce the following:

THEOREM 21. *Let $X$ be an $A_3$-space. Then the following are equivalent*:

1. *$X$ is a $\sigma$-space.*
2. ONE *does not have a winning strategy in* $\mathsf{G}(\Sigma^0_2)$.

When $\mathcal{F}$ is the power set of $X$ we obtain an instance of our game which is naturally associated with the notion of a $Q$-space. We also have very limited information about the exact relation of our game to $Q$-spaces.

LEMMA 22. *For a $Q$-space $X \subseteq \mathbb{R}$, the following are equivalent*:

1. *$X$ is an $A_3$-set.*
2. $|X| < \mathfrak{d}$.

Applying Theorem 2, we see that:

THEOREM 23. *For a set $X \subseteq \mathbb{R}$ such that $|X| < \mathfrak{d}$, the following are equivalent*:

1. *$X$ is a $Q$-space.*
2. ONE *does not have a winning strategy in* $\mathsf{G}(\mathcal{P}(X))$.

It is not clear that the cardinality hypothesis is really needed:

PROBLEM 2. *Could there be a subset of the real line which is a $Q$-space, and for which* ONE *has a winning strategy in* $\mathsf{G}(\mathcal{P}(X))$?

PROBLEM 3. *Is there a $Q$-space $X$ for which* ONE *has a winning strategy in* $\mathsf{G}(\mathcal{P}(X))$? *Or even in* $\mathsf{G}([X]^{\leq\aleph_0})$?

**3. Strategies for player Two.** Every $Q$-space is a $\sigma$-space and every $T_1$ $\sigma$-space is a $\lambda$-space. Similarly, if TWO has a winning strategy in the game $\mathsf{G}(\mathcal{P}(X))$, then TWO has a winning strategy in $\mathsf{G}(\Sigma^0_2)$. If $X$ is a

$T_1$-space, then the latter implies that Two has a winning strategy in the game $\mathsf{G}([X]^{\leq\aleph_0})$.

THEOREM 24. *If* $(X,\tau)$ *is a hereditarily Lindelöf space, then the following are equivalent*:

1. $X$ *is countable.*
2. Two *has a winning strategy in the game* $\mathsf{G}([X]^{\leq\aleph_0})$.
3. Two *has a winning strategy in the game* $\mathsf{G}([X]^{<\aleph_0})$.

Proof. The proof that 1 implies 2 and that 2 implies 3 is easy. We show that the negation of 1 implies the negation of 3. Assume that $X$ is uncountable. Let $\sigma$ be a strategy for Two. For $\tau \in {}^{\omega}\omega \setminus \{\emptyset\}$, define by recursion over the length of $\tau$ a set $C_\tau \in [X]^{<\aleph_0}$ as follows: First pick $\{C_{(n)} : n \in \omega\}$ such that $\{\sigma(C_{(n)}) : n \in \omega\}$ is an open cover of $X$. Given $C_\tau$, choose $\{C_{\tau\frown n} : n \in \omega\}$ in such a way that $\{\sigma(C_{\tau|1},\ldots,C_\tau,C_{\tau\frown n}) : n \in \omega\}$ is an open cover of $\sigma(C_{\tau|1},\ldots,C_\tau)$. This is possible since $X$ is hereditarily Lindelöf.

Then the set

$$C = \bigcup_{\tau\in{}^{<\omega}\omega\setminus\{\emptyset\}} C_\tau$$

is a countable subset of $X$. Using the uncountability of $X$ we fix a point $y \in X \setminus C$. Here is how One now defeats Two's strategy $\sigma$: Choose $n_0$ such that $y \in \sigma(C_{(n_0)})$, then choose $n_1$ such that $y \in \sigma(C_{(n_0)}, C_{(n_0,n_1)})$, then choose $n_2$ such that $y \in \sigma(C_{(n_0)}, C_{(n_0,n_1)}, C_{(n_0,n_1,n_2)})$, and so on. We find a sequence $(n_0, n_1, \ldots, n_k, \ldots)$ of integers such that for each $k$ we have:

1. $y \in \sigma(C_{(n_0)},\ldots,C_{(n_0,\ldots,n_k)})$, and yet
2. $y \notin C$.

Thus, Two loses this play. ∎

**4. Undetermined games and Set Theory.** Theorem 10 and Theorem 24 combined show that there is subspace $X$ of the real line such that neither player has a winning strategy in the game $\mathsf{G}([X]^{\leq\aleph_0})$.

It is not possible to show in ZFC that there is a subspace of the real line so that the corresponding game $\mathsf{G}(\Sigma^0_2)$ is not determined. On the one hand, A. W. Miller has proved in [12] that it is relatively consistent with ZFC that every subset of the real line which is a $\sigma$-space is countable. Under these circumstances we see that for a set $X$ of real numbers One has a winning strategy in $\mathsf{G}(\Sigma^0_2)$ if, and only if, $X$ is uncountable; Two has a winning strategy in the game $\mathsf{G}(\Sigma^0_2)$ if, and only if, $X$ is countable. On the other hand, we have:

PROPOSITION 25. *If $X$ is a Sierpiński set of real numbers, then*:

1. $\mathsf{G}([X]^{\leq\aleph_0})$ *is undetermined.*
2. $\mathsf{G}(\Sigma^0_2)$ *is undetermined.*
3. $\mathsf{G}(\Pi^0_1)$ *is undetermined.*

Proof. It follows from Theorem 24 that TWO does not have a winning strategy in any of these games. If ONE has a winning strategy in one of these games, then ONE has a winning strategy in $\mathsf{G}(\Sigma^0_2)$. But by Corollary 6 this would be the case only if there were a $\Sigma^0_2$-subset of a Sierpiński set that is not a $\mathsf{G}_\delta$-set. But suppose that $Y$ is a $\Sigma^0_2$-subset of a Sierpiński set $X$. Then there is a Lebesgue measurable set of reals $Z$ such that $Z \cap X = Y$. Then there is a $\mathsf{G}_\delta$-set $G$ of reals, containing $Z$ and of the same measure as $Z$. Therefore $G \cap X \setminus Y$ is countable and we see that $Y$ is a $\mathsf{G}_\delta$-subset of $X$ (in fact we have shown that for any Lebesgue measurable set $Z$ of reals, $Z \cap X$ is a relative $\mathsf{G}_\delta$-set in $X$). ■

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